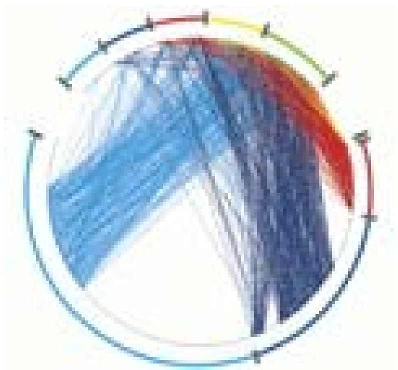
Lecture 22&23: Dynamical Systems & Synchronization



Dynamical systems

- **Dynamics** is the study of systems that evolve with time.
- The same framework of dynamics can be applied to biological, chemical, electrical, mechanical systems.
- Typically expressed as differential equation(s) (or discrete difference equation but we will focus on ODEs)
- Systems that evolve with time and they have a "memory"
- State at time t depends upon the state at a slightly earlier time
- They are therefore inherently *deterministic*: state is determined by the earlier state

Where x is going in future time

$$\frac{dx}{dt} = f(x)$$

Is determined by where x is now

Linear vs Nonlinear terms

- **Linear Terms: one that is first degree in its <u>dependent</u> variables and derivatives**
  - **x is 1st degree and therefore a linear term**
  - **xt is 1st degree in x and therefore a linear term**
  - **x^2 is 2nd degree in x and there <u>not</u> a linear term**
- **Nonlinear Terms: any term that contains higher powers, products and transcendentals of the <u>dependent</u> variable is nonlinear**
  - x^2, e^x, $x(x+1)^{-1}$ all nonlinear terms
  - $\sin x$ nonlinear term

Linear vs Nonlinear terms

- Linear

$$\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}$$

2nd order
not
2nd degree

$$\sin t\left(\frac{dy}{dt}\right)$$

- Nonlinear

$$\left(\frac{dy}{dt}\right)^2$$

$$\sin x\left(\frac{dy}{dt}\right)$$

$$xy$$

$$x^3$$

Linear vs Nonlinear equations

linear equation: consists of a sum of linear terms

$y = x + 2$
$y(t) + x(t) = N$
$dy/dt = x + \sin t$

nonlinear equation: all other equations

$y + x^2 = 2$
$x(t) * y(t) = N$
$dy / dt = xy + \sin x$

most nonlinear differential equations are impossible to solve analytically! So what do we do???

Linear vs Nonlinear equations

system 1

$$\frac{dy_1}{dt} = 2y_1 + y_2$$

$$\frac{dy_2}{dt} = y_1 + 3y_2$$

system 2

$$\frac{dy_1}{dt} = 2y_1y_2 + y_2$$

$$\frac{dy_2}{dt} = y_1y_2 + 3y_2$$

linear system: system of linear equations

nonlinear system: system of equations containing at least 1 nonlinear term

we can use tools such as Laplace transformations to assist in solving linear systems of differential equations
can't use for nonlinear systems!!

What is nonlinear conceptually?

- Nonlinearity implies interactions!!

$$y = 2x_1 + x_2$$

The impact of x_1 is always the same

$$y = 2x_1x_2 + x_2$$

The impact of x_1 on y depends on the value of x_2

There is an *interaction* between x_1 and x_2

Linear vs Nonlinear terms

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So,

- Dynamical system
  - A system of one or more variables which evolve in time according to a given rule
  - Two types of dynamical systems:
    - Differential equations: time is continuous

$$\frac{dX}{dt} = F(X,t)$$

    - Difference equations (iterated maps): time is discrete

$$X(t+\Delta t) = F(X(t))$$

$$X_{n+1} = F(X_n)$$

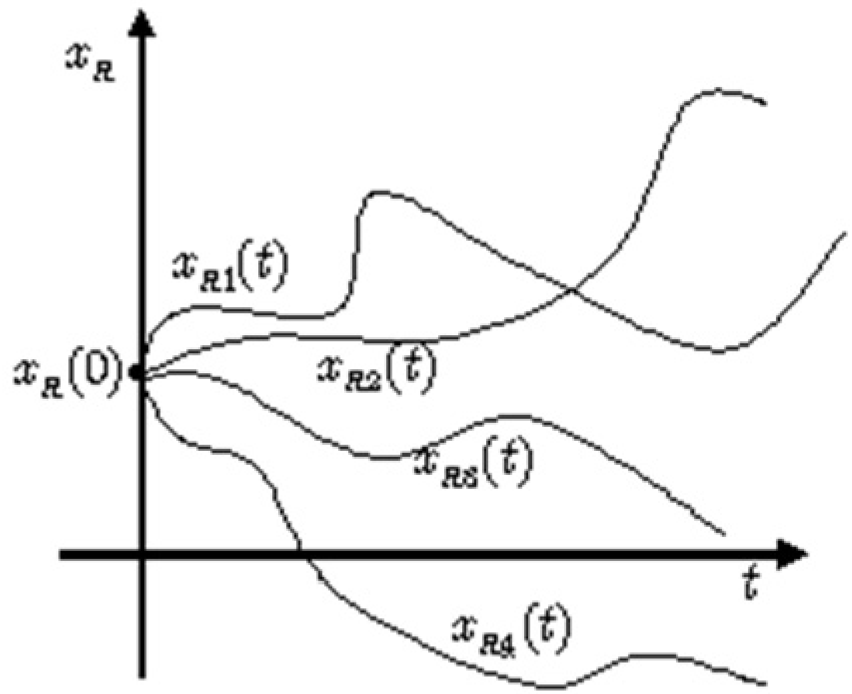
So,

- Linear vs. nonlinear
  - A linear dynamical system is one in which the rule governing the time-evolution of the system involves a linear combination of all the variables.
    - EXAMPLE: $$\frac{dX}{dt} = AX + B$$
  - A nonlinear dynamical system is simply…

    … not linear

So,

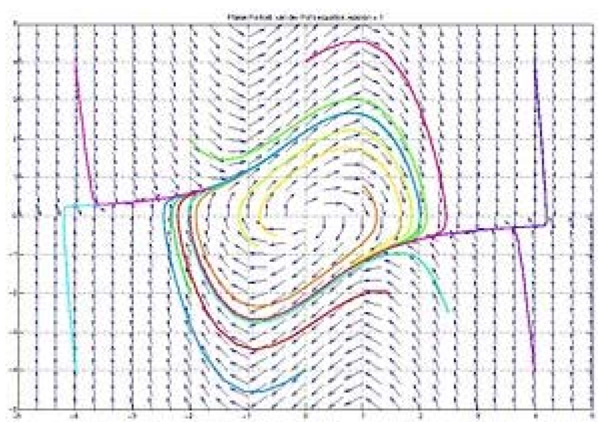
- Usually we study the trajectories of the states in dynamical systems:
- Trajectory: The path a moving object follows through space as a function of time.



1-D ODEs

- $dx/dt = f(x,t)$
- If the system is time-invariant: $dx/dt = f(x)$
- Fixed points: where the derivative is zero (no variation in the system) $\rightarrow dx/dt = f(x^*) = 0$
- A fixed point can be **stable** or **unstable**

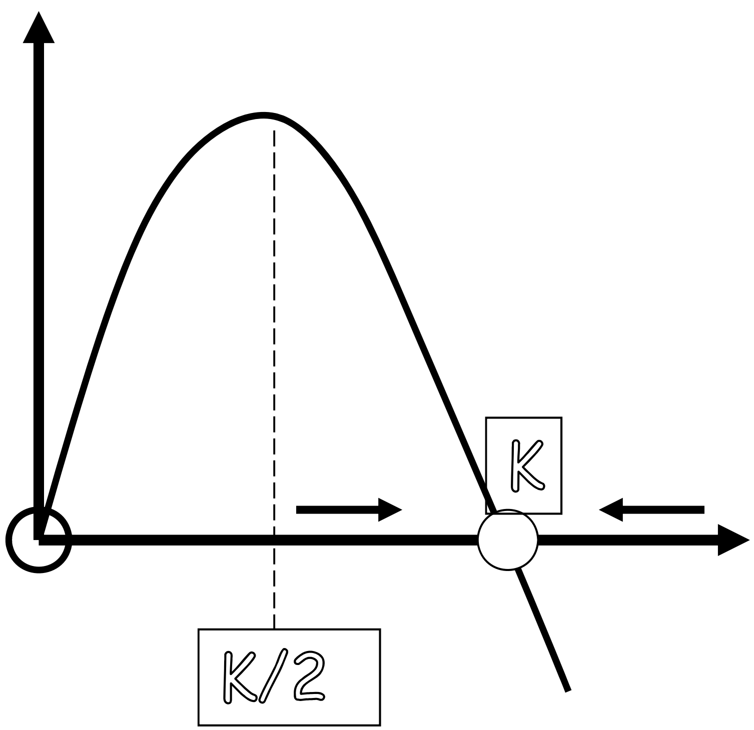
  stable: $f'(t) < 0$ unstable: $f'(t) > 0$
- A usual method for analysis is **phase portrait:**
- Plotting dx/dt vs x

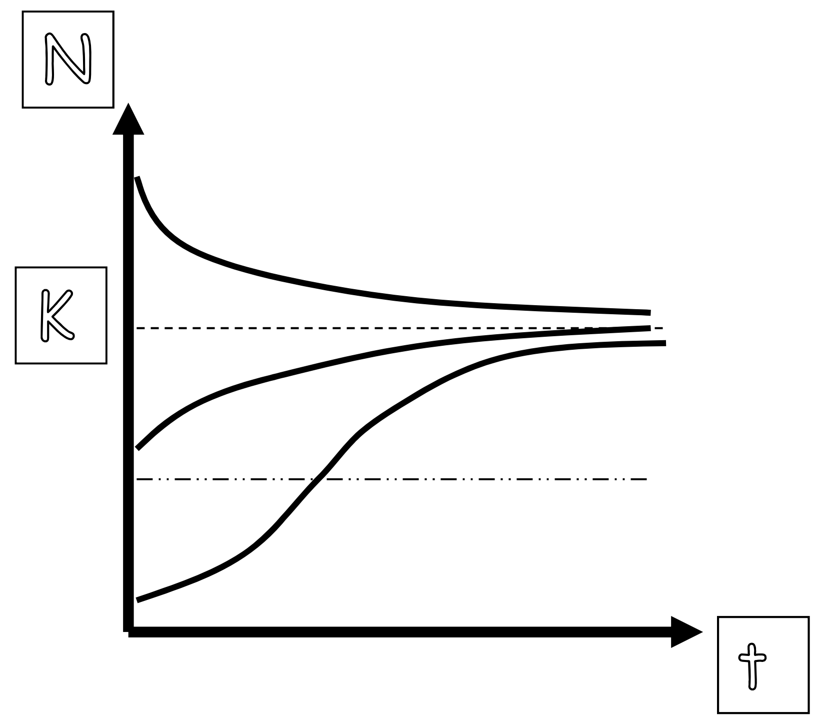


Example

- A simple model for growth

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$





Linearization

- Linearization is a technique for analyzing stability of nonlinear systems:
- x^* : fixed point, $dy/dt = yf'(x^*)$: linear zed version
- stable: $f'(x^*) < 0$ unstable: $f'(x^*) > 0$
- However, the results are locally valid (around the fixed point)

Numerical solution of ODEs

- An approximate solution is obtained based on difference equations:
- Euler's method:

$$X(n+1) = x(n) + f(x(n))\Delta t$$

- Improved Euler's method:

$$x_1(n+1) = x(n) + f(x(n))\Delta t$$

$$x(n+1) = x(n) + 0.5[\, f(x(n)) + f(x_1(n+1))]\Delta t$$

- Runge-Kutta methods:

$$k_1 = f(x(n))\Delta t,\ k_2 = f(x(n)+0.5\, k_1)\Delta t$$

$$k_3 = f(x(n)+0.5\, k_2)\Delta t,\ k_4 = f(x(n)+ k_3)\Delta t$$

$$x(n+1) = x(n) + 1/6[k_1+2k_2+2k_3+k_4]\Delta t$$

Bifurcation

- Consider a parameter dependent system
- If change in parameter
  - Structurally stable: no significant change
  - Bifurcation: sudden change in dynamics

Transcritical Bifurcation

- Consider the ODE

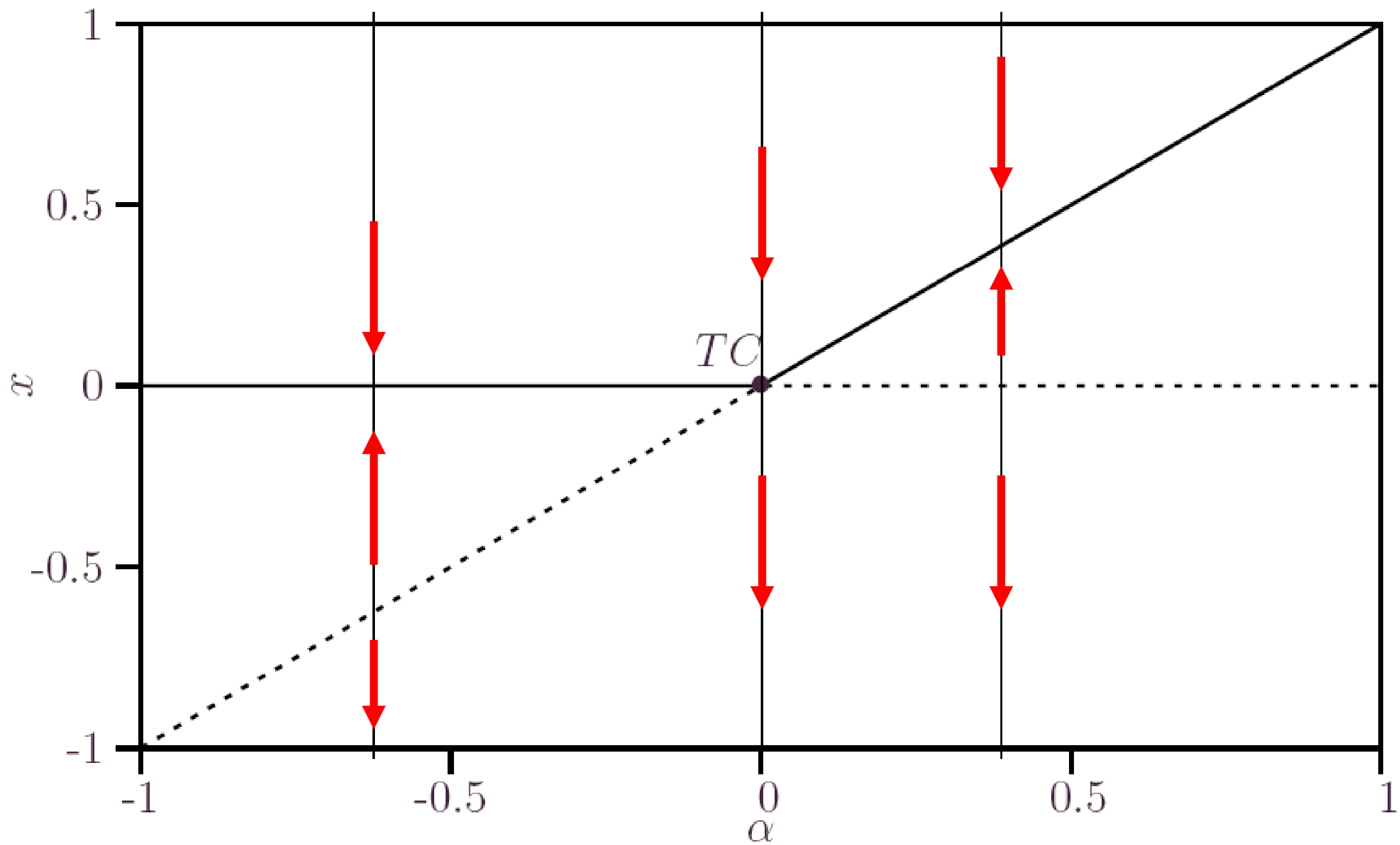
$$\dot{x} = x(\alpha - x)$$

- Two equilibria

$$x = 0\ , \quad x = \alpha$$

- Example: $a = 1$
- Equilibria: $x = 0$, $x = 1$
- Derivative: $-2x + a$
- Stability
  - $x = 0 \Rightarrow f'(x) > 0$ (unstable)
  - $x = a \Rightarrow f'(x) < 0$ (stable)

Transcritical Bifurcation



Transcritical bifurcation point α = 0

2-D systems

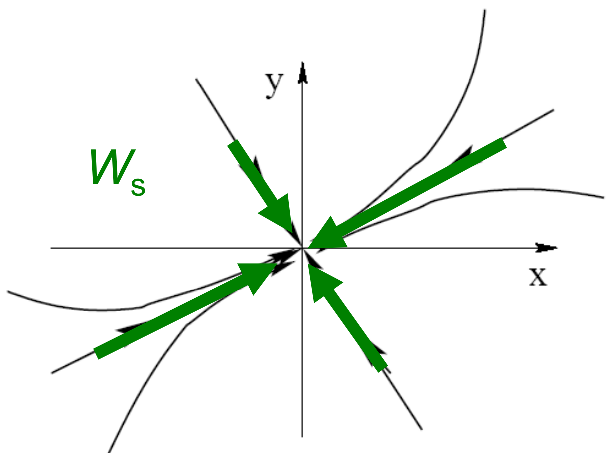
- Consider 2D ODE:

$$dx/dt = f(x,y)$$

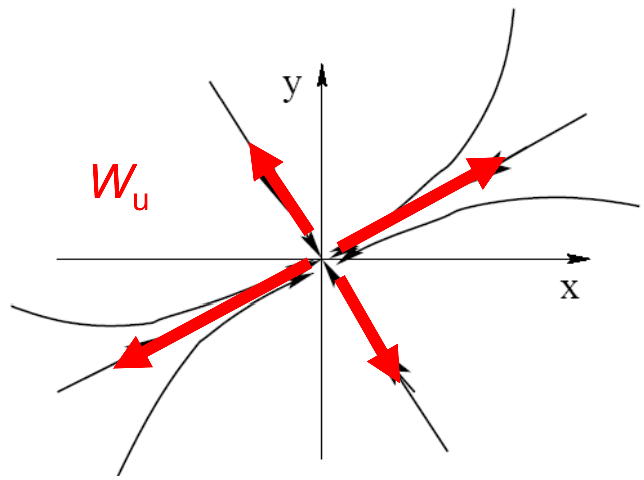
$$dy/dt = f(x,y)$$

- Different kinds of analysis for 2D ODE systems
  - Equilibria: determine type(s)
  - Transient or long term behaviour
- Different types of equilibria
- Stability
  - Stable
  - Unstable
  - Saddle
- Convergence type
  - Node
  - Spiral (or focus)

Equilibria: nodes



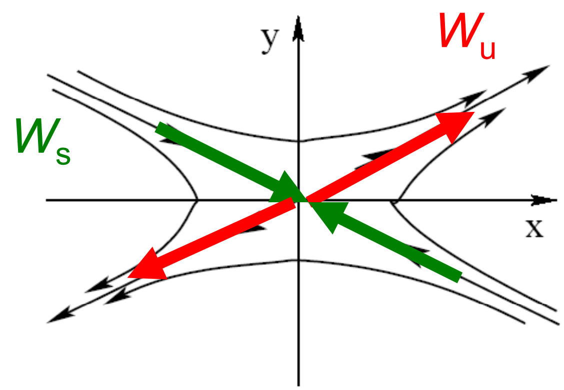
Stable node



Unstable node

Node has two (un)stable manifolds

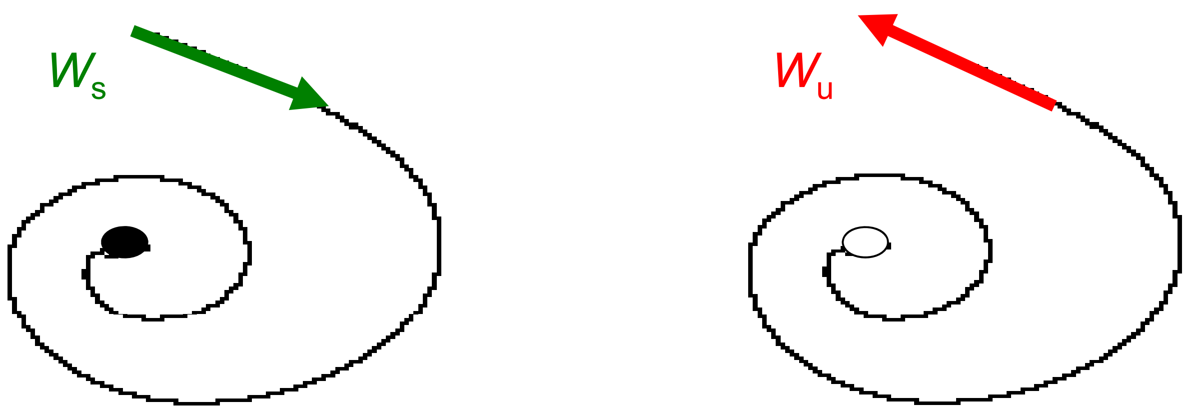
Equilibria: saddle



Saddle point

Saddle has one stable & one unstable manifold

Equilibria: foci



Stable spiral

Unstable spiral

Spiral has one (un)stable (complex) manifold

Equilibria: determination

- How do we determine the type of equilibrium?
- Linearisation of point
- Eigenfunction
- Linearisation of equilibrium in more than one dimension → partial derivatives
- Jacobian:

$$\mathbf{J} = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}$$

Eigenfunction

- Determine eigenvalues (λ) and eigenvectors (v) from Jacobian

$$\mathbf{J}\mathbf{v} = \lambda\mathbf{v}$$

- Of course there are two solutions for a 2D system

$$\begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \lambda \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

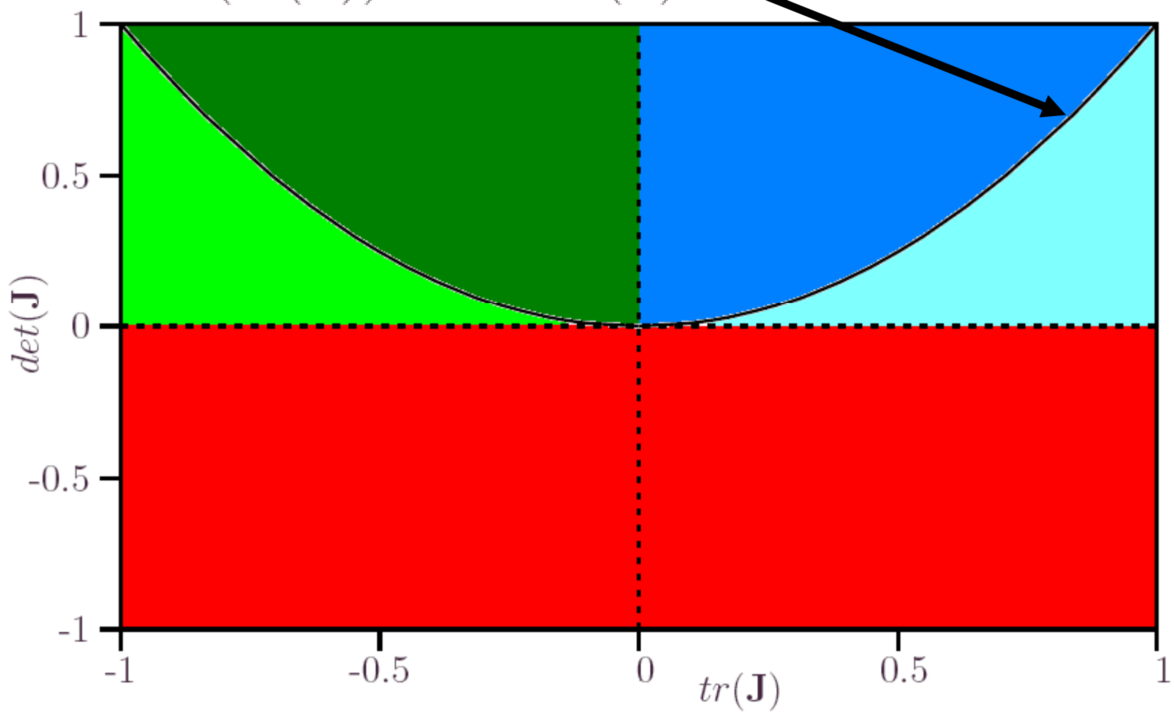
- If $\lambda < 0$ ➔ stable, $\lambda > 0$ ➔ unstable
- If two λ complex pair ➔ spiral

Determinant & trace

- Alternative in 2D to determine equilibrium type (much less computation)

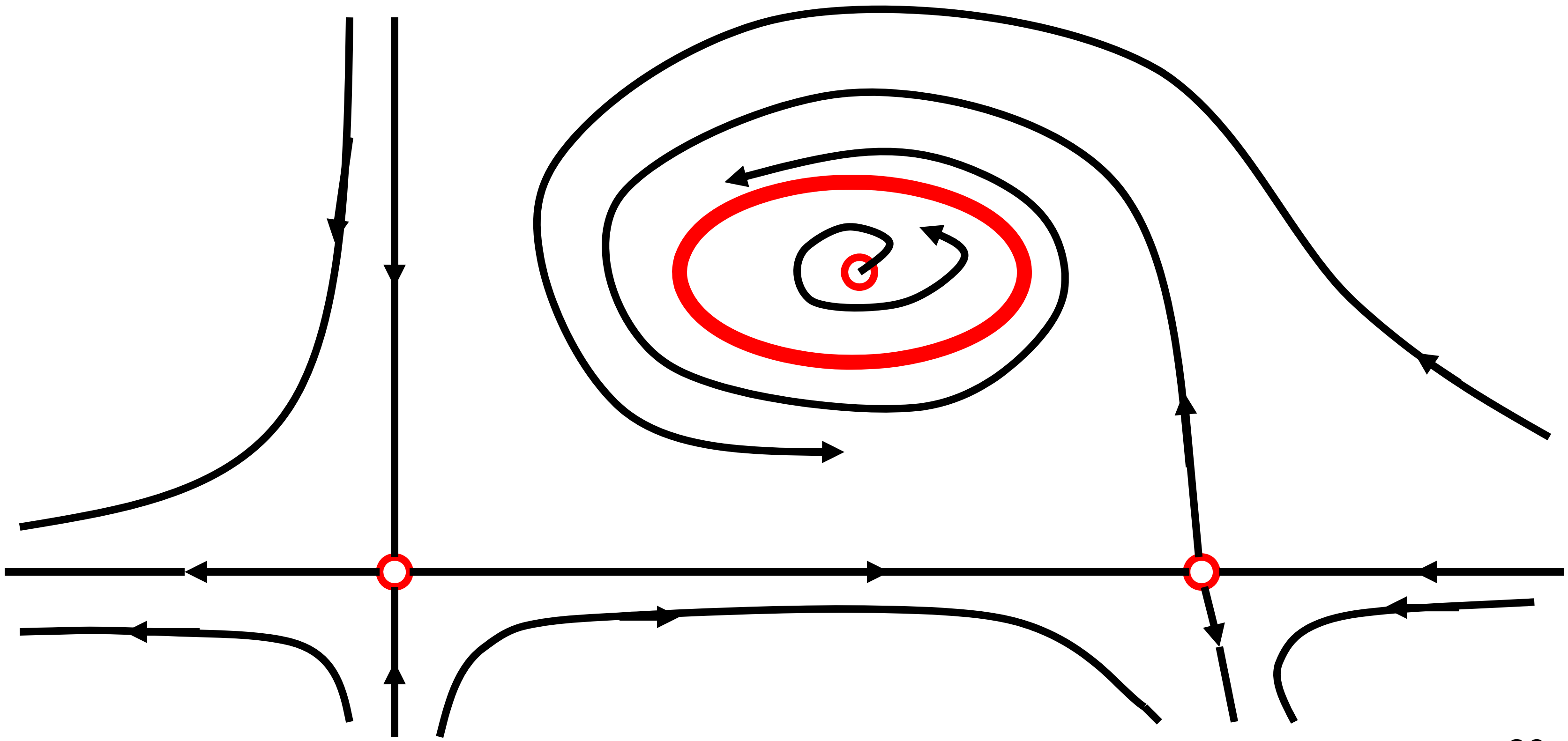
$$det(\mathbf{J}) = \frac{df}{dx}\frac{dg}{dy} - \frac{df}{dy}\frac{dg}{dx} \qquad tr(\mathbf{J}) = \frac{df}{dx} + \frac{dg}{dy}$$

$(tr(\mathbf{J}))^2 < 4det(\mathbf{J})$



Saddle
Stable node
Stable spiral
Unstable spiral
Unstable node

Typical phase portrait of 2-D Nonlinear Systems



Linear Stability Analysis

- For nonlinear systems, we study the qualitative behavior near the fixed points
  - 1) Finding the fixed points
  - 2) Linearizing the system near the fixed points
  - 3) Classifying the fixed points

Finding the fixed points

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

- NullClines:

$$x \text{ nullcline} \quad : \qquad \frac{dx}{dt} = f(x, y) = 0$$

$$y \text{ nullcline} \quad : \qquad \frac{dy}{dt} = g(x, y) = 0$$

- The intersections of nullclines are the fixed points

Example: Lotka-Volterra competition

- Rabbits x(t) vs. Sheep y(t)
    - Rabbits reproduce more
    - Sheep reproduce less

logistic growth :

$$\frac{dx}{dt} = r_x x(1 - \frac{x}{k_x})$$

$$\frac{dy}{dt} = r_y x(1 - \frac{x}{k_y})$$

$$r_x > r_y, k_x > k_y$$

- Sheep are stronger
- Rabbits are weaker

logistic growth + competition :

$$\frac{dx}{dt} = r_x x(1 - \frac{x}{k_x} - c_y y)$$

$$\frac{dy}{dt} = r_y y(1 - \frac{x}{k_y} - c_x x)$$

$$c_y > c_x$$

Lotka-Volterra competition

$$\frac{dx}{dt} = r_x x(1 - \frac{x}{k_x} - c_y y)$$

$$\frac{dy}{dt} = r_y y(1 - \frac{x}{k_y} - c_x x)$$

- Numerical example:

$$\frac{dx}{dt} = x(3 - x - 3y)$$

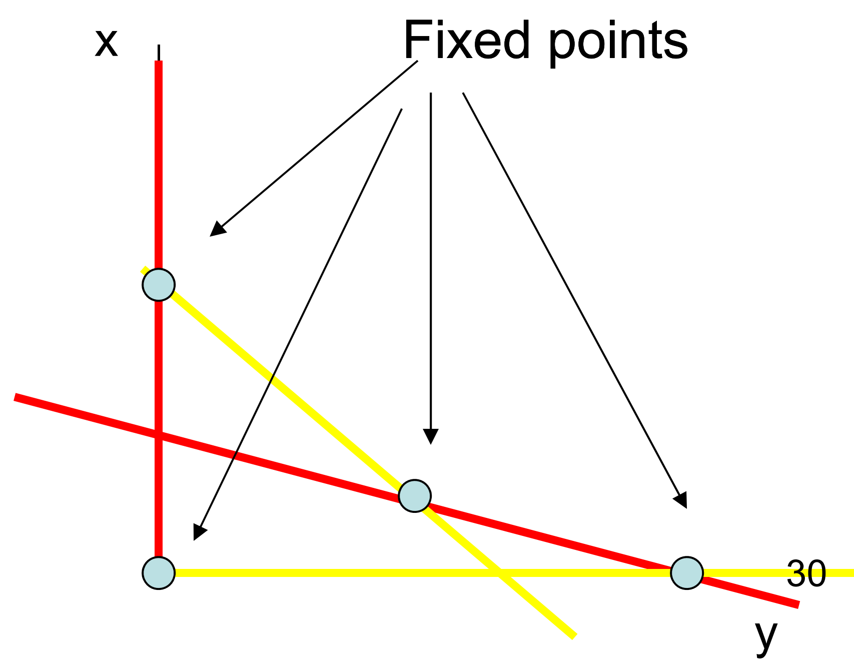
$$\frac{dy}{dt} = y(2 - x - y)$$

x nullclines :

$$\frac{dx}{dt} = 0 \Rightarrow x(3 - x - 3y) = 0 \Rightarrow \begin{cases} x = 0 \\ x = 3 - 3y \end{cases}$$

y nullclines :

$$\frac{dy}{dt} = 0 \Rightarrow y(2 - x - y) = 0 \Rightarrow \begin{cases} y = 0 \\ x = 2 - y \end{cases}$$



Linearization

- Equations: $$\frac{dx}{dt} = f(x,t)$$ $$\frac{dy}{dt} = g(x,t)$$
- Fixed point: (x_0, y_0) $\left(\frac{dx_0}{dt} = 0, \frac{dy_0}{dt} = 0\right)$
  - Intersection of nullclines
- Perturbation: $$x = x_0 + \varepsilon$$ $$y = y_0 + \eta$$
- Stability: $$\varepsilon(t) \underset{t\to\infty}{\longrightarrow} 0, \eta(t) \underset{t\to\infty}{\longrightarrow} 0$$

Linearization

$$x = x_0 + \varepsilon$$

$$\varepsilon = x - x_0$$

$$\frac{d\varepsilon}{dt} = \frac{d(x - x_0)}{dt} = \frac{dx}{dt} = f(x, y)$$

$$\frac{d\varepsilon}{dt} = f(x, y) = f(x_0 + \varepsilon, y_0 + \eta) = f(x_0, y_0) + \varepsilon \frac{\partial f(x_0, y_0)}{\partial x} + \eta \frac{\partial f(x_0, y_0)}{\partial y} + O(2)$$

$$\frac{d\varepsilon}{dt} \approx \varepsilon \frac{\partial f(x_0, y_0)}{\partial x} + \eta \frac{\partial f(x_0, y_0)}{\partial y}$$

$$Simillarly:$$

$$\frac{d\eta}{dt} \approx \varepsilon \frac{\partial g(x_0, y_0)}{\partial x} + \eta \frac{\partial g(x_0, y_0)}{\partial y}$$

Linearization

$$\frac{d\varepsilon}{dt} \approx \varepsilon \frac{\partial f(x_0, y_0)}{\partial x} + \eta \frac{\partial f(x_0, y_0)}{\partial y}$$

$$\frac{d\eta}{dt} \approx \varepsilon \frac{\partial g(x_0, y_0)}{\partial x} + \eta \frac{\partial g(x_0, y_0)}{\partial y}$$

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_0, y_0)} . \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_0, y_0)}$$

The Jacobian Matrix: 2D equivalent of derivitive

Lotka-Volterra competition

$$\frac{dx}{dt} = x(3 - x - 3y)$$
$$\frac{dy}{dt} = y(2 - x - y)$$

Fixed Points :

(0,0)

(0,2)

(1.5,0.5)

(3,0)

Jacobian matrix :

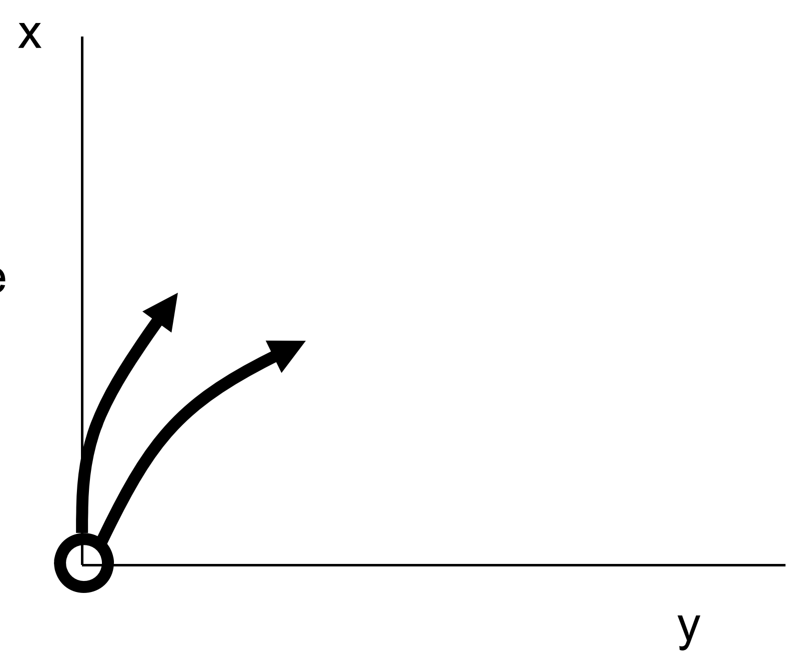
$$A = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}$$

Lotka-Volterra competition

Fixed Point : Jacobian matrix : eigenvalues

$(0,0)$ $\quad A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda = 3,2$

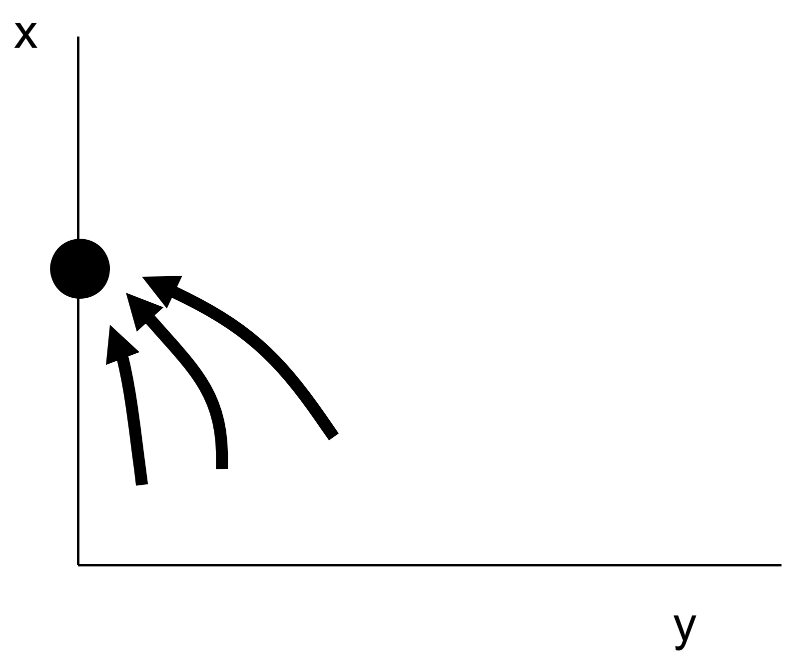
Unstable node



Lotka-Volterra competition

Fixed Point :	Jacobian matrix :	eigenvalue s
(0,2)	$A = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$	$\lambda = -1, -2$

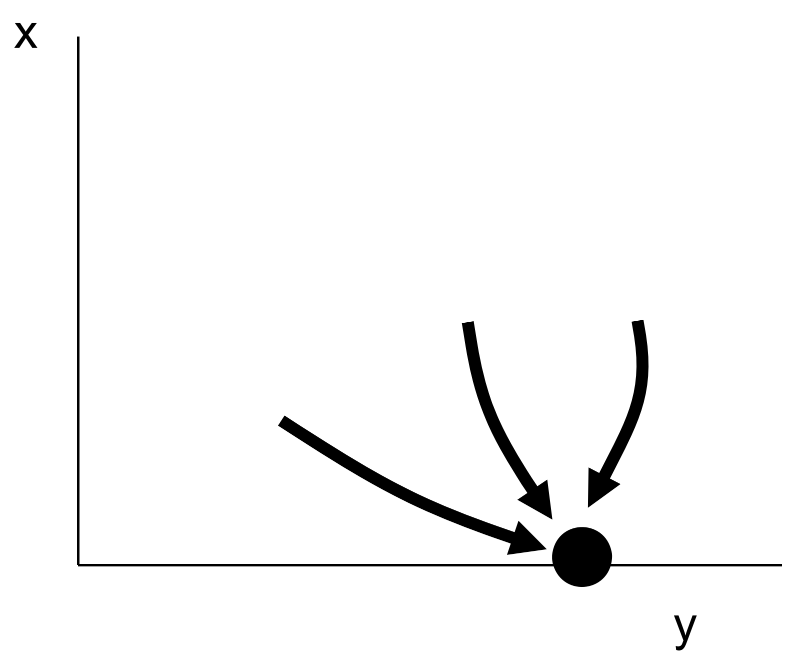
Stable Node



Lotka-Volterra competition

Fixed Point :	Jacobian matrix :	eigenvalues
(3,0)	$A = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$	$\lambda = -3, -1$

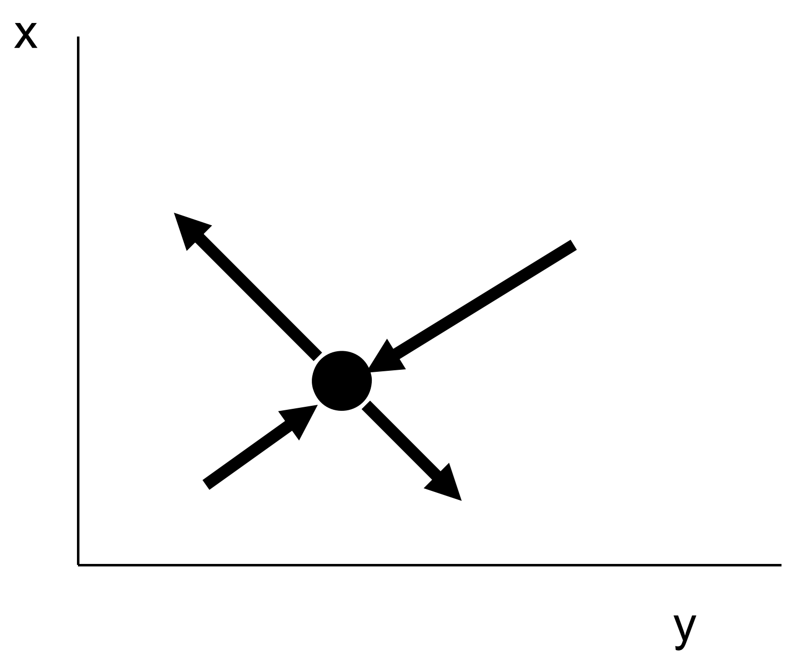
Stable Node



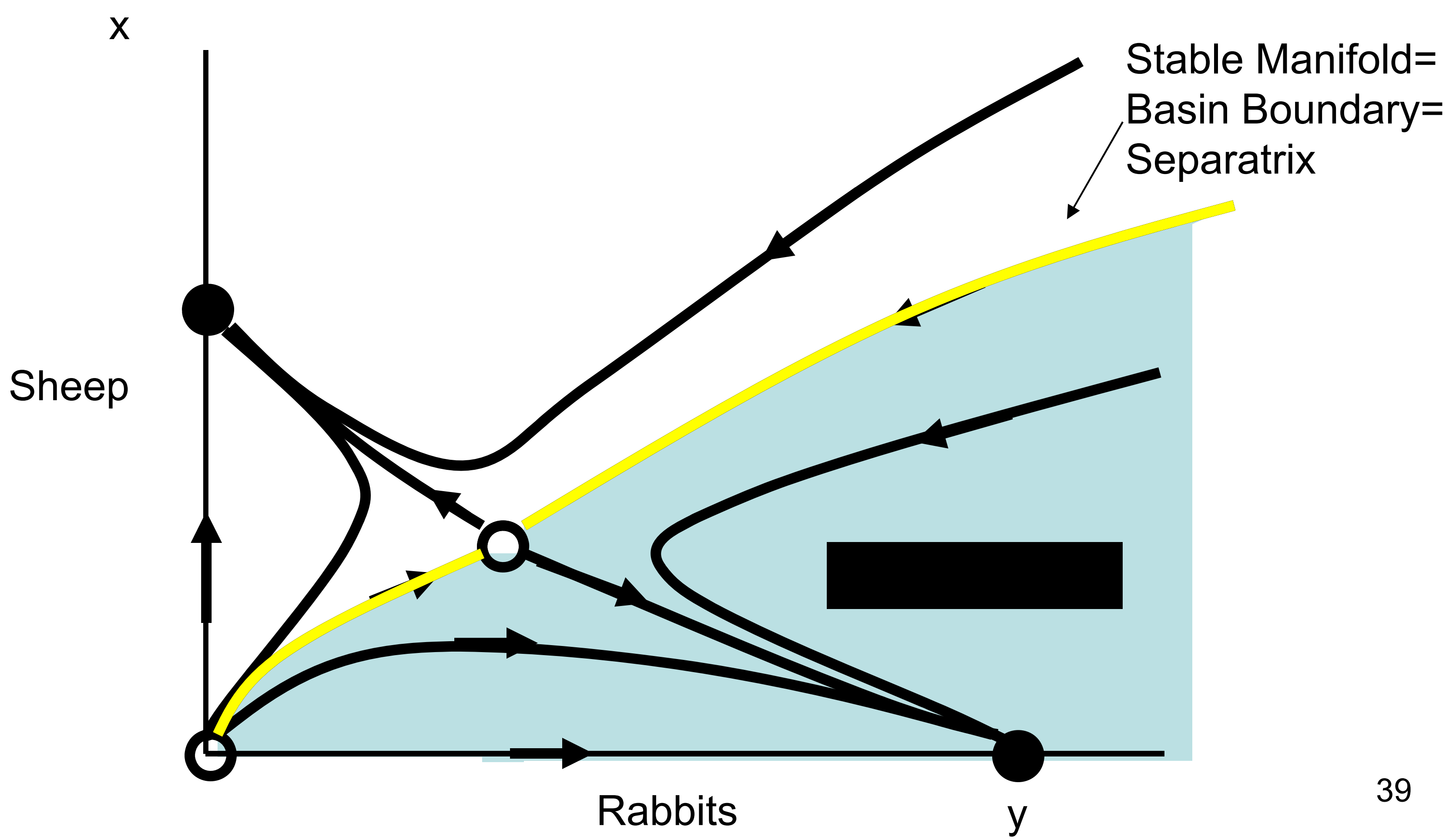
Lotka-Volterra competition

Fixed Point :	Jacobian matrix :	eigenvalues
(1,1)	$A = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$	$\lambda = -1 \pm \sqrt{2}$

Saddle Point



Phase Portrait of the Lotka-Volterra system

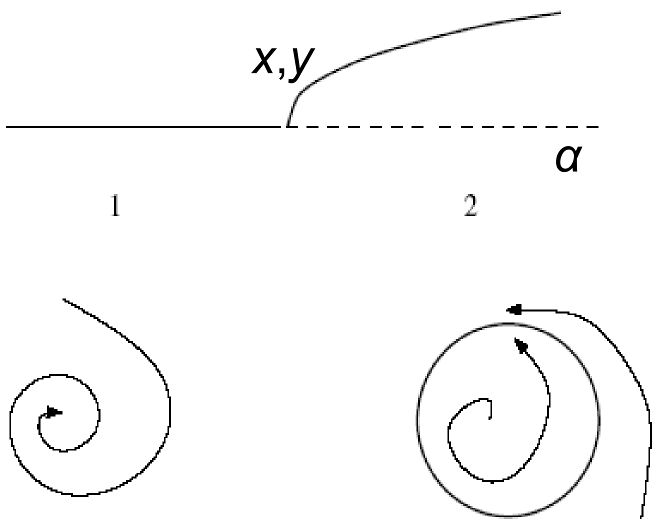


Hopf bifurcation

Andronov-Hopf bifurcation:

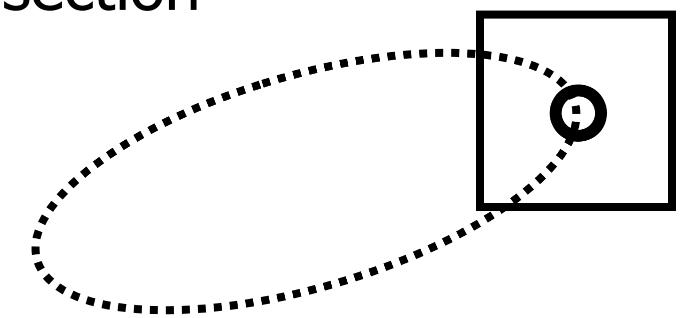
- Transition from stable spiral to unstable spiral (equilibrium becomes unstable)
- Or vice versa
- Periodic orbit (limit cycle), stable
- Or vice verse, respectively
- Conditions:

$$\Re(\lambda) = 0,\ \Im(\lambda) \neq 0,\ \Im(\lambda_1) = -\Im(\lambda_2)$$

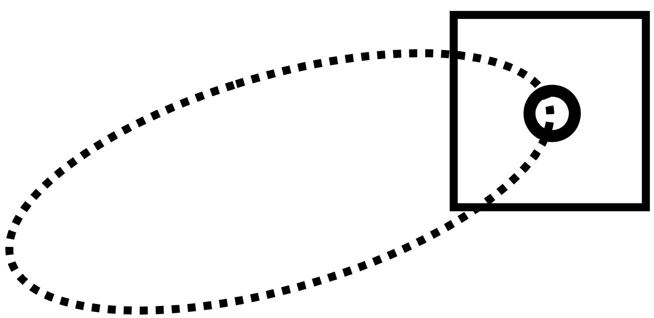


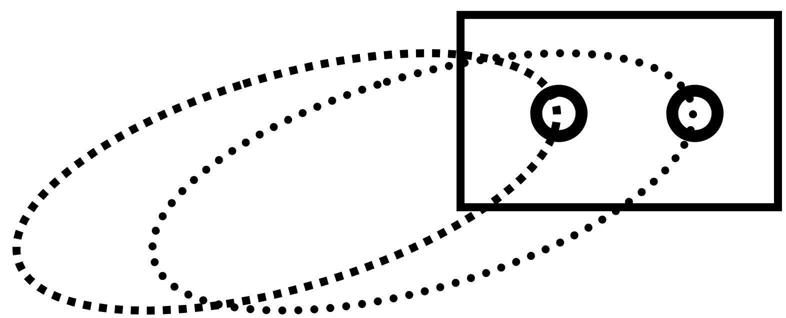
Limit cycle bifurcations

- For limit cycles same bifurcations as for equilibria
- Imagine cross section



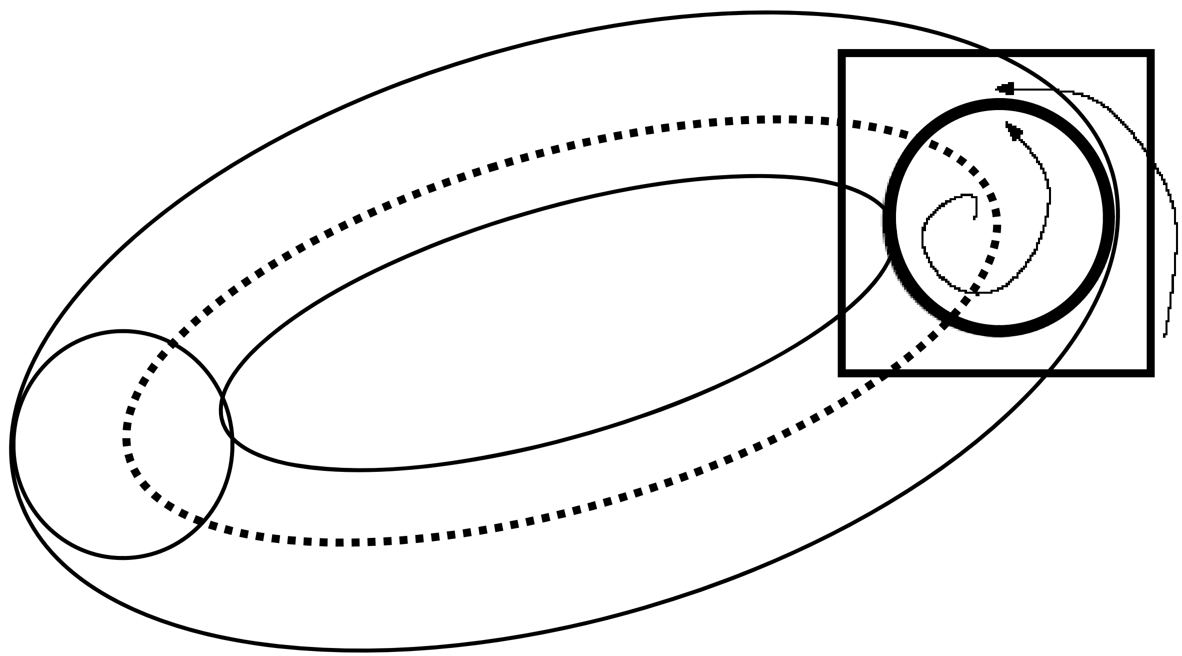
- Transcritical ➔ Cycle on axis (mostly)
- Tangent ➔ Birth or destruction cycle(s)





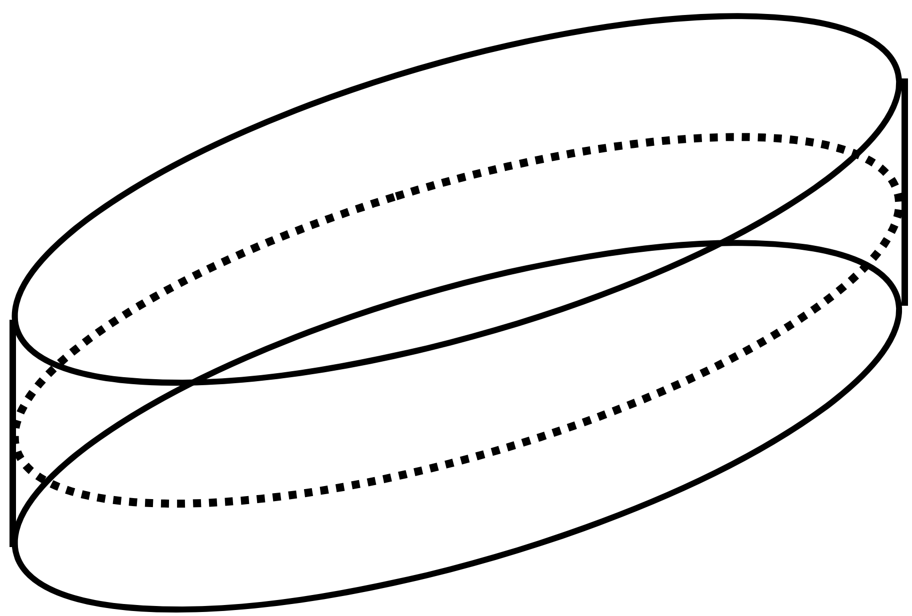
Limit cycle bifurcations

- Hopf (called Neimark-Sacker) ➔ Torus



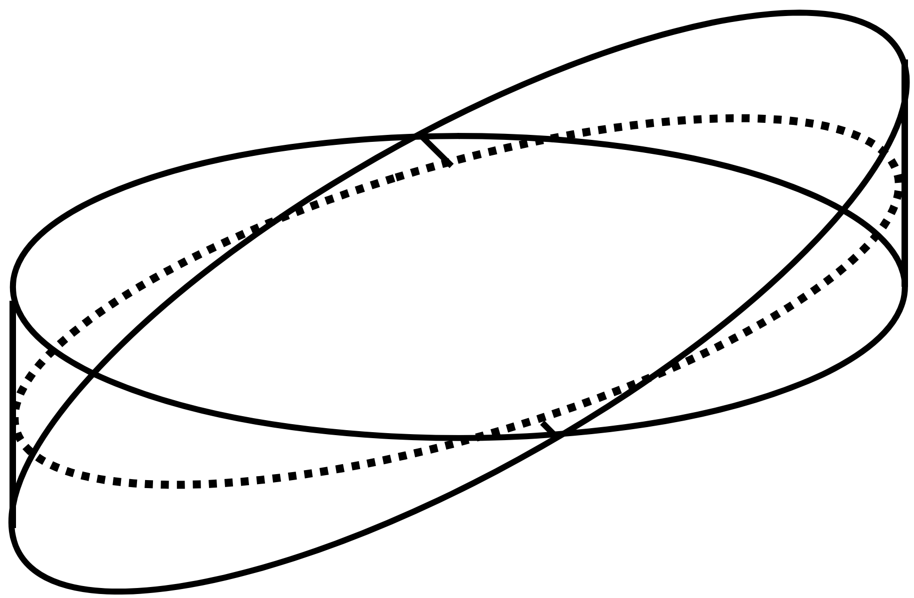
Limit cycle bifurcations

- Flip bifurcation
- Manifold around cycle



Flip bifurcations

- Manifold twisted



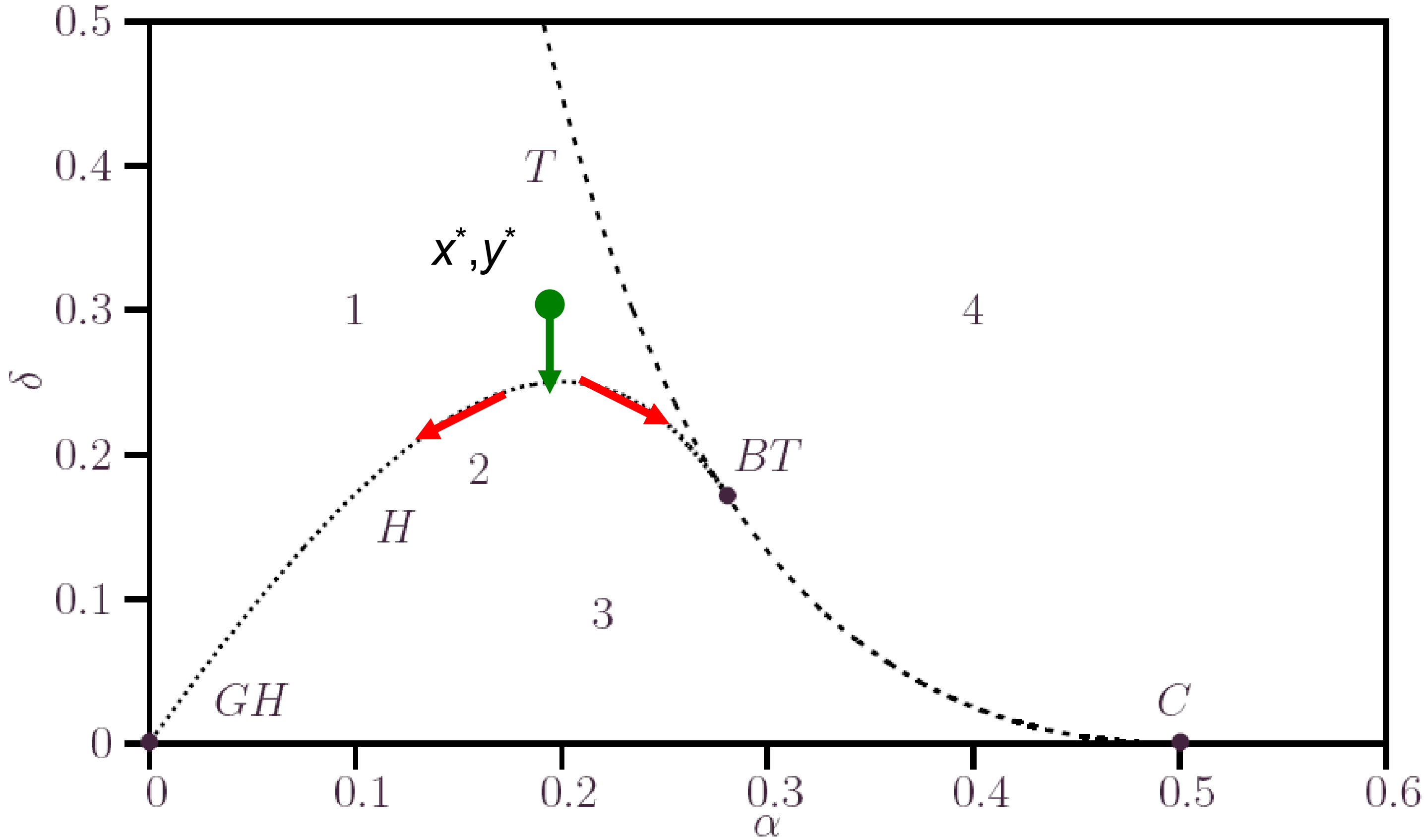
Example

- Bazykin model

$$\begin{aligned} \dot{x} &= x - \frac{xy}{1+\alpha x}, \\ \dot{y} &= \frac{xy}{1+\alpha x} - 2y - \delta y^2 \end{aligned}$$

- Calculate equilibrium
- Vary one parameter until a bifurcation is encountered

Bazykin

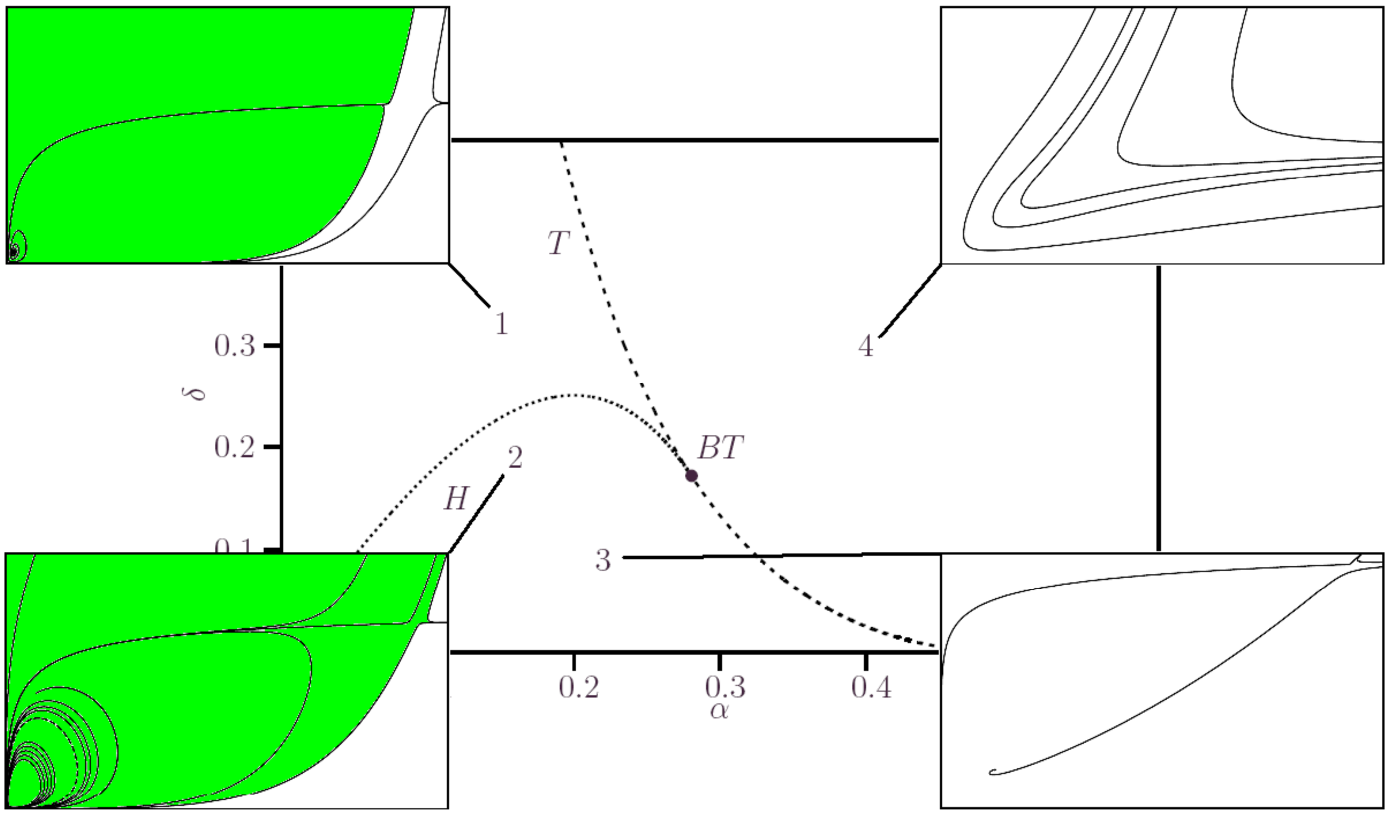


Continuation in two-parameter space

Bazykin: dynamics

Stable node: coexistence

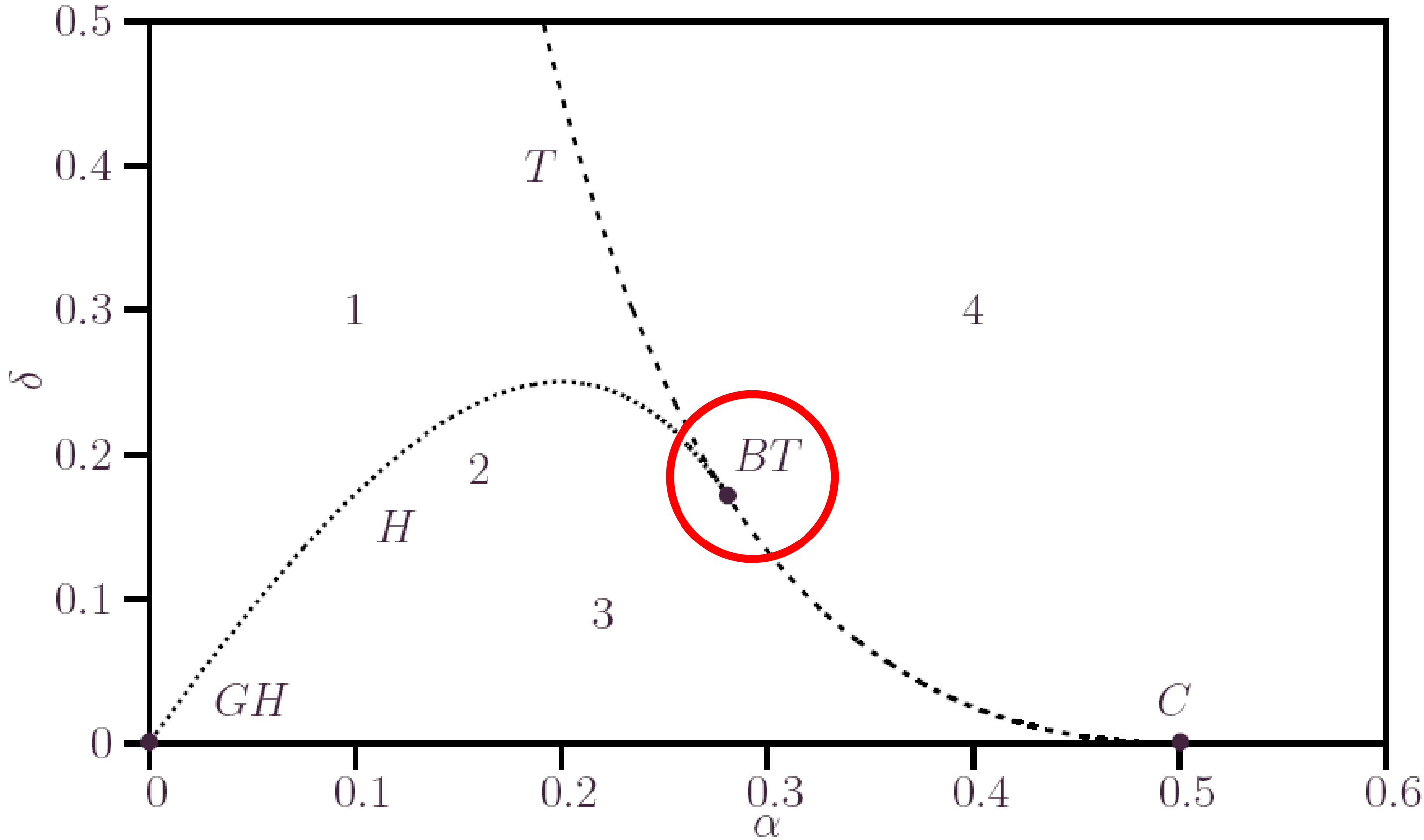
No positive equilibria: extinction



Stable cycle: coexistence

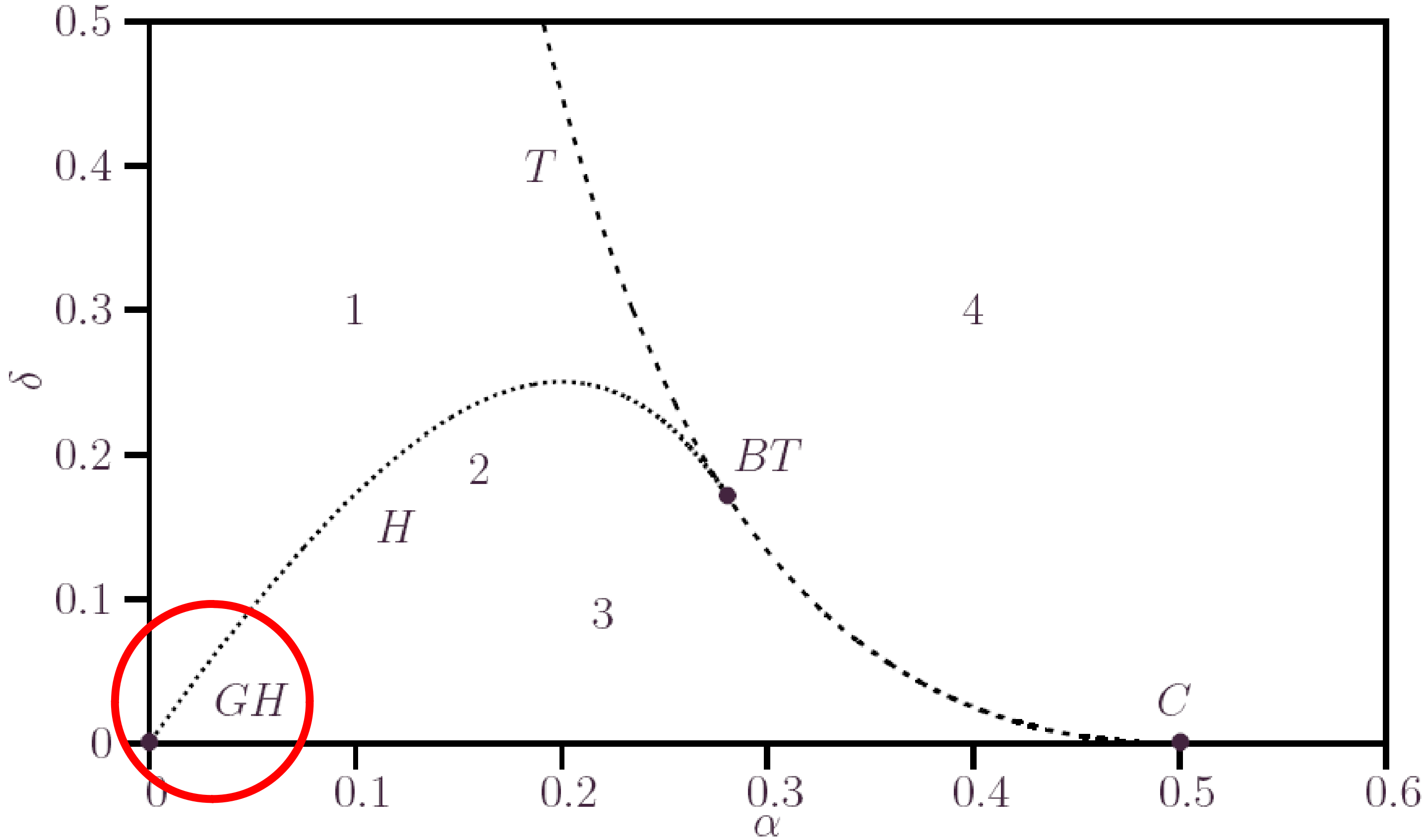
Unstable equilibria: extinction

Bazykin: BT point



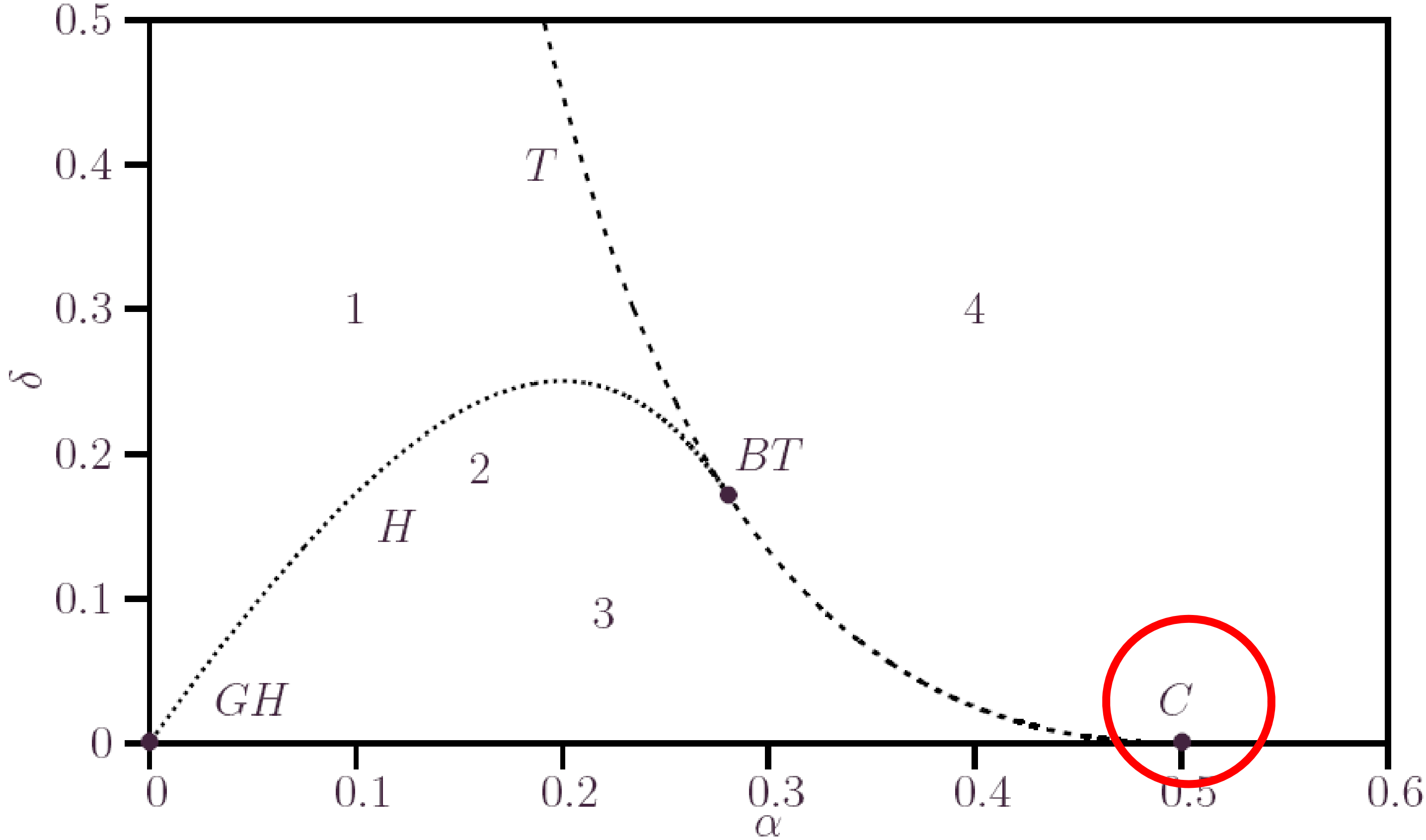
Bogdanov-Takens point → tangent & Hopf

Bazykin: GH point



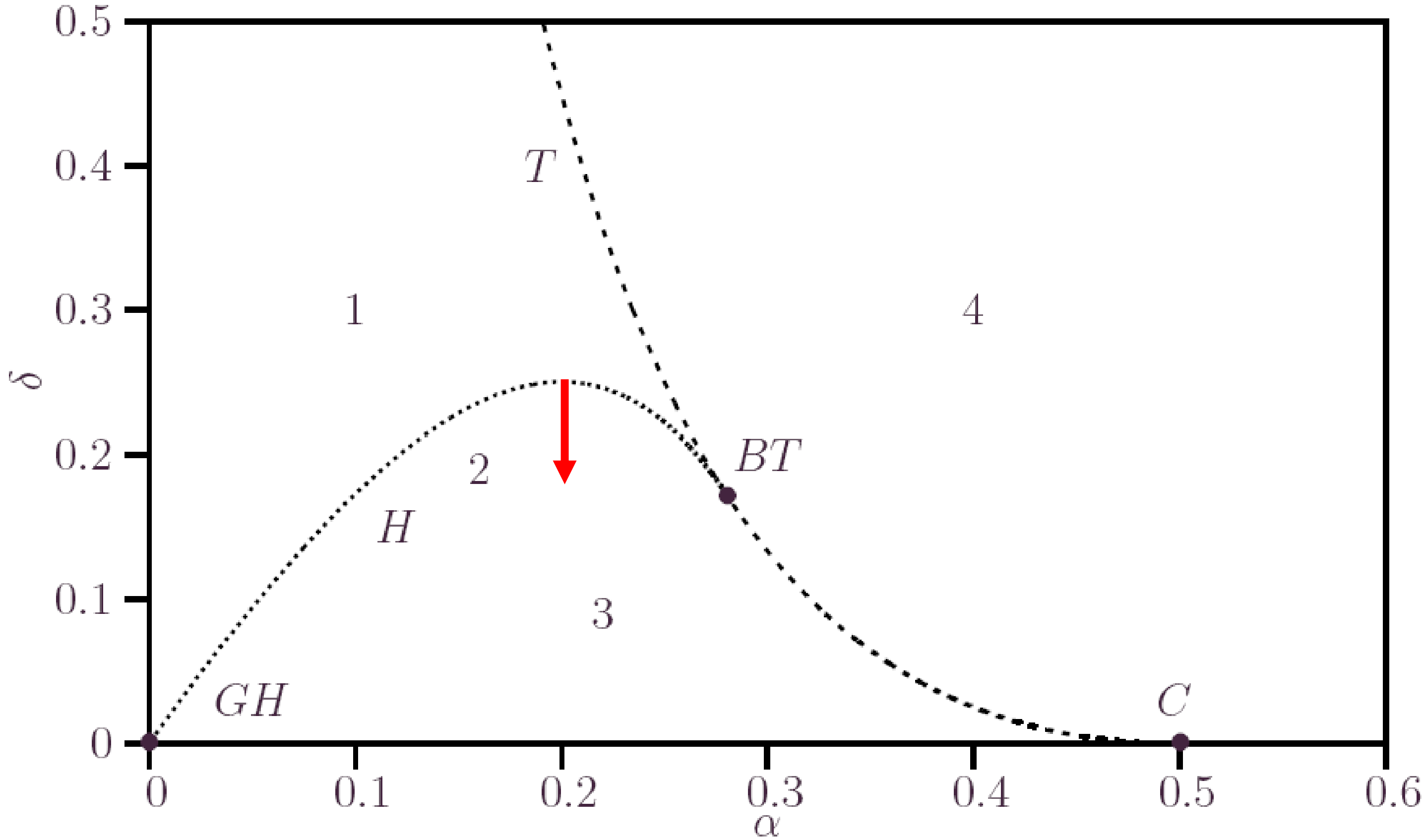
Bautin point → transition Hopf from stable to unstable point

Bazykin: cusp point



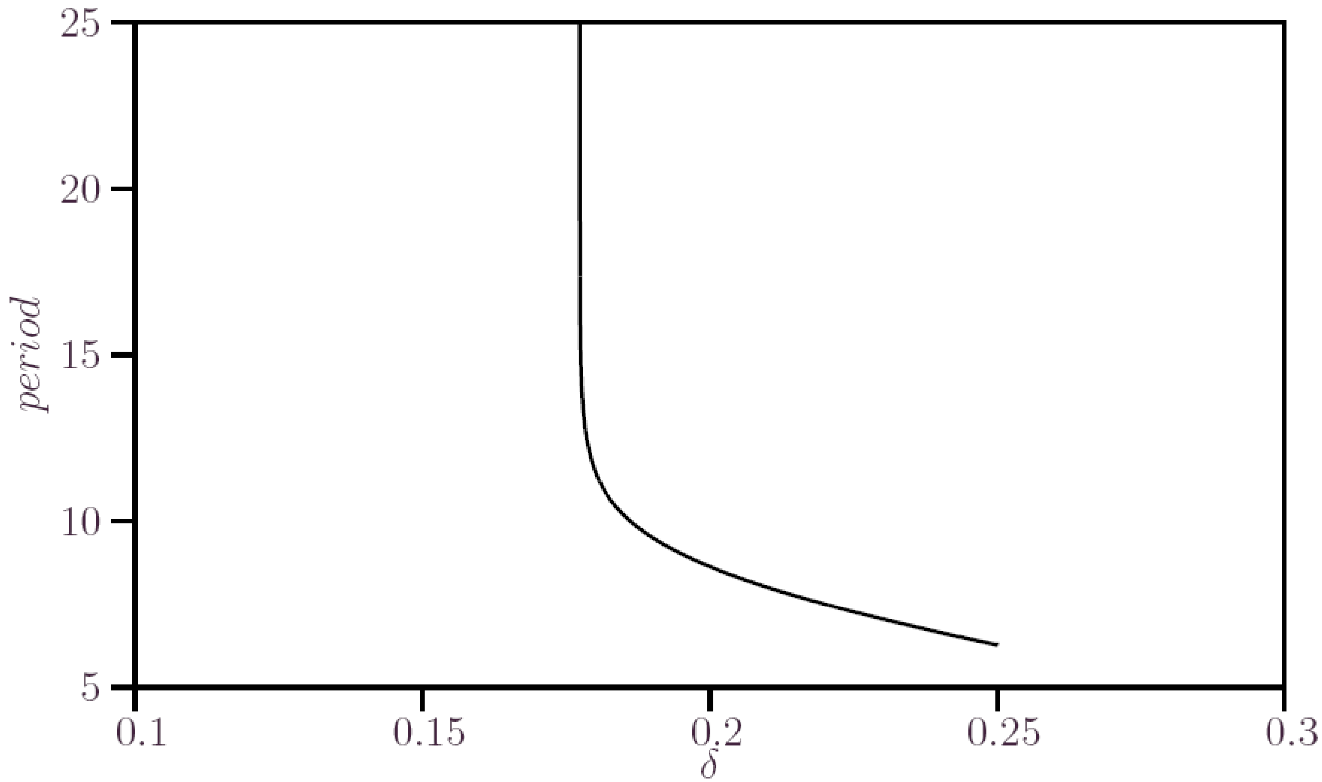
Cusp point → collision two tangent points

Bazykin: homoclinic



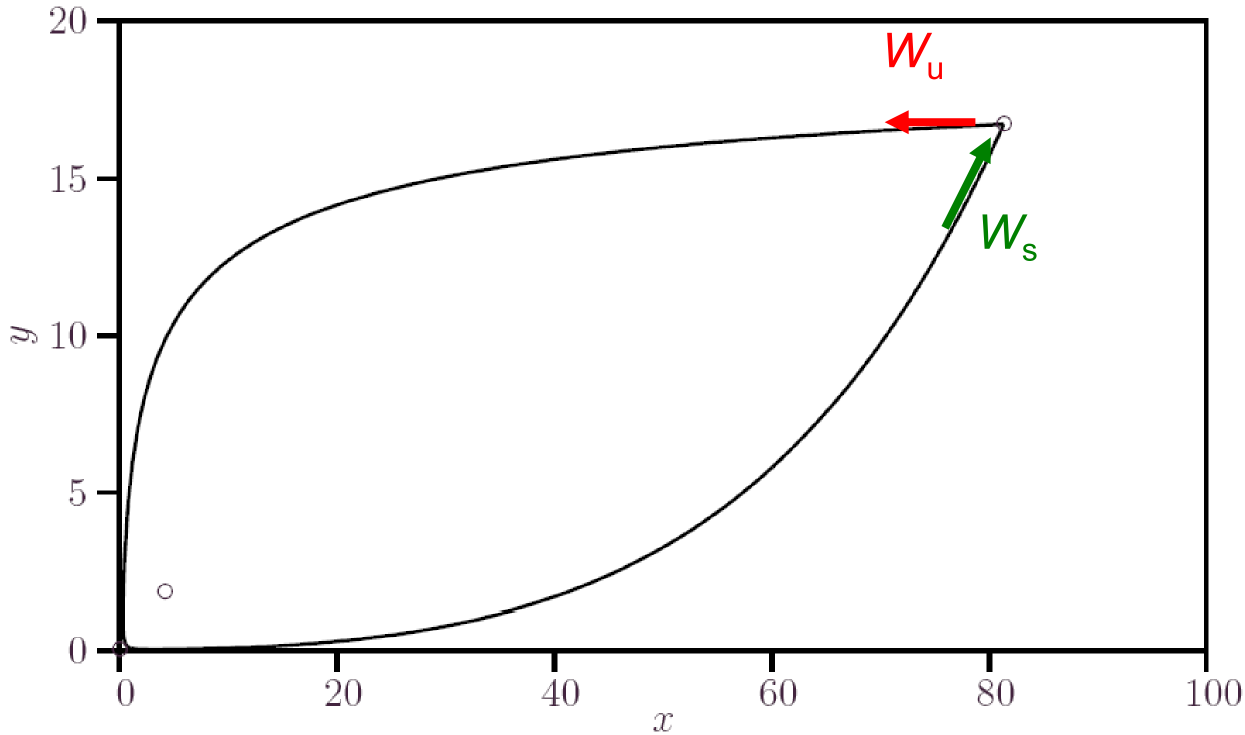
Starting at Hopf continue cycle. What happens?

Bazykin: homoclinic



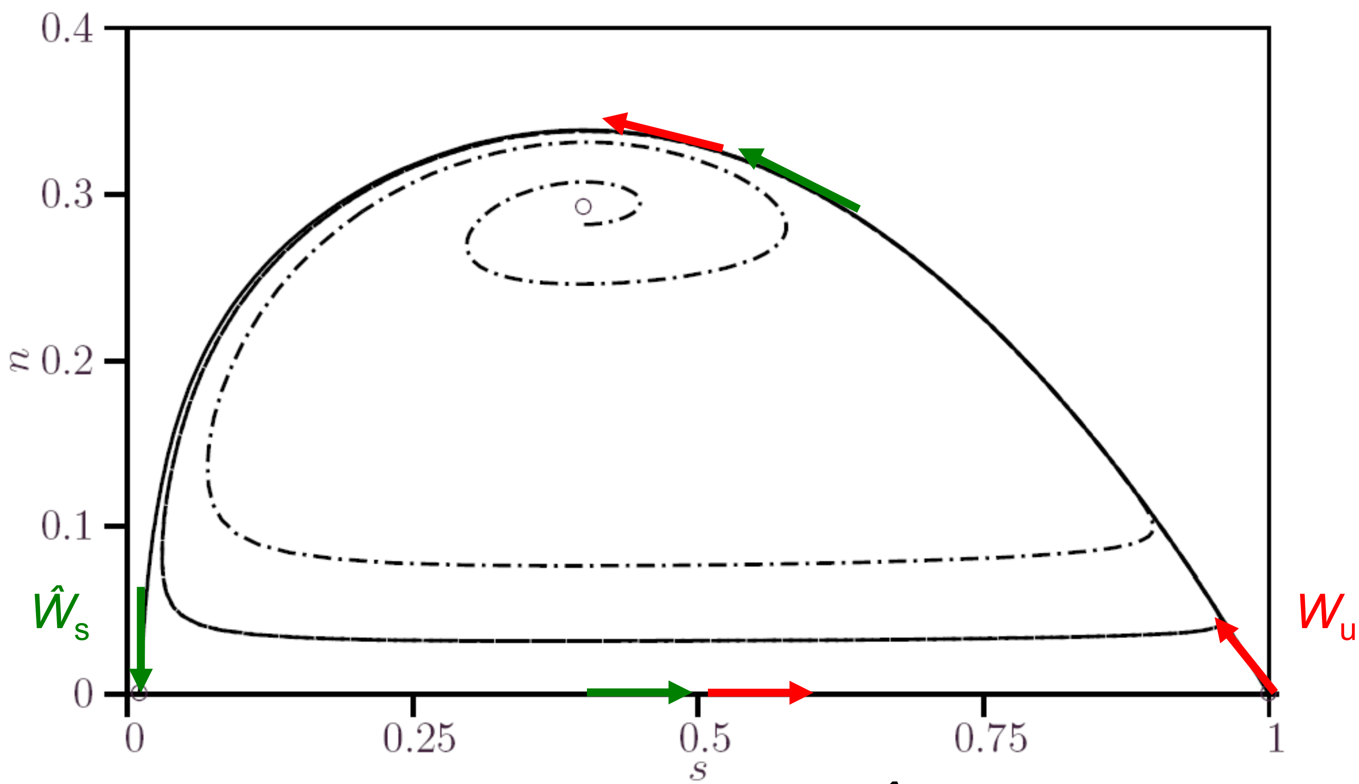
Limit cycle period to infinity. Why?

Homoclinic connection



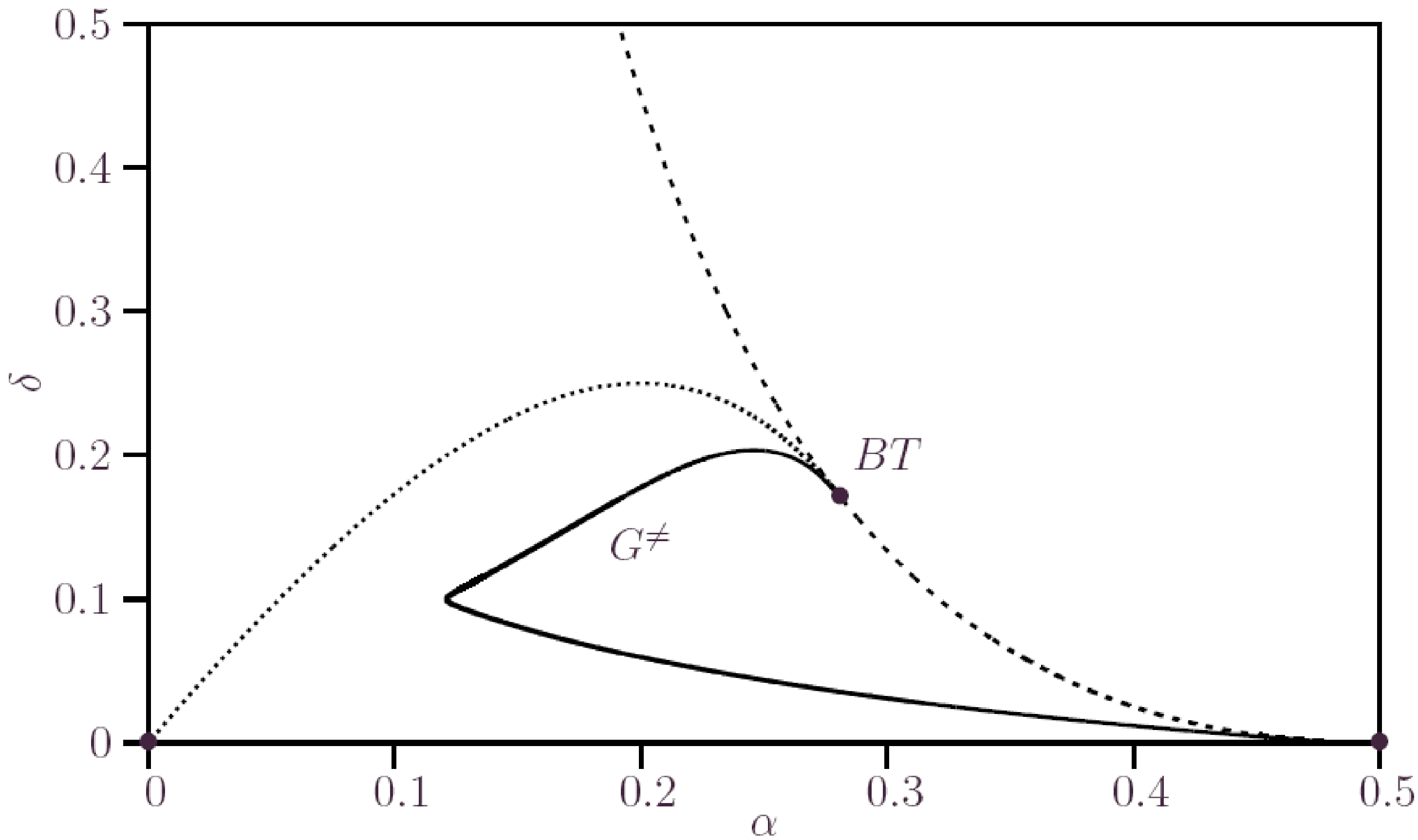
Homoclinic connecting orbit: $W_u = W_s$
Time to infinity near equilibrium

Heteroclinic connection



Heteroclinic connecting orbit: $W_u = \hat{W}_s$

Bazykin: homoclinic



Lyapunov method

- Basic idea:
- If the total energy of a mechanical (or electrical) system is continuously dissipated, then the system, where linear or nonlinear, must eventually settle down to an equilibrium point.
- **Definition:** A scalar continuous function V(x) is said to be locally positive definite if V(0) = 0 and

$$\mathbf{x} \neq \mathbf{0} \quad => \quad V(\mathbf{x}) > 0$$

- If V(0) and the above property holds over the whole state space, then V(x) is said to be globally positive definite.

Lyapunov method

- V(**x**) represents an implicit function of time t.
- Assuming that V(**x**) is differentiable:

$$\dot{V} = \frac{dV(\mathrm{x})}{dt} = \frac{\partial V}{\partial \mathrm{x}}\dot{x} = \frac{\partial V}{\partial \mathrm{x}}\mathrm{f}(\mathrm{x})$$

- **Definition:** If, the function V(**x**) is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of system is negative definite, i. e.,

$$\dot{V}(\mathrm{x}) \le 0$$

  then V(**x**) is said to be a Laypunov function for the system.
- **If a Lyapunov function can be found for a system, it is globally asymptotically stable.**

Example

A simple pendulum with viscous damping

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

Consider a locally positive definite

$V(\mathrm{x}) = (1\text{-}\cos\theta) + \dfrac{\dot{\theta}^2}{2}$:total energy of the pendulum

–The origin is a stable equilibrium point

$$\dot{V}(\mathrm{x}) = \dot{\theta}\sin\theta + \dot{\theta}\ddot{\theta} = -\dot{\theta}^2 \leq 0$$

Chaos

- Do not confuse chaotic with random:

Random:

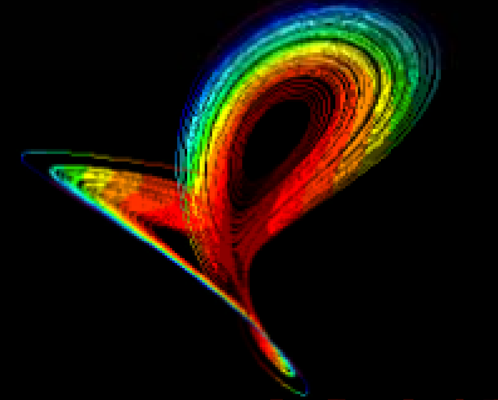
- irreproducible and unpredictable

Chaotic:

- deterministic - same initial conditions lead to same final state… **but the final state is very different for small changes to initial conditions**
- ***difficult or impossible to make long-term predictions***

Chaos: history

2.2 Lorenz Chaos



- In 1963 Lorenz was trying to improve weather forecasting
- Using a computer, he discovered the first chaotic attractor
- Three variables (*x*, *y*, *z*) define convection of the atmosphere
- Changing in time, these variables give a trajectory in a 3D space
- From all starts, trajectories settle onto a strange, chaotic attractor
- Right and left flips occur as randomly as heads and tails
- Prediction is impossible

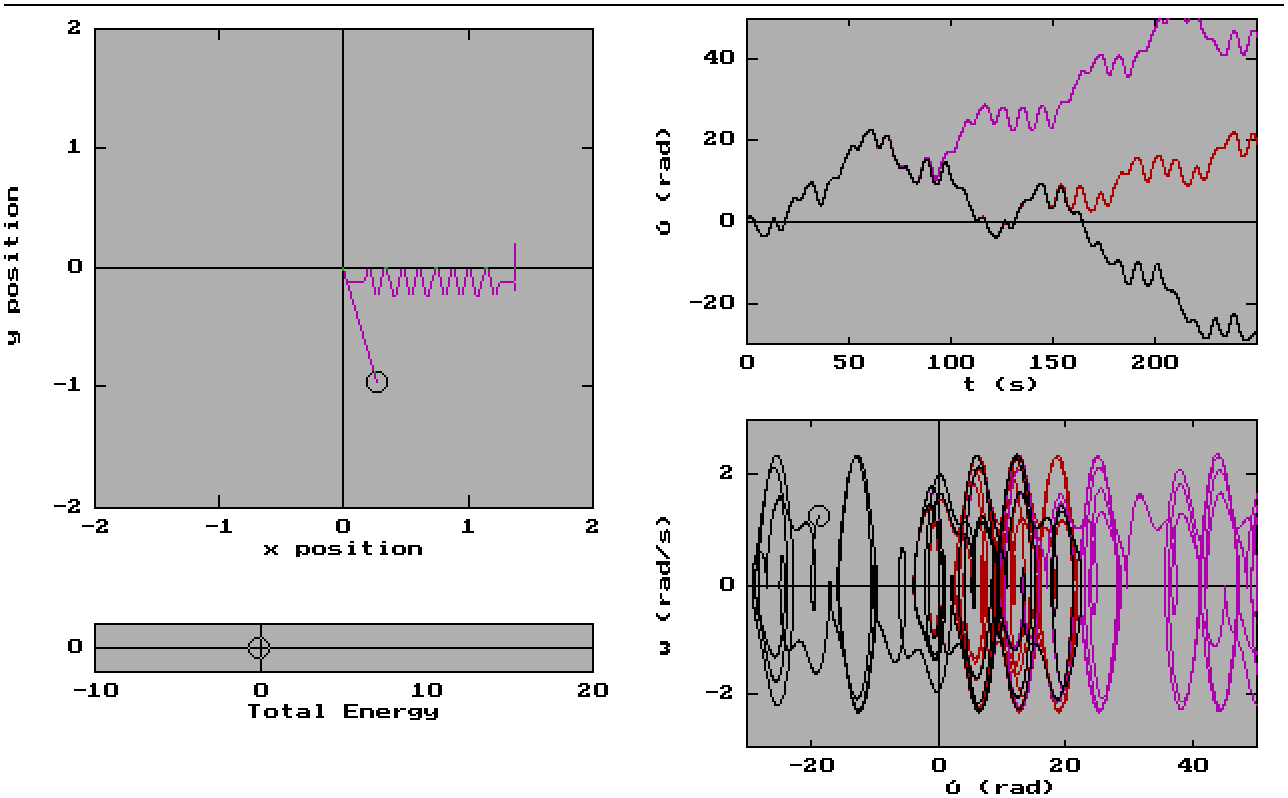
$$x' = -10\,(x - y)$$

$$y' = 28\,x - y - x\,z$$

$$z' = x\,y - (8/3)\,z$$

An example: chaotic pendulum

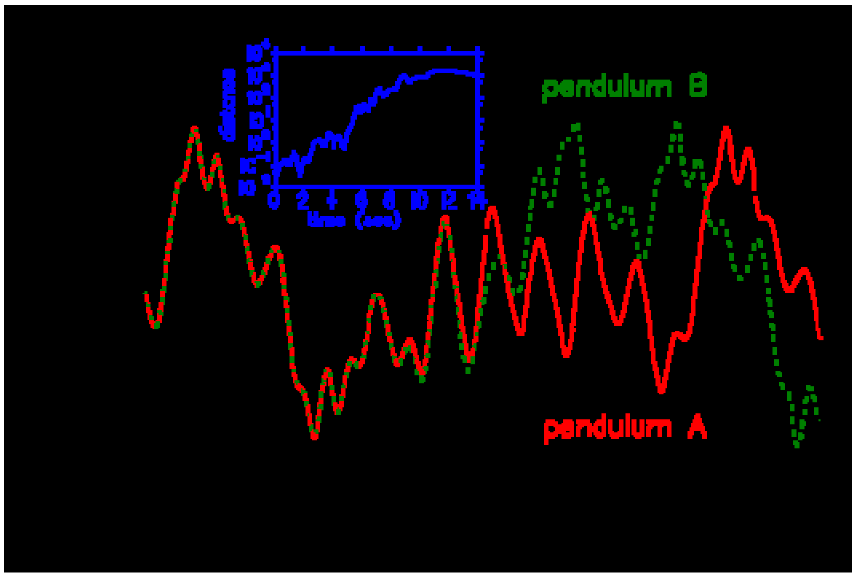
- starting at 1, 1.001, and 1.000001 rad:



Attractor

- A non-wandering set may be stable or unstable
- **Lyapunov stability:** Every orbit starting in a neighborhood of the non-wandering set remains in a neighborhood.
- **Asymptotic stability:** In addition to the Lyapunov stability, every orbit in a neighborhood approaches the non-wandering set asymptotically.
- **Attractor:** Asymptotically stable minimal non-wandering sets.
- **Basin of attraction:** is the set of all initial states approaching the attractor in the long time limit.
- **Strange attractor (or chaos):** attractor which exhibits a sensitive dependence on the initial conditions.

Sensitive to initial condition



pendulum A: φ = -140°, $d\varphi/dt = 0$
pendulum B: φ = -140°1', $d\varphi/dt = 0$

- Definition: A set S exhibits *sensitive dependence* if $\exists r>0$ s.t. $\forall \varepsilon>0$ and $\forall x \in S$ $\exists y$ s.t $|x-y|<\varepsilon$ and $|x_n-y_n|>r$ for some n.

The sensitive dependence of the trajectory on the initial conditions is a key element of deterministic chaos!

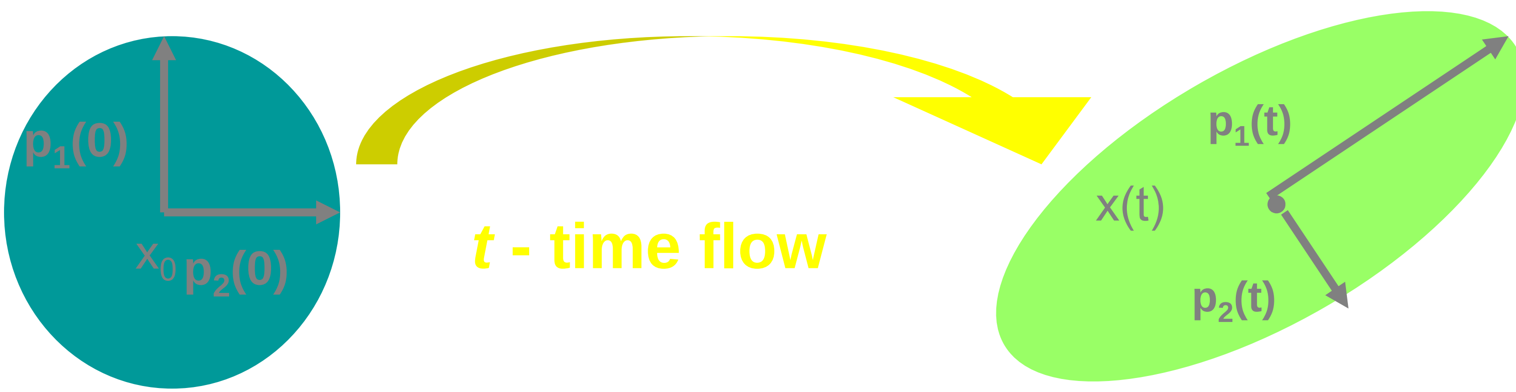
Lyapunov exponent

- A quantitative measure of the sensitive dependence on the initial conditions is **the Lyapunov exponent λ.**
- It is the averaged rate of divergence (or convergence) of two neighboring trajectories in the phase space.
- Actually there is a whole spectrum of Lyapunov exponents. Their number is equal to the dimension of the phase space. If one speaks about *the* Lyapunov exponent, the largest one is meant.

Lyapunov exponent

- Given a continuous dynamical system in an *n*-dimensional phase space, we monitor the long-term evolution of an *infinitesimal n*-sphere of initial conditions.
- The sphere will become an n-ellipsoid due to the locally deforming nature of the flow.
- The *i*-th one-dimensional Lyapunov exponent is then defined as following:

$$\boxed{\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_i(0)}}$$



Lyapunov exponent

- Order: $\lambda_1 > \lambda_2 > \ldots > \lambda_n$
- The linear extent of the ellipsoid grows as $2^{\lambda_1 t}$
- The area defined by the first 2 principle axes grows as $2^{(\lambda_1+\lambda_2)t}$
- The volume defined by the first 3 principle axes grows as $2^{(\lambda_1+\lambda_2+\lambda_3)t}$ and so on…
- The sum of the first *j* exponents is defined by the long-term exponential growth rate of a *j*-volume element.
- Any continuous time-dependent DS without a fixed point will have ≥1 zero exponents.
- The sum of the Lyapunov exponents must be negative in dissipative DS ⇒ ∃ at least one negative Lyapunov exponent.
- A positive Lyapunov exponent reflects a “direction” of *stretching* and *folding* and therefore determines chaos in the system.

The sign of Lyapunov exponent

- 1D maps: $\exists!\ \lambda_1 = \lambda$:
  - $\lambda = 0$ – a marginally stable orbit;
  - $\lambda < 0$ – a periodic orbit or a fixed point;
  - $\lambda > 0$ – chaos.
- 3D continuous dissipative DS: $(\lambda_1, \lambda_2, \lambda_3)$
  - (+,0,-) – a strange attractor (chaos);
  - (0,0,-) – a two-torus;
  - (0,-,-) – a limit cycle;
  - (-,-,-) – a fixed point.

The sign of Lyapunov exponent

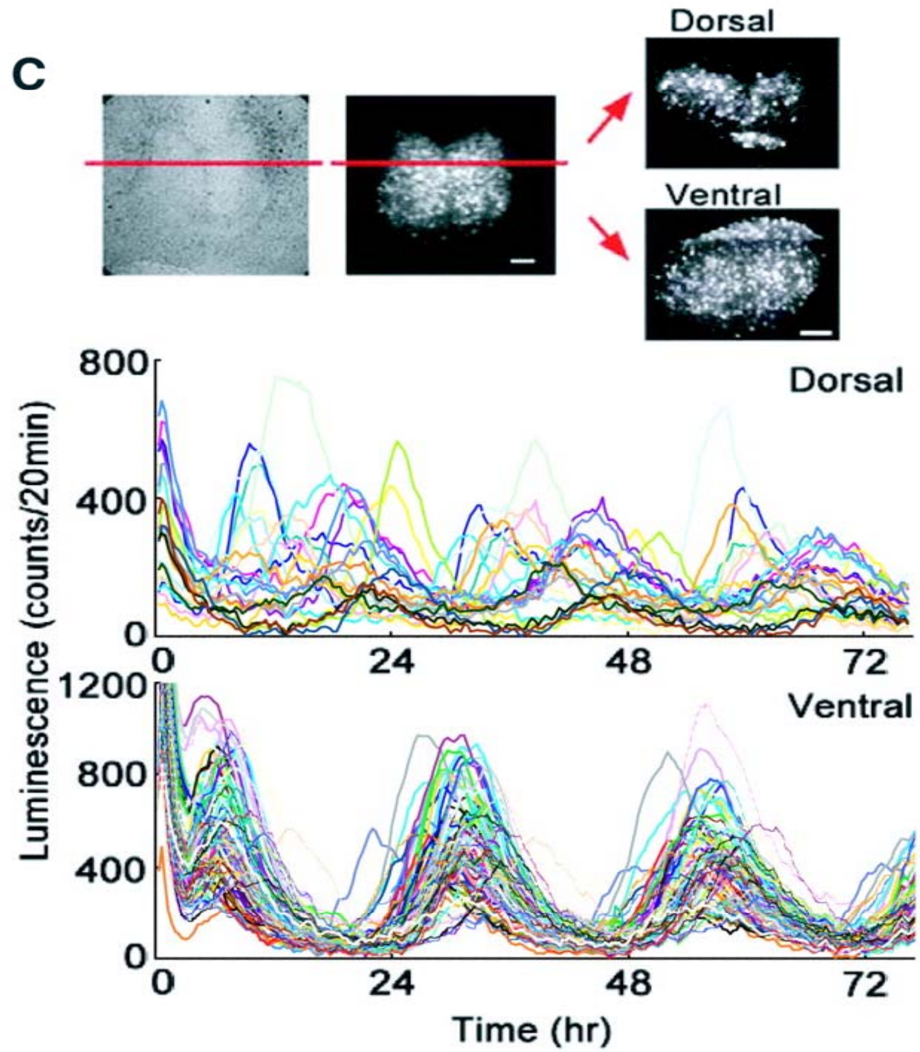
- **$\lambda < 0$ -** the system attracts to a fixed point or stable periodic orbit. These systems are non conservative (dissipative) and exhibit *asymptotic stability*.
- **$\lambda = 0$ -** the system is neutrally stable. Such systems are conservative and in a steady state mode. They exhibit *Lyapunov stability*.
- **$\lambda > 0$ -** the system is chaotic and unstable. Nearby points will diverge irrespective of how close they are.

Synchronization

- Cellular clocks in the brain
- Pacemaker cells in the heart
- Pedestrians on a bridge
- Electric circuits
- Laser arrays
- Oscillating chemical reactions
- Bubbly fluids
- Neutrino oscillations
- Parkinson’s disease

Male fireflies flashing in unison

Circadian Rhythm

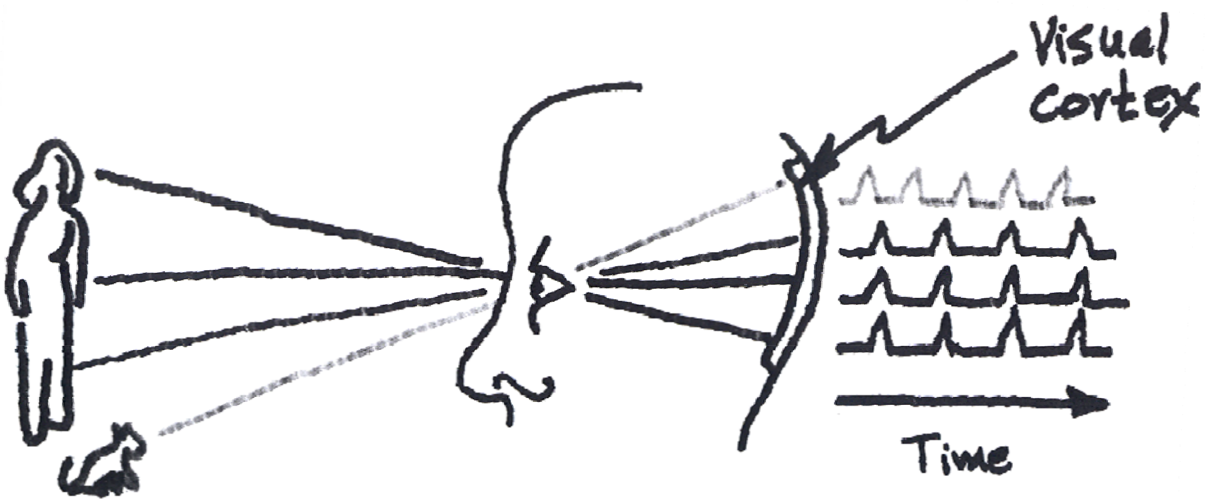


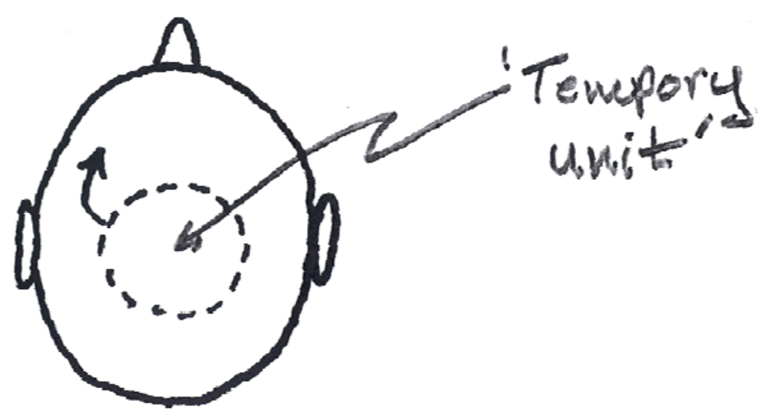
Incoherent

Coherent

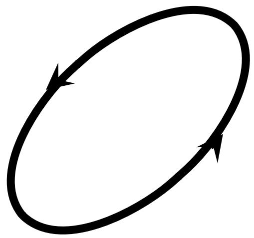
Yamaguchi et al., Science 302 p.1408 (2003).

Synchrony in the brain

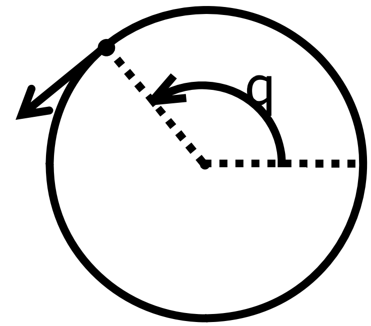




Coupled phase oscillators



Change of variables



Limit cycle in phase space

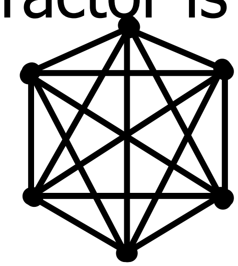
$$d\theta/dt = \omega$$

Many such 'phase oscillators': $\dfrac{d\theta_i}{dt} = \omega_i$; i=1,2,…,N »1

Couple them: $$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} k_{ij}(\theta_j - \theta_i)$$

$$k_{ii}(\phi) = 0, \quad k_{ij}(\phi) = k_{ij}(\phi \pm 2\pi)$$

Assumption: Attraction to limit cycle attractor is 'strong'.



$N = 6:$

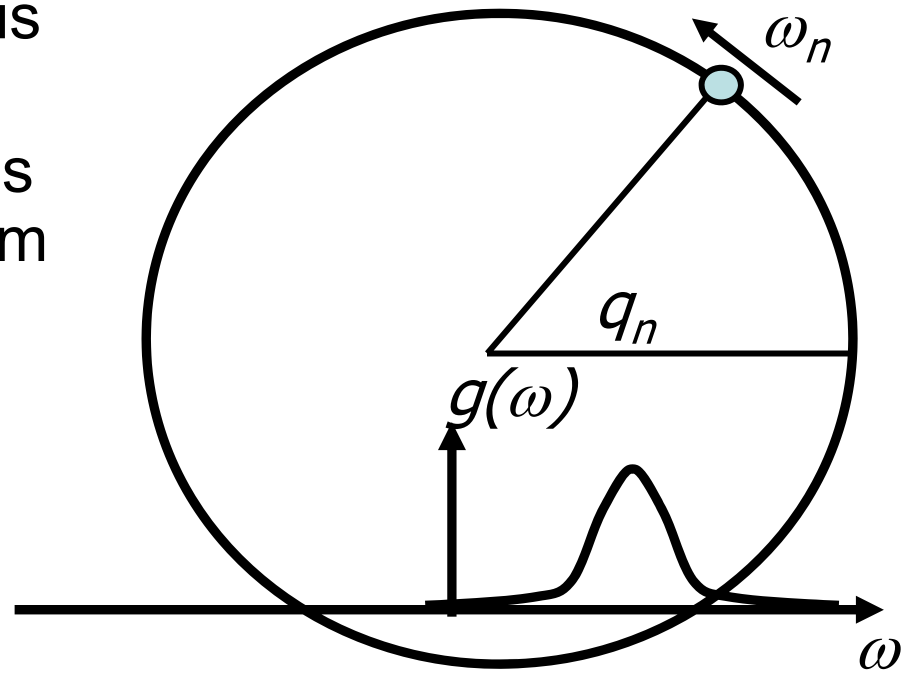
Global coupling

Kuramoto 1975: $k_{ij}(\phi) = k\sin(\phi)$

(H.Daido: PRL 1994)

Framework

- N oscillators described only by their phase q. N is very large.
- The oscillator frequencies are randomly chosen from a distribution g(ω) with a single local maximum.
- We assume the mean frequency is zero



Kuramoto model: all-to-all coupling

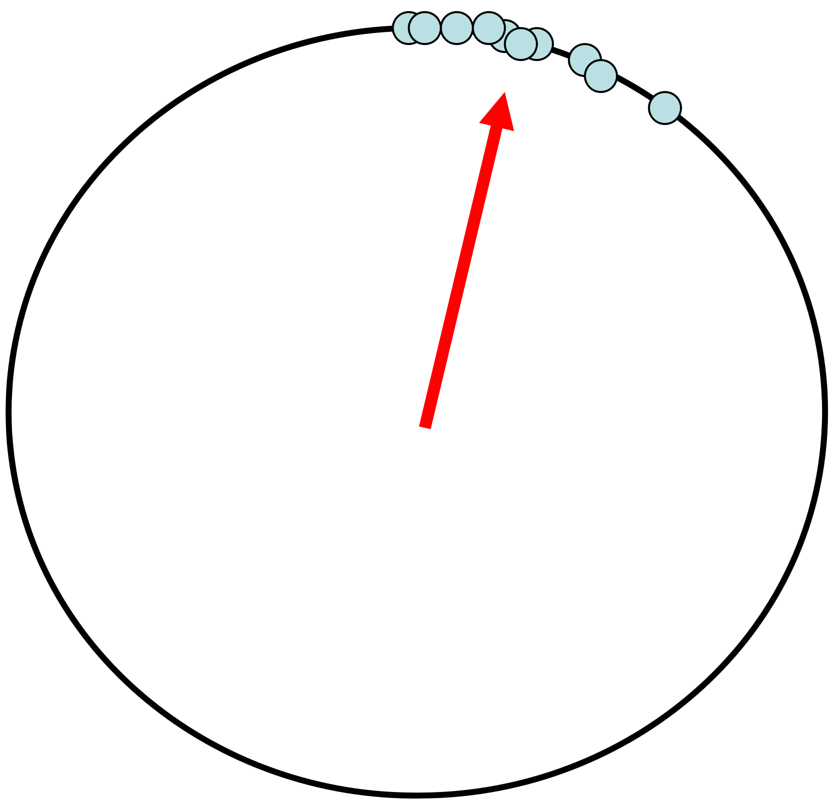
$$\frac{d\theta_n}{dt} = \omega_n + k \sum_{m=1}^{N} \sin(\theta_m - \theta_n)$$

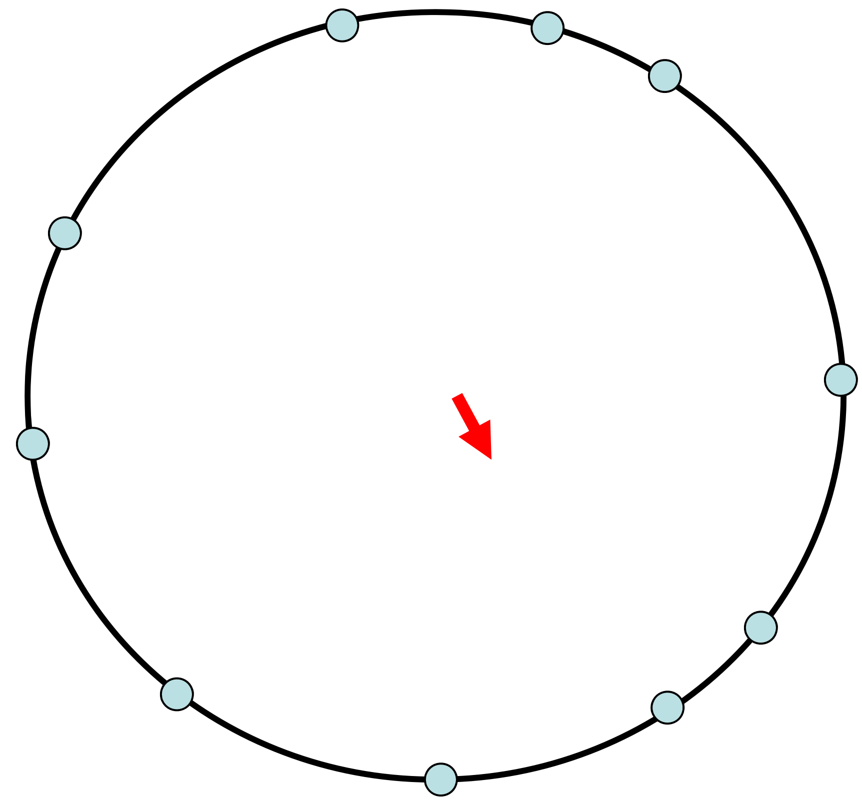
$n = 1, 2, \ldots, N$ $\quad$ k = (coupling constant)

- Assumes sinusoidal *all-to-all* coupling.
- Macroscopic coherence of the system is characterized by

$$r = \left| \frac{1}{N} \sum_{m=1}^{N} \exp(i\theta_m) \right| = \text{“order parameter”}$$

Order parameter measures coherence



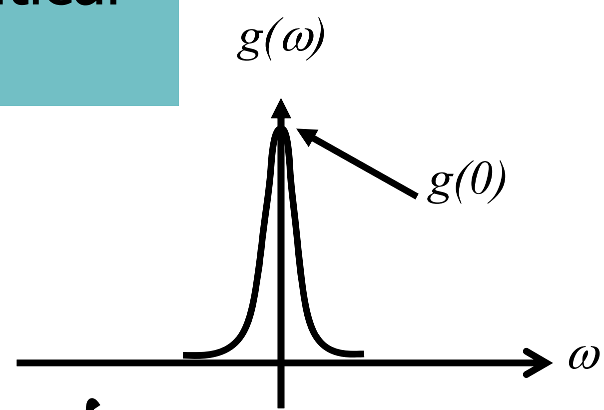


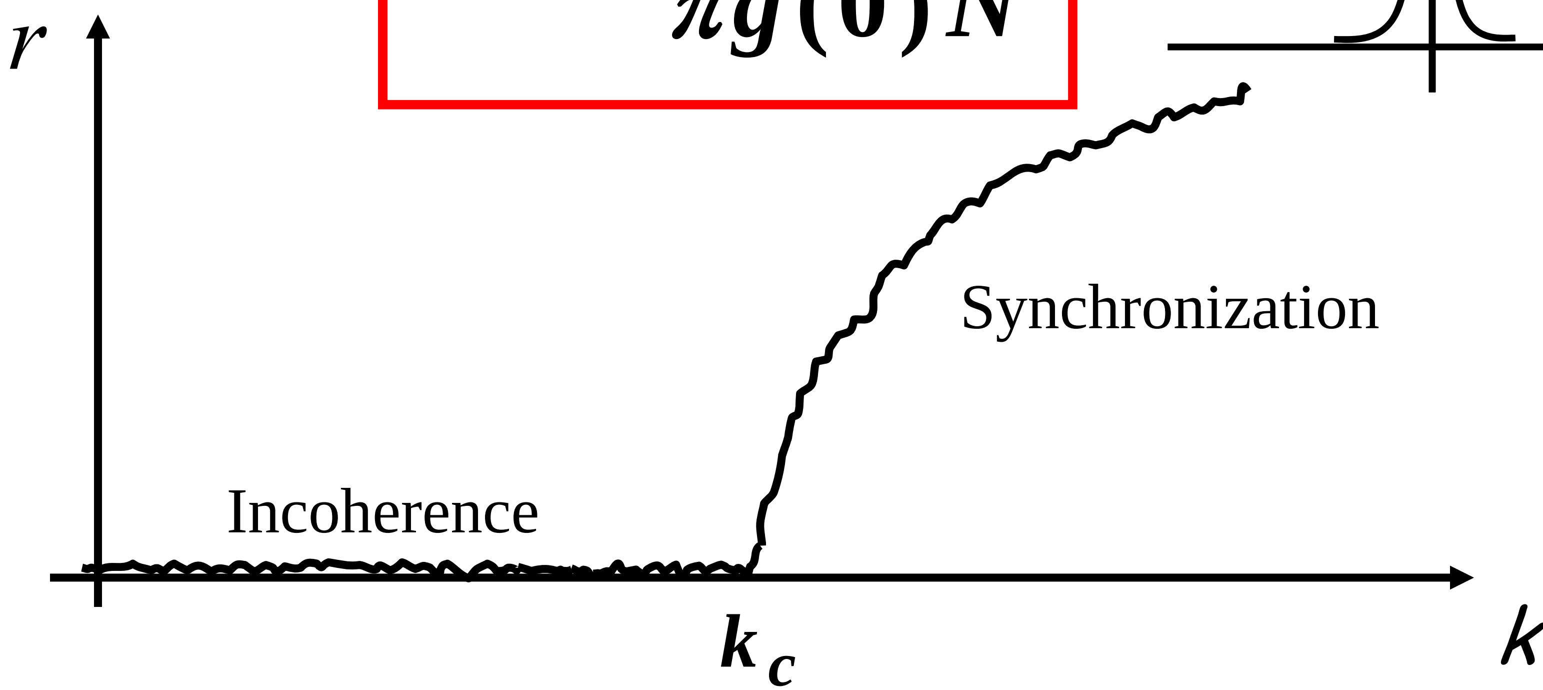
$r \approx 1$ $\qquad\qquad$ $r \approx 0$

Results of Kuramoto model

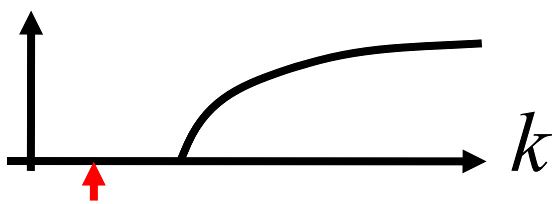
There is a transition to synchrony at a critical value of the coupling constant.

$$\boxed{k_c = \frac{2}{\pi g(0) N}}$$

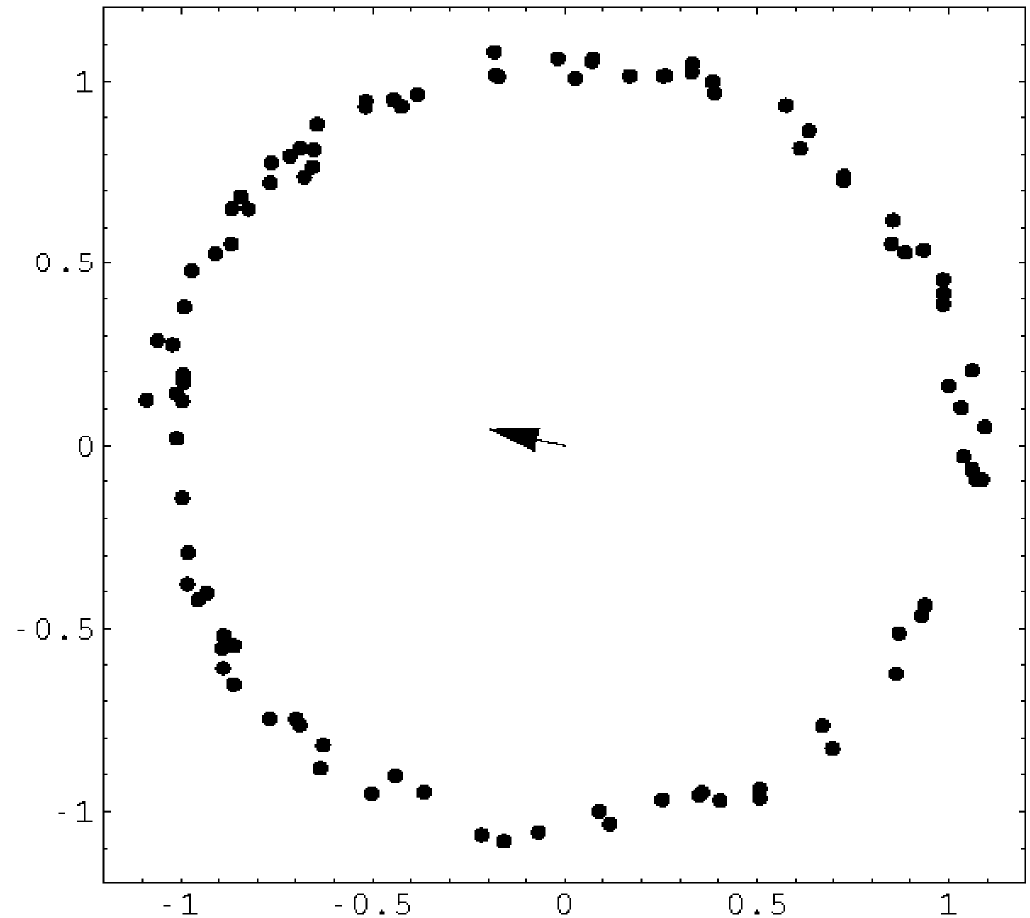




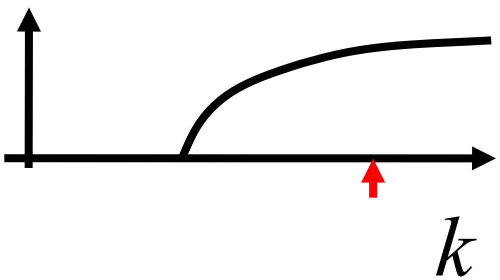
Example for incoherent case



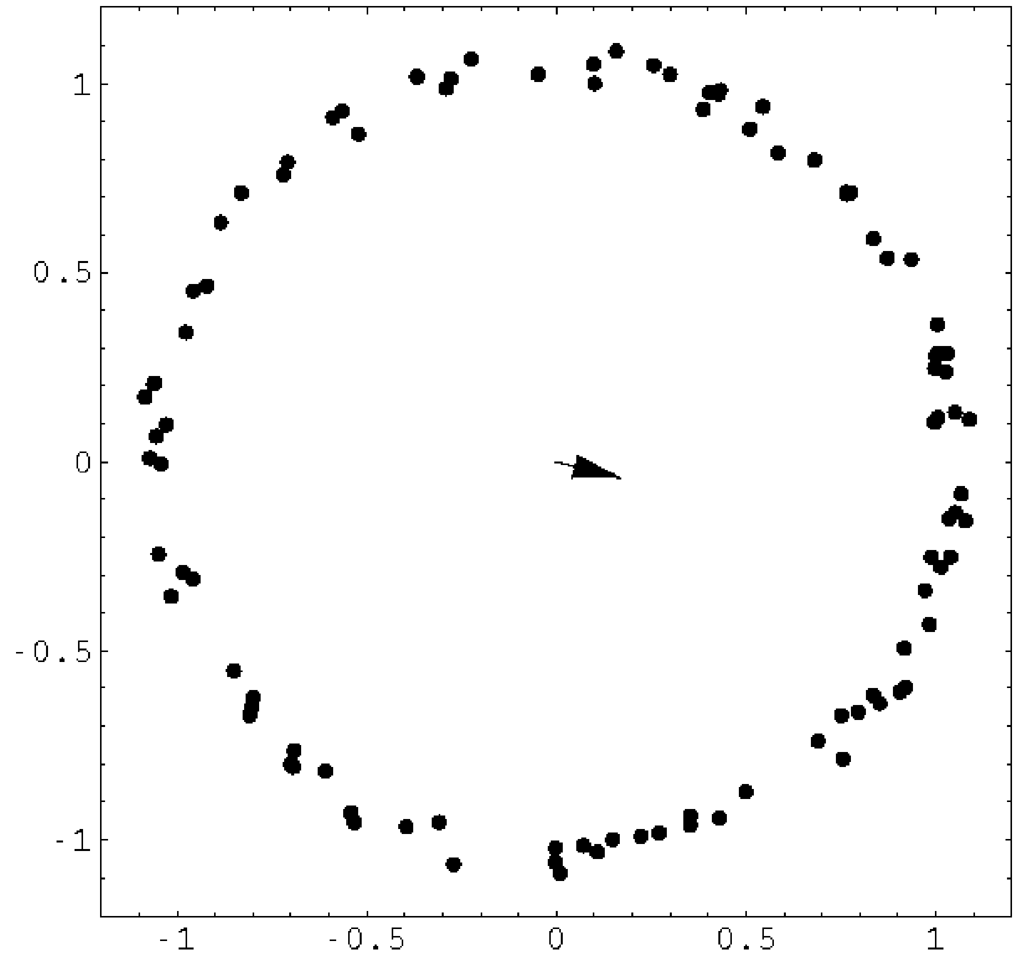
$k < k_C$



Example for coherent case



$$k > k_C$$



Derivation of k_c

$$\boxed{\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}$$

$$\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) = \frac{1}{N}\,\mathrm{Im}\left\{\sum_{j=1}^{N} e^{i(\theta_j - \theta_i)}\right\}$$

$$= \mathrm{Im}\left\{e^{-i\theta_i}\underbrace{\left(\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}\right)}_{r\,e^{i\psi}}\right\} = \mathrm{Im}\left[r\,e^{i(\psi-\theta)}\right] = r\sin(\psi - \theta_i)$$

"The order parameter"

$$\boxed{\begin{gathered} d\theta_i/dt = \omega_i + k\,r\sin(\psi - \theta_i) \\ r\,e^{i\psi} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \end{gathered}}$$

N $\rightarrow\infty$

Introduce the distribution function *F(q,w,t)*

$$F(\theta,\omega,t)\,d\omega\, d\theta =$$

[the fraction of oscillators with phases in the range *(q,q+dq)* and freqs. in the range *(w,w+dw)*]

$$\int_0^{2\pi} F\,d\theta \equiv g(\omega)$$

Conservation of number of oscillators:

$$\frac{\partial F}{\partial t}+\frac{\partial}{\partial\theta}\left[\frac{d\theta}{dt}F\right]+\frac{\partial}{\partial\omega}\left[\overset{0}{\cancel{\frac{d\omega}{dt}}}F\right]=0$$

$$\boxed{\begin{gathered}\frac{\partial F}{\partial t}+\frac{\partial}{\partial\theta}\left\{\left[\omega+k\,r\sin(\psi-\theta)\right]F\right\}=0\\ re^{i\psi}=\int\int_0^{2\pi}F\,e^{i\theta}\,d\theta\, d\omega\end{gathered}}$$

Incoherent solution

$$F(\theta,\omega,t) = \frac{g(\omega)}{2\pi}$$ (uniform distribution in angle)

$$\int_0^{2\pi} F\,\frac{d\theta}{2\pi} = 0 \quad \Rightarrow \quad r = 0$$

$$\cancelto{0}{\frac{\partial F}{\partial t}} + \cancelto{0}{\frac{\partial}{\partial \theta}\{\omega F\}} = 0 \ \checkmark$$

Q. Is it stable?

A. Yes, for $k<k_c$. No, for $k > k_c$.

Stability analysis

$$F(\theta,\omega,t) = g(\omega)/2\pi + \underbrace{f(\theta,\omega)e^{st}}_{\text{perturbation}}, \quad r(t) = \hat{r}\,e^{st}.$$

$$\mathrm{Re}(s) > 0 \quad \Rightarrow \quad \text{unstable.} \quad \mathrm{Re}(s) < 0 \quad \Rightarrow \quad \text{stable.}$$

Dispersion relation

$$\boxed{s f + \omega \frac{\partial f}{\partial \theta} = k\,\hat{r}\,\frac{g(\omega)}{2\pi}\cos(\psi - \theta) \Rightarrow \hat{r} e^{i\psi} = \iint d\omega\, d\theta\, f(\theta,\omega) e^{i\theta}}$$

$$\cos(\psi-\theta) = \tfrac{1}{2}\left(e^{i(\psi-\theta)} + e^{-i(\psi-\theta)}\right)$$

Look for a solution of the form

$$f = f_1 e^{i(\psi-\theta)} + f_2 e^{-i(\psi-\theta)}$$

Get

$$f = \frac{g(\omega)}{4\pi} k\,\hat{r}\left(\frac{e^{i(\psi-\theta)}}{s-i\omega} + \frac{e^{-i(\psi-\theta)}}{s+i\omega}\right)$$

$$D(s) \equiv 1 - \frac{k}{2}\int_{-\infty}^{+\infty} \frac{g(\omega)}{s-i\omega} d\omega \quad , \quad \mathrm{Re}(s) > 0$$

$$D(s) = 0 \quad \Rightarrow \quad \text{determines } s$$

Kuramoto model on a network

The network is introduced by means of a matrix A

$$\boxed{\frac{d\theta_n}{dt} = \omega_n + k\sum_{m=1}^{N} A_{nm}\sin(\theta_m - \theta_n)}$$

m is not connected to n $\Longleftrightarrow$ $A_{nm} = 0$.
PDF of frequencies symmetric about 0.

The nonzero elements of A can have any positive or negative value and correspond to the interaction strength at each link.

Order parameter description

Local order parameter for node *n*

$$r_n \exp(i\psi_n) \equiv \sum_{m=1}^{N} A_{nm} \left\langle \exp(i\theta_m(t)) \right\rangle_t$$

where $\langle \ldots \rangle_t$ = time average.

Global order parameter: $r = \sum_{n=1}^{N} r_n \Big/ \sum_{n=1}^{N} d_n$

where d_n is the node degree: $d_n = \sum_{m=1}^{N} A_{nm}$

PROBLEM: Find *r* vrs. *k*

Order parameter from the dynamics

$$\dot{\theta}_n = \omega_n + k \sum_{m=1}^{N} A_{nm} \sin(\theta_m - \theta_n)$$

and

$$r_n e^{i\psi_n} \equiv \sum_{m=1}^{N} A_{nm} \left\langle e^{i\theta_m(t)} \right\rangle_t$$

yield

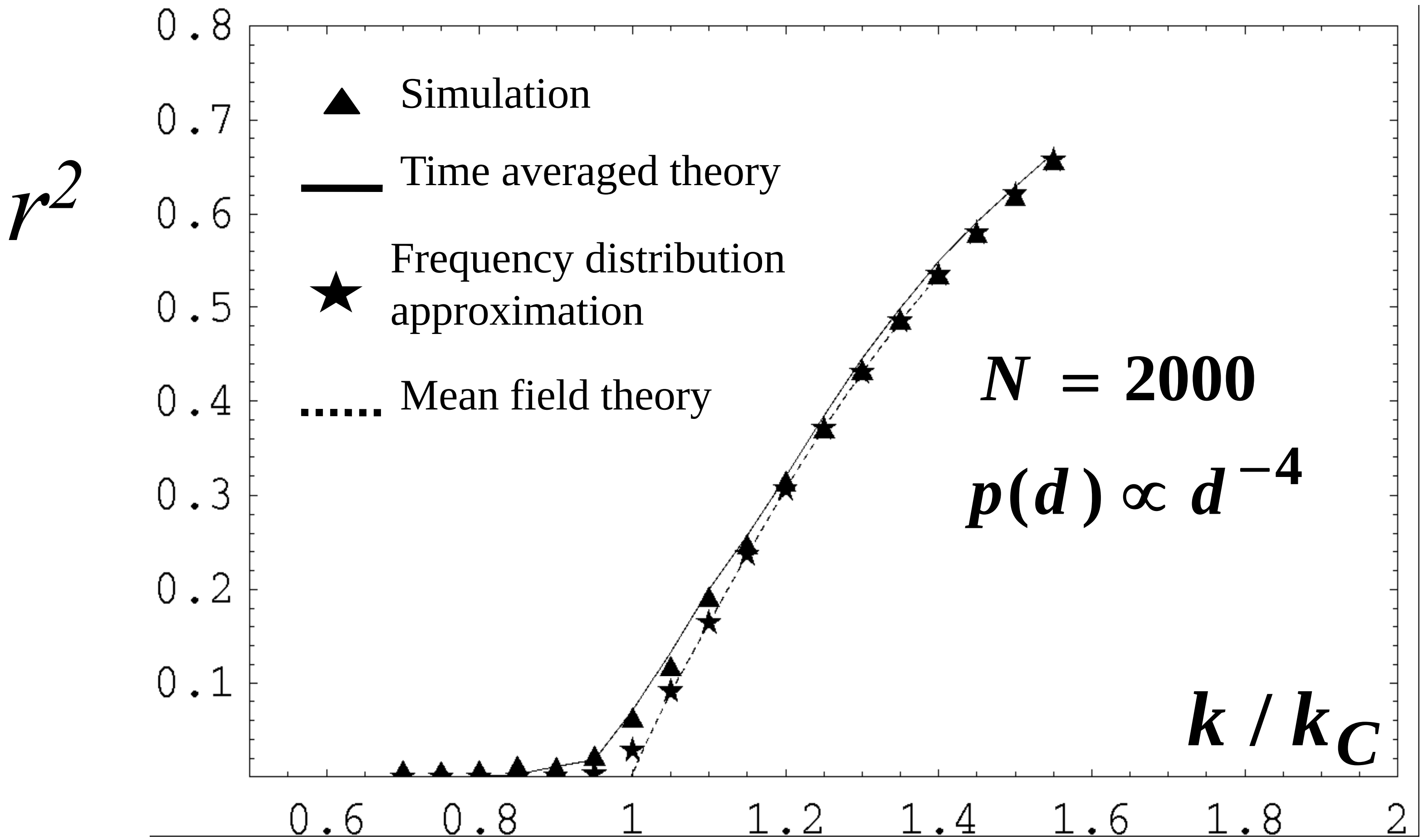
$$\dot{\theta}_n = \omega_n - k r_n \sin(\theta_n - \psi_n) - k h_n(t)$$

where

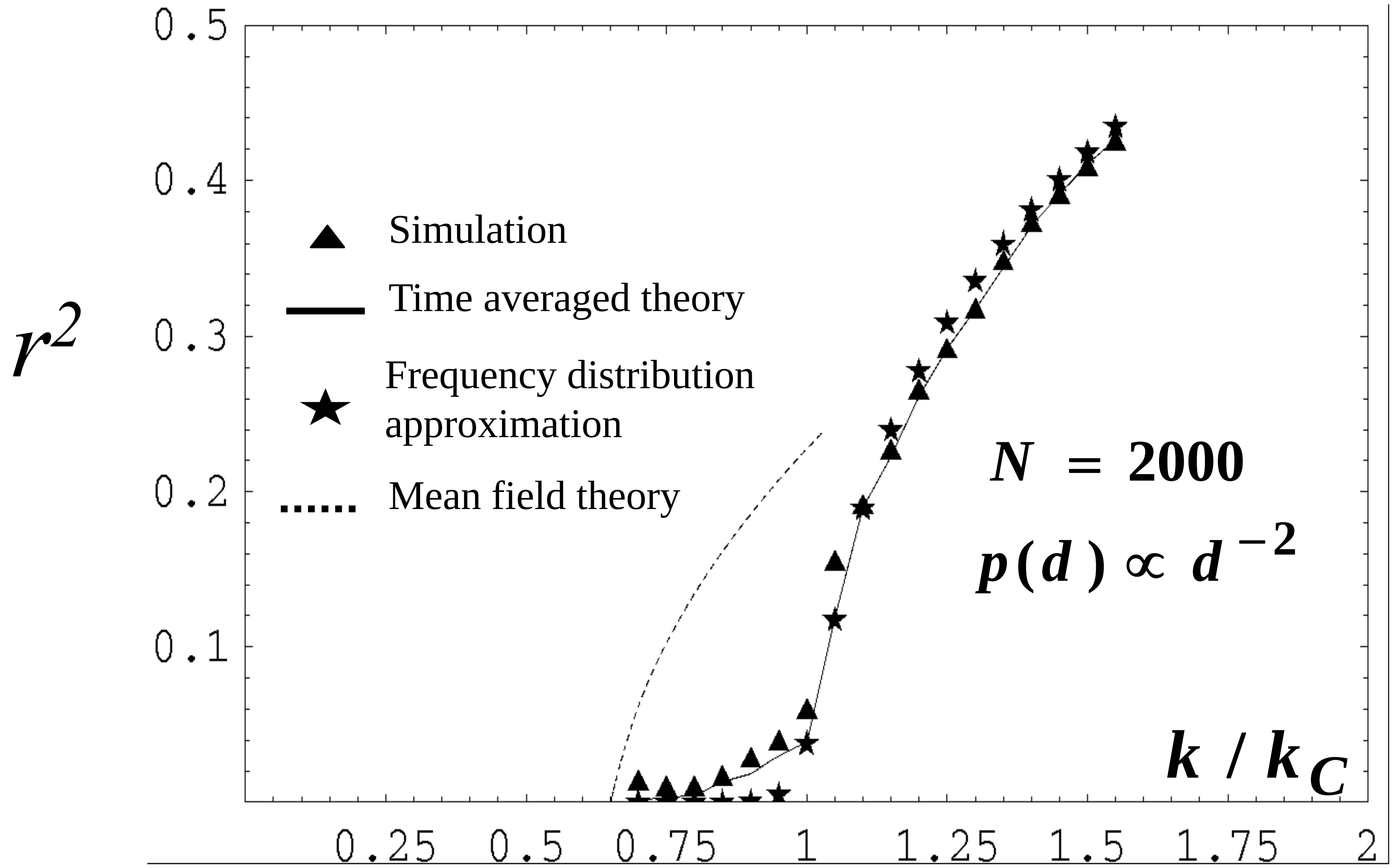
$$h_n(t) = \mathrm{Im}\left\{ e^{-i\theta_n} \sum_{m=1}^{N} A_{nm} \left(\left\langle e^{i\theta_m} \right\rangle_t - e^{i\theta_m} \right) \right\}$$

Example: scale-free networks

We prescribe a degree distribution of the form $p(d) \propto d^{-\gamma}$, $d \geq 50$, and $N = 2000$.



Example: scale-free networks



Nets with general node dynamics

Uncoupled node dynamics:

$$d\,\underline{x}_i(t)/dt = \underline{F}_i(\underline{x}_i(t)) \qquad \text{(continous time)}$$

or

$$\underline{x}_i^{(n+1)} = \underline{M}_i(\underline{x}_i^{(n)}) \qquad \text{(discrete time)}$$

Could be periodic or chaotic.
Kuramoto is a special case: $\underline{x}_i \to \theta_i\,,\ \underline{F}_i \to \omega_i$

Types of chaos synchronization

Complete Synchronization

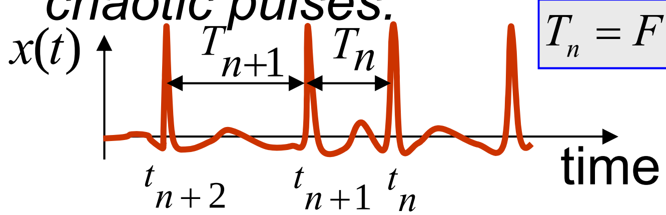
- *Identical synchronous chaotic oscillations.*

$$\lim_{t\to\infty}\left|x(t)-y(t)\right|=0$$

- *Generalized synchronized chaos.*

$$\lim_{t\to\infty}\left|x(t)-\Phi\big(y(t)\big)\right|=0$$

- *Threshold synchronization of chaotic pulses.*

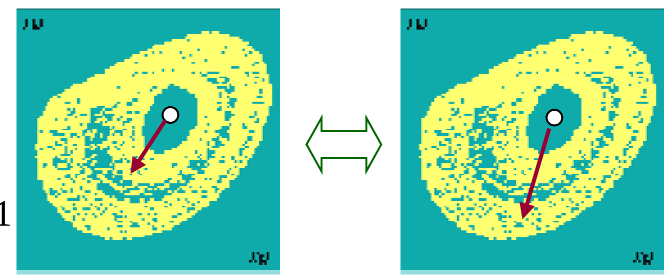


$$\boxed{T_n = F(T_{n+1})}$$

$$x(t)\Rightarrow t_n^{(x)},\; y(t)\Rightarrow t_n^{(y)} \qquad t_n^{(x)}=t_n^{(y)}+\Delta$$

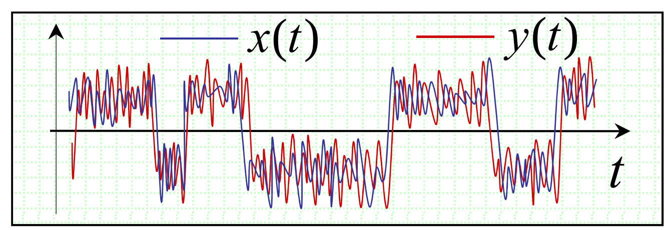
Partial Synchronization

- *Phase Synchronization.*

A_1,φ_1  A_2,φ_2

$$\left|\varphi_1-\varphi_2\right|<const$$

- *Synchronization of switching.*

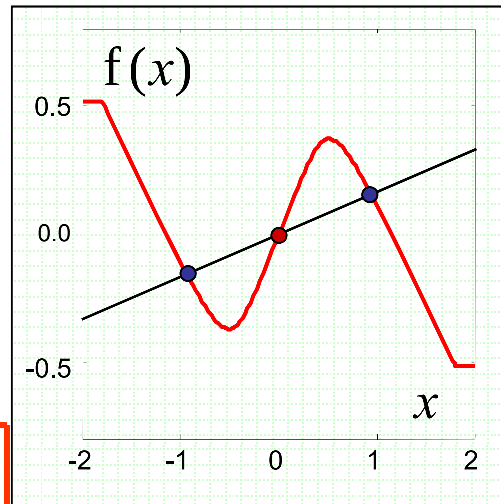


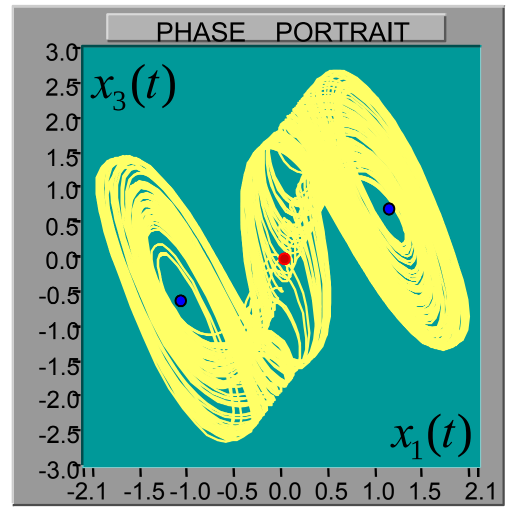
- *Others*

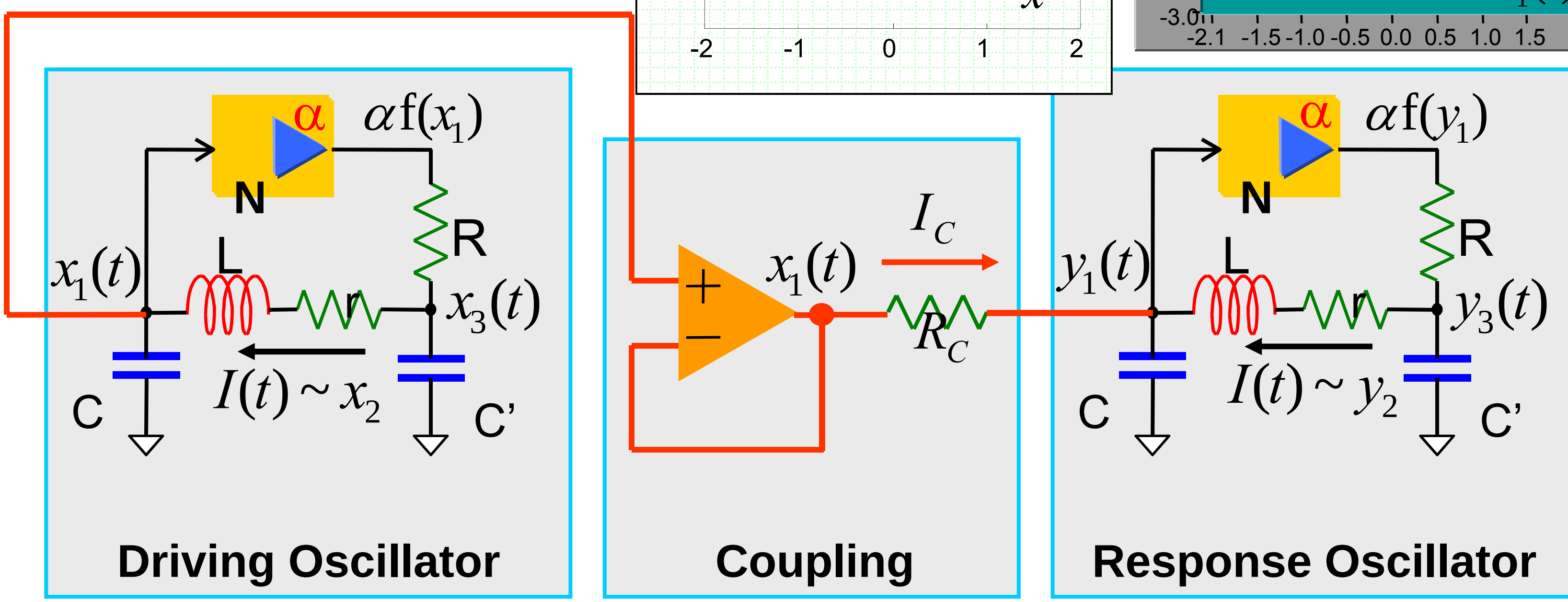
Synchronization of chaos in electrical circuits

Unidirectional coupling

$$I_C = \frac{1}{R_C}\left(x_1(t) - y_1(t)\right)$$







Synchronization Manifold

The model:

$$\dot{x}_1 = x_2 \qquad \dot{y}_1 = y_2 \boxed{+ g(x_1 - y_1)}$$

$$\dot{x}_2 = -x_1 - \delta x_2 + x_3 \qquad \dot{y}_2 = -y_1 - \delta y_2 + y_3$$

$$\dot{x}_3 = \gamma[\alpha\, \mathrm{f}(x_1) - x_3] - \sigma x_2 \qquad \dot{y}_3 = \gamma[\alpha\, \mathrm{f}(y_1) - y_3] - \sigma y_2$$

The coupling parameter:

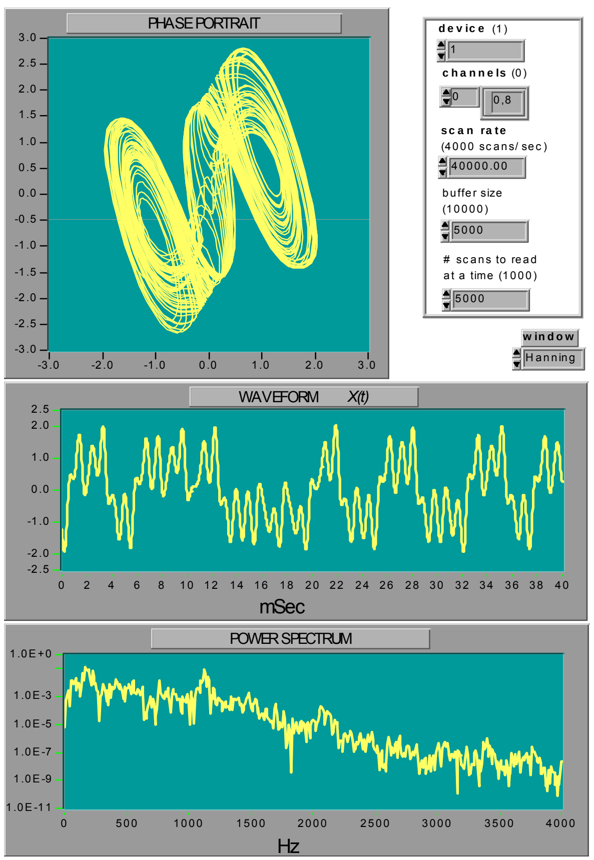
$$g = \frac{1}{Rc}\sqrt{\frac{L}{C}}$$

There exits a 3-dimensional <u>invariant manifold</u>:

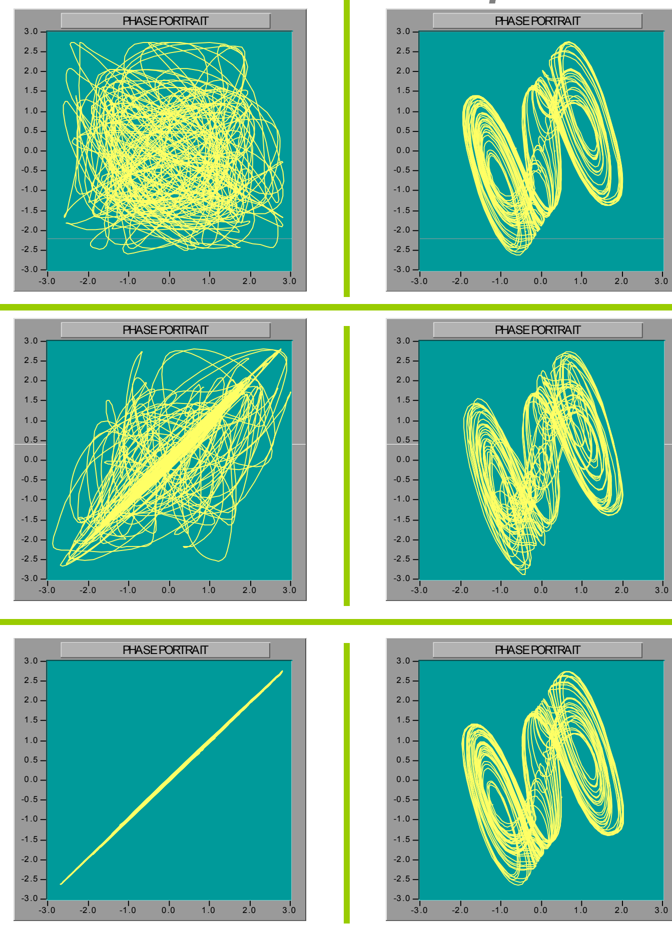
$$\boxed{\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ x_3 &= y_3 \end{aligned}}$$

Synchronization of chaos: Experiment

Driving Oscillator



Response Oscillator



Uncoupled Oscillators

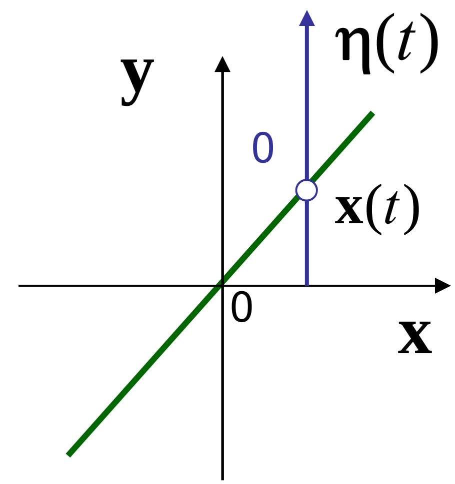
Coupling below the threshold of synchronization

Coupling above the threshold of synchronization

Stability of the Synchronization Manifold: Identical Synchronization

Driving System: $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \qquad \mathbf{x} \in \Re^{\mathbf{n}},$

Response System: $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}) + \mathbf{G}(\mathbf{x} - \mathbf{y}), \qquad \mathbf{y} \in \Re^{\mathbf{n}}, \quad \mathbf{G}(0) = 0$



Synchronization Manifold: $\mathbf{x} = \mathbf{y}$

Perturbations transversal to the Synchronization Manifold: $\eta(t) = \mathbf{y}(t) - \mathbf{x}(t)$

Linearized Equations for the transversal perturbations:

$$\dot{\eta}(t) = [\mathbf{DF}(\mathbf{x}(t)) + \mathbf{DG}(0)] \times \eta, \qquad DF(\mathbf{x}(t))_{ij} = \frac{\partial F_i}{\partial x_j}$$

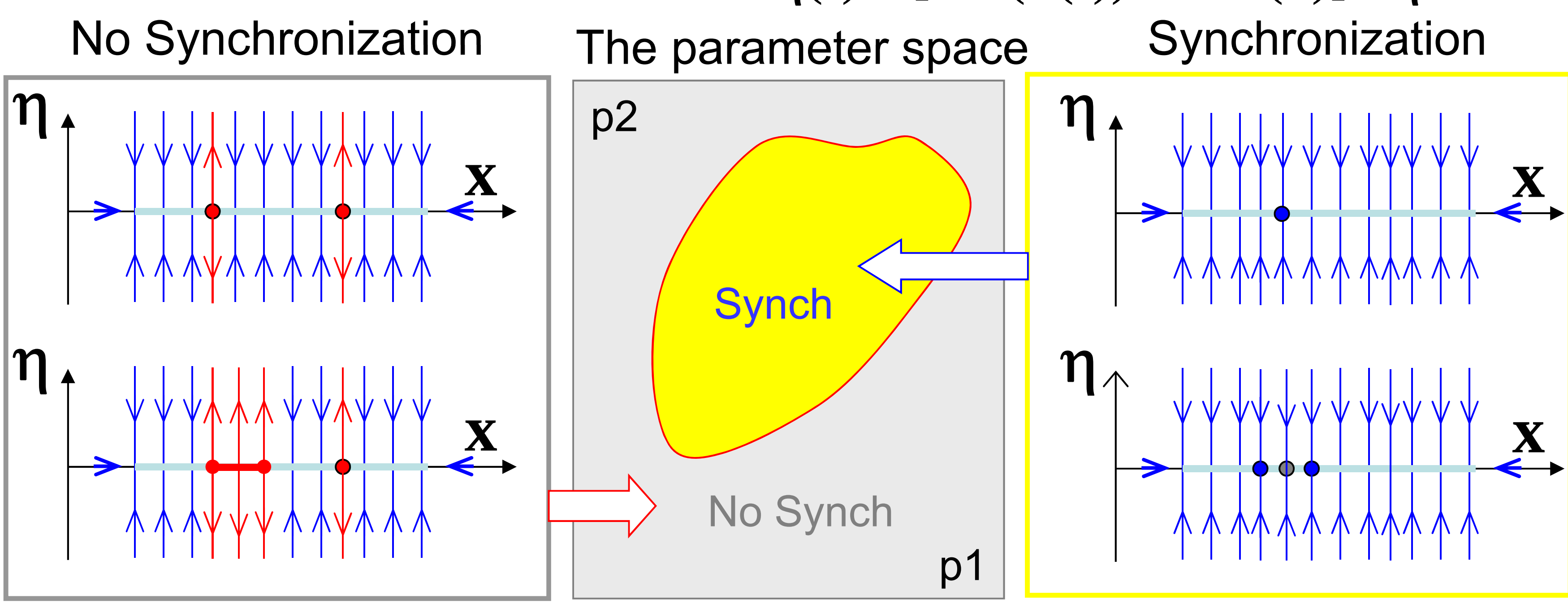
Chaos Synchronization Regime

A regime of dynamical behavior should have a qualitative feature that is an invariant for this regime.

Consider dynamics in the phase space $(\mathbf{x}, \boldsymbol{\eta})$

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

$$\dot{\boldsymbol{\eta}}(t) = [\mathbf{DF}(\mathbf{x}(t)) + \mathbf{DG}(0)] \times \boldsymbol{\eta}$$



- Projection of chaotic limiting set

- Limit cycles

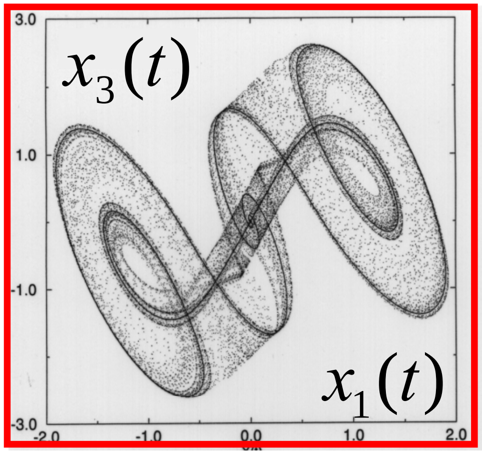
Transient trajectories

Synchronization of Chaos in Numerical Simulations

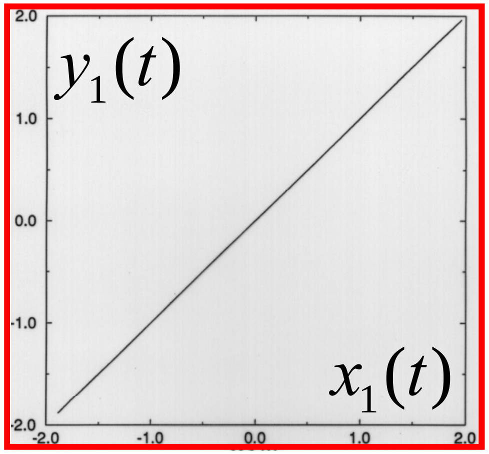
Coupling: ***g*=1.1**

Transversal Lyapunov exponent evaluated for the chaotic trajectory **x(t)** equals $\lambda_{\max}^{\perp} = -0.03$

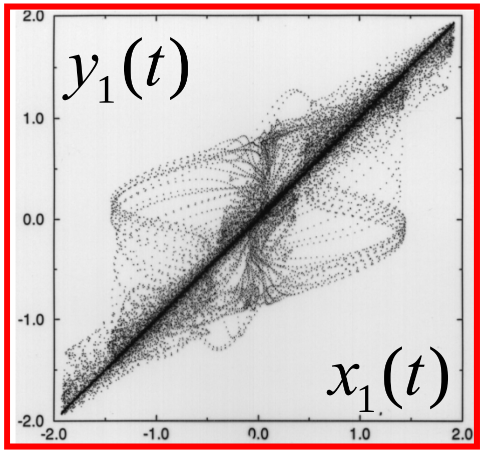
Attractor in the Driving Circuit $(\lambda_{\max} = 0.1)$



Simulation without noise and parameter mismatch



Simulation with 0.4% of parameter mismatch



Networks with N nodes

$$\dot{x}_i = f(x_i) + \sum_{k=1}^{N} D_{ik} x_k, \quad i = 1, \ldots, N$$

x_i *m* - dimensional vector

D_{ik} an m × m real matrix

Assumptions: $D_{ik} = g_{ik} H$

H an m × m real matrix

g_{ik} real number

Synchronization manifold: $x_1 = x_2 = \ldots = x_N$

Connectivity matrix: $G = (g_{ik})$ N × N real matrix

$$\sum_j g_{ij} = 0$$

Variation equation

$$\dot{x}_k = (J + l_k H)x_k, k = 1, \dots, N$$

λ_k eigenvalue of the connectivity matrix
$\lambda_1 = 0$

Master Stability Function

$$\dot{x}_i = f(x_i)$$

$$\dot{V}_k = [J + (a + i\,b)H]V_k$$

W= $\{(a,b) : l_{\max}(a,b) < 0\}$

Connectivity Matrix

$$G = (g_{ik})$$

Properties of master stability function

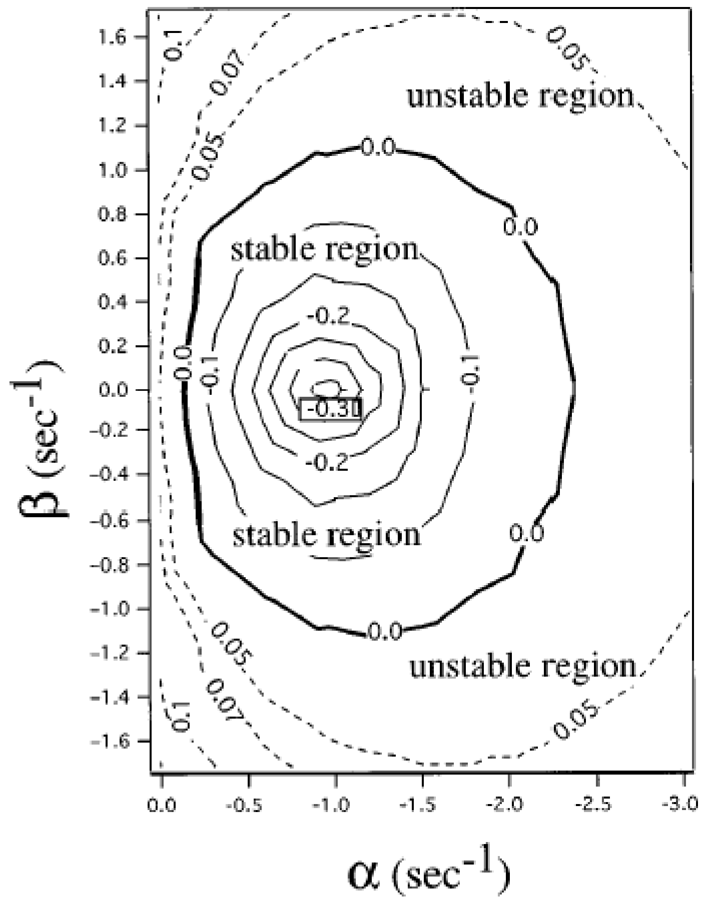
$$W = \{(a, b) : l_{\max}(a, b) < 0\}$$

- Empty set
- Ellipsoid
- Half plane

The master stability function for *x* coupling in the Rossler circuit.

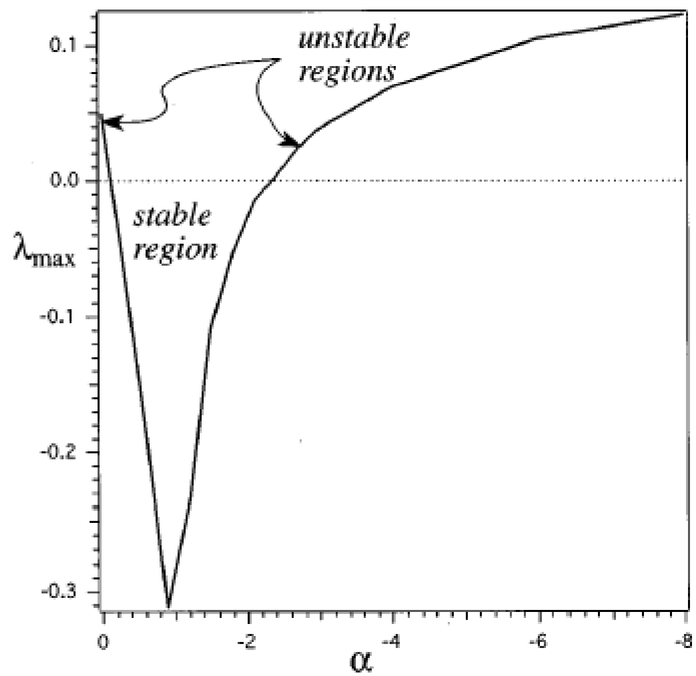
The dashed lines show contours in the unstable region.

The solid lines are contours in the stable region.



Properties of master stability function

$\lambda_{\max}(\alpha, \beta)$ versus α



Stable region:

$\lambda_{\max} \in (\alpha_1, \alpha_2)$

$\lambda_{\max} \in (-\infty, \alpha)$

$\lambda_N \le \lambda_{N-1} \le \ldots \le \lambda_2 \le \lambda_1 = 0$

$$\lambda_{\max} \in (\alpha_1, \alpha_2) \Rightarrow \frac{\lambda_N}{\lambda_2} < \frac{\alpha_1}{\alpha_2}$$

$$\lambda_{\max} \in (-\infty, \alpha) \Rightarrow \lambda_2 < \alpha$$

Indeed

The equations of the motion of the network read

$$\dot{\boldsymbol{x}}_i = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} AH(\boldsymbol{x}_i - \boldsymbol{x}_j) \quad \text{or} \quad \dot{\boldsymbol{x}}_i = F(\boldsymbol{x}_i) - \sigma \sum_{j=1}^{N} GH\boldsymbol{x}_j \quad ; \quad i = 1, \ldots, N$$

- $\boldsymbol{x}_i$: d-dimensional state vector
- N: Network size
- F: individual systems' dynamical equation
- σ: unified coupling strength
- H: Projection matrix determining by which components the systems are coupled
- $A = [a_{ij}]$: The adjacency matrix
- $G = [g_{ij}]$: The Laplacian matrix
  - Zero row-sum
  - Positive diagonal elements

The dynamical network synchronizes globally (and completely), if starting from any initial condition

$$\left\| \boldsymbol{x}_i(t) - \boldsymbol{x}_j(t) \right\| \xrightarrow[t \to \infty]{} 0 \quad \forall \ i, j = 1, \ldots N$$

Master stability function

The variational equations can be diagonalized as

$$\dot{\eta}_i = JF\eta_i - \sigma\lambda_i H\eta_i \quad ; \quad i = 1, 2, \ldots, N$$

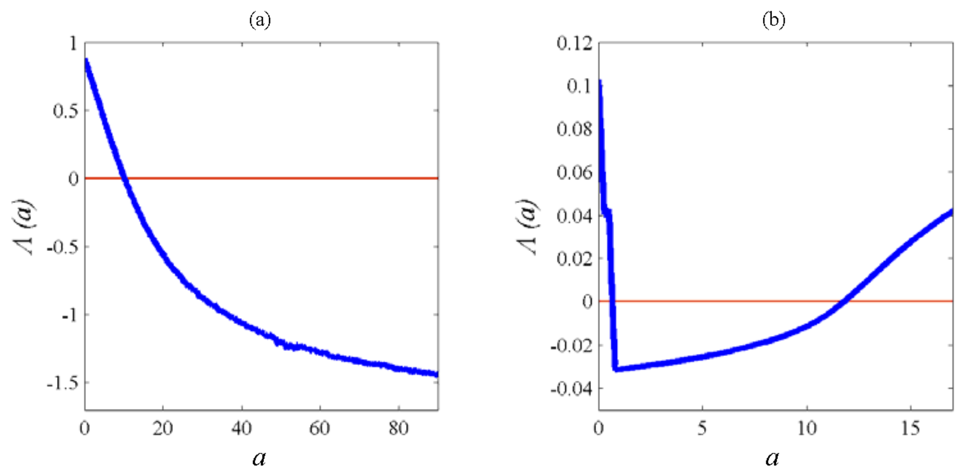
- λ_i: eigenvalues of G
- $\lambda_1 = 0 < \lambda_2 < \ldots < \lambda_N$ (λ_1 is associated with the synchronization manifold)
- MSF: The largest Lyapunov exponent of the equation $\dot{\eta}_i = JF\eta_i - aH\eta_i$
- necessary condition for the local stability of the synchronization (Pecora and Carroll 1998)

Type I systems:

MSF is negative within interval (a_1, ∞)

Type II systems:

MSF is negative within interval (a_1, a_2)



a) x-coupled Lorenz oscillators (type I)
b) x-coupled Rössler oscillators (type II)

Another measure

The network synchronizes globally if (sufficient condition) (Belykh et al, 2004)

$$\boxed{\sigma_{ij}(t) > \frac{a}{N} b_{ij} \quad ; \quad \text{for } i, j = 1, \ldots, N \text{ and } \forall t}$$

- σ_{ij}: coupling strength of the link between the i-th and j-th nodes
- a: double coupling strength sufficient for global synchronization of two systems
- b_{ij}: CGS-score for the link between the i-th and j-th nodes

$$b_{ij} = \sum_{u=1}^{n-1} \sum_{v>u; e_{ij} \in P_{uv}}^{n} | P_{uv} |$$

- $|P_{uv}|$: length of path P_{uv} between the u-th and v-th nodes

CGS gives the sufficient strength for each link guaranteeing the global synchronization

Synchronizability measures

- Large range of synchronizing coupling parameter
  - The smaller the λ_N/λ_2 the better the synchronizability
- The cost of synchronization C_{syn} (the sum over the coupling strengths)
  - For the same effort, the larger the λ_2 the better the synchronizability
- The time to synchronize T_{syn}

$$E(t) = \frac{2}{N(N-1)} \sum_{i<j} \left\| \boldsymbol{x}_i(t) - \boldsymbol{x}_j(t) \right\|^2$$

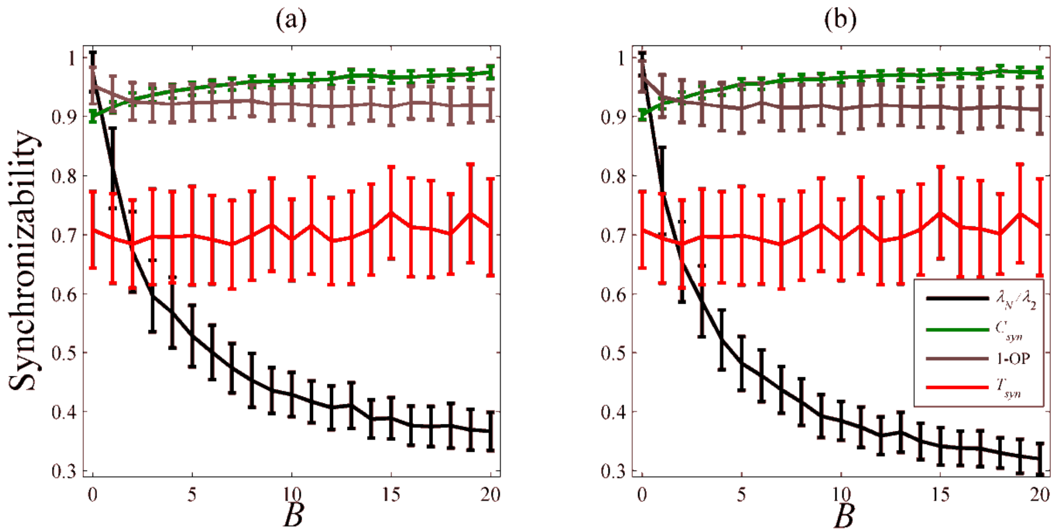
$E(T_{\text{syn}}) = \varepsilon$ (e.g. $\varepsilon = 1\text{e-}5$) and $E(t) < \varepsilon$ for $t > T_{\text{syn}}$

- The degree of phase synchronization: the order parameter (OP)

$$\Phi = \left\langle \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\varphi_j(t)} \right| \right\rangle_t$$

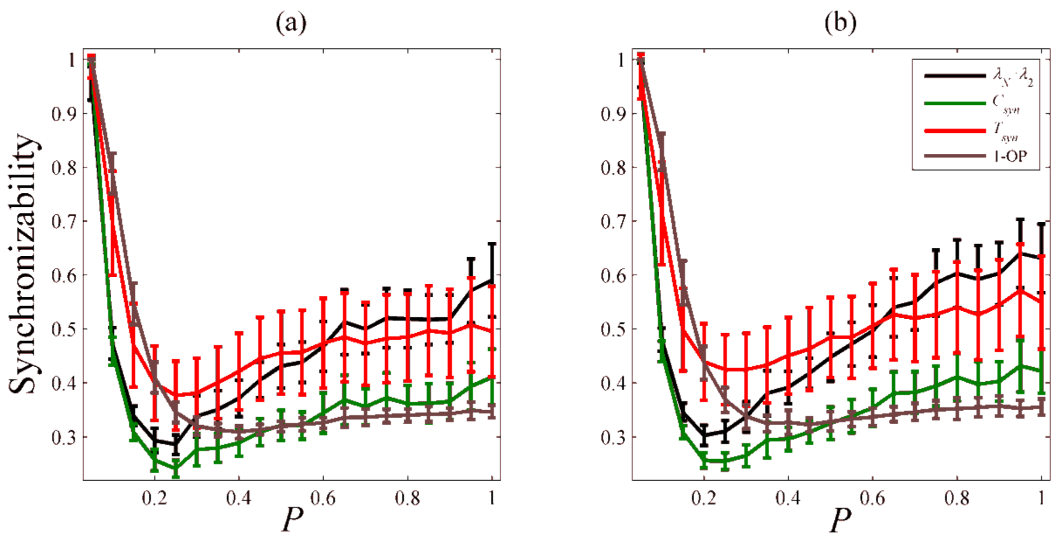
φ_j: instantaneous phase of the j-th oscillator

Synchronizability measures



Scale-free networks with average degree <*k*> = 6 and a) *N* = 1000, b) *N* = 2000

The bigger the B the less heterogeneous the network



Watts-Strogatz networks with average degree <*k*> = 6 and a) *N* = 1000, b) *N* = 2000

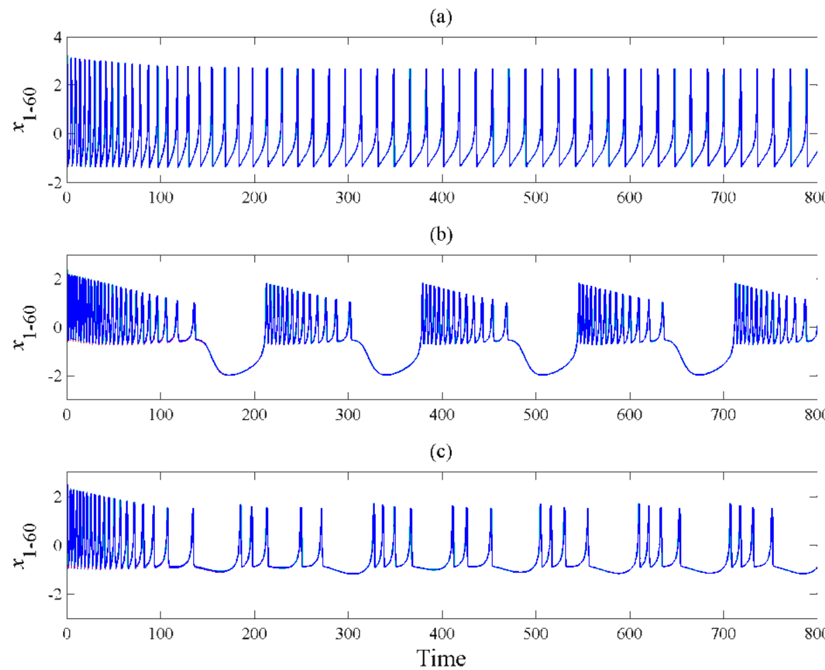
P is rewiring probability

Example: HR neurons

Hindmarsh-Rose neuron model

$$\begin{cases} \dot{x} = y + ax^2 - x^3 - z + I \\ \dot{y} = 1 - dx^2 - y \\ \dot{z} = \mu\left(b\left(x - x_0\right) - z\right) \end{cases}$$

a) Spiking mode
b) Bursting mode
c) Chaotic mode



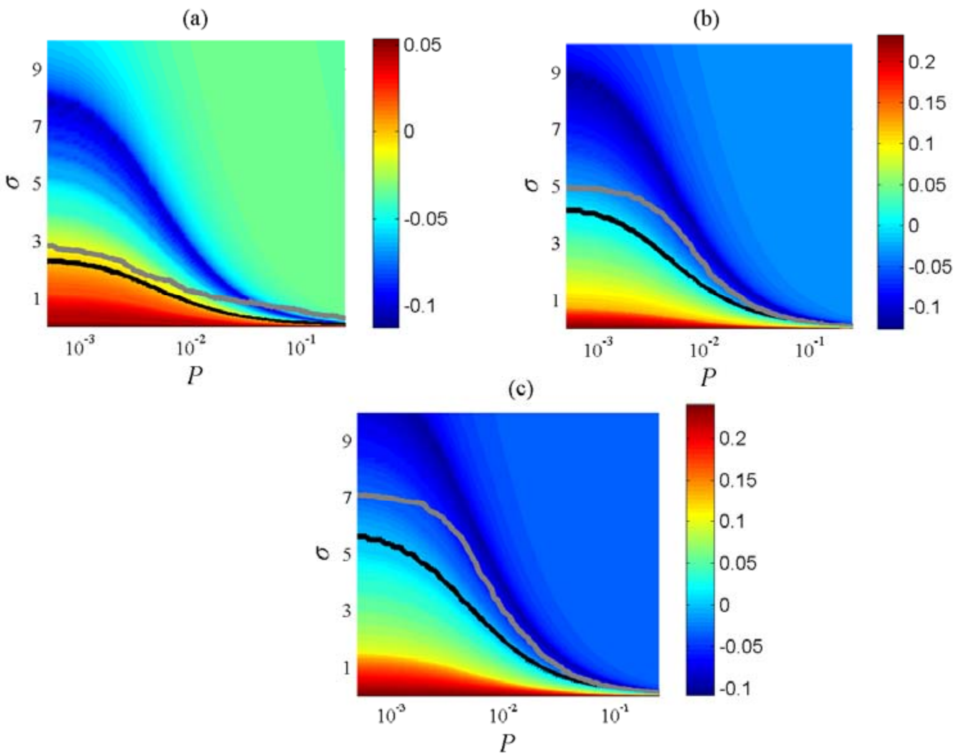
Newman-Watts network model

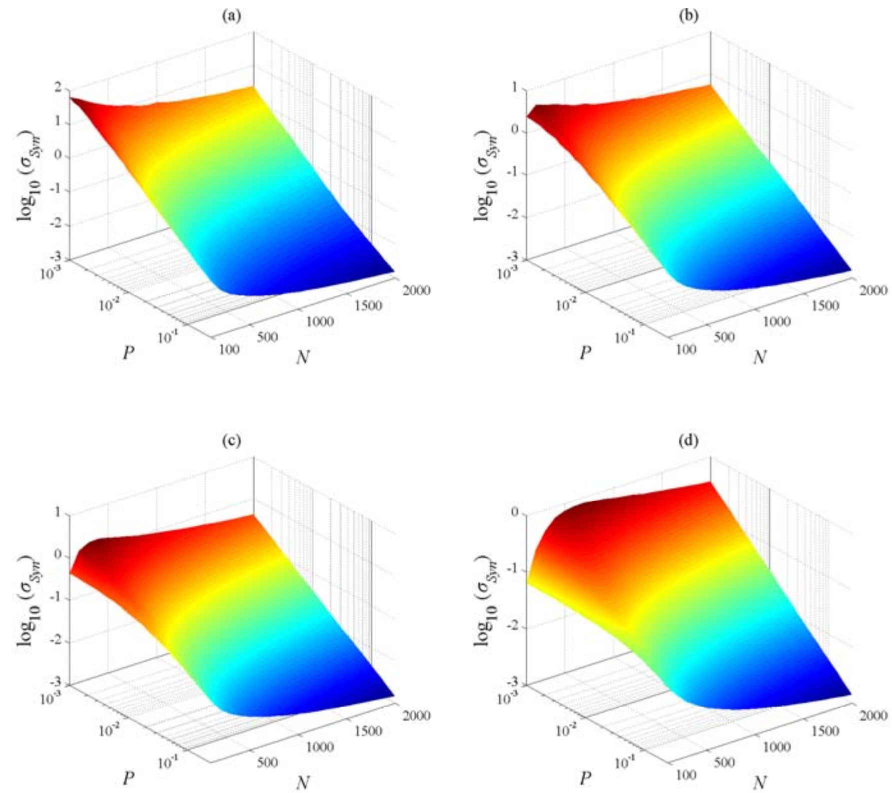
- Starting with a m-regular graph
- Connecting the unconnected nodes with probability

Meta Newman-Watts (clustered) network model

- k clusters
- Each cluster as NW network with dense intra-cluster connections
- Sparse inter-cluster connections

Example: HR neurons

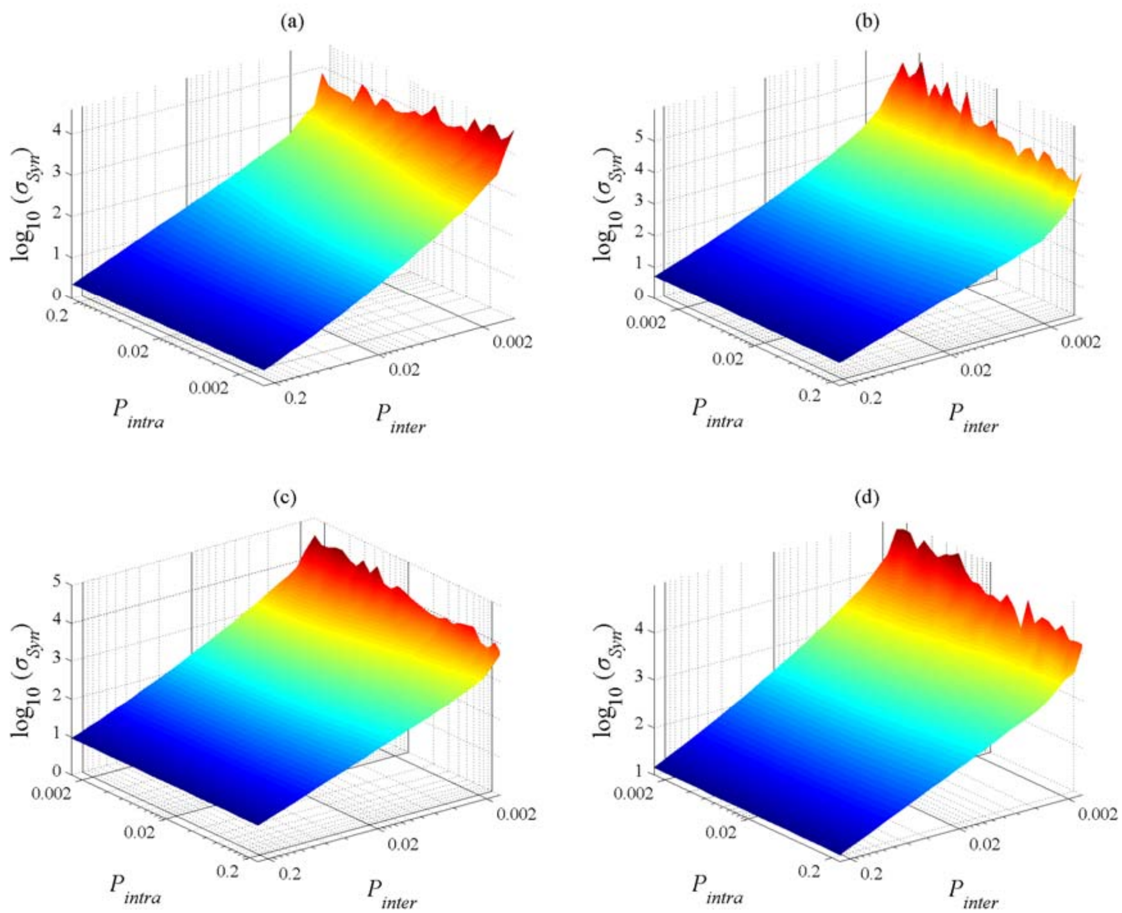




N = 60, m = 3
MSF results
lines show the synchronizing parameter
black line: MSF, gray line: Numerical calculation
a) spiking, b) bursting, c) chaotic

only bursting mode
calculations based on MSF
a) m = 1, b) m = 5, c) m = 10, d) m = 20
$\sigma_{Syn} = \alpha P^{-\gamma}$, where $\gamma \sim (1,2)$ and $\alpha \sim (0.0001,0.1)$

Example: HR neurons



only bursting mode, clusters with N_k = 50,
calculations based on MSF
a) k = 20, b) k = 30, c) k = 40, d) k = 50
P_{intra} has almost no effect; Power-law dependence on P_{inter}

Example: HR neurons

Fast threshold modulation model for the chemical synapses

- current I_{ji} injected from presynaptic cell i to the postsynaptic cell j

$$I_{ji} = \sigma_{ch}\left(V_s - x_j\right)\Theta\left(x_i\right)\ ;\ \Theta\left(x_i\right) = \frac{1}{1+\exp\left\{-\lambda\left(x_i - \theta_s\right)\right\}}$$

The network equations read

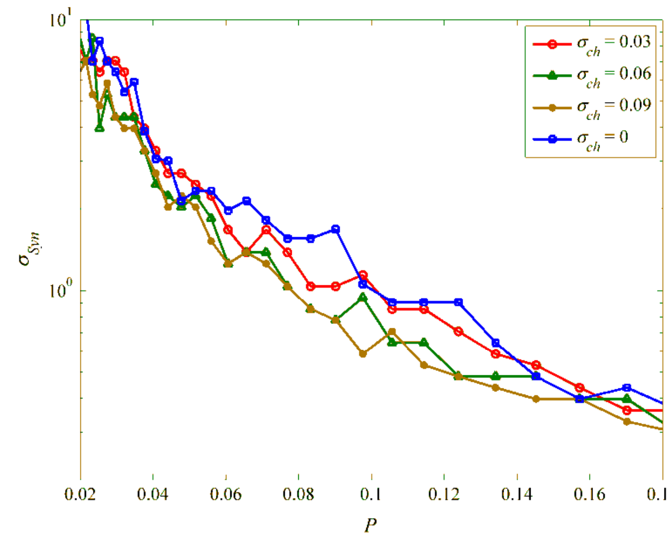
$$\begin{cases} \dot{x}_i = y_i + ax_i^2 - x_i^3 - z_i + I + \sigma\sum_{j=1}^{N} a_{ij}\left(x_j - x_i\right) + \sigma_{ch}\left(V_s - x_i\right)\sum_{j=1}^{N} c_{ij}\dfrac{1}{1+\exp\left\{-\lambda\left(x_j - \theta_s\right)\right\}} \\ \dot{y}_i = 1 - dx_i^2 - y_i \\ \dot{z}_i = \mu\left(b\left(x_i - x_0\right) - z_i\right) \end{cases}$$

Electrical coupling

- NW networks N = 40 and m = 1

Excitatory chemical coupling

- random networks such that each neuron receives input from two other neurons (Belykh et al 2005)



synchronizing electrical coupling electrical coupling has the main role chemical coupling has complementary role

Readings

- http://en.wikipedia.org/wiki/Dynamical_system
- http://en.wikipedia.org/wiki/Linear_dynamical_system
- http://en.wikipedia.org/wiki/Stability_theory
- http://en.wikipedia.org/wiki/Chaos_theory
- A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Synchronization in complex networks, Physics Reports 469 (2008) 93153.