



Eigenvalue – Eigenvector

Department of Computer Engineering

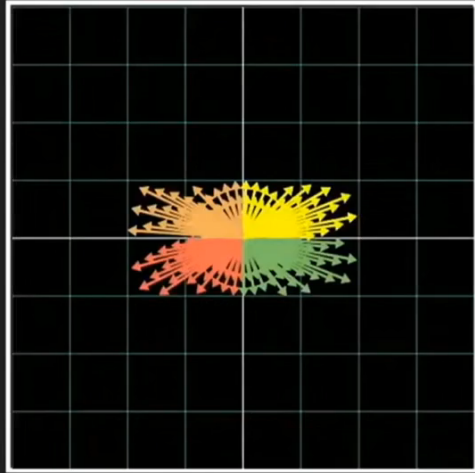
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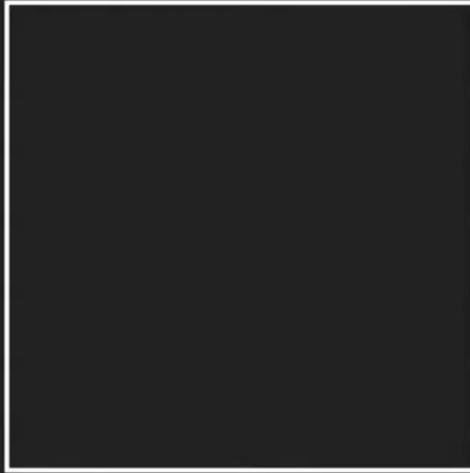
Review

Diagonal Matrix: **Stretching** each axis differently



$$\begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 2.0 \end{bmatrix}$$

vector is arrow



01

Introduction



Motivation

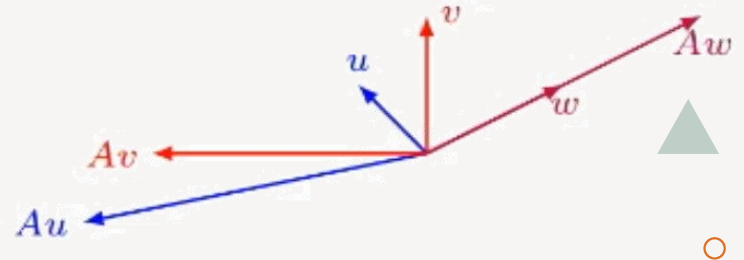
□ $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$

$v = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

$w = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Aw = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

□ Vector “w” keeps the straight, but changes the scale.



Definition

Definition

An **eigenvector** of a square $n \times n$ matrix A is nonzero vector v such that $Av = \lambda v$ for some scalar λ . A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution v of $Av = \lambda v$; such an v is called an *eigenvector corresponding to λ* .

- An eigenvector must be nonzero, by definition, but an eigenvalue may be zero.

Example

□ $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \lambda = 2$

- Show that 7 is an eigenvalue of matrix B, and find the corresponding eigenvectors.

$$B = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

Eigenspace

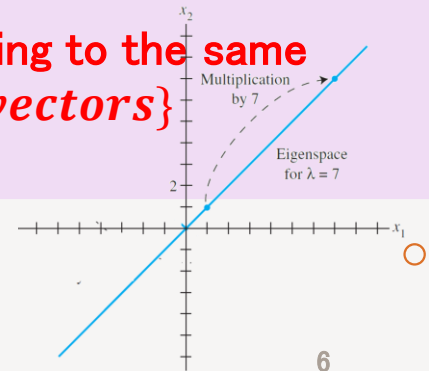
Note

λ is an eigenvalue of an $n \times n$ matrix:

$$Av = \lambda v \Rightarrow (A - \lambda I)v = 0$$

The set of all solutions of above is just the null space of the matrix $A - \lambda I$. So this set is the *subspace of \mathbb{R}^n* and is called the **eigenspace of A corresponding to λ** . The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ .

Eigenspace: A vector space formed by eigenvectors corresponding to the same eigenvalue and the origin point. *span{corresponding eigenvectors}*



Definitions

Note

□ $Av = \lambda v \Rightarrow Av - \lambda vI = 0 \Rightarrow (A - \lambda I)v = 0 \quad v \neq 0$

- $v \in N(A - \lambda I)$
- $A - \lambda I$ must be singular.
- Proof that for finding the eigenvalue we should solve the determinate zero equation. Look at nullspace, rank and nullity theorem, singular matrix, and det zero!

□ **Characteristic polynomial** $\det(A - \lambda I)$

□ **Characteristic equation** $\det(A - \lambda I) = 0$

□ If λ is an eigenvalue of A , then the subspace $E_\lambda = \{\text{span}\{v\} \mid Av = \lambda v\}$ is called the **eigenspace** of A associated with λ . (This subspace contains all the span of eigenvectors with eigenvalue λ , and also the zero vector.)

□ **Eigenvector is basis for eigenspace.**

□ Set of all eigenvalues of matrix is $\sigma(A)$ named **spectrum of a matrix**

Definitions

Note

- Instead of $\det(A - \lambda I)$, we will compute **$\det(\lambda I - A)$** . Why?
 - $\det(A - \lambda I) = (-1)^n \det(\lambda I - A)$
 - Matrix $n \times n$ with real values has eigenvalues.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ , find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X .



Example

Example

Find eigenvalues and eigenvectors, eigenspace (E), and *spectrum* of matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$:

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$\left. \lambda_1 = 1 \right\} \Rightarrow q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left. \lambda_2 = 2 \right\} \Rightarrow q_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvalues={1,2}

Eigenvectors= $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$E_1(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $E_2(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\sigma(A) = \{1, 2\}$

$$AQ = QA \Rightarrow \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

02

Eigenvalues



Expanding the Characteristic equation of A to polynomial form

Theorem

To have (1) scalar for largest degree instead of $|A - \lambda I|$, consider $|\lambda I - A|$

$$f(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0 \quad \text{Proof?}$$

- The n roots of this polynomial are eigenvalues!
 - $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$
- What is c_{n-1} ?
 - $c_{n-1} = -\text{trace}(A)$
- What is c_0 ?
 - $c_0 = \det(-A) = (-1)^n \det(A)$


Sum and Product of eigenvalues

Theorem

If A is an $n \times n$ matrix, then the sum of the n eigenvalues of A is the trace of A .

(coefficient c_{n-1} in expanded characteristic equation)

Other view: $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$


$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

Proof?

Theorem

If A is an $n \times n$ matrix, then the product of the n eigenvalues is the determinant of A .

(coefficient c_0 in expanded characteristic equation)

Proof?

Determinant and Eigenvalue

Theorem

$$0 \in \sigma(A) \Leftrightarrow |A|=0$$

Proof?



Conclusion: The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

- ❑ The number 0 is not an eigenvalue of A .
- ❑ The determinant of A is not zero.



An Important Theorem!

Theorem

The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal. For the diagonal matrix the eigenvectors are e_i s. For upper/lower matrices, Q matrix of $AQ = QA$ will be upper/lower triangular matrix.

Proof?

Real Eigenvalues of different matrices

- Projection matrix
 - 0, 1
 - If $\text{rank}(P)=r$ with n columns, what are the repetition of the eigenvalues?
 - 0: $n-r$ 1: r
- Reflection matrix
 - 1, -1
- Permutation matrix
 - 1, -1

Characteristic Equation

Example

Find the eigenvalues with their repetition and eigenvectors:

□ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

□ The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$.

□ $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

□ $C = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

□ $D = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

Eigenvalues of matrix products

Theorem

The nonzero Eigenvalues of AB equal to the nonzero eigenvalues of BA .



Why Diagonalization?



Conclusion from pervious theorems

- Theorem “The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal.” can leads to if we have matrix A and B that $D = B^{-1}AB$ be a diagonal matrix:

$$\det(\lambda I - A) = \det(\lambda I - B^{-1}AB)$$

Proof?

Similarity and Diagonalizable

Definition

Two n -by- n matrices A and B are called **similar** if there exists **an invertible n -by- n matrix Q** such that

$$A = Q^{-1}BQ$$



Definition

A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix D : $D = Q^{-1}AQ$, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D .



03

Similarity



Relation between similar matrix and change of basis!

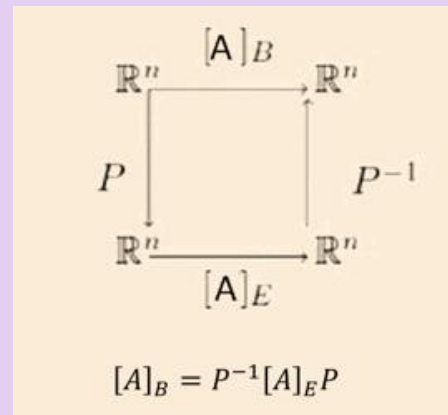
Note

- A square matrix for a linear transform

$$A: n \times n \quad T: R^n \rightarrow R^n \Rightarrow \mathbf{Aa = b} \quad a, b \in R^n$$

$$\left. \begin{array}{l} a = P\bar{a} \\ b = P\bar{b} \end{array} \right\} \Rightarrow AP\bar{a} = P\bar{b} \Rightarrow \underbrace{P^{-1}AP}_{\bar{A}} \bar{a} = \bar{b} \Rightarrow \bar{A}\bar{a} = \bar{b}$$

- Linear transform in new basis $\bar{A} = P^{-1}AP$
- \bar{A} is the standard matrix of linear transform in new basis.
- **Similarity Transformation**



Think!

Warnings

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B , then $B = EA$ for some invertible matrix E .) Row operations on a matrix usually change its eigenvalues.

❑ A matrix is a similarity invariant, meaning it remains unchanged under a similarity transformation.

❑ Why trace is a similarity invariant?

❑ Why rank is a similarity invariant?

Facts

Theorem

- *Similar matrices have:*
 - same determinant
 - equal characteristic equations
 - same trace
 - same rank
 - inverse of A and B are similar (if exists)

Proof?

Find matrix Q in similarity formula

Note

Two n -by- n matrices A and B are called **similar** if there exists an invertible n -by- n matrix Q such that $A = Q^{-1}BQ$. One solution for Q is the matrix whose columns are the eigenvectors of B .

Example

Find the similarity matrix of A

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$B = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

04

Diagonalization



Diagonalizable

Definition

A matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D .

Theorem

An $n \times n$ matrix A is diagonalizable **if and only if** A has n linearly independent **eigenvectors**.

Corollary

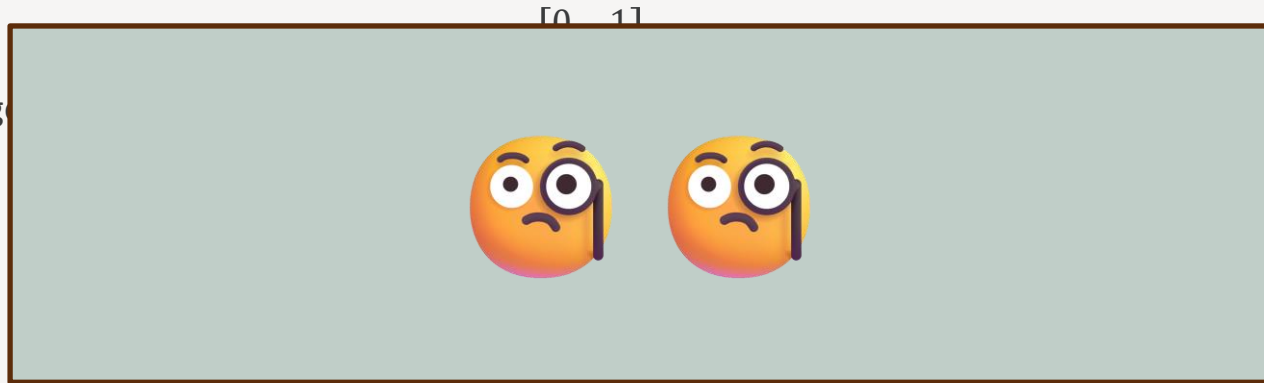
□ An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalizable and Non-Diagonalizable Matrices

- Distinct eigenvalues \rightarrow eigenvectors are Linear Independent
- Duplicate eigenvalues \rightarrow 🙄 🙄

- Not all matrices are diagonalizable.
 - Example:

- The diag




Diagonalizable and Non-Diagonalizable Matrices

$$A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$$

□ For matrix

- Its eigenvalues are -2, -2 and -3 (repeated eigenvalues)

$$AS = SD$$


$$\begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \boxed{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & -4 & 3 \\ 0 & 6 & -9 \end{pmatrix}$$

Diagonal Matrix

S is not invertible!

Diagonalizable and Non-Diagonalizable Matrices

$$B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$$

□ For matrix

So what's going on here?

$$\begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix} \begin{pmatrix} 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 0 & -6 & -9 \end{pmatrix}$$

Diagonal Matrix

R is invertible!

Diagonalizable and Non-Diagonalizable Matrices

- Details for matrix A:
 - (i) For the eigenvalue -3 , we have

$$\begin{pmatrix} 3 & -6 & -4 \\ 5 & -8 & -6 \\ -6 & 9 & 7 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

- (ii) For the repeated eigenvalue -2 , we have

$$\begin{pmatrix} 2 & -6 & -4 \\ 5 & -9 & -6 \\ -6 & 9 & 6 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which equally straightforwardly gives the eigenvector

$$\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}.$$

Diagonalizable and Non-Diagonalizable Matrices

(i) For the eigenvalue -3 , we have

- Details for matrix B:

$$\begin{pmatrix} 7 & 8 & -2 \\ -3 & -3 & 1 \\ 9 & 12 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which, as before, straightforwardly gives the eigenvector

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

(ii) This time, for the repeated eigenvalue -2 , we have

$$\begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, here things are different, because all three of the rows of this matrix may be reduced to the equation

$$3X + 4Y - Z = 0.$$

Diagonalizable and Non-Diagonalizable Matrices

This represents a **plane** in 3D space, and any vector in this plane is an eigenvector. We may therefore form our diagonalising matrix S out of

- Details for matrix B :

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

together with any two non-parallel vectors of the form

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

that satisfy

$$3X + 4Y - Z = 0;$$

that is, that are perpendicular to the vector

$$\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}.$$

Both of the choices

$$S = \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix},$$

$$S = \begin{pmatrix} 5 & 3 & 2 \\ -3 & -3 & -1 \\ 3 & -3 & 3 \end{pmatrix}$$

will work fine, as will infinitely many others.

Diagonalizable and Non-Diagonalizable Matrices

- General considerations

1. In general, any n by n matrix whose eigenvalues are distinct can be diagonalized.
2. If there is a repeated eigenvalue, whether or not the matrix can be diagonalized depends on the eigenvectors.
 - (i) If there $k < n$ eigenvectors (up to multiplication by a constant), then the matrix cannot be diagonalized.
 - (ii) If the unique eigenvalue corresponds to an eigenvector e , but the repeated eigenvalue corresponds to an entire plane, then the matrix can be diagonalised, using e together with any two vectors that lie in the plane.
3. If all n eigenvalues are repeated, then things are much more straightforward: the matrix can't be diagonalized unless it's already diagonal.

Power of matrix

Example


Find A^n ?



Conclusion

Another Notation

- With similarity transformation Q , matrix A changed to a diagonal matrix $diag(\lambda_1, \lambda_2)$
- Matrix A has n linear independent eigenvectors



- $[Aq_1 \ Aq_2 \ \cdots \ Aq_n] = \underbrace{[q_1 \ q_2 \ \cdots \ q_n]}_Q \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_\Lambda$

- $A [q_1 \ q_2 \ \cdots \ q_n] = Q\Lambda \Rightarrow AQ = Q\Lambda$

- $\Lambda = Q^{-1}AQ^T$

- $A = Q\Lambda Q^{-1}$

