Linear Transformation

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Introduction

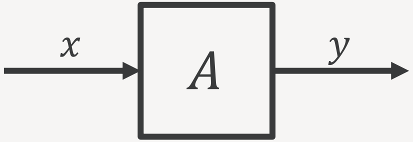
Linear Transformation

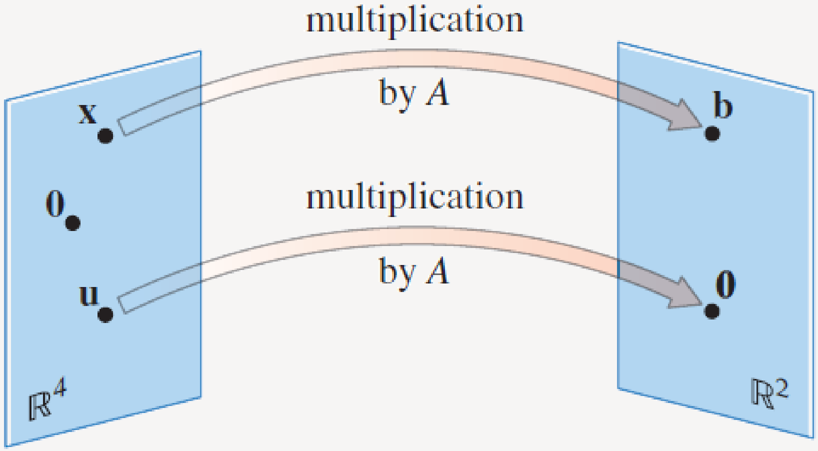
- Matrix is a linear transformation: map one vector to another vector

$$A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m}: \qquad y_{m\times 1} = A_{m\times n}x_{n\times 1}$$

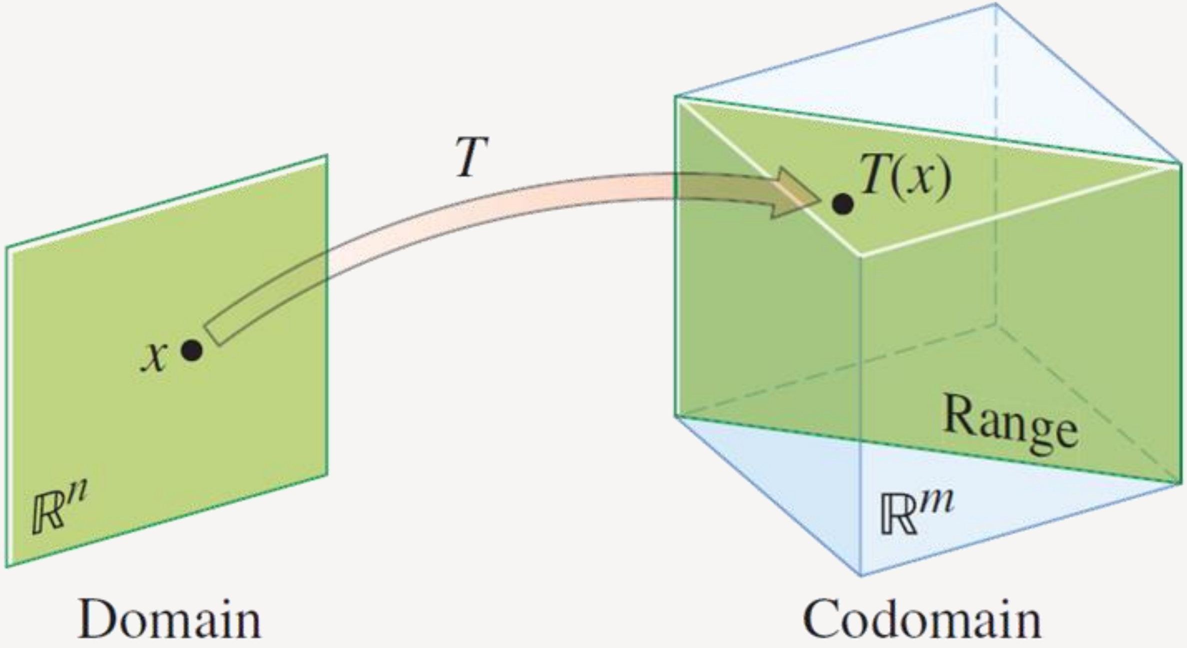
$$A : \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$$

- Input-output





Linear Transformation



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$

02

Linear Transformation (Linear Map)

Linear Mapping

Definition

Let V and W be vector spaces over the field $\mathbb{F}$. A linear transformation (or a linear map) from V into W is a function $\mathbf{T}: V \rightarrow W$ that satisfies following properties for all x, y in V and all scalars a in $\mathbb{F}$:

$$T(x+y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x)$$

Notes

- ❑ $T(0) = 0$
- ❑ Transformation preserves linear combinations

$$T(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1(\mathrm{T}(\mathrm{x}_1)) + \cdots + \alpha_n(\mathrm{T}(\mathrm{x}_\mathrm{n}))$$

Linear Mapping

Notes

- ❑ The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
- ❑ The set of linear maps from V to V is denoted by $\mathcal{L}(V)$.
  In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$

Linear Mapping

Theorem

Let $(v_1, \ldots, v_n)$ be a ordered basis of finite-dimensional vector space V over the field $\mathbb{F}$ and $(w_1, \ldots, w_n)$ an arbitrary list of any vectors in W. If we define following linear map, it is unique.

$$T : V \to W \qquad \text{such that } T(v_i) = w_i.$$

Proof

Linear Mapping

Example

Which are linear mapping?

- ❑ **zero** map $0 : V \to W$
- ❑ **identity** map $I : V \to V$
- ❑ Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be the **differentiation** map defined as $T_{\mathcal{P}(z)} = \acute{\mathcal{P}}(z)$
- ❑ $\boxed{\text{Let } T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ be the map given by } T(x, y) = (x - 2y, 3x + y)}$
- ❑ $T(x) = e^x$
- ❑ $T : \mathbb{F} \to \mathbb{F}$ given by $T(x) = x - 1$

Algebraic Operations on L(V,W)

Definition

Let S and $T \in L(V,W)$ and $\lambda \in \mathbb{F}$. The sum $S+T$ and the product λT are the linear maps from V to W defined by:

$$(S+T)(v) = Sv + Tv \text{ and } (\lambda T)(v) = \lambda(Tv)$$

For all $v \in V$.

Theorem

With the addition and scalar multiplication as defined above, $L(V,W)$ is a vector space.

Proof

Review: Vector Space Properties

- Addition of vector space $(x + y)$

  - **Commutative** $x + y = y + x \; \forall x, y \in V$
  - **Associative** $(x + y) + z = x + (y + z) \; \forall x, y, z \in V$
  - **Additive identity** $\exists \mathbf{0} \in V$ such that $x + \mathbf{0} = x, \forall x \in V$
  - **Additive inverse** $\exists (-x) \in V$ such that $x + (-x) = 0, \forall x \in V$

Review: Vector Space Properties

- Action of the scalars field on the vector space (αx)

  - **Associative** $\alpha(\beta x) = (\alpha\beta)x \quad \forall \alpha, \beta \in F; \forall x \in V$

  - **Distributive over ……**

    scalar addition: $(\alpha + \beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in F; \forall x \in V$

    vector addition: $\alpha(x + y) = \alpha x + \alpha y \quad \forall \alpha \in F; \forall x, y \in V$

  - **Scalar identity** $1x = x \quad \forall x \in V$

Linear Mapping

Definition

Let $T \in L(U, V)$ and $S \in L(V, W)$, then the product $ST \in L(U, W)$ is defined by:

$$(ST)(u) = S(Tu)$$

For all $\mathrm{u} \in U$.

Notes

Note that ST is defined only when T maps into the domain of S. You should verify that ST is indeed a linear map from U to W whenever T $\in \mathcal{L}$(U, V) and S $\in \mathcal{L}$(U, V).

Linear Mapping

Notes

Multiplication of linear maps is not commutative.

Example

$$D \in L\big(P(R)\big) \ \ as \ D\big(P(x)\big) = P'(x)$$

$$T \in L\big(P(R)\big) \ \ as \ T\big(P(x)\big) = x^2 P(x)$$

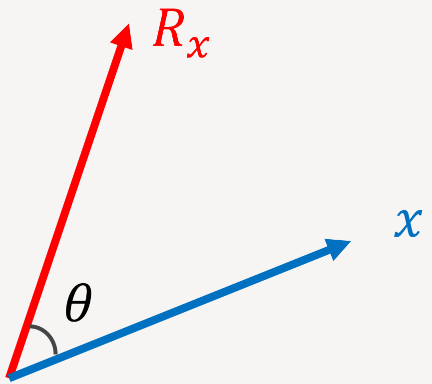
$$TD \neq DT$$

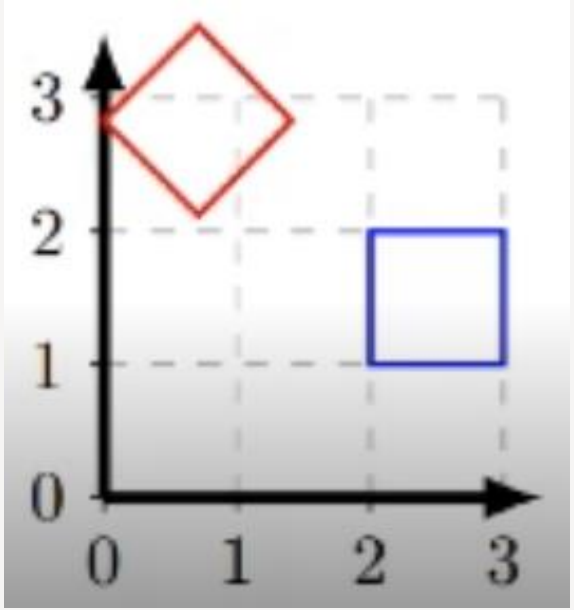
03

Rotation-Projection-Reflection

Rotation with θ degree

- $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Why?



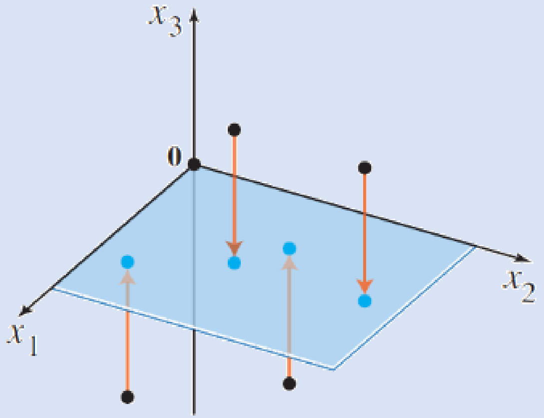


Projection

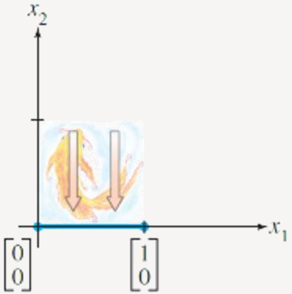
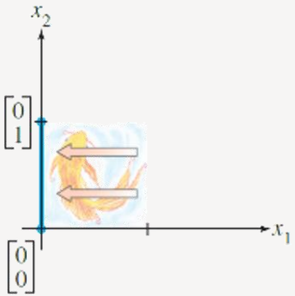
Example

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ projects points in $\mathbb{R}^3$ onto the x_1x_2–plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$



Projection

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1–axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2–axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Projection

Definition

Suppose that V is a vector space and $P : V \rightarrow V$ is a linear transformation.

a) If $P^2 = P$ then P is called a **projection**. Why?

b) If V is an inner product space and $P^2 = P = P^n$ then P is called an **orthogonal projection**.

We furthermore say that P **projects onto** $\text{range}(P)$.

- ❑ Projection of vector v on:
  - ❑ Two orthogonal vectors
  - ❑ Two non-orthogonal vectors

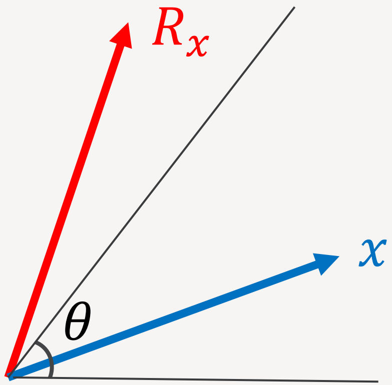
Projection on θ Line

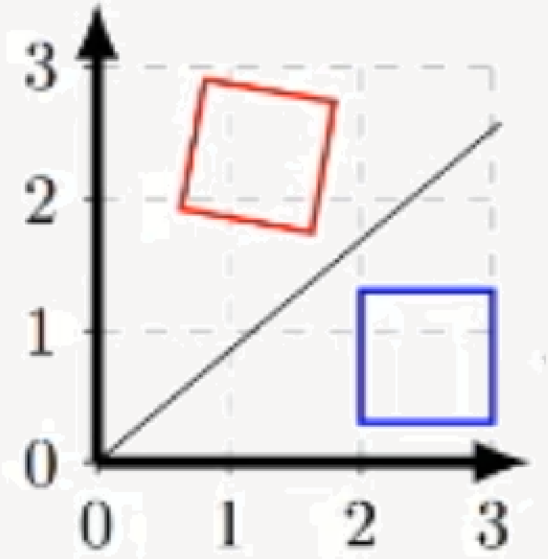
$$P = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$ Why?

$$P^2 = P$$

Reflection in θ Line

- $R = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

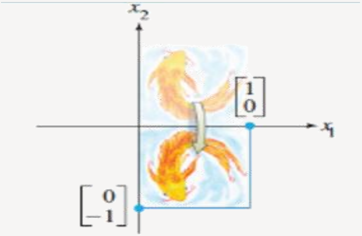
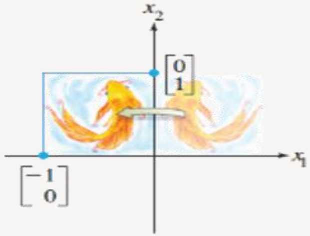
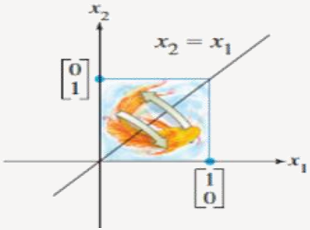




Why?

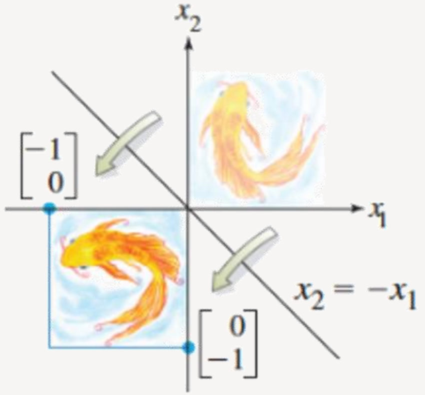
$$R^2 = I$$

Reflection in θ Line

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1–axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2–axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

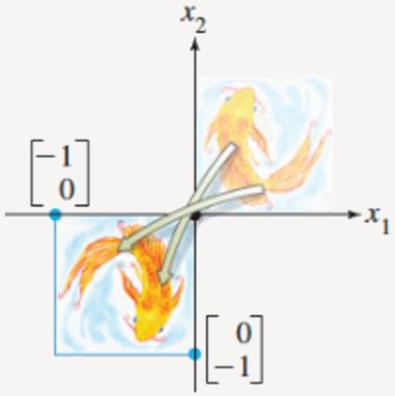
Reflection in θ Line

Reflection through the line $x_2 = -x_1$



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Reflection through the origin

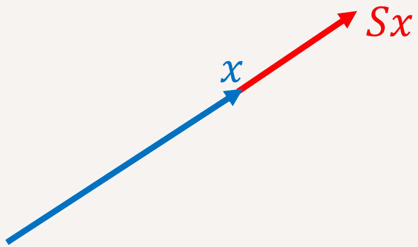


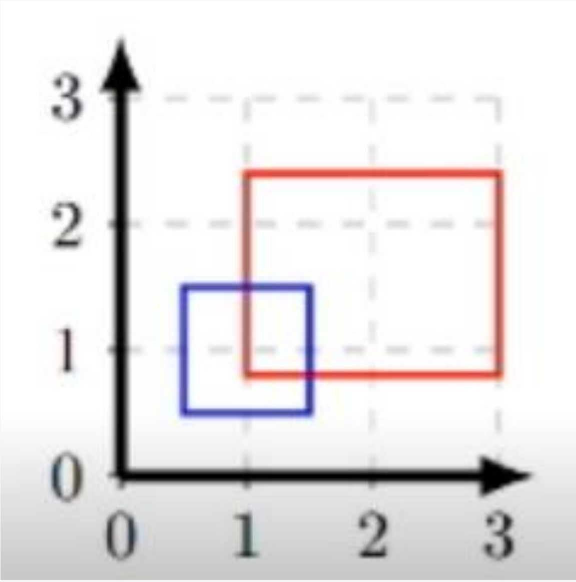
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Applications

Uniform Scaling

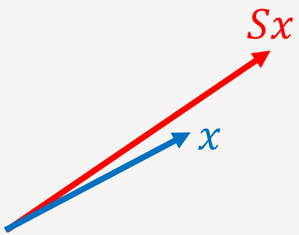
- $S = sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

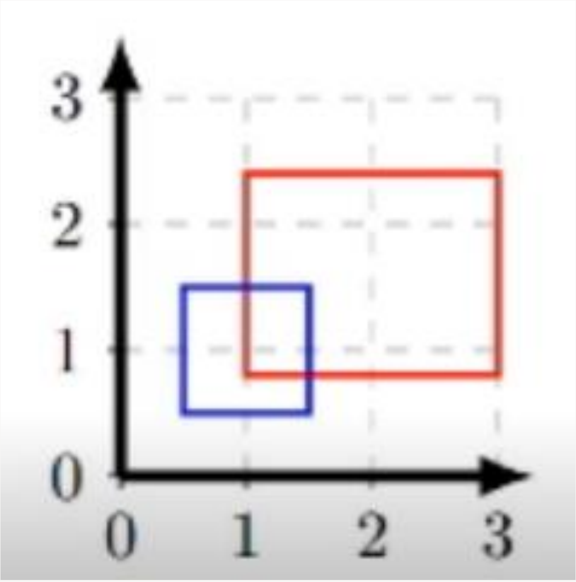




Non-uniform Scaling

- $S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$





Shearing

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

A typical shear matrix is of the form

$$S = \begin{pmatrix} 1 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

sheep

sheared sheep

Shearing

A shear parallel to the x axis results in $\acute{x} = x + \lambda y$ and $\acute{y} = y$.

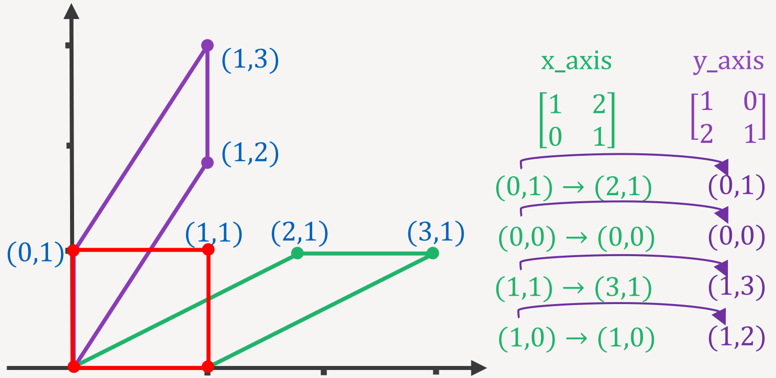
In matrix form:

$$\begin{pmatrix} \acute{x} \\ \acute{y} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Similarly, a shear parallel to the y axis has $\acute{x} = x$ and $\acute{y} = y + \lambda x$.

In matrix form:

$$\begin{pmatrix} \acute{x} \\ \acute{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$



Shearing

Note

$$D_{(n-1)\times n} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

$$D: \mathbb{R}^n \to \mathbb{R}^{n-1} \quad \Rightarrow \quad D\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

Example

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 - 0 \\ 3 - (-1) \\ 2 - 3 \\ 5 - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$$

Selectors

- an m × n selector matrix: each row is a unit vector (transposed)

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix}$$

multiplying by A selects entries of x:

$$Ax = (x_{k_1}, x_{k_2}, \dots, x_{k_m})$$

- $A : \mathbb{R}^n \to \mathbb{R}^m \quad \Rightarrow \quad A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{k_1} \\ x_{k_2} \\ \vdots \\ x_{k_m} \end{bmatrix}$

Selectors

Example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

- ❑Selecting first and last elements of vector:
- ❑Reversing the elements of vector:

Slicing

- Keeping m elements from r to s (m=s-r+1)

$$\begin{bmatrix} 0_{m\times(r-1)} & I_{m\times m} & 0_{m\times(n-s)} \end{bmatrix}$$

Example

- Slicing two first and one last elements:

$$\begin{bmatrix} -1 \\ 2 \\ 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Down Sampling

- Down sampling with k: selecting one sample in every k samples

Example

$\mathrm{K} = 2?$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \end{bmatrix}$$

Applications

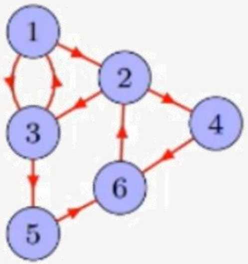
$(i)\ \sin 2A = 2\sin A\cos A$

$(ii)\ \cos 2A = \cos^2 A - \sin^2 A$

- Rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow R^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

- Adjacency matrix



$$A = \begin{array}{c} \begin{array}{cccccc} n1 & n2 & n3 & n4 & n5 & n6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

04

Non-Linear Map

Norms

Is norm a linear map?

- First, the triangle inequality defines: $\| x + y \| \leq \| x \| + \| y \|$. Whereas the first requirement for linear mappings demands: $T(x + y) = T(x) + T(y)$. The problem here is in the ≤ condition, which means adding two vectors and then taking the norm can be less than the sum of the norms of the individual vectors. Such condition is, by definition, not allowed for linear mappings.

- Second, the positive definite defines: $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$. Put simply, norms have to be a positive value. For instance, the norm of $\|-x\| = \|x\|$, instead of $\|-x\|$. But, the second property for linear mappings requires $\|-\alpha x\| = -\alpha \|x\|$. Hence, it fails when we multiply by a negative number (i.e., it can preserve the negative sign).

Translations

Is translation a linear map?

- Translation is a geometric transformation that moves every vector in a vector space by the same distance in a given direction. Translation is an operation that matches our everyday life intuitions: move a cup of coffee from your left to your right, and you would have performed translation in R3 space.
- $T: R^2 \to R^3$

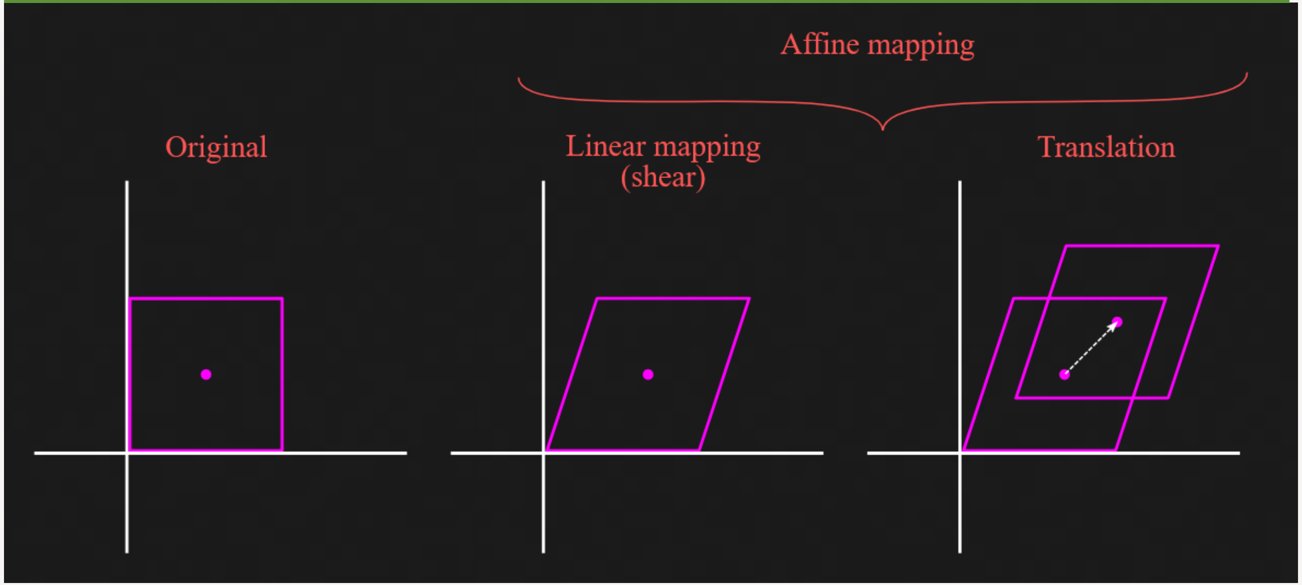
$$T_v = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

Affine Mappings

- linear mapping + translation

$$M(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

where A is a linear mapping or transformation and $\mathbf{b}$ is the translation vector.



05

Null Spaces and Ranges

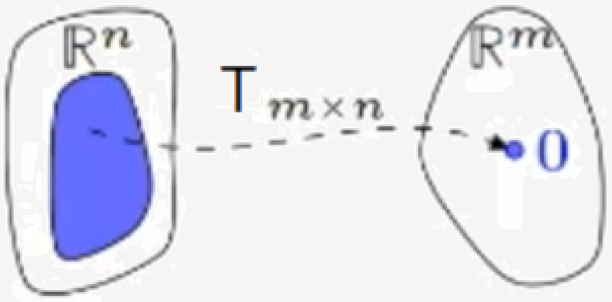
Null Space

Definition

Let $T: V \rightarrow W$ be a linear map. Then the **null space or kernel of T** is the set of all vectors in V that map to zero:

$$N(T) = Null(T) = \{v \in V \mid Tv = 0\}$$

- $Nullity(T) := \ Dim(Null(T))$



Null Space

Theorem

Suppose $T \in L(V, W)$. Then null T is a subspace of V.

Proof

Theorem

Suppose $T \in L(V, W)$. Then null T is vector space.

Null Space

Example

Find Null Space T?

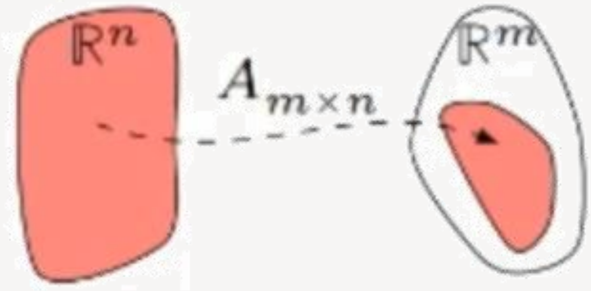
- **zero** map $0 : V \to W$
- Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be the **differentiation** map defined as $T_{\mathcal{P}(z)} = \acute{\mathcal{P}}(z)$
- Let $T : C^3 \to C$ be the map given by $T(x, y, z) = x + 2y + 3z$
- $T(P(x)) = x^2 P(x)$
- $T \in L(\mathbb{F}^\infty)$ given by $T(x_1, x_2, \dots) \to (x_2, x_3, \dots)$
- When is Nullity(T) = 0 ?

Range

Definition

Let $T: V \to W$ be a linear map. Then the **range of T** is the subset of W consisting of those vectors that are equal to Tv for some $v \in V$:

$$range(T) = \{T(v) | v \in V\}$$



Range

Theorem

Suppose $T \in L(V, W)$. Then range T is a subspace of W.

Proof

Theorem

Suppose $T \in L(V, W)$. Then range T is vector space.

Range

Example

Find Range T?

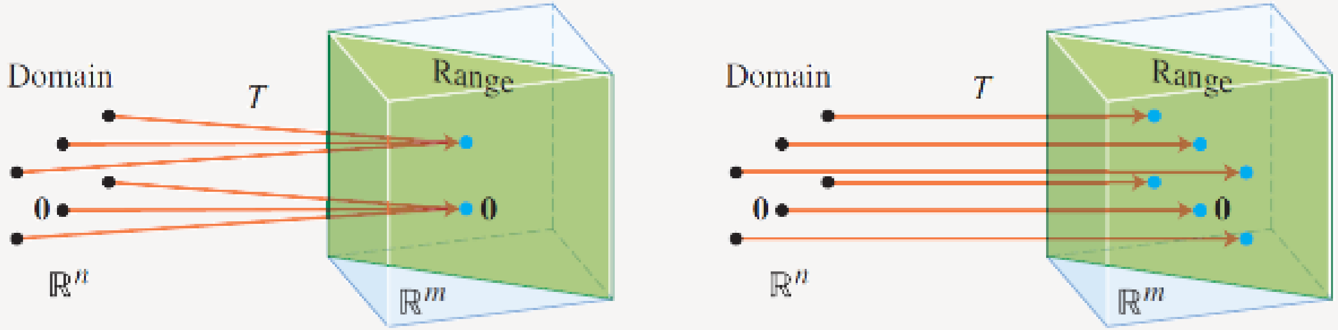
- ❑ **zero** map $0 : V \to W$
- ❑ Let $T : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ be the **differentiation** map defined as $T_{\mathcal{P}(z)} = \acute{\mathcal{P}}(z)$

06

One-to-one (Injective)

One-to-One Mapping

- A mapping T : $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one (injective)** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$



Injective and homogeneous linear

Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation T(x) = 0 has only the trivial solution.

Proof

One-to-One and Null Space

Theorem

Let $T: V \rightarrow W$ be a linear transformation. Then T is one−to−one if and only if the equation Null(T)={0} (Nullity(T)=0!).

Proof

One-to-One and Null Space

Example

Let T be the linear transformation whose standard matrix is

$$A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one–to–one mapping?

One-to-One Linear Transformation

Important

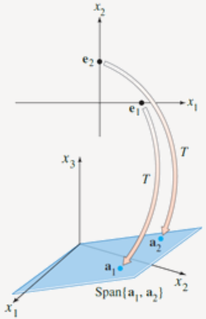
Let $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

b. T is one–to–one if and only if the columns of A are linearly independence.

Example

Let T$(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one–to–one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?



One-to-One Transformations

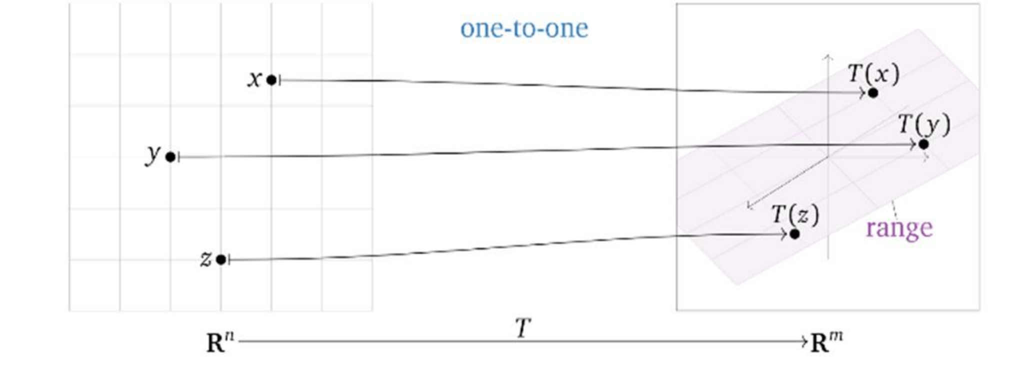
Definition

One-to-one transformations: A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if, for every vector b in $\mathbb{R}^m$, the equation T(x) = b has at most one solution x in $\mathbb{R}^n$.

Remark

Here are some equivalent ways of saying that T is one-to-one:

- For every vector b in $\mathbb{R}^m$, the equation T(x) = b has zero or one solution x in $\mathbb{R}^n$.
- Different inputs of T have different outputs.
- If T(u) = T(v) then u = v.

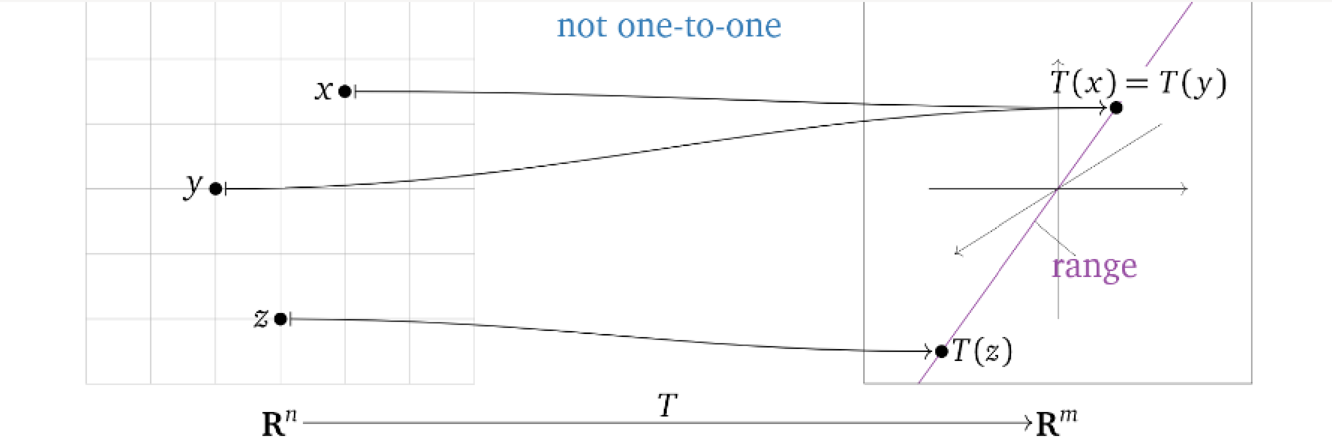


One-to-One Transformations

Remark

Here are some equivalent ways of saying that T is not one–to–one:

- There exist some vector b in $\mathbb{R}^m$ such that the equation T(x) = b has more than one solution x in $\mathbb{R}^n$.
- There are two different inputs of T with the same output.
- There exist vectors u, v such that $u \neq v$ but T(u) = T(v).



One-to-One Transformations

Theorem

Let A be an m $\times$ n matrix and let T(x) = Ax be the associated matrix transformation. The following statements are equivalent:

1. T is one-to-one.
2. For every b in $\mathbb{R}^m$, the equation T(x) = b has at most one solution.
3. For every b in $\mathbb{R}^m$, the equation T(x) = b has a unique solution or is inconsistent.
4. Ax = 0 has only the trivial solution.
5. The columns of A are linearly independent.
6. A has a pivot in every column.
7. The range of T has dimension n.

One-to-One Transformations

Important

Wide matrices do not have one–to–one transformations.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is an one–to–one matrix transformation, what can we say about the relative sizes of n and m?

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every column, it must have at least as many rows as columns:

$n \leq m.$

This says that for instance, $\mathbb{R}^3$ is **too big** to admit a one–to–one linear transformation into $\mathbb{R}^2$.

Note that there exist tall matrices that are not one–to–one, for example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

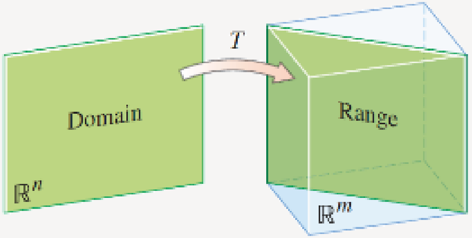
Does not have a pivot in every column.

06

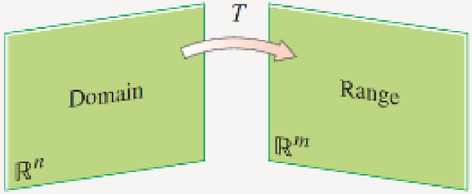
Onto (Surjective) Linear Transformation

Onto Mapping

- A mapping T : $\mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto (surjective)** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in R^n



T is *not* onto $\mathbb{R}^m$



T is onto $\mathbb{R}^m$

Onto (surjective) Transformation

Definition

A transformation $T : V \to W$ is onto if, for every vector b in W, the equation T(x) = b has at least one solution x in V. It range equals W.

Note

Here are some equivalent ways of saying that T is onto:

- The range of T is equal to the codomain of T.
- Every vector in the codomain is the output of some input vector.

Onto (surjective) Transformation

Example

Which one is surjective?

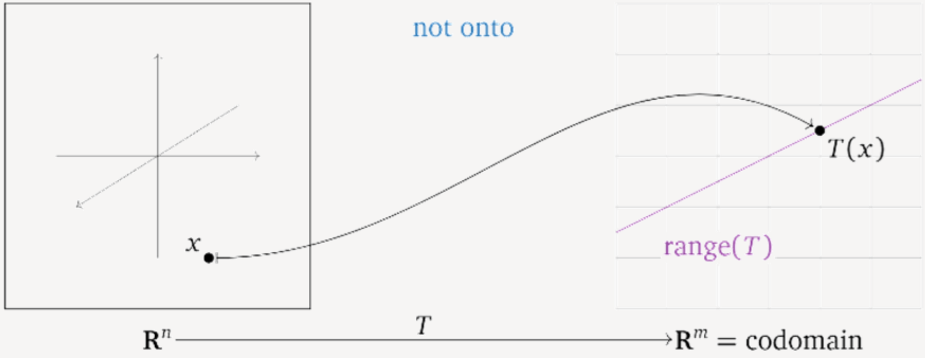
- ❑ $D \in L\big(P_5(R)\big)$ defined by $DP = P'$
- ❑ $S \in L(P_5(R), P_4(R))$ defined by $SP = P'$

Onto (surjective) Transformation

Note

Here are some equivalent ways of saying that T is not onto:

- The range of T is smaller to the codomain of T.
- There exists a vector b in $\mathbb{R}^m$ such that the equation T(x) = b does not have a solution
- There is a vector in the codomain that is not the output of any input vector.



Onto (surjective) Transformation

Theorem

Let A be an $m \times n$ matrix and let T(x) = Ax be the associated matrix transformation. The following statement are equivalent:

- T in onto.
- T(x) = b has at least one solution for every b in $\mathbb{R}^m$.
- Ax = b is consistent for every b in $\mathbb{R}^m$.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot in every row.
- The range of T has dimension m.

Onto (surjective) Transformation

Important

Tall matrices do not have onto transformations.

The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every row, it must have at least as many columns as rows: $m \leq n$.
This says that for instance, $\mathbb{R}^2$ is **too small** to admit an onto linear transformation to $\mathbb{R}^3$**.**
Note that there exist wide matrices that are not onto, for example,

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

Does not have a pivot in every row.

Solution

The reduction row echelon form of A is :

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

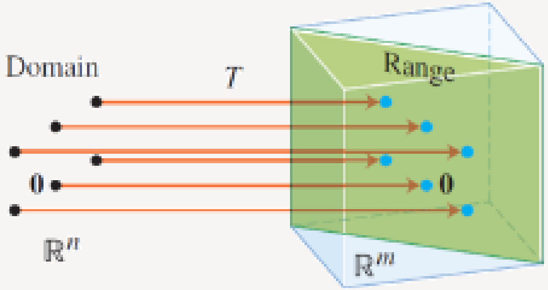
There is not a pivot in every row, so T is not onto. The range of T is the column space of A which is equal to

$$span\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}\right\} = span\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

since all three columns of A are collinear. Therefore, any vector not on the line through $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is not in the range of T. for instance, if b = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then T(x) = b has no solution.

Comparison

A is an m × n matrix, and T: $\mathbb{R}^n \to \mathbb{R}^m$ is the matrix transformation T(x) = Ax.



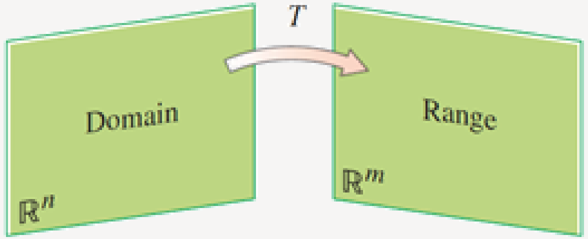
T is one-to-one

$T(x) = b$ has *at most one solution* for every b.

The columns of A are linearly independent.

A has a pivot in every column.

The range of T has dimension n.



T is onto

$T(x) = b$ has *at least one solution* for every b.

The columns of A span $\mathbf{R}^m$.

A has a pivot in every row.

The range of T has dimension m.

One-to-One and Onto

Important

One-to-one is the same as onto for square matrices. We observed that a square has a pivot in every row if and only if it has a pivot in every column. Therefore, a matrix transformation T from $\mathbb{R}^n$ to itself is one-to-one if and only if it is onto : in this case, the two notations are equivalent.

Conversely, by this note, if a matrix transformation T: $\mathbb{R}^m \to \mathbb{R}^n$ is both one-to-one and onto, then
m = n.

Note that in general, a transformation T is both one-to-one and onto if and only if T(x) = b has exactly one solution for all b in $\mathbb{R}^m$.

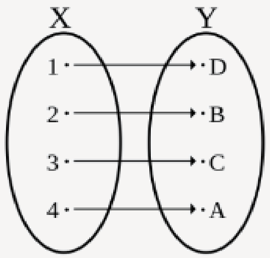
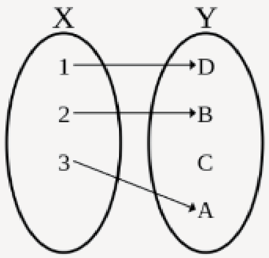
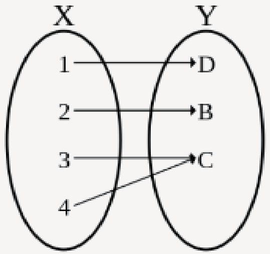
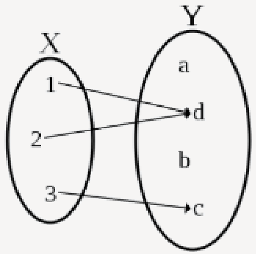
Bijective

Note

- One-to-one and onto.
- If and only if every possible image is mapped to by exactly one argument.

Conclusion onto

One-to-one

	surjective	non-surjective
injective	 bijective	 injective-only
non-injective	 surjective-only	 general

Machine learning application

- The central problem in machine learning and deep learning is to meaningfully transform data; in other words, to learn useful representations of the input data at hand – representations that get us to the expected output.

07

Fundamental Theorem of Linear Maps

dim V = dim null T + dim range

Theorem

Let V be a finite-dimensional vector space and $T \in L(V, W)$. Then rang T is finite-dimensional and

$$\mathrm{Dim}(V) = Nullity(T) + Dim(range(T))$$

Proof

$\dim V = \dim \operatorname{null} T + \dim \operatorname{range}$

Corollary

Linear map to a lower-dimensional space is not injective.

Proof

Corollary

Linear map to a higher-dimensional space is not surjective

Proof

$\dim V = \dim \operatorname{null} T + \dim \operatorname{range}$

Example

Is T injective or not?

$$T: \mathbb{F}^4 \to \mathbb{F}^3$$
$$T(x_1, x_2, x_3, x_4) = (\sqrt{7}x_1 + \pi x_2 + x_4, 97x_1 + 3x_2 + 2x_3, x_2 + 6x_3 + 7x_4)$$

08

Invertible Linear Maps

Invertible, Inverse

Definition

A linear map $T \in \mathrm{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathrm{L}(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.

A linear map $S \in \mathrm{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an inverse of T (note that the first I is the identity operator on V and the second I is the identity operator on W).

Inverse is unique

Theorem

An invertible linear map has a unique inverse.

Definition

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

Example

Find the inverse of $T(x, y, z) = (-y, x, 4z)$

Invertibility

Theorem

A linear map is invertible if and only if it is injective and surjective.

Theorem

Suppose that V and W are finite-dimensional vector spaces, dim V = dim W, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible} \Longleftrightarrow T \text{ is injective} \Longleftrightarrow T \text{ is surjective.}$$

09

Isomorphic

Whether Vector Spaces Are Isomorphic

Definition

- An isomorphism is an invertible linear map.
- Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

Isomorphisms

Definition

Suppose V and W are vector spaces over the same field. We say that V and W are isomorphic, denoted by $V \cong W$, if there exists an invertible linear transformation

T: $V \to W$ (called an isomorphism from V to W).

If T: $V \to W$ is an isomorphism then so is $T^{-1}: W \to V$.

If T: $V \to W$ and S: $W \to X$ are isomorphism then so is $S \circ T: V \to X$.

in particular, if $V \cong W$ and W $\cong X$ then $V \cong X$.

Theorem

Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

Isomorphisms

Example

Show that the vector space V = span(e^x, xe^x, x^2e^x) and $\mathbb{R}^3$ are isomorphic.

The standard way to show that two space are isomorphic is to construct an isomorphism between them. To this end, consider the linear transformation T: $\mathbb{R}^3 \rightarrow V$ defined by

$$T(a,b,c) = ae^x + bxe^x + cx^2e^x.$$

It is straightforward to show that this function is linear transformation, so we just need to convince ourselves that it is invertible. We can construct the standard matrix $[T]_{B\leftarrow E}$, where $E = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$:

$$[T]_{B\leftarrow E} = \big[[T(1,0,0)]_B, [T(0,1,0)]_B, [T(0,0,1)]_B\big]$$

$$= \big[[e^x]_B, [xe^x]_B, [x^2e^x]_B\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $[T]_{B\leftarrow E}$ is clearly invertible (the identity matrix is its own inverse), T is invertible too and is thus an isomorphism.

Resources

- Chapter 1: Advanced Linear and Matrix Algebra, Nathaniel Johnston
- Chapter 6: Linear Algebra David Cherney
- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares