



Graph Modeling- Part1

CE642: Social and Economic Networks
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01

Random Walk

What is a Random Walk

- Given a graph and a starting point (node), we select a neighbor of it at random, and move to this neighbor;
- Then we select a neighbor of this node and move to it, and so on;
- The (random) sequence of nodes selected this way is a **random walk** on the graph

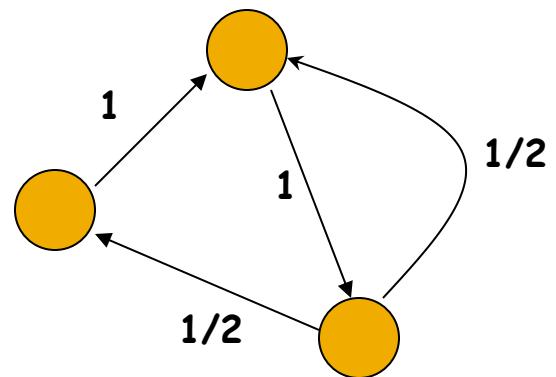
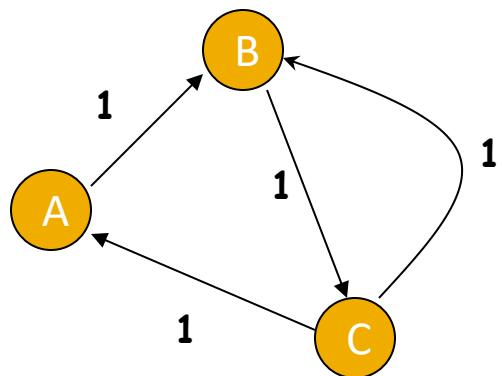
An example

0	1	0
0	0	1
1	1	0

Adjacency matrix A

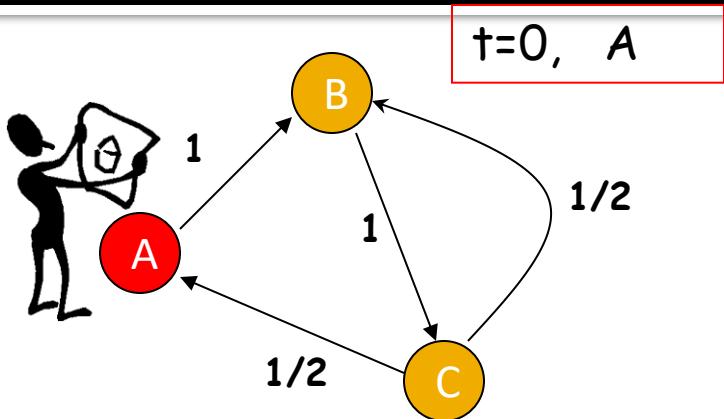
0	1	0
0	0	1
1/2	1/2	0

Transition matrix P



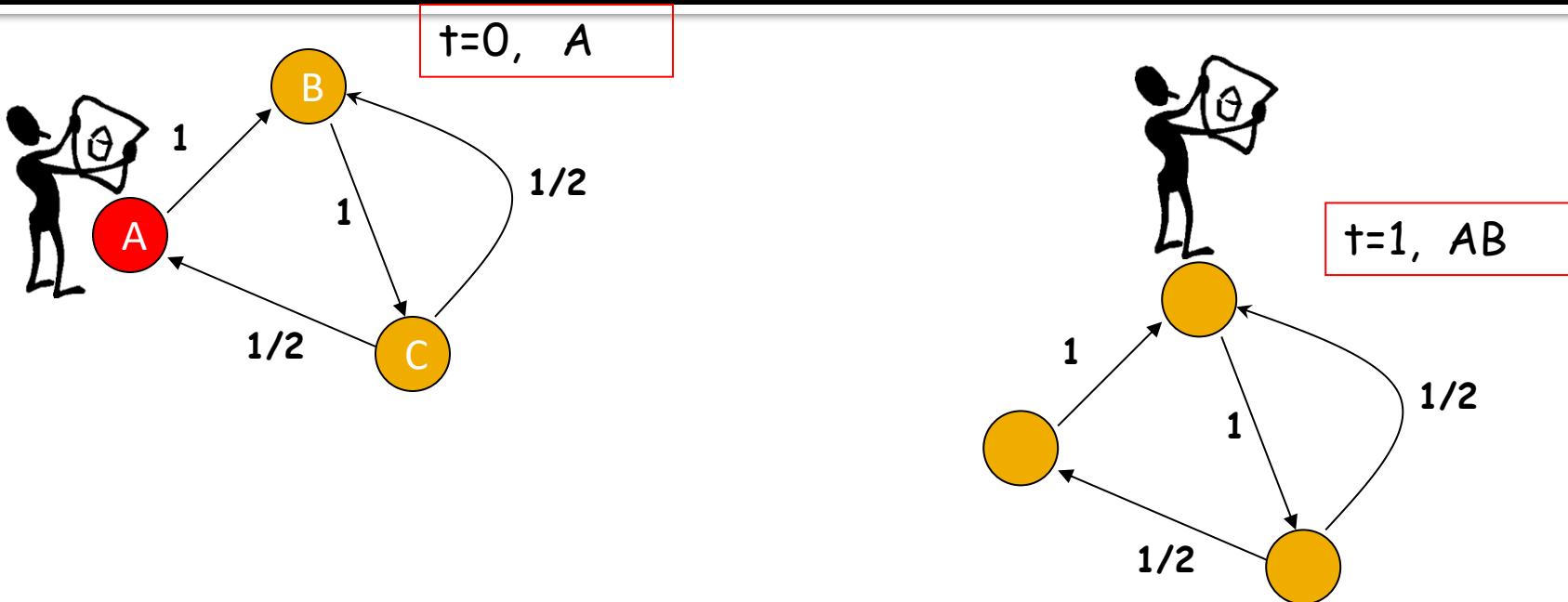
Slide from Purnamitra Sarkar, *Random Walks on Graphs: An Overview*

An example



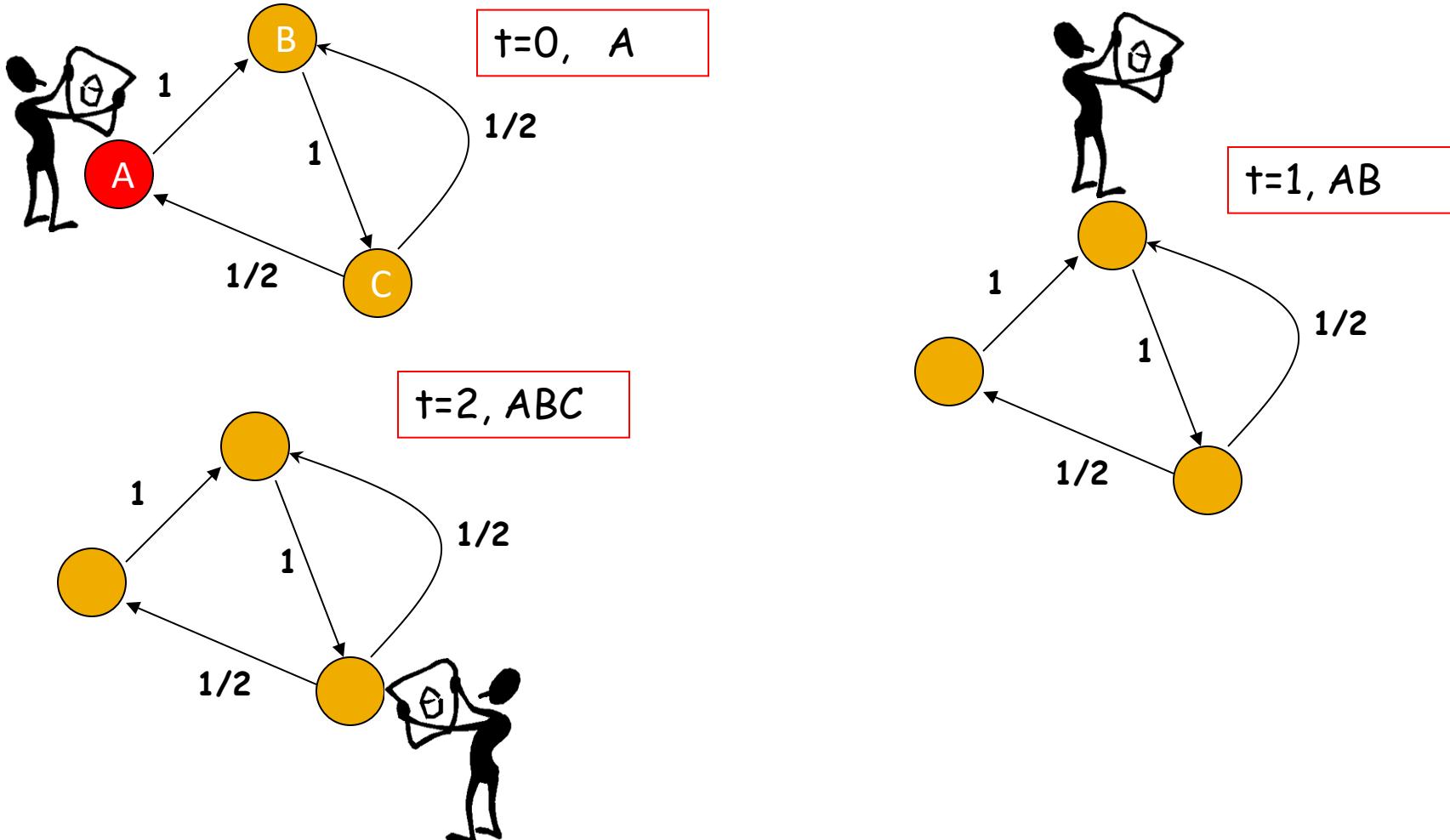
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An example



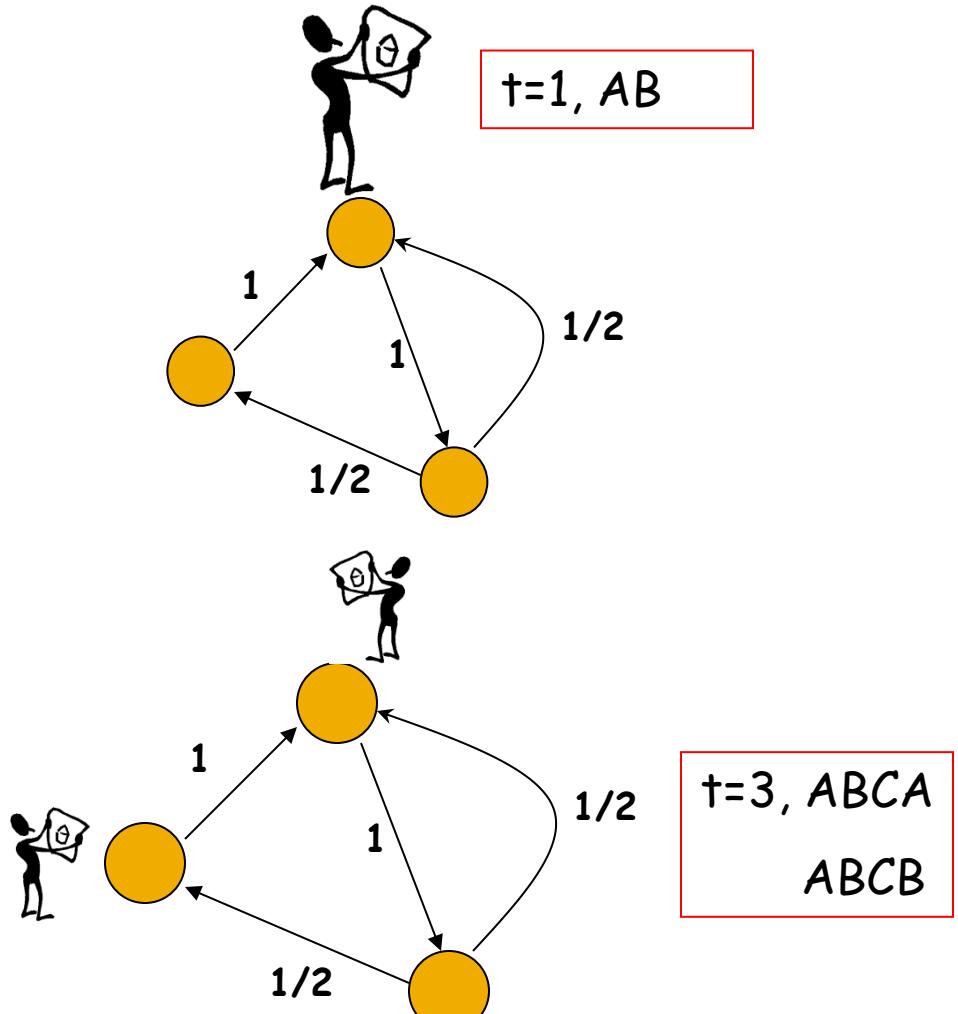
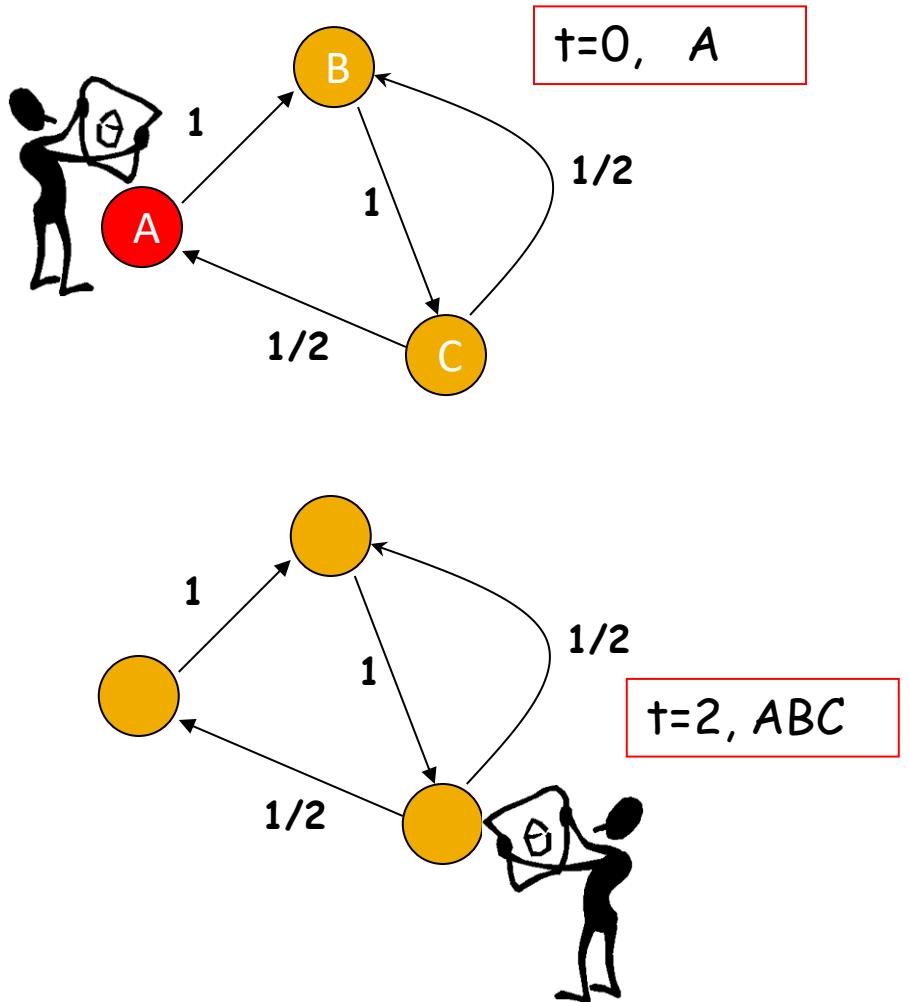
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An example



Slide from Purnamitra Sarkar, *Random Walks on Graphs: An Overview*

An example



Slide from Purnamitra Sarkar, *Random Walks on Graphs: An Overview*

Why are random walks interesting?

- When the underlying data has a natural graph structure, several physical processes can be conceived as a random walk

Data	Process
WWW	Random surfer
Internet	Routing
P2P	Search
Social network	Information percolation

Random walks: definitions

- $n \times n$ **Adjacency matrix A**.
 - $A(i,j)$ = weight on edge from i to j
 - If the graph is undirected $A(i,j)=A(j,i)$, i.e. A is symmetric
- $n \times n$ **Transition matrix P**.
 - P is row stochastic
 - $P(i,j) = \text{probability of stepping on node } j \text{ from node } i = A(i,j)/\sum_i A(i,j)$
- $n \times n$ **Laplacian Matrix L**.
 - $L(i,j)=\sum_i A(i,j)-A(i,j)\Rightarrow L=D-A$
 - **Symmetric positive semi-definite for undirected graphs??**
 - **Singular??**

Laplacian Matrix

- Positive semi-definite for undirected graphs.

$$\forall \mathbf{x} \in \mathbb{R}^n \quad \mathbf{x}^T \mathbf{L} \mathbf{x} \geq 0$$

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \mathbf{x}^T \mathbf{D} \mathbf{x} - \mathbf{x}^T \mathbf{A} \mathbf{x}$$

- $\mathbf{x}^T \mathbf{D} \mathbf{x} = \sum_i \deg(i) x_i^2$
- $\mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{i,j} A(i,j) x_i x_j$

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \sum_i \deg(i) x_i^2 - \sum_{i,j} A(i,j) x_i x_j$$

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \frac{1}{2} \sum_{i,j} A(i,j) (x_i - x_j)^2$$

$$A(i,j) \geq 0 \quad \rightarrow \quad \mathbf{x}^T \mathbf{L} \mathbf{x} \geq 0$$

Laplacian Matrix

■ Singular

$$\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$$

$$L \cdot \mathbf{1} = (D - A)\mathbf{1} = D\mathbf{1} - A\mathbf{1}$$

- $D\mathbf{1} = \deg(i)$
- $A\mathbf{1} = \text{sum of neighbors} = \deg(i)$

$$L \cdot \mathbf{1} = 0$$

So, zero is the eigenvalue, and eigenvalues multiplication is the determinant. Therefore, $\det(L)=0$.

Probability Distributions

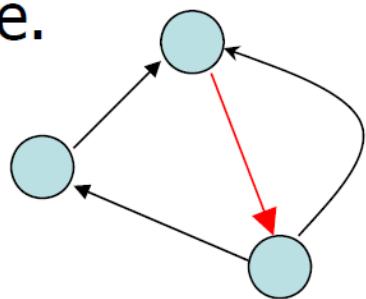
- $x_t(i)$ = probability that the surfer is at node i at time t
- $x_{t+1}(i) = \sum_j (\text{Probability of being at node } j) * \Pr(j \rightarrow i)$
 $= \sum_j x_t(j) * P(j,i)$
- $x_{t+1} = x_t P = x_{t-1} * P * P = x_{t-2} * P * P * P = \dots = x_0 P^t$
- What happens when the surfer keeps walking for a long time?
- Stationary distribution:
 - When the surfer keeps walking for a long time
 - When the distribution does not change anymore, i.e. $x_{T+1} = x_T$
 - For “well-behaved” graphs this does not depend on the start distribution!!!

What is a stationary distribution?

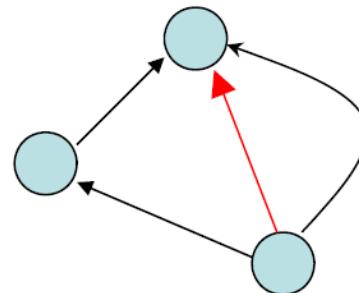
- The stationary distribution at a node is related to the amount of time a random walker spends visiting that node.
- Remember that we can write the probability distribution at a node as
 - $x_{t+1} = x_t P$
- For the stationary distribution v_0 we have
 - $v_0 = v_0 P$
- So, that's just the left eigenvector of the transition matrix!
- Interesting questions:
 - Does a stationary distribution always exist? Is it unique? (Yes, if the graph is "well-behaved")
 - What is "well-behaved"?
 - How fast will the random surfer approach this stationary distribution? (Mixing Time!)

Well-behaved graphs

- **Irreducible:** There is a path from every node to every other node.

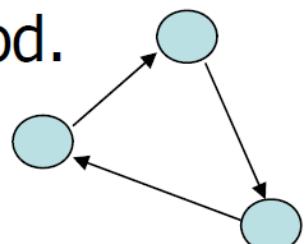


Irreducible

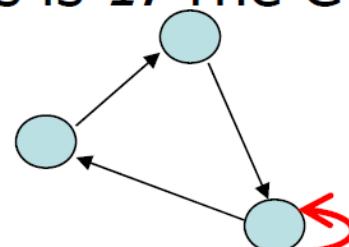


Not irreducible

- **Aperiodic:** The GCD of all cycle lengths is *1*. The GCD is also called period.



Periodicity is 3

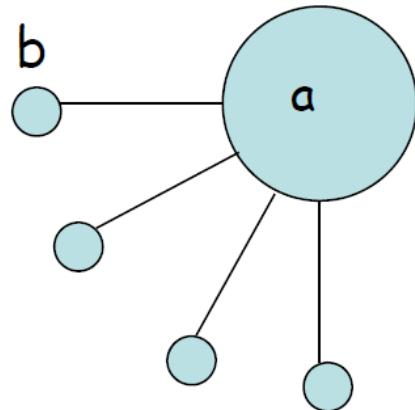


Aperiodic

Perron Frobenius Theorem

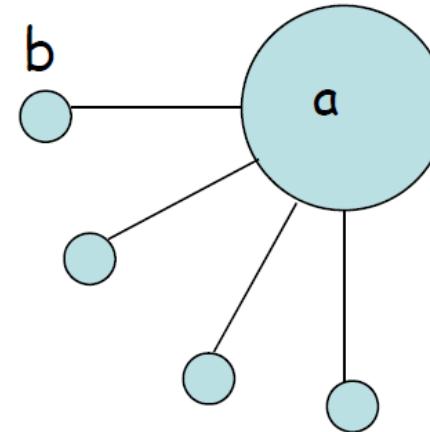
- If a markov chain is irreducible and aperiodic, then the largest eigenvalue of the transition matrix will be equal to **1** and all the other eigenvalues will be strictly less than **1**.
 - Let the eigenvalues of P be $\{\sigma_i \mid i=0:n-1\}$ in non-increasing order of σ_i .
 - $\sigma_0 = 1 > \sigma_1 > \sigma_2 \geq \dots \geq \sigma_n$
- These results imply that **for a well behaved graph there exists an unique stationary distribution.**
- The pagerank uses these results.
- We know that
 - A connected undirected graph is irreducible
 - A connected non-bipartite undirected graph has a stationary distribution proportional to the degree distribution!
 - Makes sense, since larger the degree of the node more, likely a random walk is to come back to it.

Proximity measures from random walks



- How long does it take to hit node b in a random walk starting at node a ? **Hitting time**.
- How long does it take to hit node b and come back to node a ? **Commute time**.

Hitting and Commute times



- Hitting time from node i to node j
 - Expected number of hops to hit node j starting at node i
 - Is **not** symmetric. $h(a,b) \neq h(b,a)$
 - $h(i,j) = 1 + \sum_{k \in \text{nbs}(A)} p(i,k)h(k,j)$
- Commute time between node i and j
 - Is expected time to hit node j and come back to i
 - $c(i,j) = h(i,j) + h(j,i)$
 - Is symmetric. $c(a,b) = c(b,a)$

Random graphs

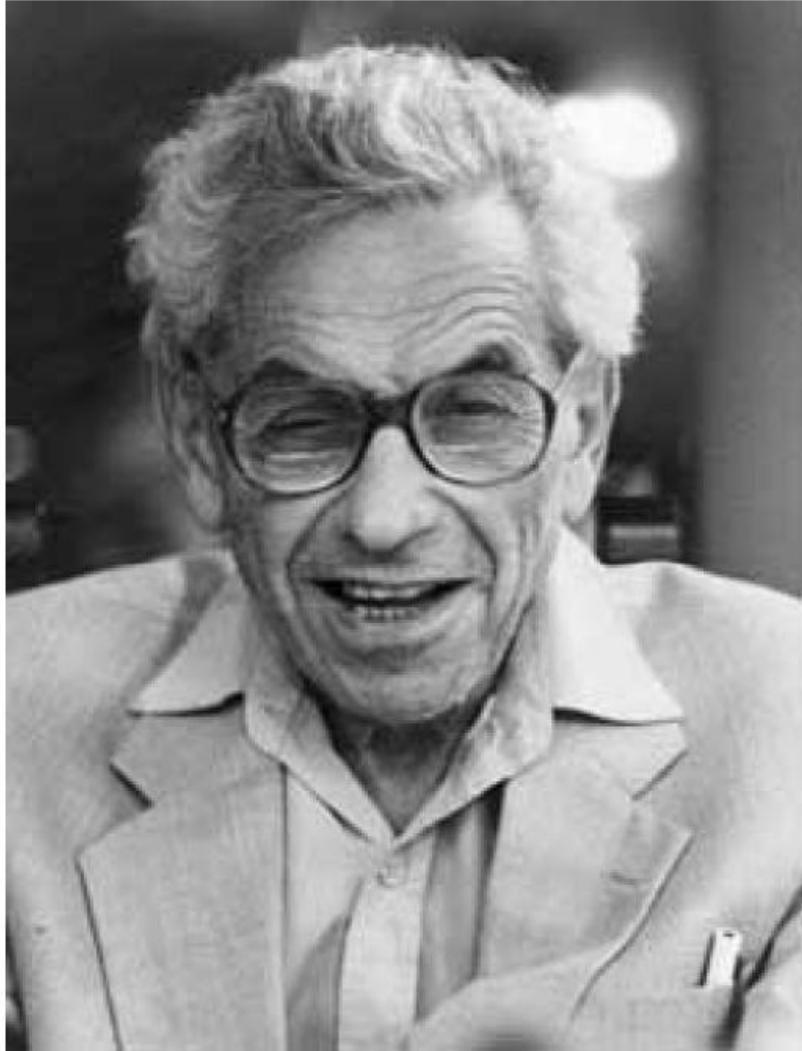
- A deterministic model D defines a single graph for each value of n (or t)
- A randomized model R defines a probability space $\langle G_n, P \rangle$ where G_n is the set of all graphs of size n , and P a probability distribution over the set G_n (similarly for t)
 - we call this a family of random graphs R , or a random graph R



02

Erdös-Renyi Random graphs

Erdös-Renyi Random graphs



Paul Erdös (1913-1996)

You may have heard about Erdös number!
What is your Erdös number?

Erdös-Renyi Random graphs

For generation of Erdös-Renyi network, one of the following methods is used:

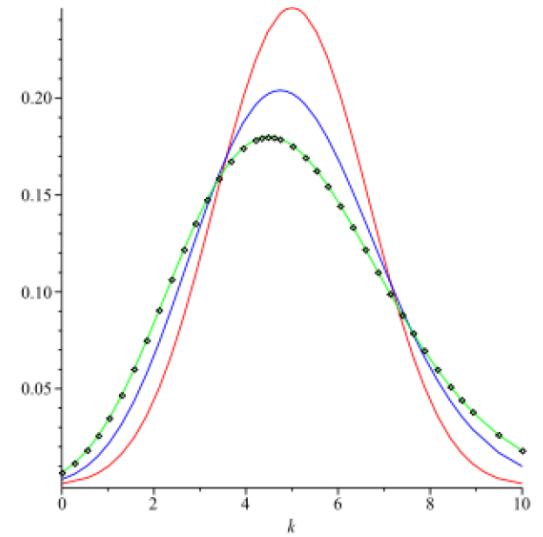
1. The $G_{n,p}$ model
 - **input:** the number of vertices n , and a parameter p , $0 \leq p \leq 1$
 - **process:** for each pair (i,j) , generate the edge (i,j) independently with probability p
2. Related, but not identical: The $G_{n,m}$ model
 - **process:** select m edges uniformly at random

Erdös-Renyi Random graphs

- $G(n, p)$:
 - Consider a set of nodes $N = \{1, 2, \dots, n\}$
 - Connect each pair i, j of nodes with probability p
 - The expected number of edges: $\binom{n}{2}p$
 - The expected degree of nodes: $(n - 1)p$
- $G(n, M)$:
 - Choose M edges out of all $\binom{n}{2}$ pair of nodes: $\binom{\binom{n}{2}}{M}$ choices
 - Number of edges: M
 - The expected degree of nodes: $\frac{M}{\binom{n}{2}} \times (n - 1) = \frac{2M}{n}$

Binomial Distribution

- Binomial Distribution:
 - Consider a sequence of Bernoulli trials. What is the probability of m heads out of n flips? $P(d)$ is:
 - Expected number of heads: np
 - The variance: $npq = np(1-p)$
 - Standard deviation: $\sqrt{np(1 - p)}$
- Binomial distribution can be approximated by $\lambda = np$ for large n
$$P(d) \approx \frac{e^{-\lambda} \lambda^d}{d!}$$
- Highly concentrated around the mean, with a tail that drops exponentially



Poisson Networks

- Degree distribution: Binomial distribution
 - The probability of having d neighboring edges is equal to:
- Can be approximated by $\lambda = (n - 1)p = np$ for large n

$$P(d) = \binom{n-1}{d} p^d (1-p)^{n-1-d}$$

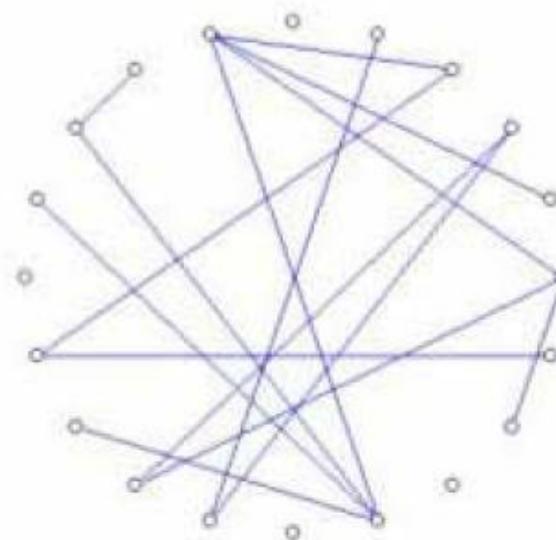
$$P(d) \approx \frac{e^{-\lambda} \lambda^d}{d!}$$

Example



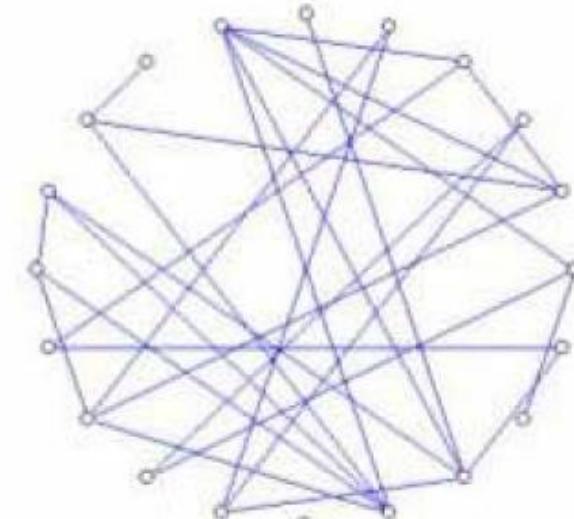
$p = 0$

(a)



$p = 0.1$

(b)



$p = 0.2$

(c)

Clustering Coefficient

Let's say a node v has degree k .

$$C_v = \frac{\text{number of links between neighbors of } v}{\binom{k}{2}}$$

$$\mathbb{E}[\text{edges among neighbors}] = \binom{k}{2} \cdot p$$

$$C_v = \frac{\binom{k}{2} \cdot p}{\binom{k}{2}} = p$$

Since $\langle k \rangle = (n - 1)p \approx np$, we get:

$$p = \frac{\langle k \rangle}{n}$$

$$C = p \sim \frac{\langle k \rangle}{n}$$

As $n \rightarrow \infty$, clustering goes to 0 in Poisson/Erdős-Rényi graphs, while **real-world networks** often **maintain high clustering**. That's why we say:
"Erdős-Rényi graphs are poor models for social networks."

Diameter

- maximum length of shortest paths
 - To estimate the maximum distance between two nodes, we think:
 - Start from any node.
 - How many steps do we need until we can reach everyone?

$$\lambda^d = n \Rightarrow d = \log_{\lambda} n = \frac{\log n}{\log \lambda}$$

Phase transition

- Starting from some vertex v perform a BFS walk
- At each step of the BFS a Poisson process with mean λ , gives birth to new nodes
- When $\lambda < 1$ this process will stop after $O(\log n)$ steps
- When $\lambda > 1$, this process will continue for $O(n)$ steps

Are real-world networks random?

- A decade ago, the most elegant theory for modelling real-world networks was based on random graphs
- But real-world networks are not random (we will see)
- However, studies on random networks provides insights into complex structures

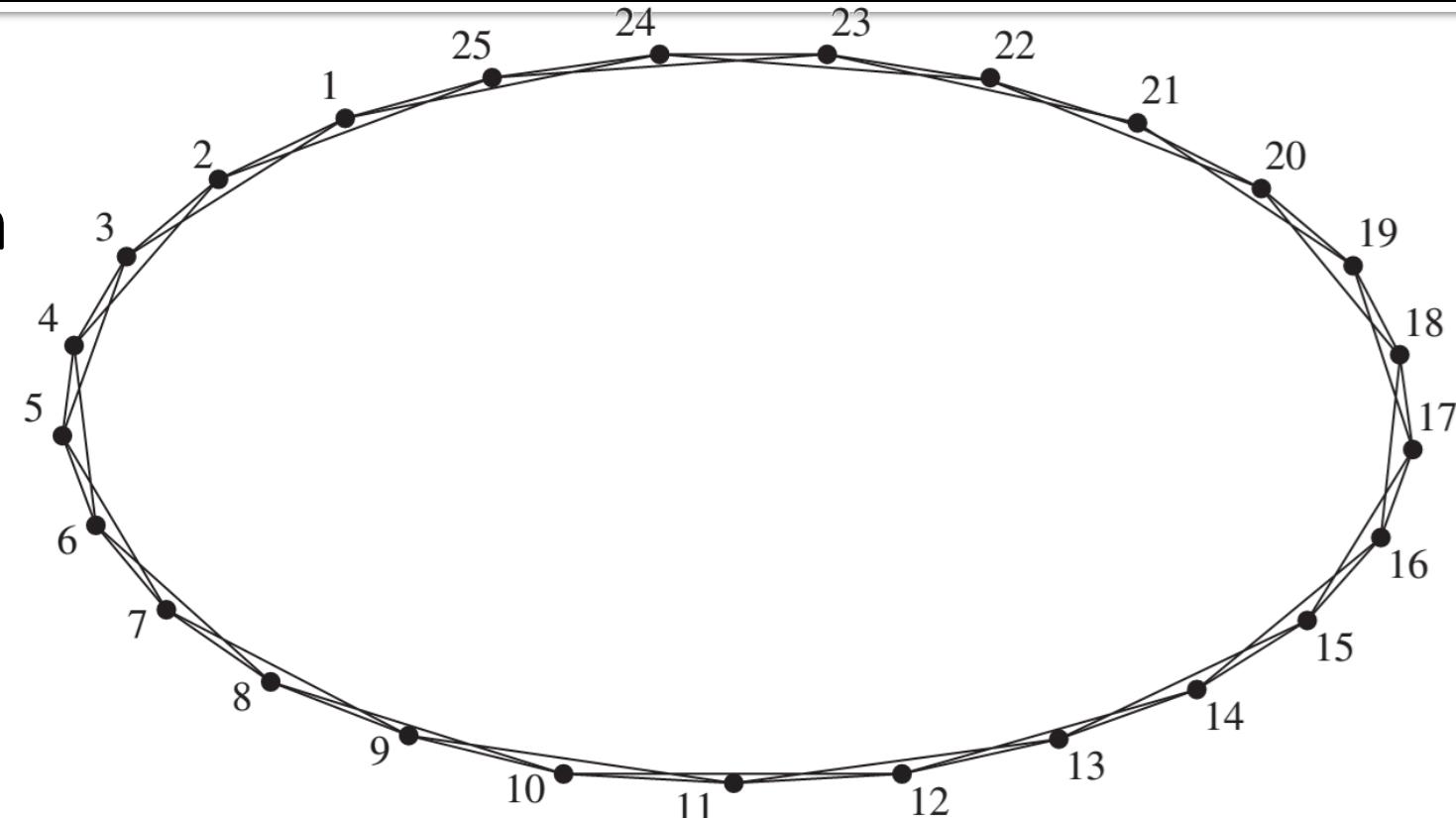


03

Watts-Strogatz Model

Watts-Strogatz Model

- Consider a n nodes cycle and connect each node to its $2m$ nearest nodes
- For $m=2$:
 - Diameter: $\frac{n}{4}$
 - Clustering Coefficient: $\frac{1}{2}$
- Diameter is high, while the clustering coefficient is also high

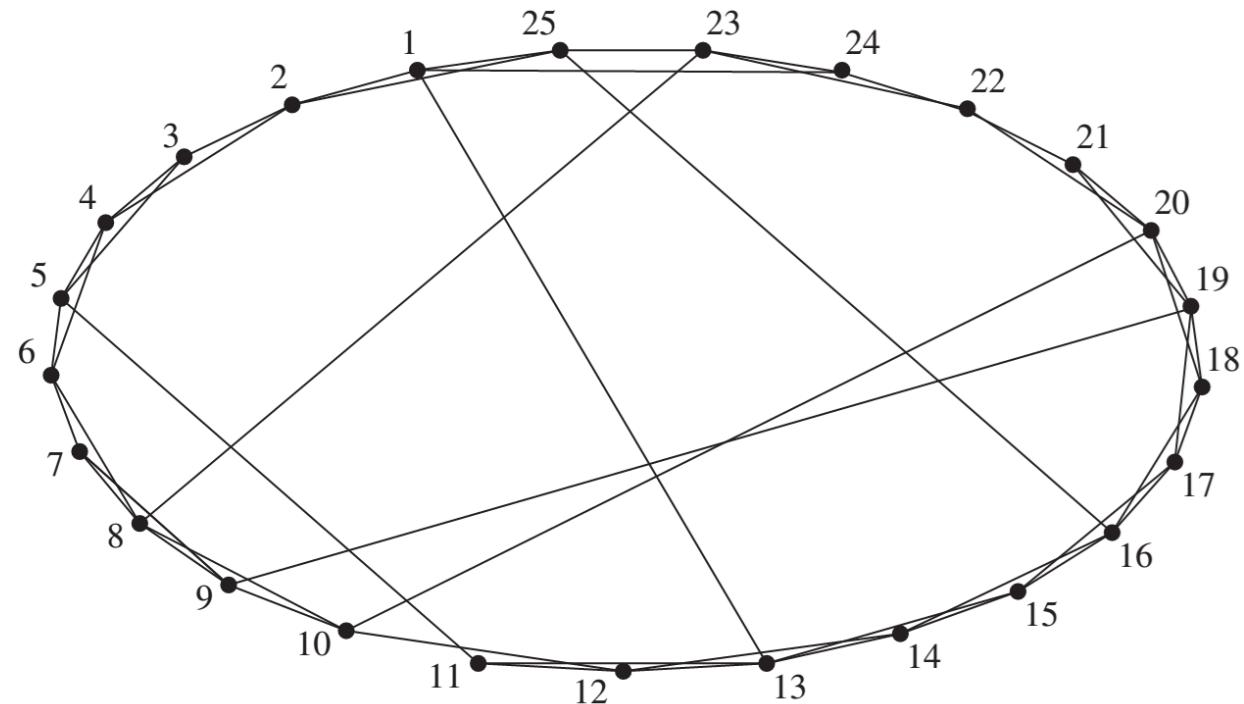


Global Clustering Coefficient

$$C = \frac{3 \times \text{number of triangles in the graph}}{\text{number of connected triples}}$$

Watts-Strogatz Model

- Watts & Strogatz show that with a few random rewiring the diameter will be decreased a lot.
- We will speak about **small-world** models deeply later

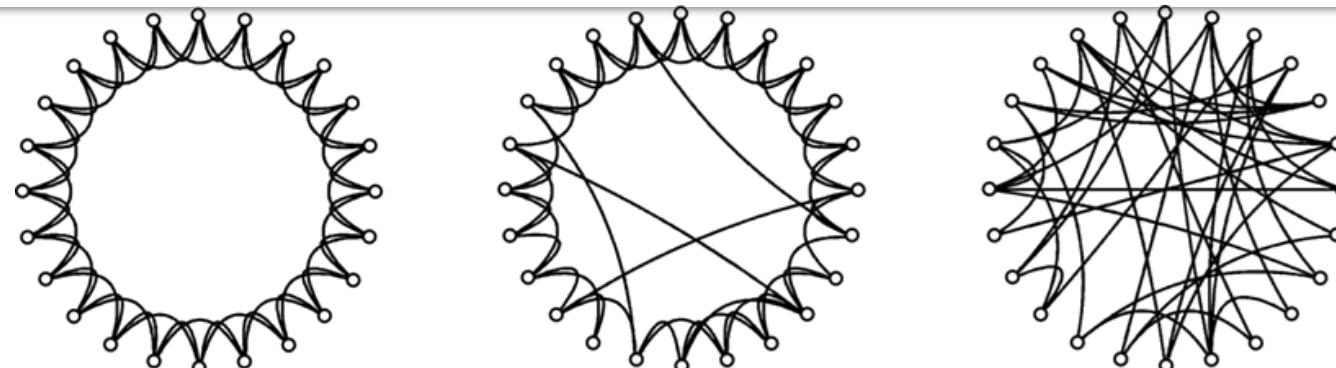


Watts-Strogatz Model

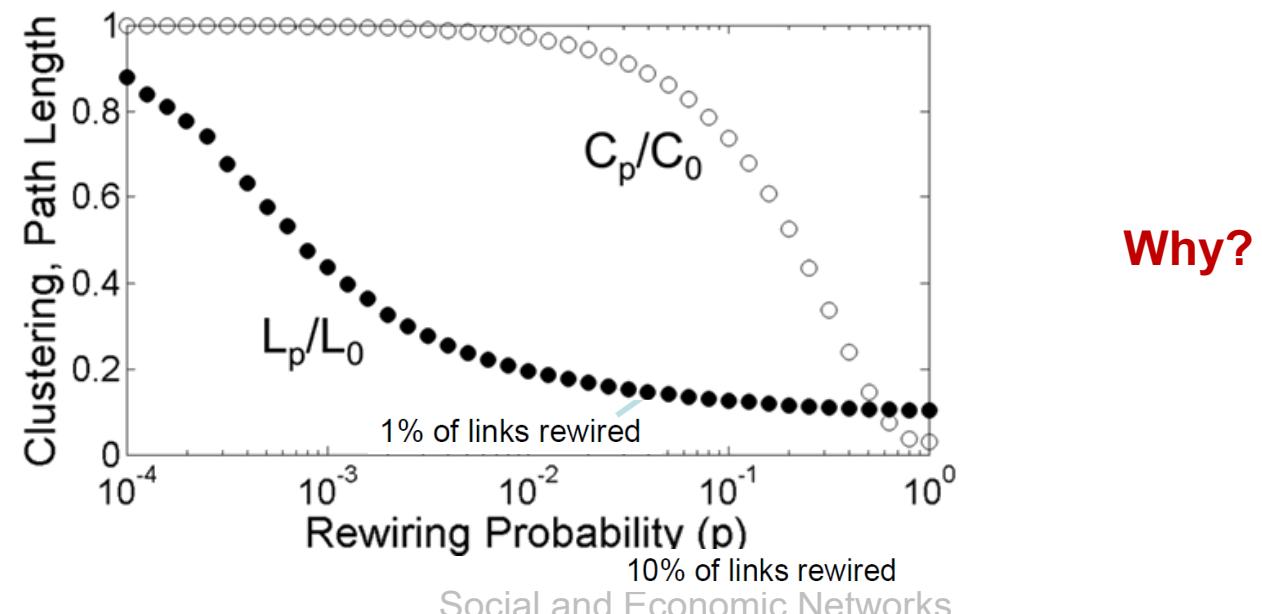
The construction algorithm:

- Consider a ring graph where each node is connected to its m nearest neighbors with undirected edges
- Choose a node and one of the edges that connects it to its nearest neighbors and then with probability P reconnect this edge to a node randomly chosen over the graph
 - provided that the duplication of edges and self-loops are forbidden
- The process is repeated until all nodes and nearest neighbor connecting edges are met
- Next, the edges that connect the nodes to their second-nearest neighbors are reconnected and the rewiring process is performed on them with the same conditions as above
- The same procedure is then repeated for the remaining edges connecting the nodes to their m nearest neighbors

Watts-Strogatz Model



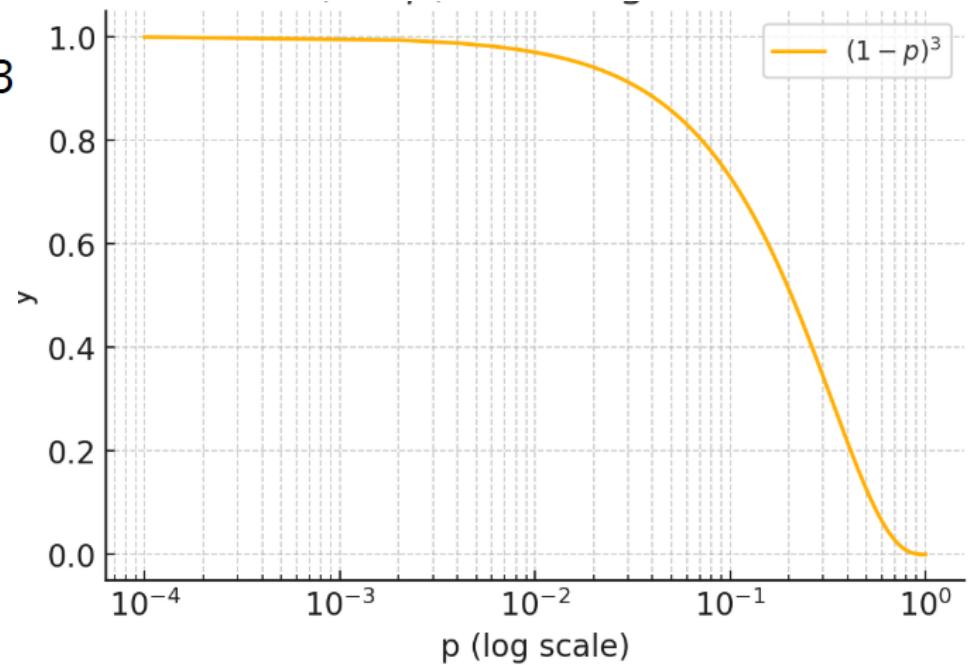
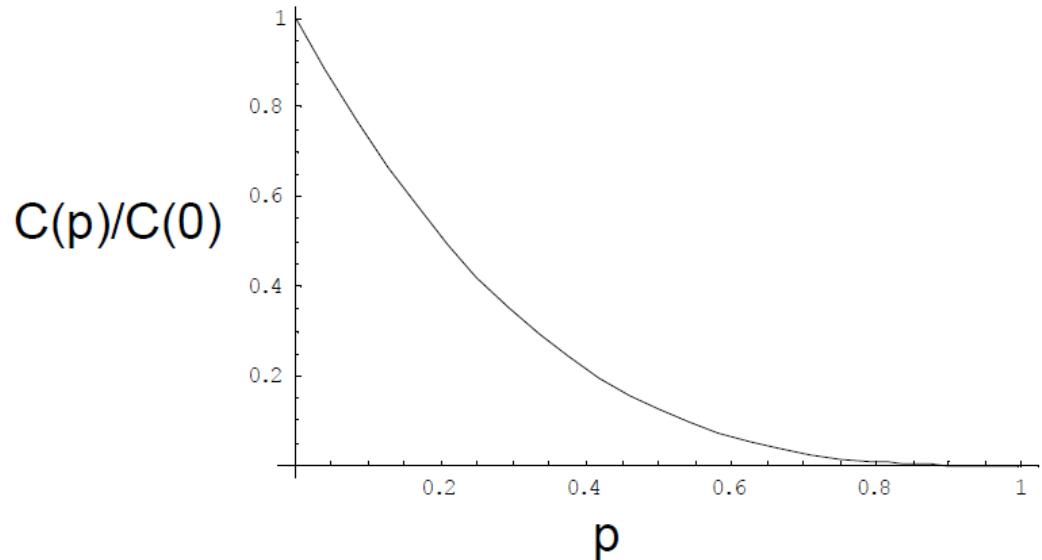
$P = 0$ increasing randomness $P = 1$



Why?

Clustering Coefficient

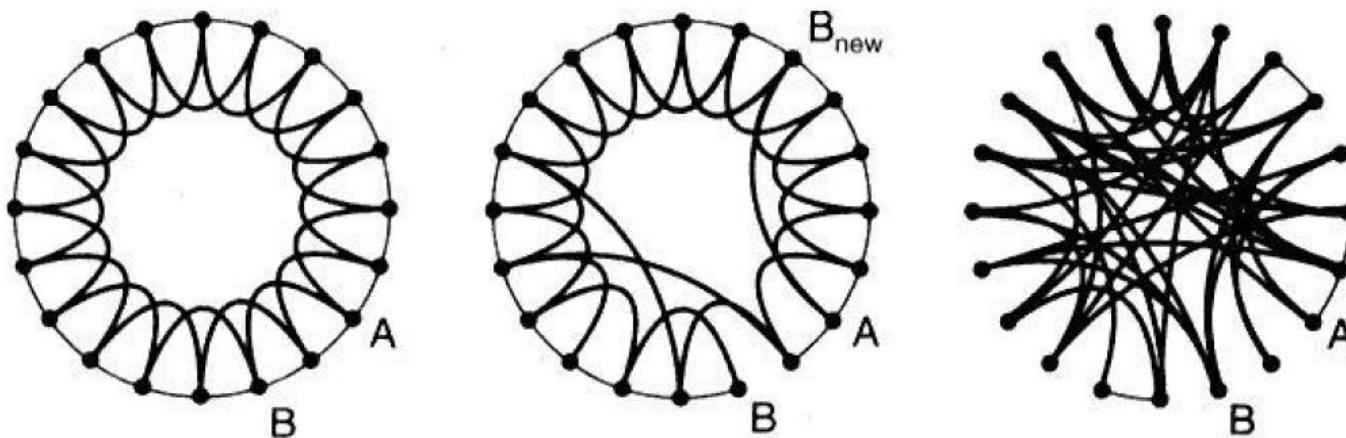
- The probability that a connected triple stays connected after rewiring
 - probability that none of the 3 edges were rewired $(1-p)^3$
 - probability that edges were rewired back to each other very small, can ignore
- Clustering coefficient = $C(p) = C(p=0)*(1-p)^3$



Watts-Strogatz Model

Reconciling two observations:

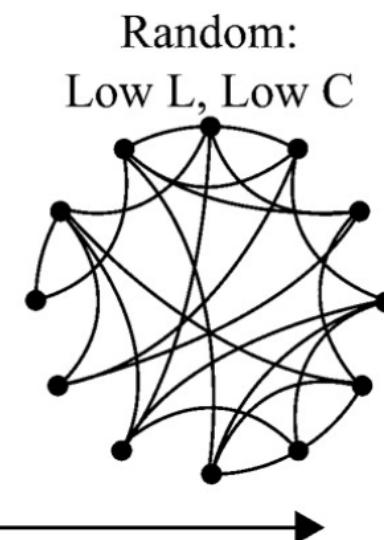
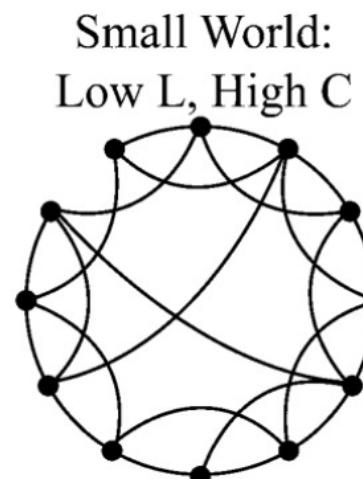
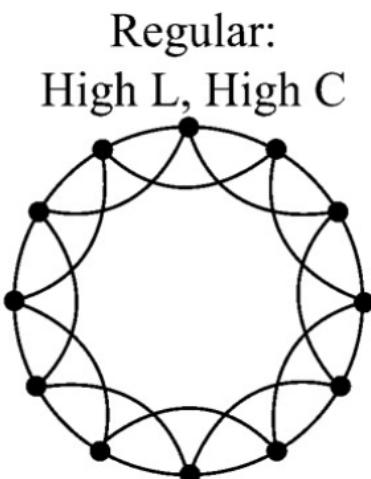
- **High clustering:** my friends' friends tend to be my friends
- **Short average paths**



Source: Watts, D.J., Strogatz, S.H.(1998) Collective dynamics of 'small-world' networks. Nature 393:440-442.

Watts-Strogatz Model

- The resulting graph is so that
 - for the value of $P = 0$ we will have the original ring graph
 - for the value of $P = 1$ produces a pure random graph
 - For some values of P between these two extremes the resulting network has small characteristics path length ,and at the same time, high clustering coefficient
 - the average degree will be $\langle k \rangle = 2m$



→ Increasingly random connectivity

Real-world networks

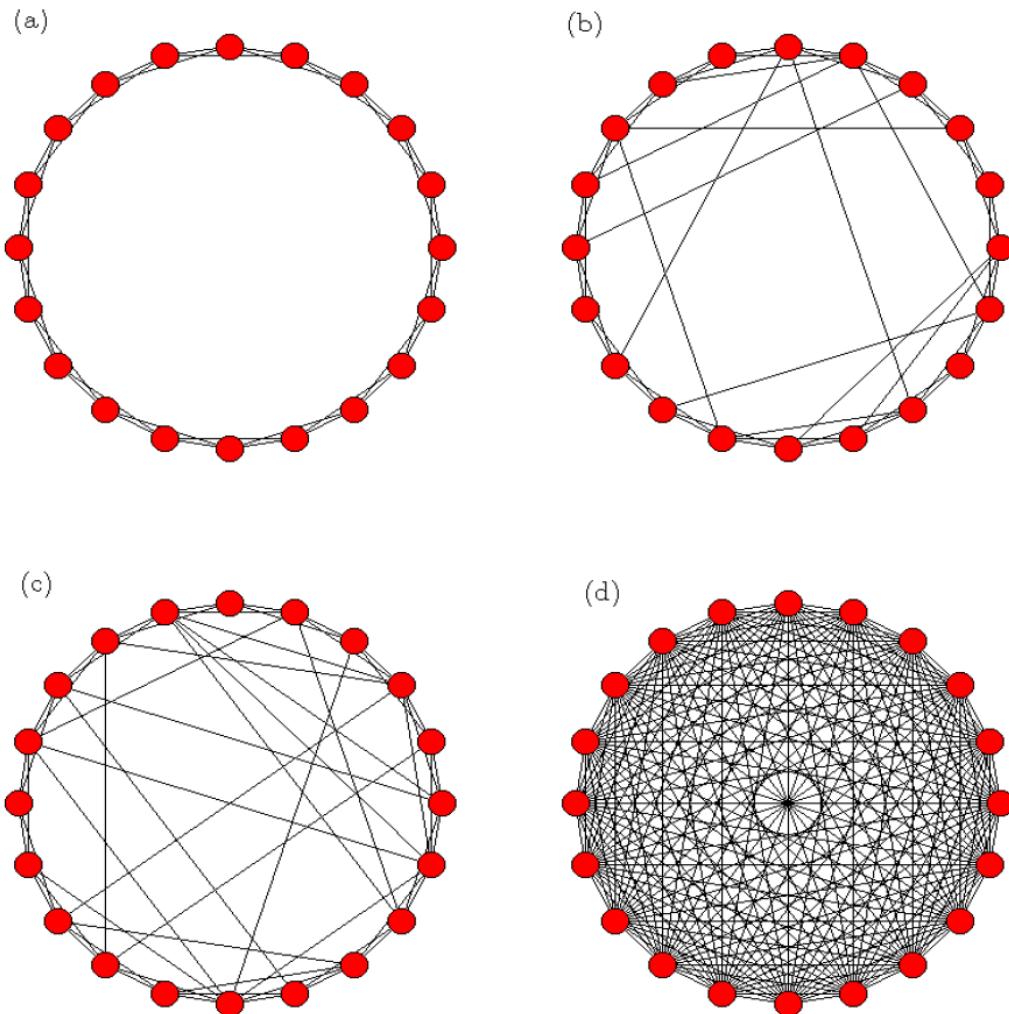
Network	size	Characteristic path length	Shortest path in fitted random graph	Clustering coefficient	Clustering in random graph
Film actors	225,226	3.65	2.99	0.79	0.00027
MEDLINE co-authorship	1,520,251	4.6	4.91	0.56	1.8×10^{-4}
E.Coli substrate graph	282	2.9	3.04	0.32	0.026
C.Elegans	282	2.65	2.25	0.28	0.05

Newman-Watts model

- Starting with a k -ring graph
- N nodes
- Non-connected nodes get connected with probability P
- $P = 1$ results in complete graph
- for some small values of P
 - small-world property
 - high transitivity
- The networks are always connected

Newman-Watts model

20 nodes in a 2-regular ring with
a) $P = 0$
b) $P = 0.05$
c) $P = 0.15$
d) $P = 1$





04

Small-world Network

Newman-Watts model

- It was Longley believed that real-world networks have random structure
- Milgram did an experiment showing the small-world property
- Watts and Strogatz showed that many real-world networks:
 - Have small characteristic path length compared to random networks
 - At the same time, have high clustering coefficient that is much larger than that of random networks
 - They are indeed small-worlds
- This discovery had huge impact on the various developments in Network fields
 - Search in complex networks
 - Communication in networks

Milgram's experiment

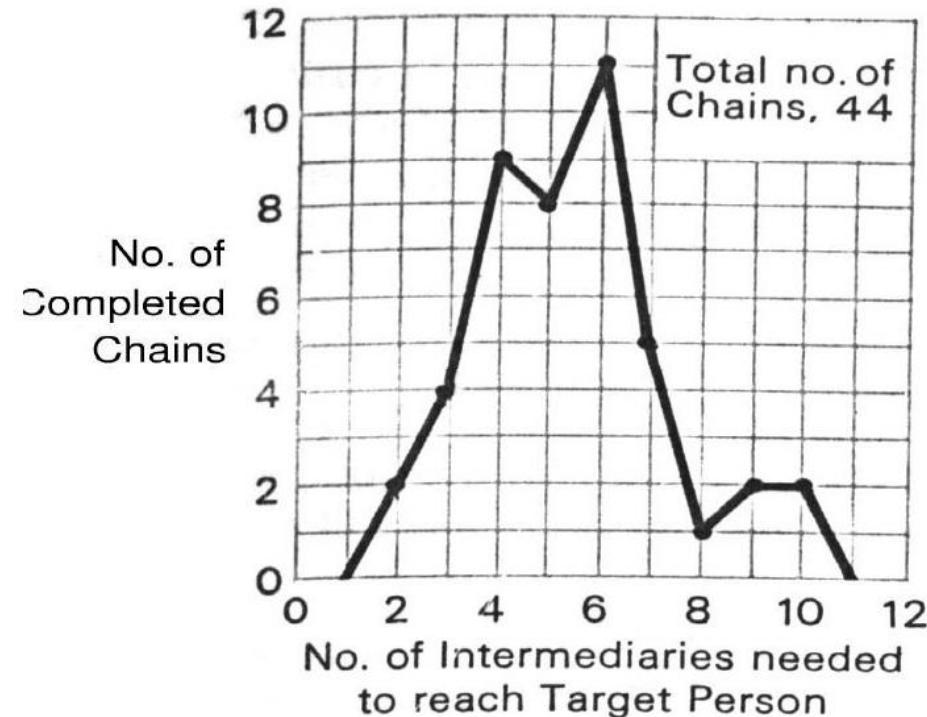
- Instructions:
 - Given a target individual (stockbroker in Boston), pass the message to a person you correspond with who is “closest” to the target.
 - 160 letters: From Wichita (Kansas) and Omaha (Nebraska) to Sharon (Mass)
 - If you do not know the target person on a personal basis, do not try to contact him directly. Instead, mail this folder to a personal acquaintance who is more likely than you to know the target person.
- Outcome:
 - 20% of initiated chains reached
 - Target average chain length = 6.5
 - “Six degrees of separation”



Milgram, *Psych Today* 2, 60 (1967)

Milgram's experiment

- “Six degrees of separation”
- The **Small World** concept in simple terms describes the fact despite their often **large size**, in most networks there is a **relatively short path between any two nodes.**



In the Nebraska Study the chains varied from two to 10 intermediate acquaintances with the median at five.

Milgram's experiment repeated

- Email experiment by Dodds, Muhamad, Watts, Science 301, (2003):
 - 18 targets
 - 13 different countries
 - More than 60,000 participants
 - 24,163 message chains
 - 384 reached their targets
 - Average path length 4.0



Source: NASA, U.S. Government; http://visibleearth.nasa.gov/view_rec.php?id=2429

Applicable to other networks?

Same pattern:

high clustering

$$C_{\text{network}} \gg C_{\text{randomgraph}}$$

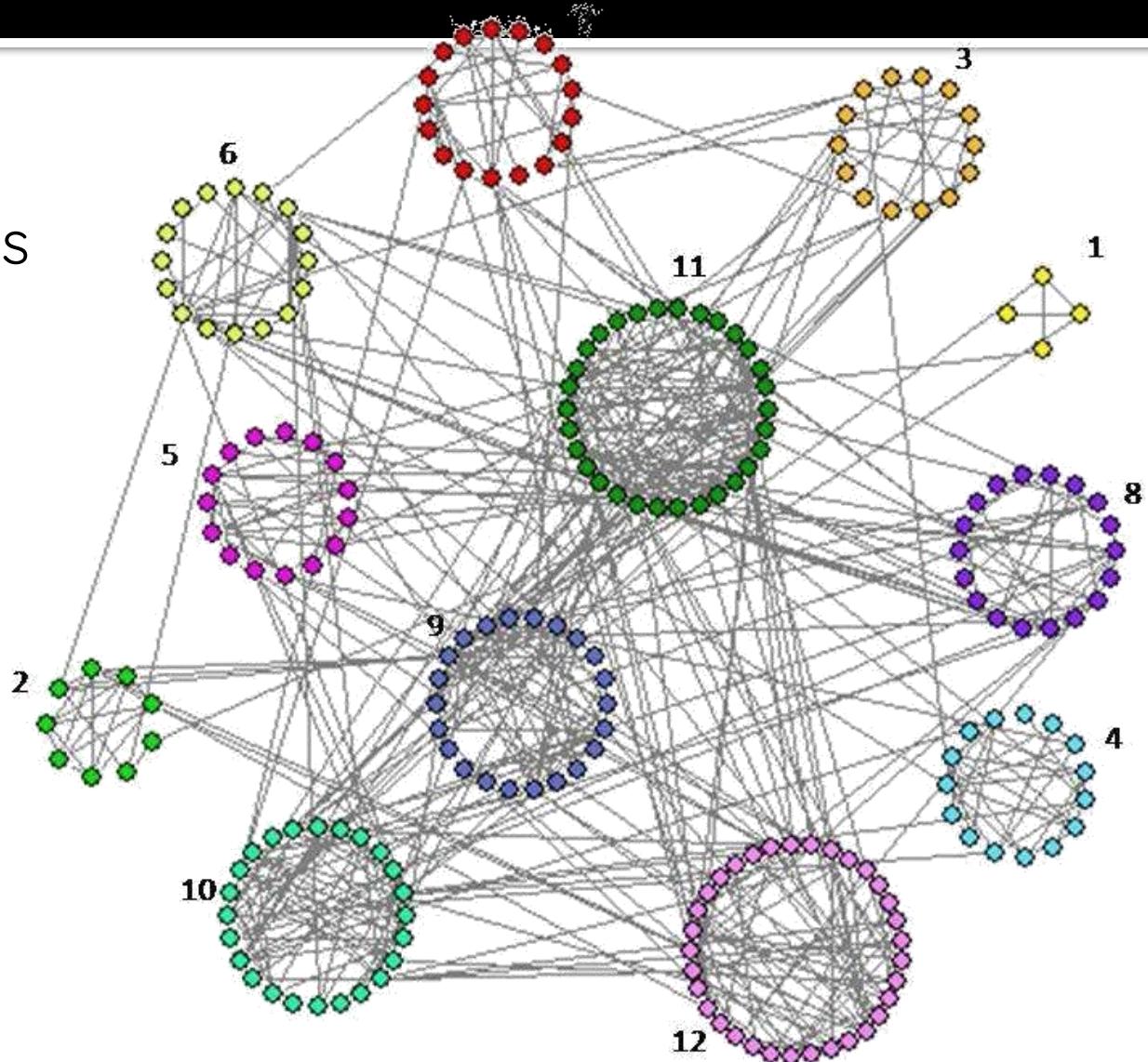
low average shortest path

$$l_{\text{network}} \approx \ln(N)$$

- of course in many social networks
- neural network of *C. elegans*,
- Human brain
- semantic networks of languages,
- actor collaboration graph
- food webs
- Power grids
- ...

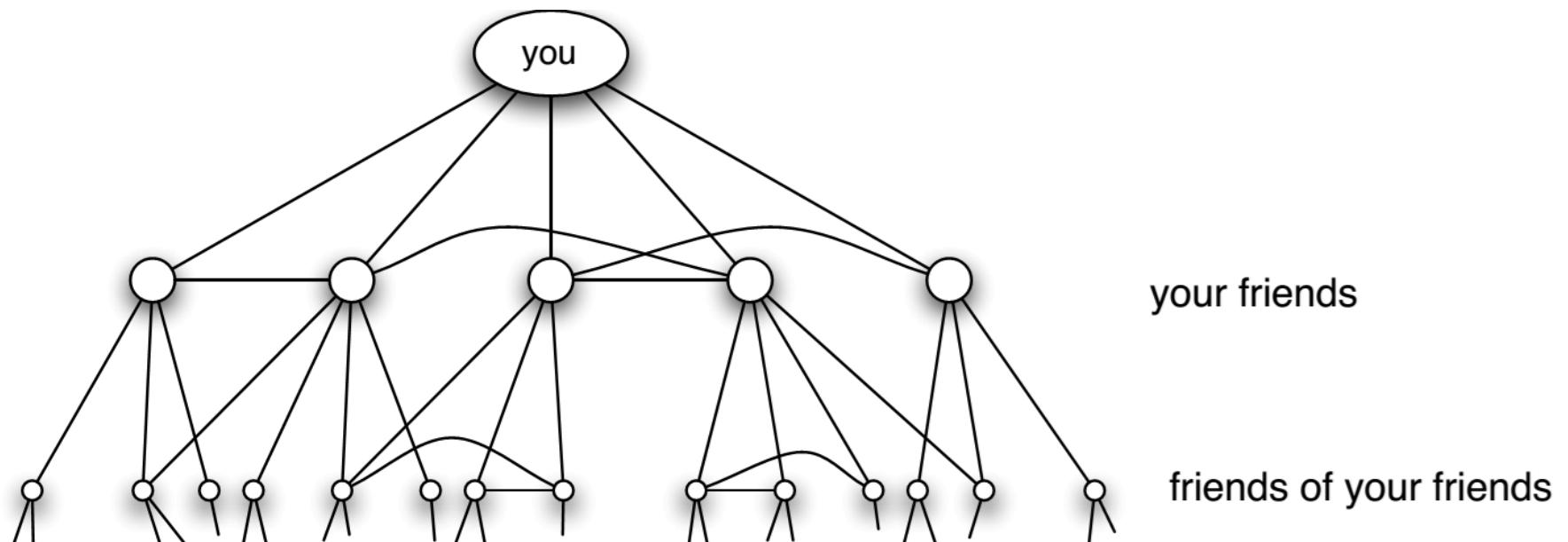
Small Worlds

- Six degrees of separation: although the number of edges is low, nodes are reachable from each other with small number of edges
- Small diameter or Small average path length
 - Weak ties to close dense communities
- Highly Clustered
 - High density of triangles
 - Homophily & prone to triadic closure



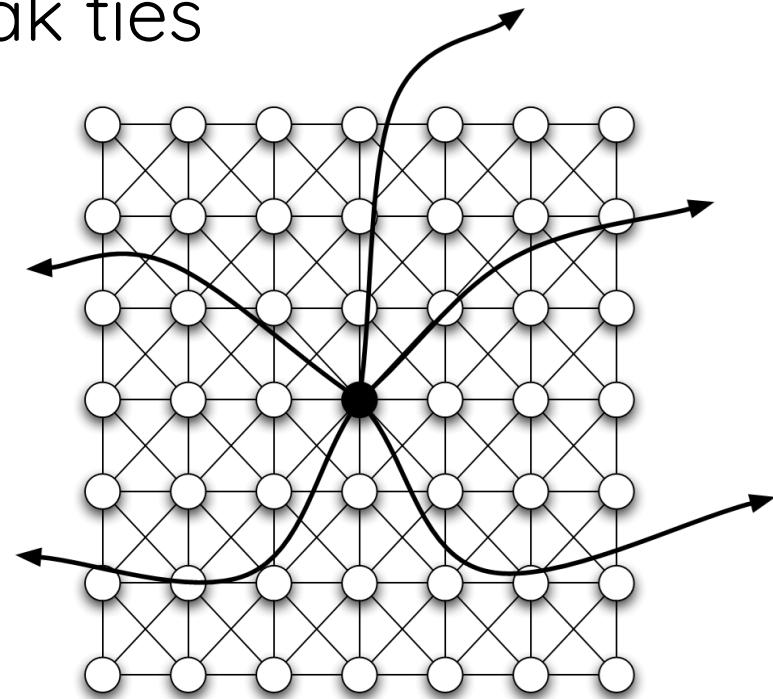
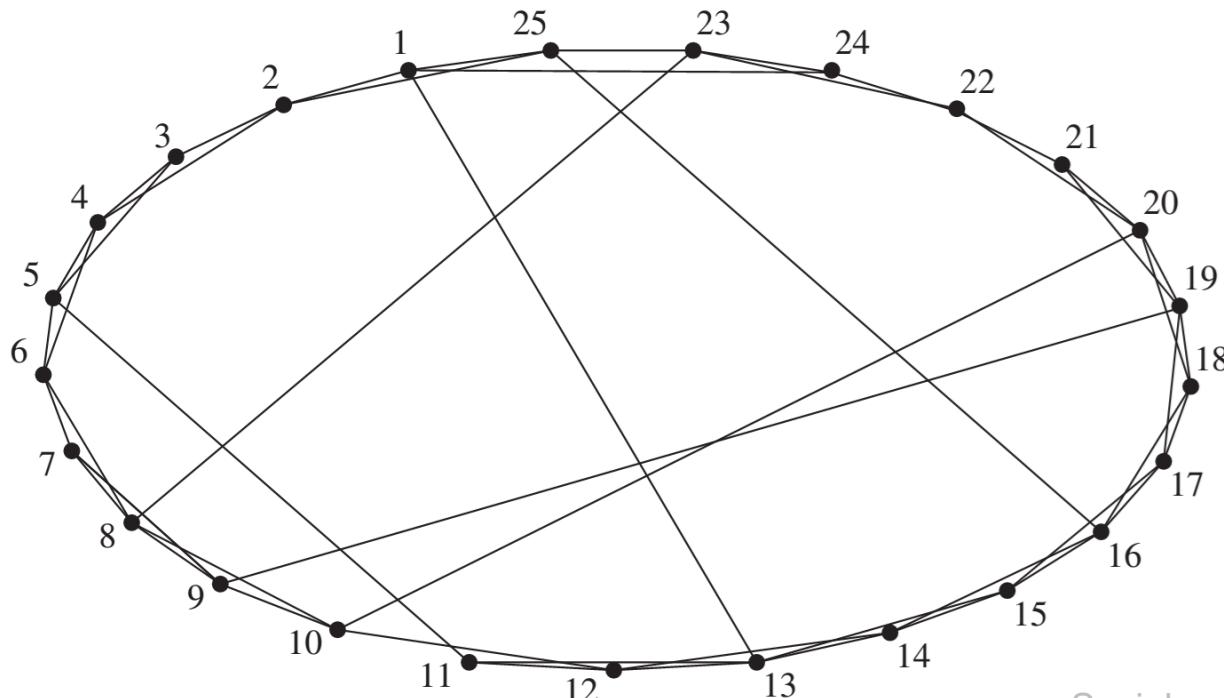
Structure + Randomness

- Structure makes shortest paths
- Random links make triads
- It is naturally incorrect!



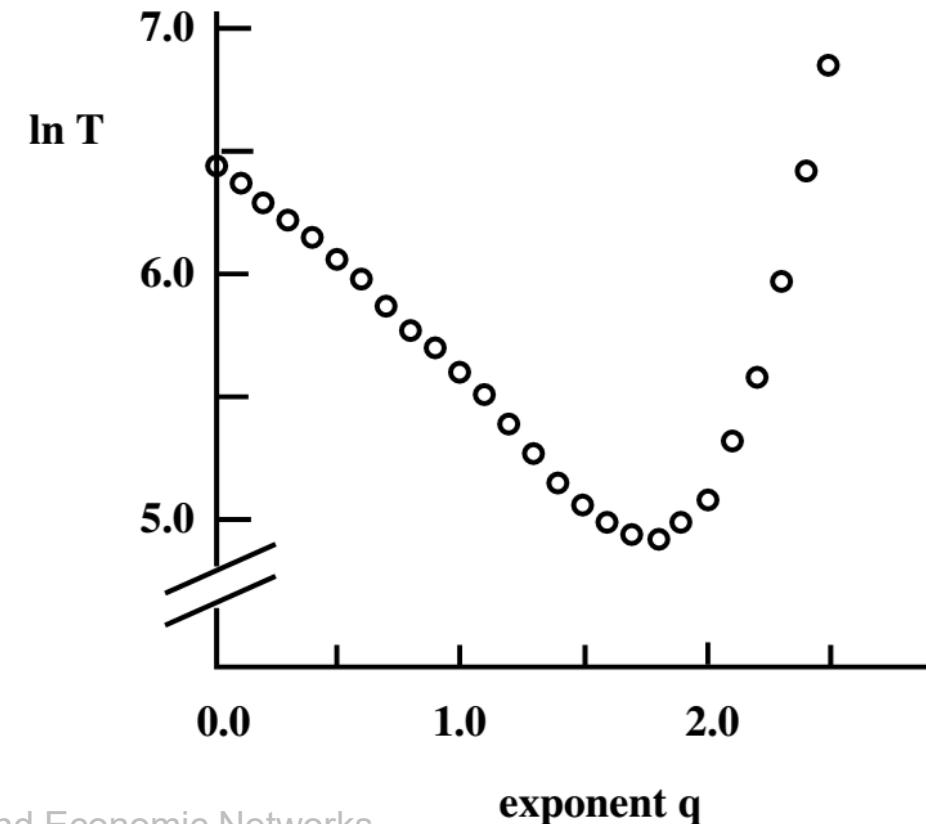
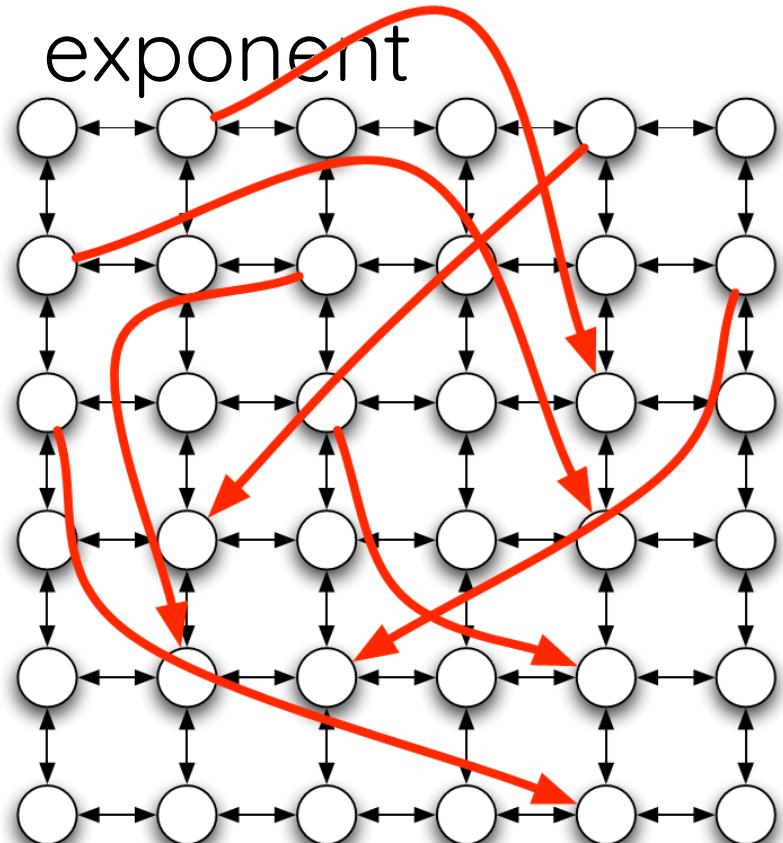
Structure + Randomness

- Watts & Strogatz model
 - Structure makes triads
 - Random links make short distances: Weak ties



Watts-Strogatz Models for Decentralized Search

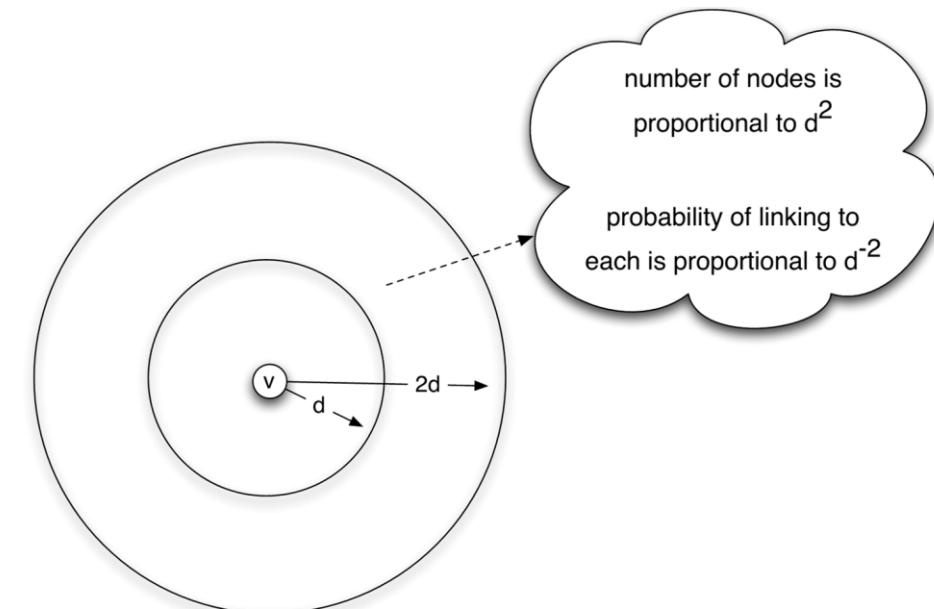
- Consider a grid with additional random links each with probability $d(v, w)^{-q}$ in which q is the clustering exponent



Watts-Strogatz Models for Decentralized Search

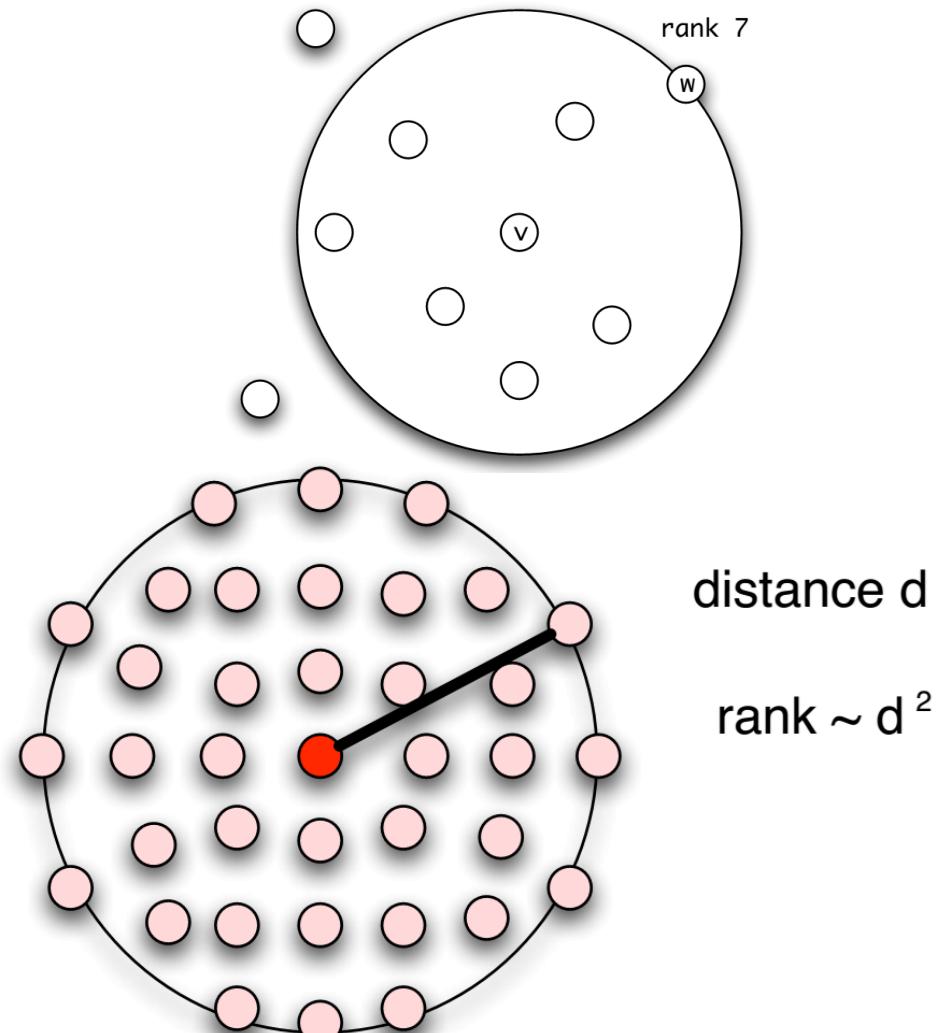
- Let's set the clustering coefficient $q = 2$
- Terms d^2 and d^{-2} cancel each other and thus the probability that a random edge links into *some node* in this ring is approximately independent of the value of d
- long-range weak ties are being formed in a way that's spread roughly uniformly over all different scales of resolution

$$\alpha\pi R^2 \times \frac{1}{R^2}$$



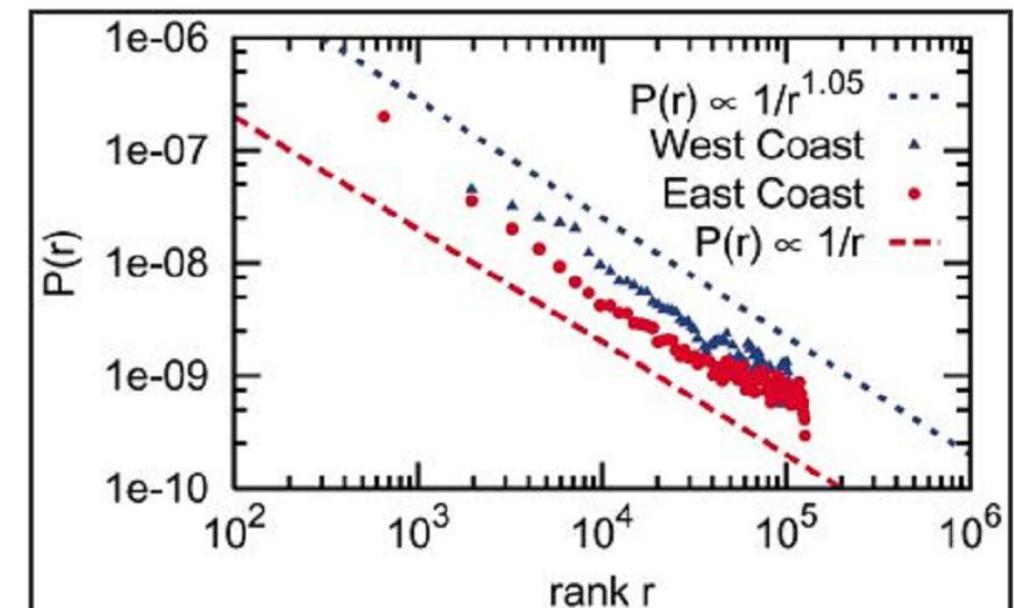
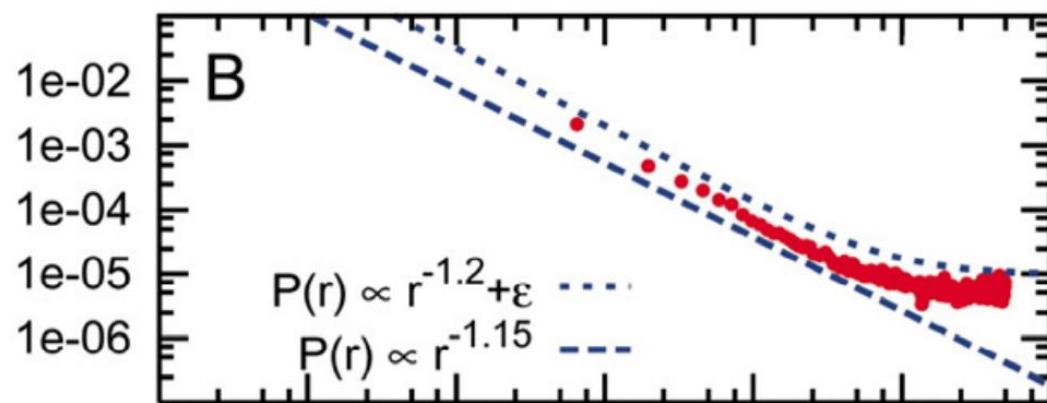
Watts-Strogatz Models for Decentralized Search

- Rank-based friendship:
 - Create (weak) random links with probability $\text{rank}(w)^{-p}$
 - What should p be to have a uniform spread of random links? rank approximately is d^2 , thus p should be approximately 1



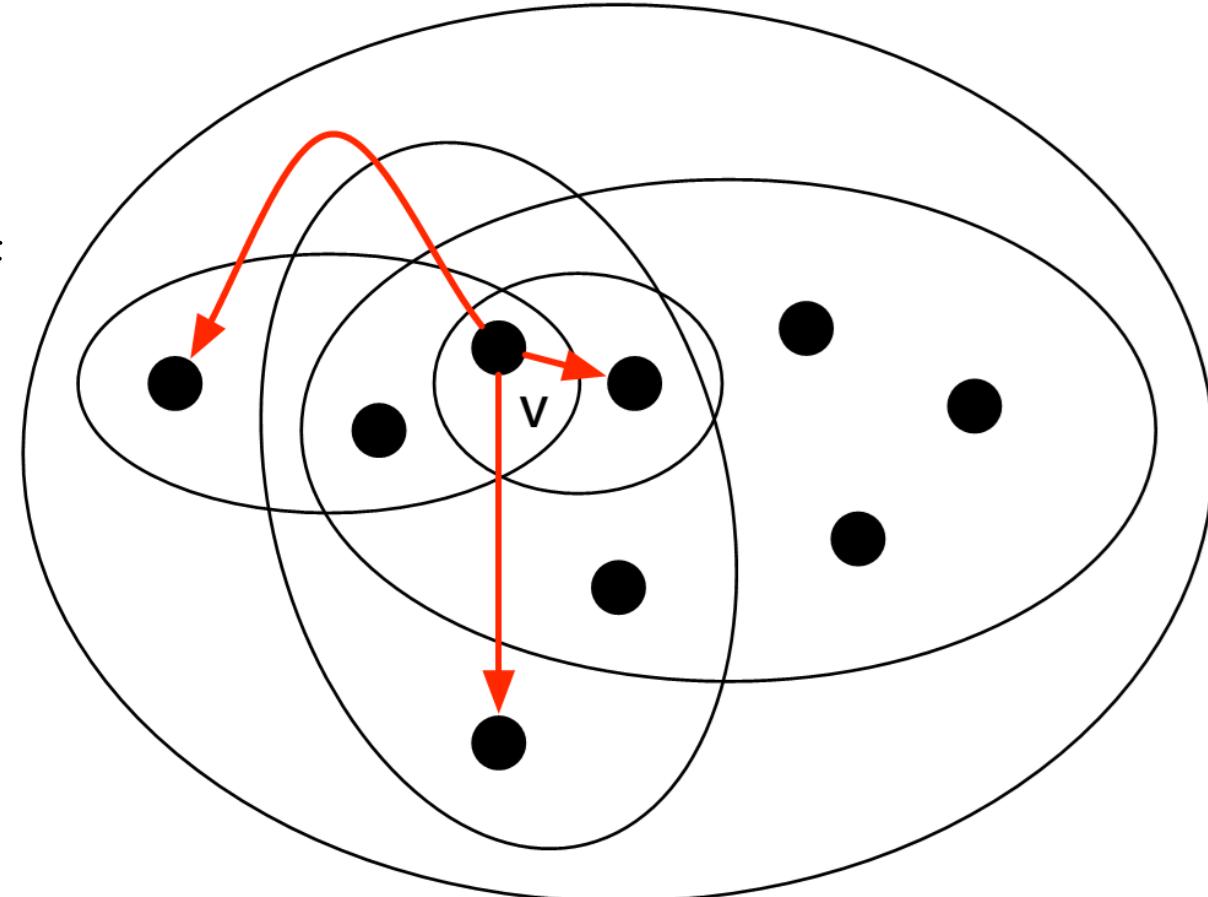
Watts-Strogatz Models for Decentralized Search

■ Some Experiments



Watts-Strogatz Models for Decentralized Search

- Foci-based friendship:
 - Define the size of the smallest focal point that include both of v and w as their distance
 - We again draw random links with probability $dis(v, w)^p$
 - If focal points are defined as the nearest nodes, we may again have $p = 1$



Watts-Strogatz Models for Decentralized Search

- Mathematical study of myopic decentralized search in a simple Watts-Strogatz model:
 - A fixed structure: a ring or a grid with empty links
 - Some additional random links with probability proportional to $d(v, w)^{-1}$ with order of outdegree is 1
 - What is the constant multiplier for link probabilities:

$$Z \leq 2 \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n/2} \right)$$

$$Z \leq 2 + 2 \log_2(n/2) = 2 + 2(\log_2 n) - 2(\log_2 2) = 2 \log_2 n$$

$$\frac{1}{Z} d(v, w)^{-1} \geq \frac{1}{2 \log n} d(v, w)^{-1}$$



05

Markov Graphs & P^* Networks

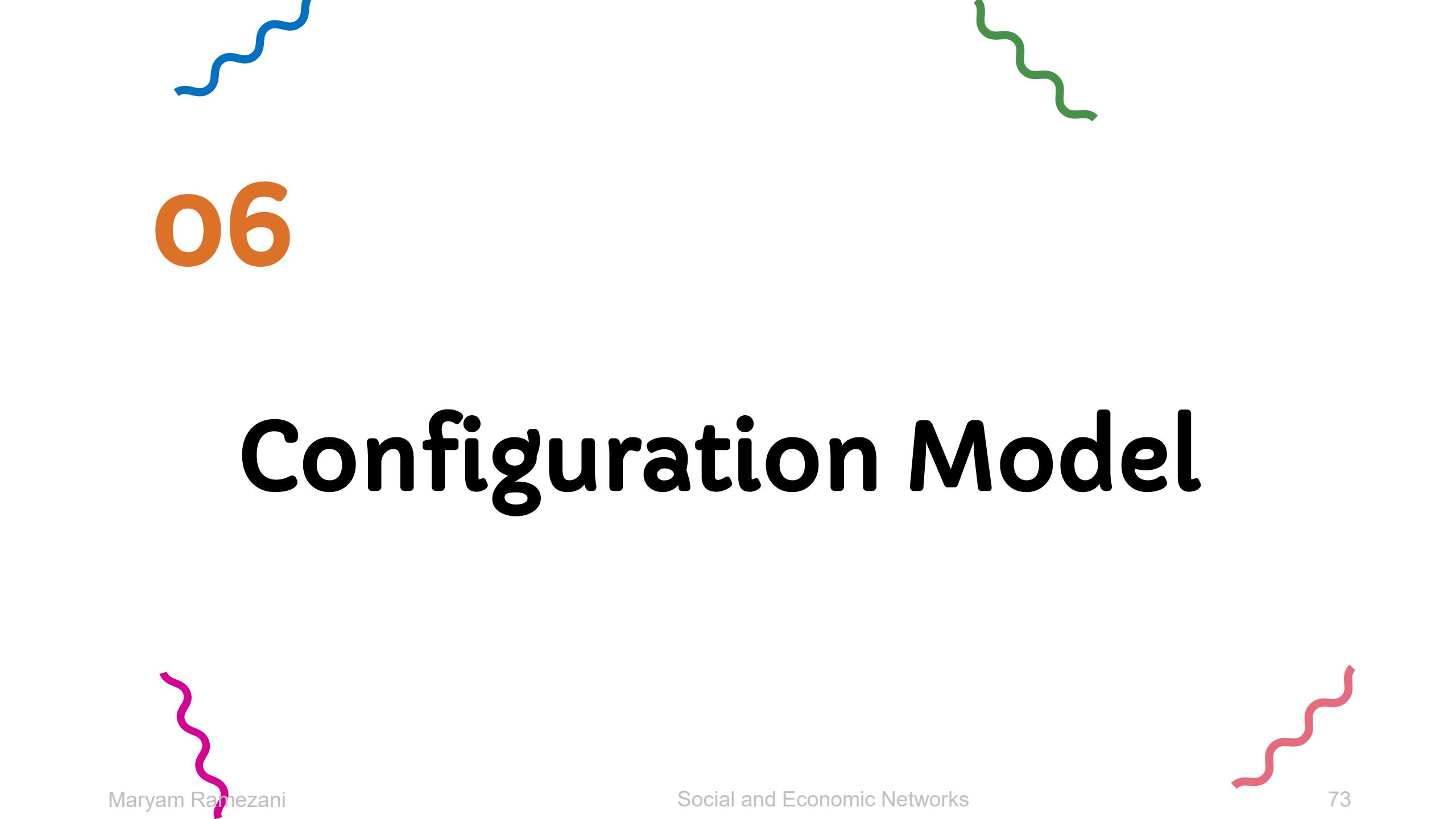
Markov Graphs & P* Networks

- Think about building a random graph in which the formation of the link ij is correlated with formation of the links jk and ik ?
- Frank & Strauss method using Clifford & Hammersley theorem:
 - Build a graph D whose nodes is the potential links in G
 - If ij and jk are linked in D , it means that there exist some sort of dependency between them
 - $C(D)$ is the set of D 's cliques
 - $I_A(G) = 1$ for $A \in C(D)$, $A \subseteq G$ (consider G as a set of edges) and else $I_A(G) = 0$
 - The probability of a given network G depends only on which cliques of D it contains:

$$\log(\Pr[G]) = \sum_{A \in C(D)} \alpha_A I_A(G) - c$$

Markov Graphs & P* Networks

- An example: a symmetric case
 - Build a random graph with controllability on the number of its edges ($n_1(G)$) and its triads ($n_3(G)$)
 - $C(D)$ consists of $n_3(G)$ triads and $n_1(G)$ edges. So, if we weight them equally, we have:
$$\log(\Pr(G)) = \alpha_1 n_1(G) + \alpha_3 n_3(G) - c$$
 - We can calibrate with different parameters to have different random networks with different number of triangles and edges.
 - $\alpha_3 = 0$ is the Poisson networks case



06

Configuration Model

The Configuration Model

- A sequence of degrees is given $(d_1, d_2, d_3, \dots, d_n)$ and we want to build a random graph having these degrees
- We generate the following sequence of numbers

$$\underbrace{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}_{d_1 \text{ entries}} \quad \underbrace{2, 2, 2, 2, 2, 2}_{d_2 \text{ entries}} \quad \dots$$
$$\underbrace{n, n, n, n, n, n, n, n, n, n, n}_{d_n \text{ entries}}.$$

- Randomly pick two number of elements and connect corresponding nodes
- The result is a multigraph

An Expected Degree Model

- Form a link between node i and node j with probability

$$p(e_{ij}) = \frac{d_i d_j}{\sum_k d_k} < 1$$

- Self links are allowed
- The expected degree of node i will be d_i
- Maximum of $d_i^2 < \sum_k d_k$

Configuration Model vs Expected Degree Model

- Consider the degree sequence $\langle k, k, \dots, k \rangle$
- In configuration model:
 - The probabilities of self links and multi links is negligible
 - The probability of a node to have degree k will converge to 1
- In expected degree model:
 - The probability of a node to have degree k will converge to

$$\frac{e^{-k} (k)^k}{k!}$$

whose maximum value is $1/2$.

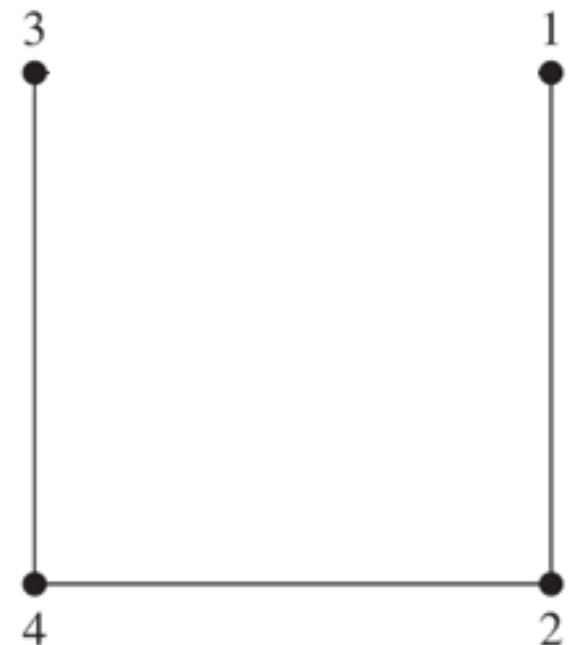
- Why?

Distribution of the Degree of Neighboring Nodes

- Consider a given graph with degree distribution $P(d)$
- A related calculation $\tilde{P}(d)$: the probability that a randomly chosen edge has a (randomly chosen) neighbor with degree d
- $P(d) = \tilde{P}(d)$?
 - $P(1) = P(2) = \frac{1}{2}$
 - $\tilde{P}(1) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$
 - $\tilde{P}(2) = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{1} = \frac{2}{3}$
- We can formulate $\tilde{P}(d)$

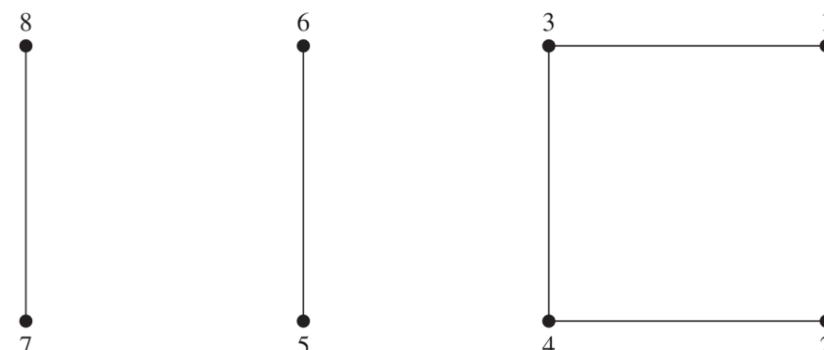
$$\tilde{P}(d) = \frac{P(d)d}{\langle d \rangle}$$

See the blackboard



Distribution of the Degree of Neighboring Nodes

- Consider the degree sequence $\langle 1, 1, 2, 2, 1, 1, 2, 2, \dots \rangle$.
Compare two cases
 - In random models such as the configuration model: The distribution of the neighboring nodes have the same distribution as $\tilde{P}(d)$ for all nodes.
 - In networks with correlation properties: The graph is highly segregated by degrees



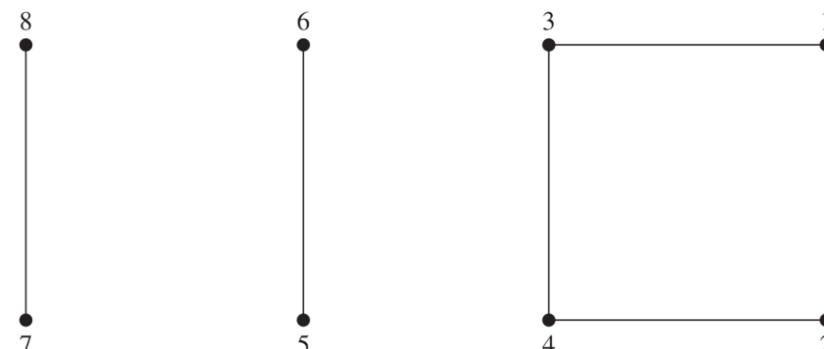


07

Preferential Attachment

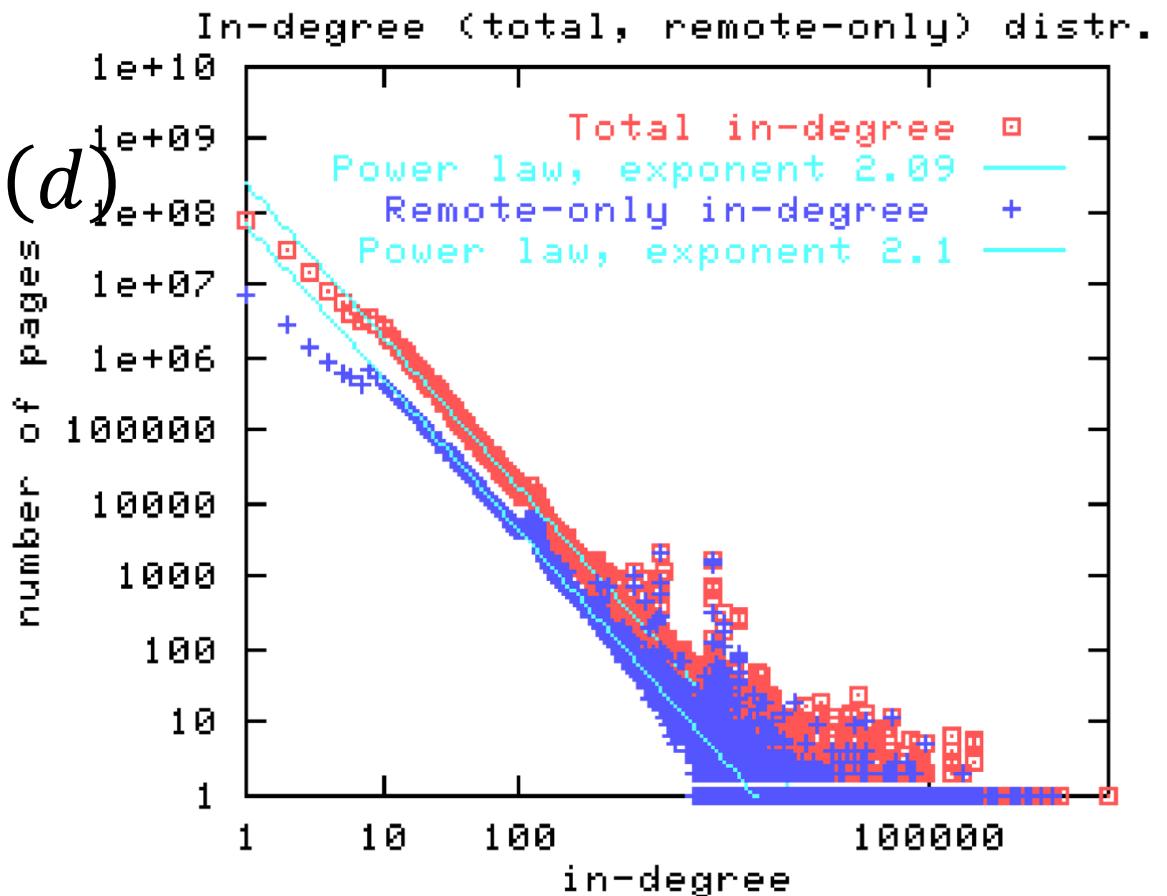
Distribution of the Degree of Neighboring Nodes

- Consider the degree sequence $\langle 1, 1, 2, 2, 1, 1, 2, 2, \dots \rangle$.
Compare two cases
 - In random models such as the configuration model: The distribution of the neighboring nodes have the same distribution as $\tilde{P}(d)$ for all nodes.
 - In networks with correlation properties: The graph is highly segregated by degrees



Power Law Degree Distribution

- $P(d) = cd^{-\gamma}$
- $\log(P(d)) = \log(c) - \gamma \log(d)$
- Features:
 - Scale-free
 - Fat tail



Richer-Get-Richer & Preferential Attachment

- In many scenarios, richers have more opportunity to get richers
 - More money for investment
 - Lower risks
 - More reputation to be involved in activities
 -
- Preferential Attachment: richer-get-richer effect in network creation
 - The probability that page L experiences an increase in popularity is directly proportional to L 's current popularity.
 - In the sense that links are formed “preferentially” to pages that already have high popularity

Preferential Attachment Models

- Devise models to simulate preferential attachment processes
- A basic growing model:
 - Nodes are born over time and indexed by their date of birth $i \in \{0, 1, 2, \dots, t, \dots\}$
 - Upon birth each new node forms m links with pre-existing nodes
 - It attaches to nodes with probabilities proportional to their degrees.
 - the probability that an existing node i receives a new link:
- The interesting fact is that these models leads to networks with power-law degree distribution

Growing Models

- A network model dealing with adding newborn nodes instead of statically having the whole network
- Consider a variation of the Poisson random setting
 - Start with a complete network of $m+1$ nodes
 - Each newborn node choose m nodes from the existing ones and links to them
- A natural study of degree distribution:
 - The expected degree of a node born at time i , at time t :

$$m + \frac{m}{i+1} + \frac{m}{i+2} + \dots + \frac{m}{t} = m \left(1 + \frac{1}{i+1} + \dots + \frac{1}{t} \right) \approx m \left(1 + \log \left(\frac{t}{i} \right) \right)$$

- Degree distribution:

$$m \left(1 + \log \left(\frac{t}{i} \right) \right) < d \Rightarrow i > t e^{1 - \frac{d}{m}}$$

Growing Models

- A natural study of degree distribution:
 - The nodes with expected degree less than d are those born at time $te^{1-\frac{d}{m}}$
 - This is a fraction of $1 - e^{1-\frac{d}{m}}$ of total t nodes
 - Thus
- Another way: Mean Field Approximation

$$F_t(d) = 1 - e^{-\frac{d-m}{m}}$$

Mean Field Approximation

- Using expected increase in the number of sth as its rate
- Visiting the last example with MFA:

$$\frac{dd_i(t)}{dt} = \frac{m}{t} \Rightarrow d_i(t) = m + m \log\left(\frac{t}{i}\right)$$

$$d = m + m \log\left(\frac{t}{i(d)}\right)$$

$$\frac{i(d)}{t} = e^{-\frac{d-m}{m}}$$

- With the same argumentation we have:

$$F_t(d) = 1 - e^{-\frac{d-m}{m}}$$

Basic Preferential Attachment Model

- The probability that an existing node i receives a new link:

$$m \frac{d_i(t)}{\sum_{j=1}^t d_j(t)} = m \frac{d_i(t)}{2mt} = \frac{d_i(t)}{2t}$$

- Using MFA:

$$\frac{dd_i(t)}{dt} = \frac{d_i(t)}{2t}$$

- With initial condition $d_i(0) = m$ we have:

$$d_i(t) = m \left(\frac{t}{i}\right)^{\frac{1}{2}}$$

Basic Preferential Attachment Model

- We have:

$$\frac{i_t(d)}{t} = \left(\frac{m}{d}\right)^2$$

- Thus

$$F_t(d) = 1 - m^2 d^{-2} \Rightarrow f_t(d) = 2m^2 d^{-3}$$

- If the rate changes to $\frac{d_i(t)}{\gamma t}$ we have:

$$f_t(d) = \gamma m^\gamma d^{-\gamma-1}$$

Which is a power law distribution

Hybrid Preferential Attachment Models

- Mixing Random & Preferential Attachment:

$$\frac{dd_i(t)}{dt} = \frac{\alpha m}{t} + \frac{(1 - \alpha)m d_i(t)}{2mt} = \frac{\alpha m}{t} + \frac{(1 - \alpha)d_i(t)}{2t}$$

- By solving the above differential equation we have:

$$d_i(t) = \phi_t(i) = \left(d_0 + \frac{2\alpha m}{1 - \alpha} \right) \left(\frac{t}{i} \right)^{(1-\alpha)/2} - \frac{2\alpha m}{1 - \alpha}$$

Hybrid Preferential Attachment Models

- By solving the above differential equation we have:

$$d_i(t) = \phi_t(i) = \left(d_0 + \frac{2\alpha m}{1-\alpha} \right) \left(\frac{t}{i} \right)^{(1-\alpha)/2} - \frac{2\alpha m}{1-\alpha}$$

- To have the degree distribution:

- If $d_i(t) = \phi_t(i)$ (the degree of the node with i'th birth)

$$F_t(d) = 1 - \frac{\phi_t^{-1}(d)}{t}$$

$$\phi_t^{-1}(d) = t \left(\frac{d_0 + \frac{2\alpha m}{1-\alpha}}{d + \frac{2\alpha m}{1-\alpha}} \right)^{2/(1-\alpha)}$$

$$F_t(d) = 1 - \left(\frac{m + \frac{2\alpha m}{1-\alpha}}{d + \frac{2\alpha m}{1-\alpha}} \right)^{2/(1-\alpha)}$$

Graph Properties

- A property P holds **almost surely** (or for **almost every** graph), if
$$\lim_{n \rightarrow \infty} P[G \text{ has } P] = 1$$
- Evolution of the graph: which properties hold as the probability p increases?
 - different from the evolving graphs that we will see in the future lectures
- **Threshold phenomena:** Many properties appear suddenly. That is, there exist a probability p_c such that for $p < p_c$ the property does not hold and for $p > p_c$ the property holds.



Any Question?