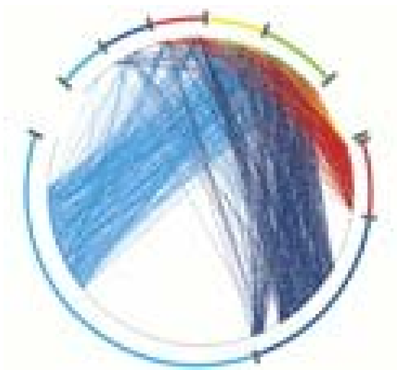
Lecture 9&10: Random Walks, Random Networks & Small-world Networks



Random walks

- When the underlying data has a natural graph structure, several physical processes can be conceived as a random walk

Data	Process
WWW	Random surfer
Internet	Routing
P2P	Search
Social network	Information percolation

- Given a graph and a starting point (node), we select a neighbor of it at random, and move to this neighbor
- Then, we select a neighbor of this node and move to it, and so on
- The (random) sequence of nodes selected this way is a random walk on the graph

Random walks: definitions

- nxn Adjacency matrix A.
  - A(i,j) = weight on edge from *i* to *j*
  - If the graph is undirected A(i,j)=A(j,i), i.e. A is symmetric

- nxn Transition matrix P.
  - P is row stochastic
  - P(i,j) = probability of stepping on node j from node i

    = $A(i,j)/\Sigma_i A(i,j)$

- nxn Laplacian Matrix L.
  - $L(i,j)=\Sigma_i A(i,j)-A(i,j)$
  - Symmetric positive semi-definite for undirected graphs
  - Singular

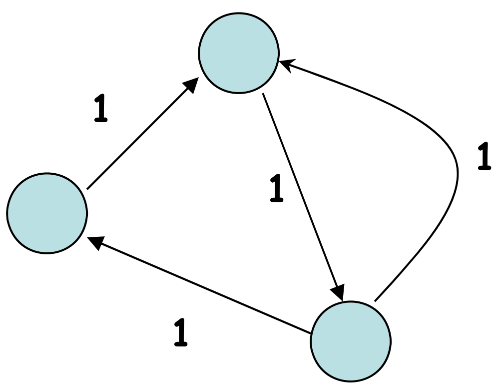
Random walks: definitions

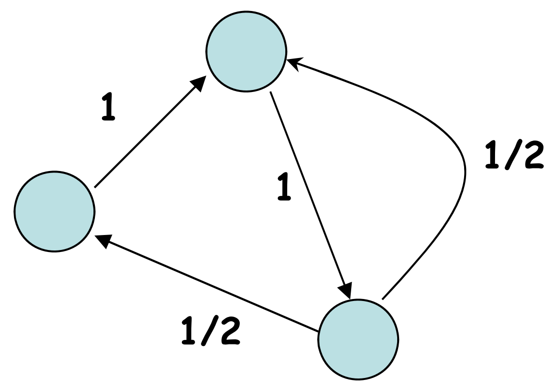
0	1	0
0	0	1
1	1	0

Adjacency matrix A

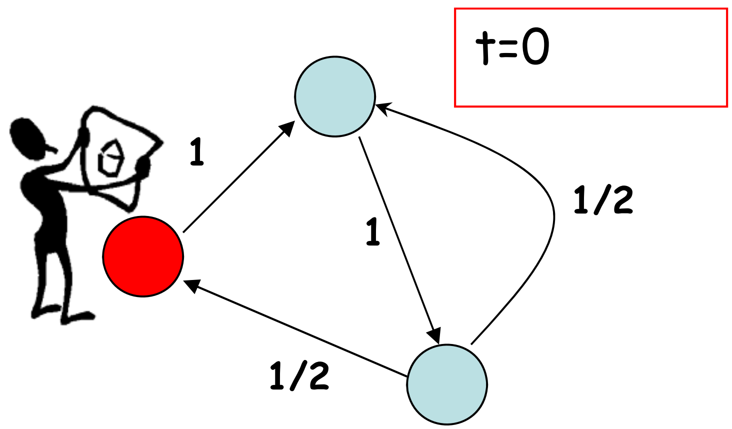
0	1	0
0	0	1
1/2	1/2	0

Transition matrix P





What is a random walk



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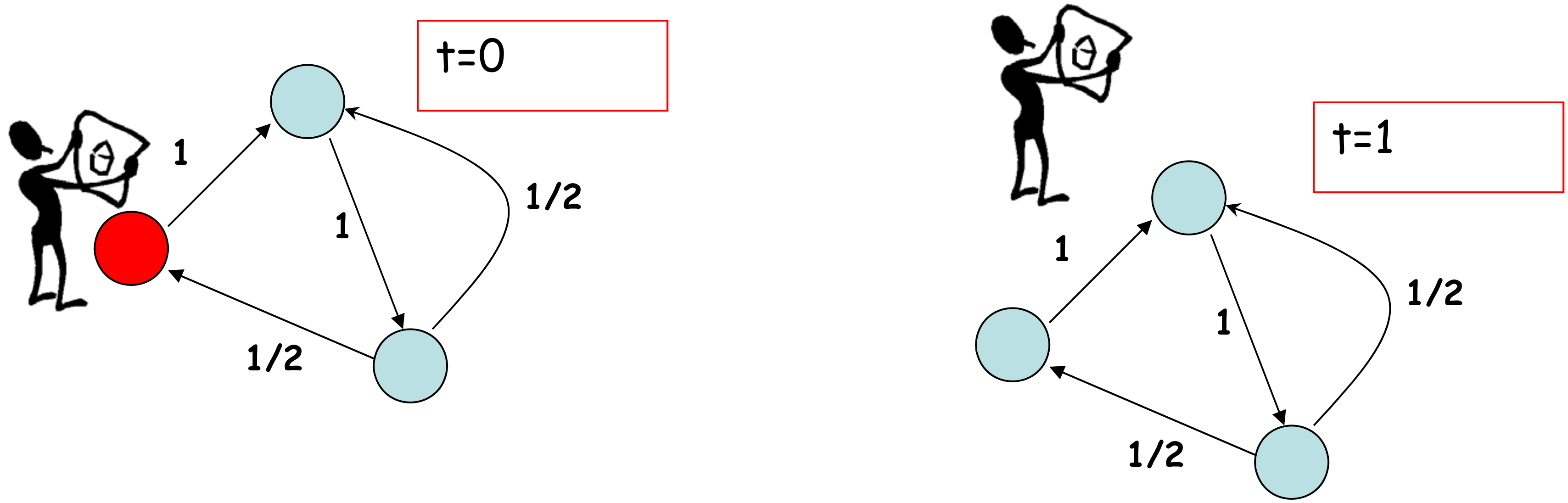

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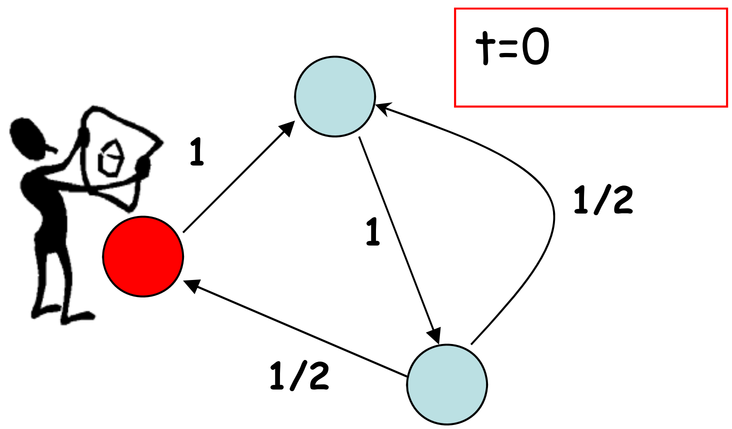
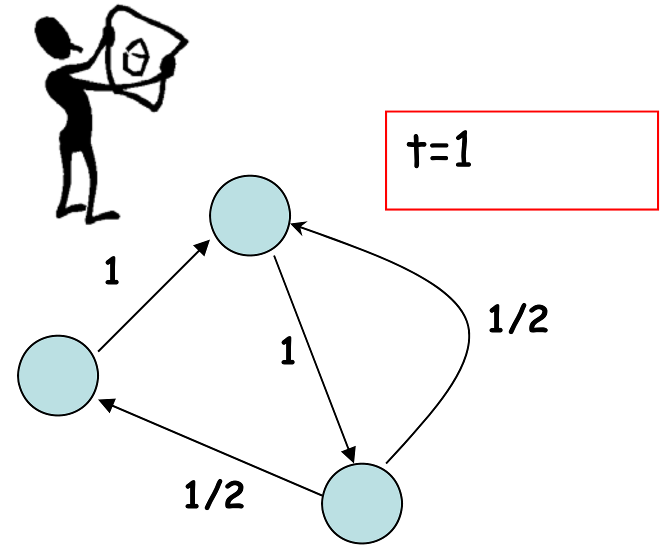
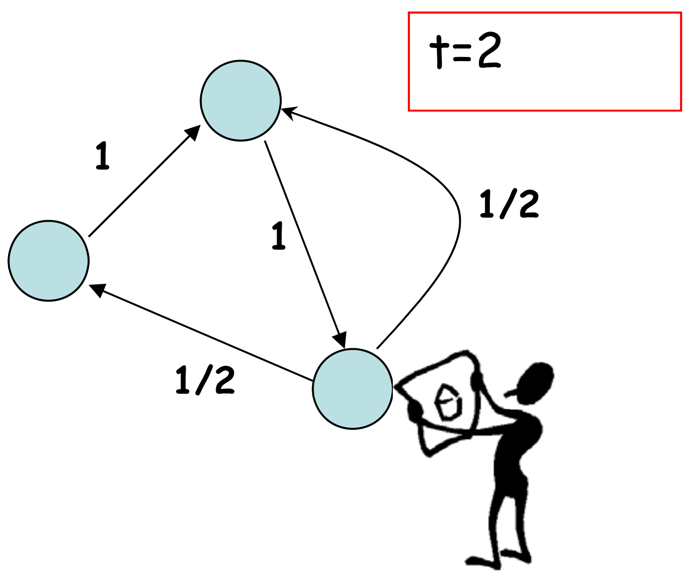
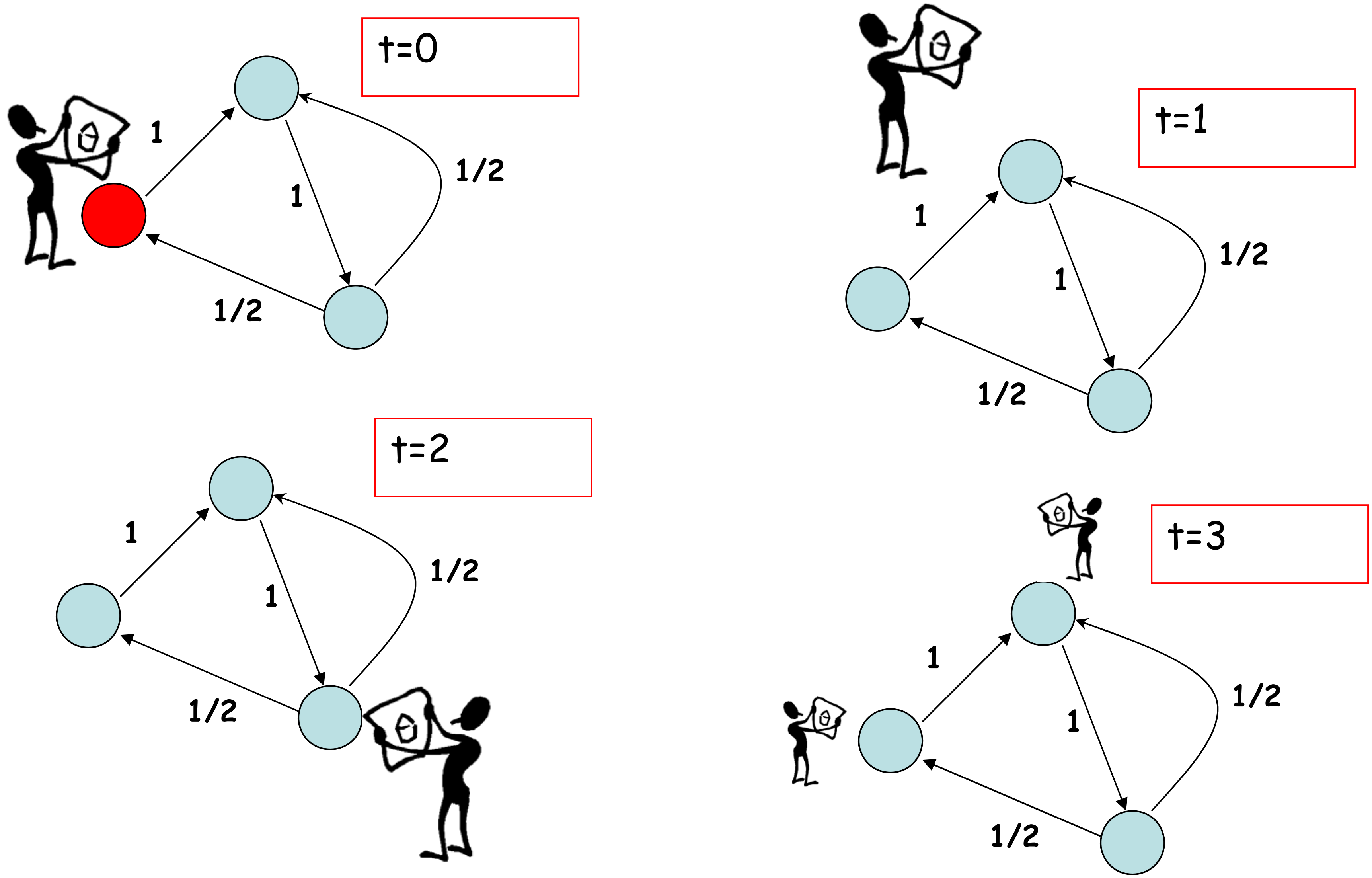






What is a random walk



Probability Distributions

- $x_t(i)$ = probability that the surfer is at node *i* at time *t*
- $x_{t+1}(i) = \sum_j$(Probability of being at node j)*Pr(j->i) $=\sum_j x_t(j)*P(j,i)$
- $x_{t+1} = x_tP = x_{t-1}*P*P = x_{t-2}*P*P*P = \ldots = x_0 P^t$

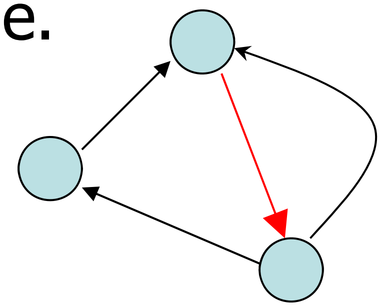
- What happens when the surfer keeps walking for a long time?
- Stationary distribution:
  - When the surfer keeps walking for a long time
  - When the distribution does not change anymore, i.e. $x_{T+1} = x_T$
  - For "well-behaved" graphs this does not depend on the start distribution!!!

What is a stationary distribution?

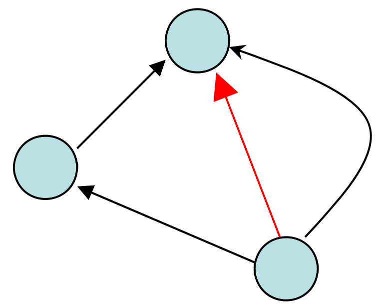
- The stationary distribution at a node is related to the amount of time a random walker spends visiting that node.
- Remember that we can write the probability distribution at a node as
  - $x_{t+1} = x_t P$
- For the stationary distribution v_0 we have
  - $v_0 = v_0 P$
- So, that's just the left eigenvector of the transition matrix!
- Interesting questions:
  - Does a stationary distribution always exist? Is it unique? (Yes, if the graph is "well-behaved")
  - What is "well-behaved"?
  - How fast will the random surfer approach this stationary distribution? (Mixing Time!)

Well-behaved graphs

- Irreducible: There is a path from every node to every other node.

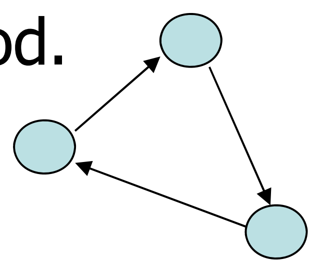


Irreducible

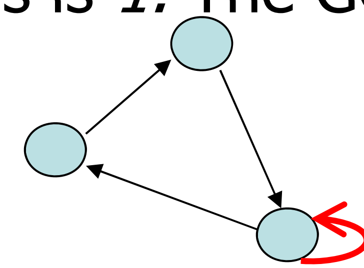


Not irreducible

- Aperiodic: The GCD of all cycle lengths is *1*. The GCD is also called period.



Periodicity is 3

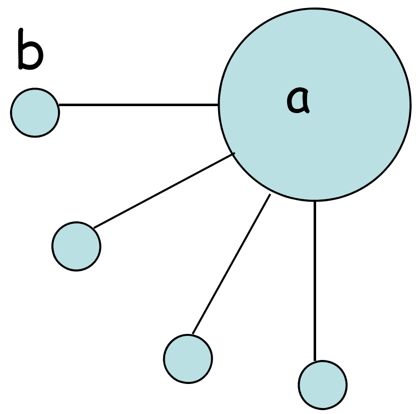


Aperiodic

Implications of the Perron Frobenius Theorem

- If a markov chain is irreducible and aperiodic, then the largest eigenvalue of the transition matrix will be equal to **1** and all the other eigenvalues will be strictly less than **1**.
  - Let the eigenvalues of P be $\{\sigma_i | i=0:n-1\}$ in non-increasing order of σ_i .
  - $\sigma_0 = 1 \mathbf{>} \sigma_1 > \sigma_2 >= \ldots\ldots >= \sigma_n$
- These results imply that for a well behaved graph there exists an unique stationary distribution.
- The pagerank uses these results.
- We know that
  - A connected undirected graph is irreducible
  - A connected non-bipartite undirected graph has a stationary distribution proportional to the degree distribution!
  - Makes sense, since larger the degree of the node more, likely a random walk is to come back to it.

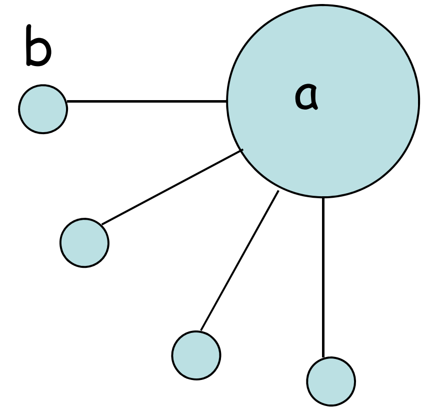
Proximity measures from random walks



- How long does it take to hit node b in a random walk starting at node a ? Hitting time.

- How long does it take to hit node b and come back to node a ? Commute time.

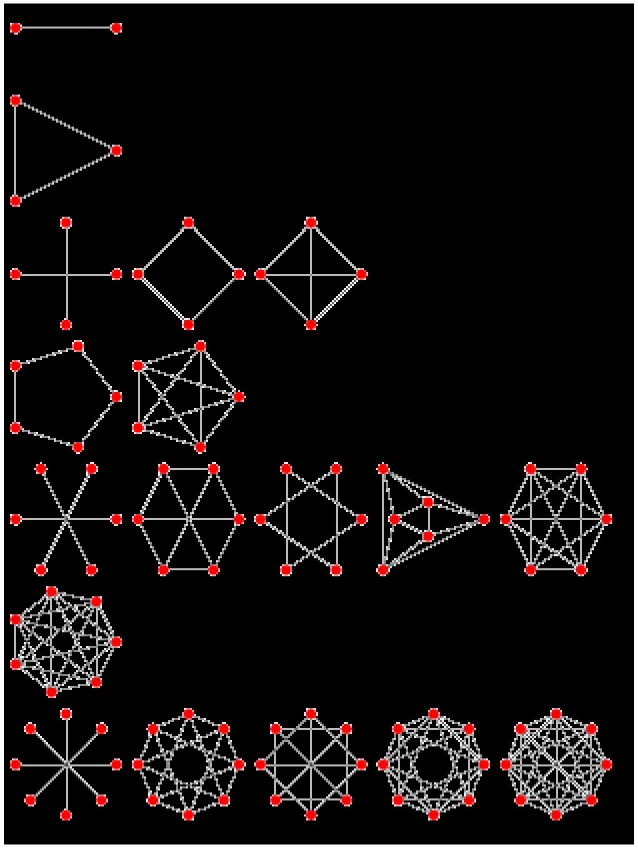
Hitting and Commute times



- Hitting time from node i to node j
  - Expected number of hops to hit node j starting at node i
  - Is not symmetric. $h(a,b) \neq h(b,a)$
  - $h(i,j) = 1 + \Sigma_{k \in nbs(A)}\, p(i,k)h(k,j)$
- Commute time between node i and j
  - Is expected time to hit node j and come back to i
  - $c(i,j) = h(i,j) + h(j,i)$
  - Is symmetric. $c(a,b) = c(b,a)$

Regular graphs

- Graphs with regular structure
  - Ring graph
  - Chain
  - Lattice
  - …
- Many of network properties can be calculated analytically
- Regular networks are indeed ordered ones



Order vs disorder

- Order
  - Abstract Algebra
  - Crystal Structures
  - Exact Symmetries
  - Group Theory
  - Regular Graphs, Lattices
- Disorder
  - Erdos–Renyi Model, Random Graphs
  - Chaos, Mixing, etc.
  - Unpredictability
  - Tossing Coins (IID Processes)
  - Ideal Gases

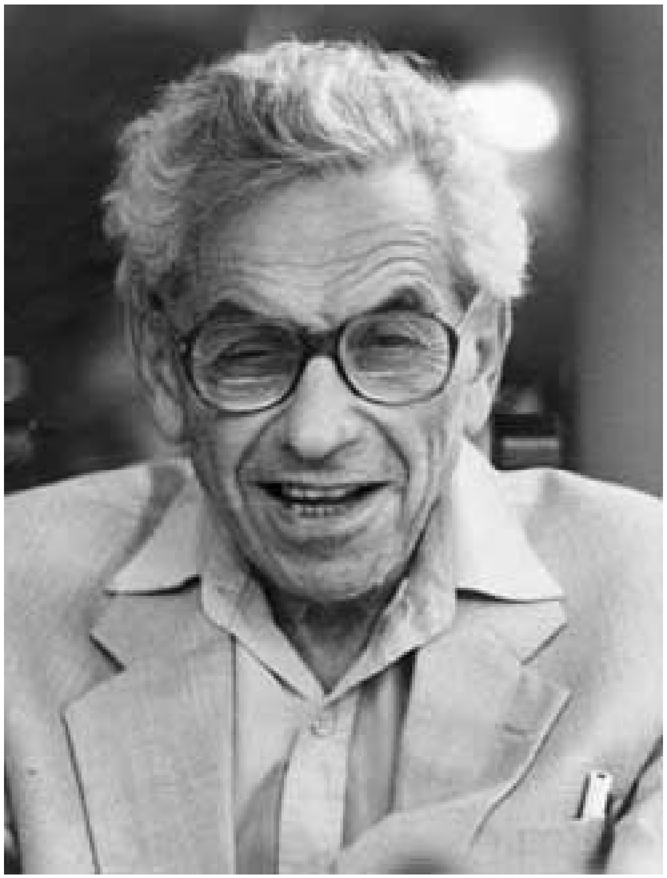
- There are well understood mathematical techniques for studying the extremes of order and disorder.
- Intermediate regions are harder. Often one starts at one extreme and then perturbs or expands off that extreme to get approximate solutions.

Random graphs

- A deterministic model D defines a single graph for each value of n (or t)

- A randomized model R defines a probability space ‹G_n,P› where G_n is the set of all graphs of size n, and P a probability distribution over the set G_n (similarly for t)
  - we call this a family of random graphs R, or a random graph R

Erdös-Renyi Random graphs



Paul Erdös (1913-1996)

You may have heard about Erdös number!
What is your Erdös number?

Erdös number

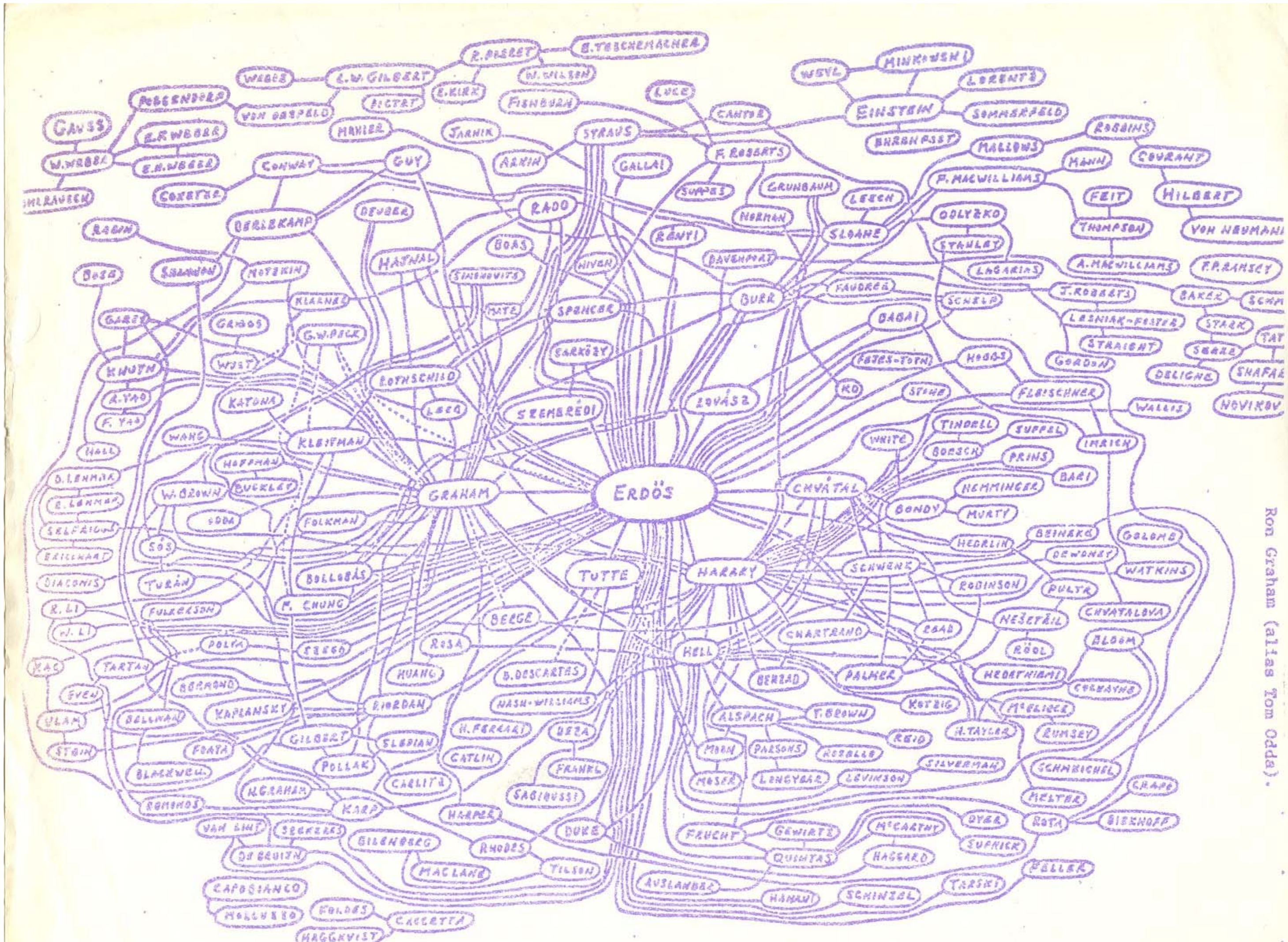


Figure 1

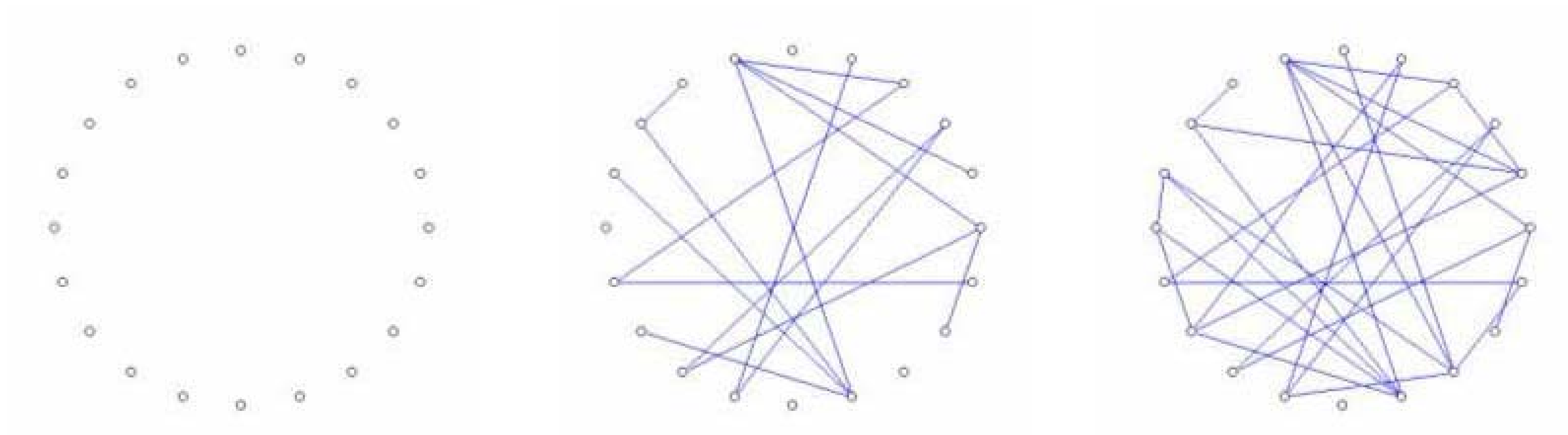
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

Erdös-Renyi Random Graphs

For generation of Erdös-Renyi network, one of the following methods is used:

1. The $G_{n,p}$ model
   - input: the number of vertices n, and a parameter p, $0 \leq p \leq 1$
   - process: for each pair (i,j), generate the edge (i,j) independently with probability p
2. Related, but not identical: The $G_{n,m}$ model
   - process: select m edges uniformly at random

Erdös-Renyi Random Graphs



$p = 0$ (a) $p = 0.1$ (b) $p = 0.2$ (c)

Graph properties

- A property P holds almost surely (or for almost every graph), if

$$\lim_{n\to\infty} P[G \text{ has } P] = 1$$

- Evolution of the graph: which properties hold as the probability p increases?
  - different from the evolving graphs that we will see in the future lectures
- Threshold phenomena: Many properties appear suddenly. That is, there exist a probability p_c such that for $p<p_c$ the property does not hold and for $p>p_c$ the property holds.

The giant component

- Let $z=np$ be the average degree
- The following statements can be proved mathematically for Erdös-Renyi networks:
  - If $z < 1$, then almost surely, the largest component has size at most $O(\ln n)$
  - if $z > 1$, then almost surely, the largest component has size $O(n)$. The second largest component has size $O(\ln n)$
  - if $z = \omega(\ln n)$, then the graph is almost surely connected.
- When $z=1$, there is a phase transition:
  - The largest component is $O(n^{2/3})$
  - The sizes of the components follow a power-law distribution.

Degree distribution

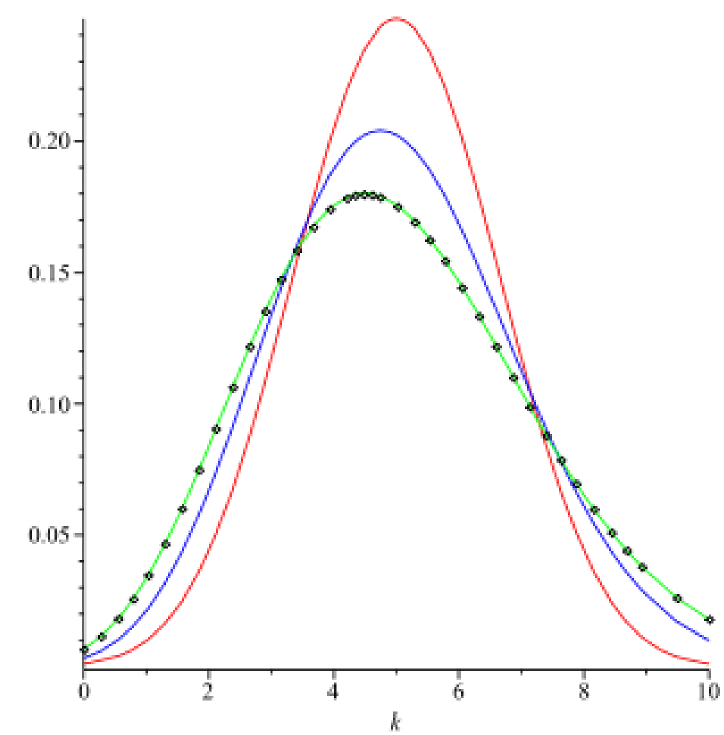
- The degree distribution follows a binomial

$$p(k) = B(n;k;p) = \binom{n}{k} p^k (1-p)^{n-k}$$

- Assuming $z=np$ is fixed, as $n \to \infty$, $B(n,k,p)$ is approximated by a Poisson distribution

$$p(k) = P(k;z) = \frac{z^k}{k!} e^{-z}$$

- Highly concentrated around the mean, with a tail that drops exponentially



Other properties

- Clustering coefficient
  - $C = z/n$
- Diameter (maximum length of shortest paths)
  - $D = \log n / \log z$
- Phase transition
  - Starting from some vertex v perform a BFS walk
  - At each step of the BFS a Poisson process with mean z, gives birth to new nodes
  - When $z<1$ this process will stop after $O(\log n)$ steps
  - When $z>1$, this process will continue for $O(n)$ steps

Are real-world networks random?

- A decade ago the most elegant theory for modelling real-world networks was based on random graphs
- But, real-world networks are not random (we will see)
- However, studies on random networks provides insights into complex structures

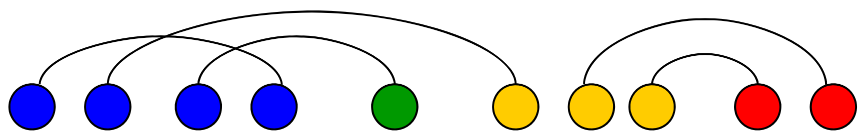
Graphs with given degree sequences

- Graphs of a predetermined degree sequence are essential in many applications
- The configuration model
  - input: the degree sequence $[d_1,d_2,\ldots,d_n]$
  - process:
    - Create d_i copies of node i
    - Take a random matching (pairing) of the copies
      - self-loops and multiple edges are allowed
- Uniform distribution over the graphs with the given degree sequence

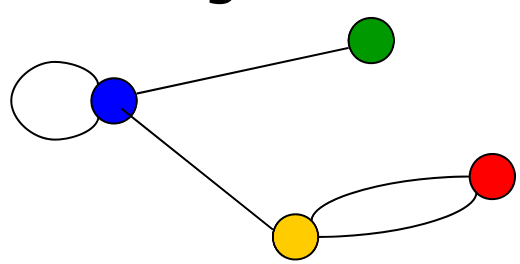
Example

- Suppose that the degree sequence is

- Create multiple copies of the nodes



- Pair the nodes uniformly at random
- Generate the resulting network



Other properties

- The giant component phase transition for this model happens when

$$\sum_{k=0}^{\infty} k(k-2)p_k = 0$$

  - where p_k: fraction of nodes with degree k

- The clustering coefficient is given by

$$C = \frac{z}{n}\left(\frac{\langle d^2 \rangle - \langle d \rangle}{\langle d \rangle^2}\right)^2$$

- The diameter is logarithmic

Graphs with given expected degree sequences

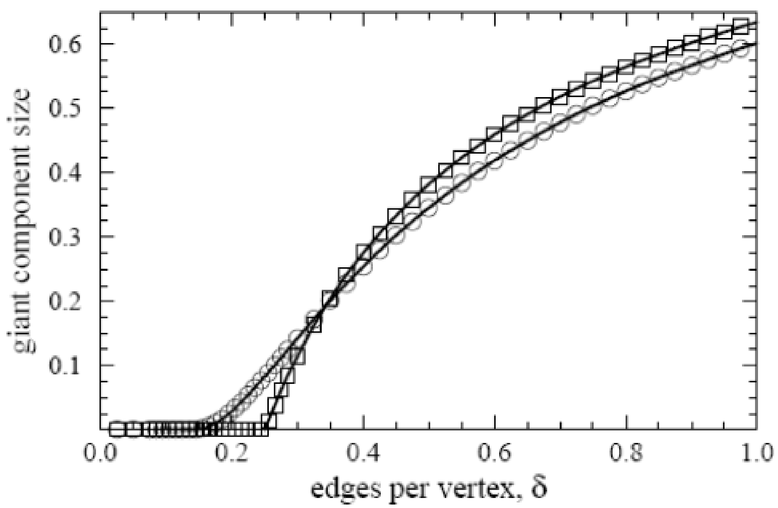
- Input:
  - the degree sequence $[d_1, d_2, \dots, d_n]$
  - m = total number of edges
- Process:
  - generate edge (i,j) with probability $d_i d_j / m$
  - preserves the expected degrees
  - easier to analyze
- However,
  - The problem is that these models are too contrived
  - It would be more interesting if the network structure emerged as a side product of a stochastic process rather than fixing its properties in advance.

A randomly grown graph

- A very simple model
  - essentially no input parameters
  - the process:
    - at each time step add a new vertex
    - with probability δ pick two vertices u,v and generate an edge
- The degree distribution is exponential

$$p_k \sim e^{-k}$$

- The randomly grown graph does not look "random"



Random geographic networks

- Spatial information of nodes is important in many real-world networks
- Nodes may not be able to communicate with any nodes communication constraints)
- For example, in wireless sensor networks nodes can communicate up to a range
- The network is indeed a geographic network where the location of nodes is also important
- Random geographic networks with application in simulating scenarios in wireless sensor networks

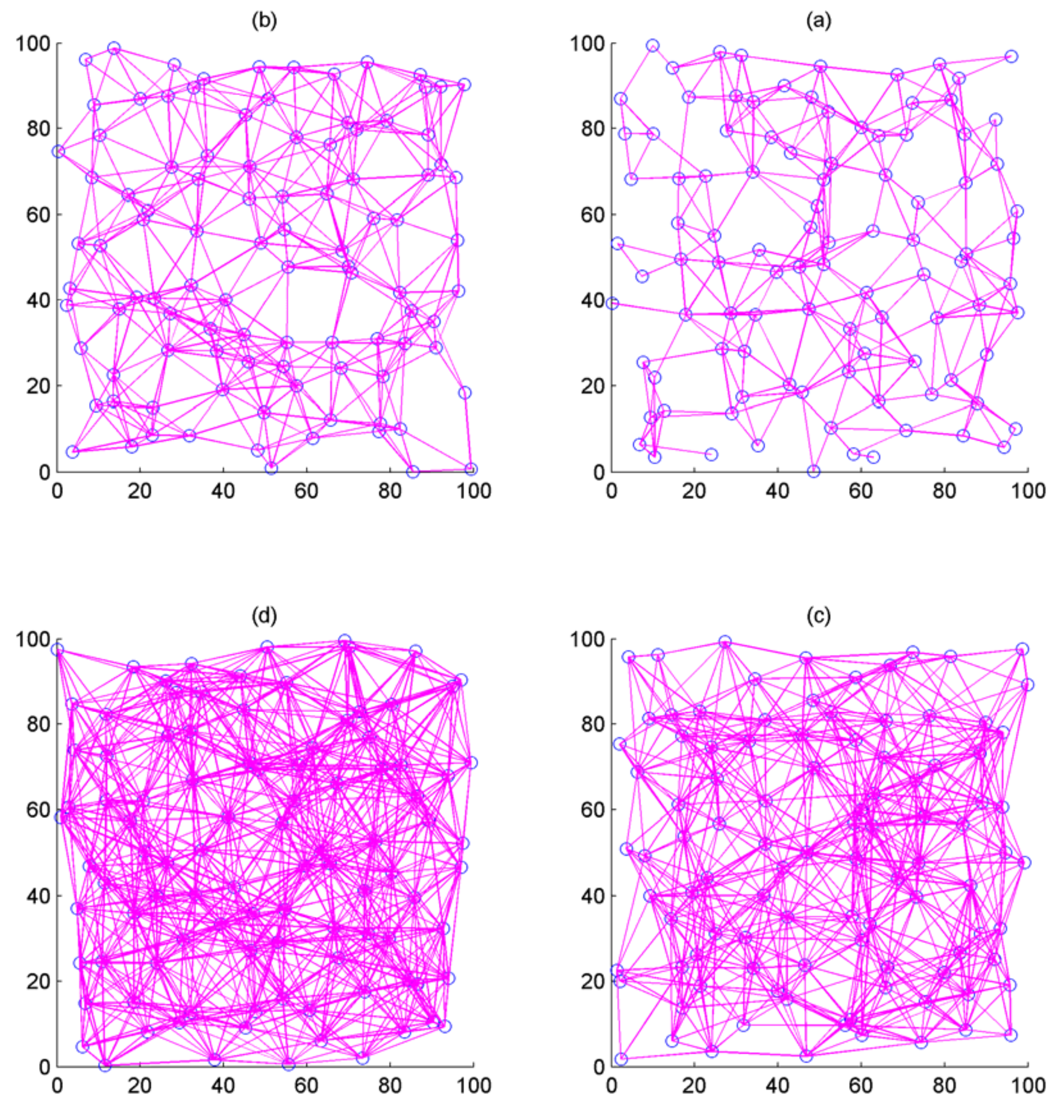
Random geographic networks

- A number of nodes are considered in a two dimensional regular lattice
- The coordinates of each node are randomly drifted around its original coordinates in the lattice
- Each node can communicate (i.e. have link) with those in its neighbourhood defined by a circle with radius *R* centred at the coordinates of the node
- The communication between them might fail with probability *P*
- A network constructed in this way is a random geographical network

Random geographic networks

100 nodes are distributed in a regular two dimensional lattice (with random drift from their initial coordinates) in a 100×100 field.

a) R = 20, P = 0.7
b) R = 20, P = 0.1
c) R = 30, P = 0.7
d) R = 30, P = 0.1



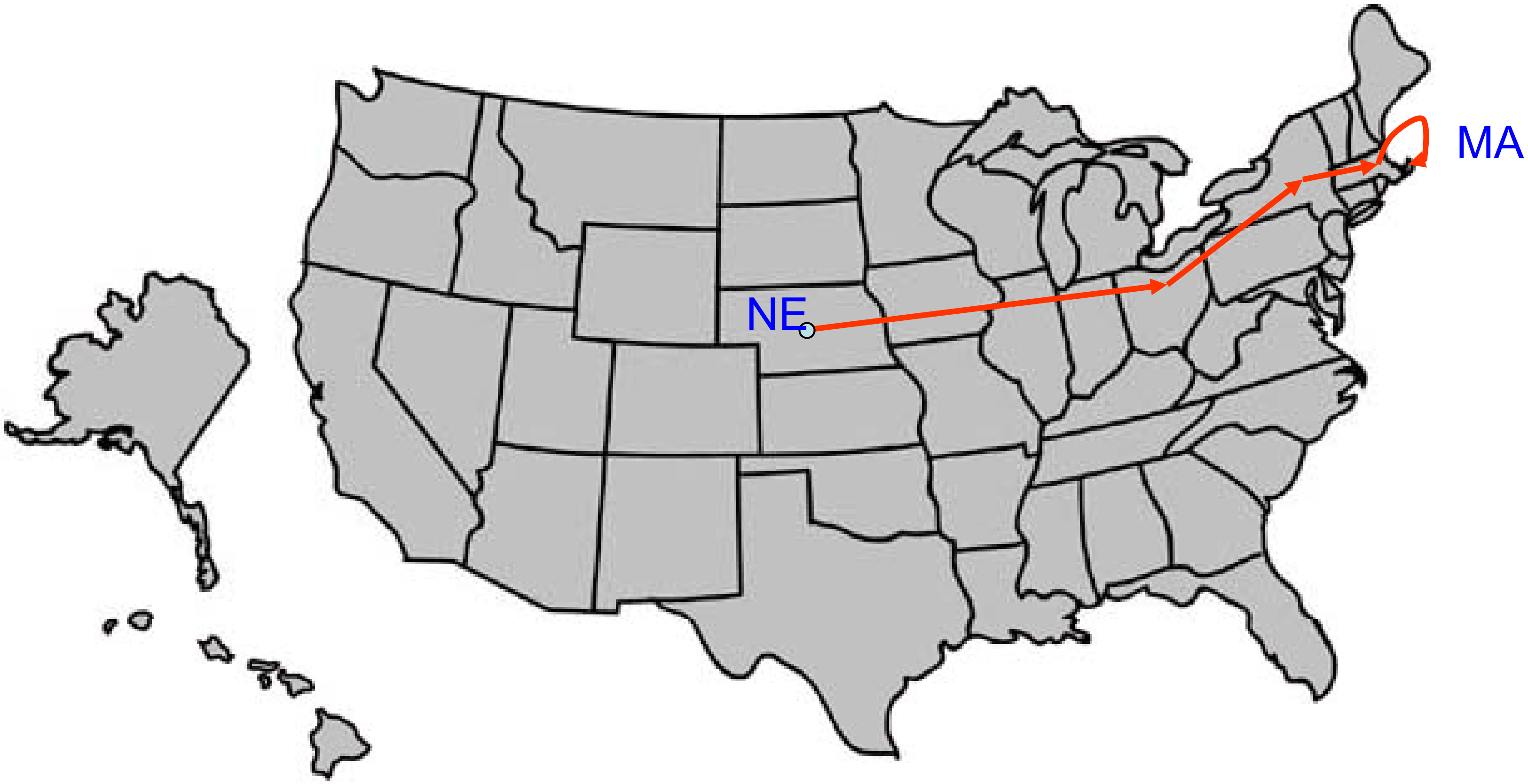
Random clustered networks

- M clusters with N nodes
- Decide how many clusters to have
- Construct a dense ER random network (either strategy) for each cluster
- With a probability P, rewire the intra-cluster links to be between inter-cluster nodes
- A random network with community structure is obtained

Small-world networks

- It was Longley believed that real-world networks have random structure
- Milgram did an experiment showing the small-world property
- Watts and Strogtaz showed that many real-world networks:
  - Have small characteristic path length compared to random networks
  - At the same time, have high clustering coefficient that is much larger than that of random networks
  - The are indeed small-worlds
- This discovery had huge impact on the various developments in Network fields
  - Search in complex networks
  - Communication in networks
  - Synchronization and consensus
  - …

Milgram’s experiment



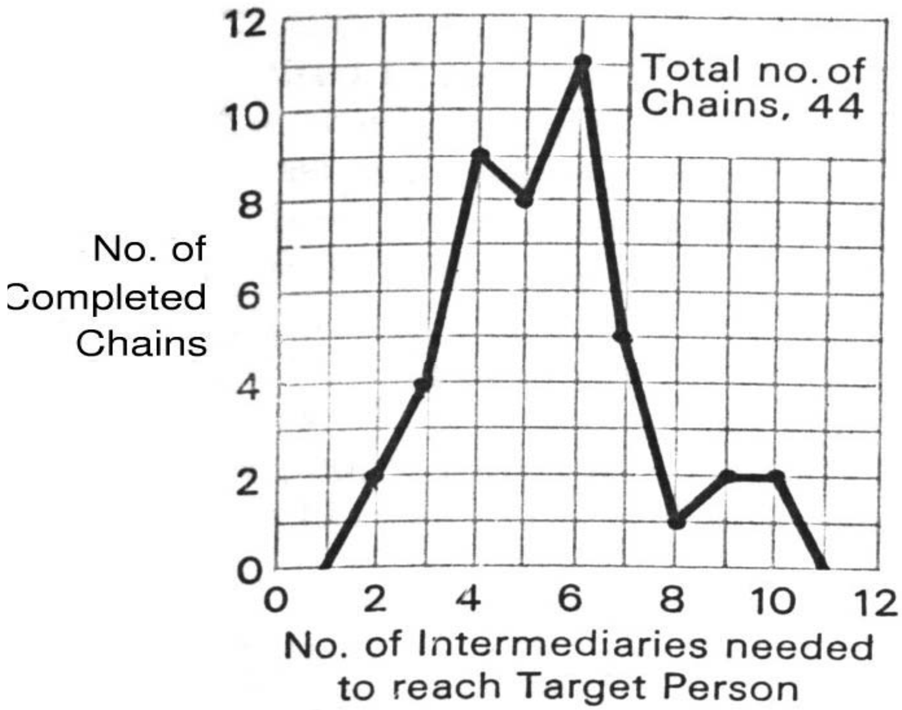
Milgram's experiment

- Instructions:
  - Given a target individual (stockbroker in Boston), pass the message to a person you correspond with who is "closest" to the target.
  - 160 letters: From Wichita (Kansas) and Omaha (Nebraska) to Sharon (Mass)
  - If you do not know the target person on a personal basis, do not try to contact him directly. Instead, mail this folder to a personal acquaintance who is more likely than you to know the target person.
- Outcome:
  - 20% of initiated chains reached
  - Target average chain length = 6.5
  - "Six degrees of separation"

Milgram, *Psych Today* **2**, 60 (1967)

Milgram's experiment

- "Six degrees of separation"
- The **Small World** concept in simple terms describes the fact despite their often **large size**, in most networks there is a **relatively short path between any two nodes**.



In the Nebraska Study the chains varied from two to 10 intermediate acquaintances with the median at five.

Milgram’s experiment repeated

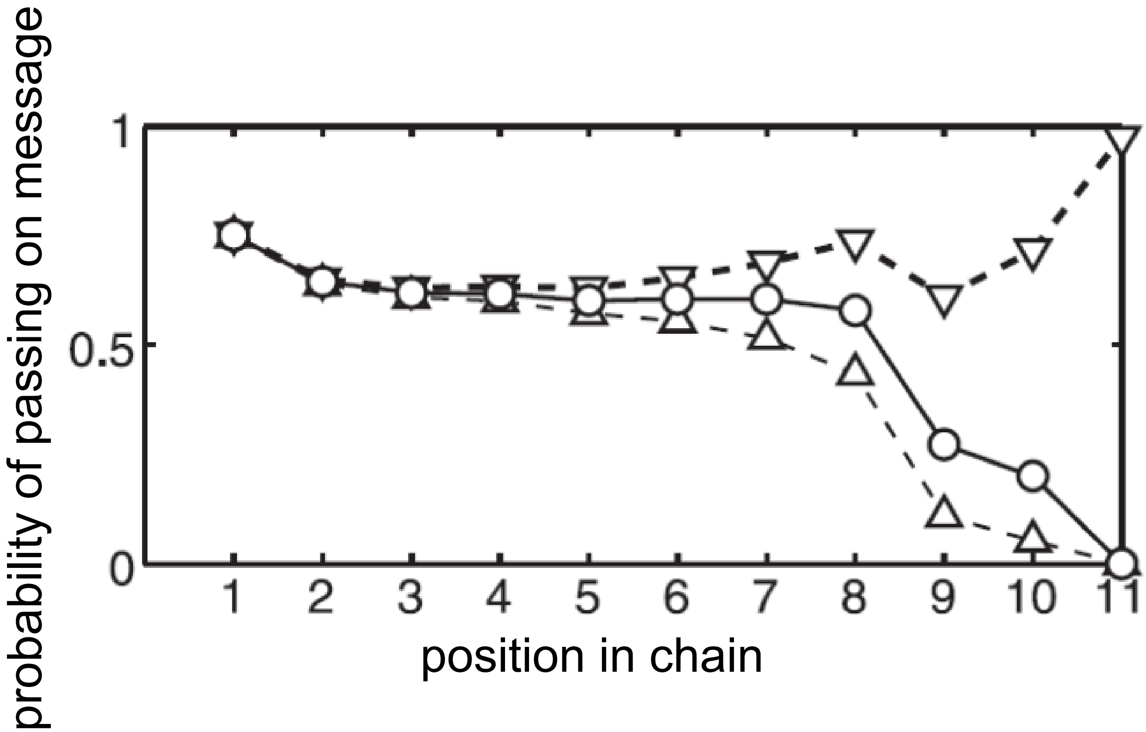
- Email experiment by Dodds, Muhamad, Watts, Science 301, (2003):
  - 18 targets
  - 13 different countries
  - More than 60,000 participants
  - 24,163 message chains
  - 384 reached their targets
  - Average path length 4.0

Source: NASA, U.S. Government; http://visibleearth.nasa.gov/view_rec.php?id=2429

Interpreting Milgram's experiment

- Is 6 is a ***surprising*** number?
  - In the 1960s? Today? Why?
- If social networks were random… ?
  - Pool and Kochen (1978) - ~500-1500 acquaintances/person
  - ~ 1,000 choices 1st link
  - ~ 1000^2 = 1,000,000 potential 2nd links
  - ~ 1000^3 = 1,000,000,000 potential 3rd links
- If networks are completely cliquish?
  - all my friends' friends are my friends
  - what would happen?
- Is 6 an ***accurate*** number?
- What bias is introduced by uncompleted chains?
  - are longer or shorter chains more likely to be completed?
  - if each person in the chain has 0.5 probability of passing the letter on, what is the likelihood of a chain being completed
    - of length 2? or of length 5?

Small-world experiment accuracy

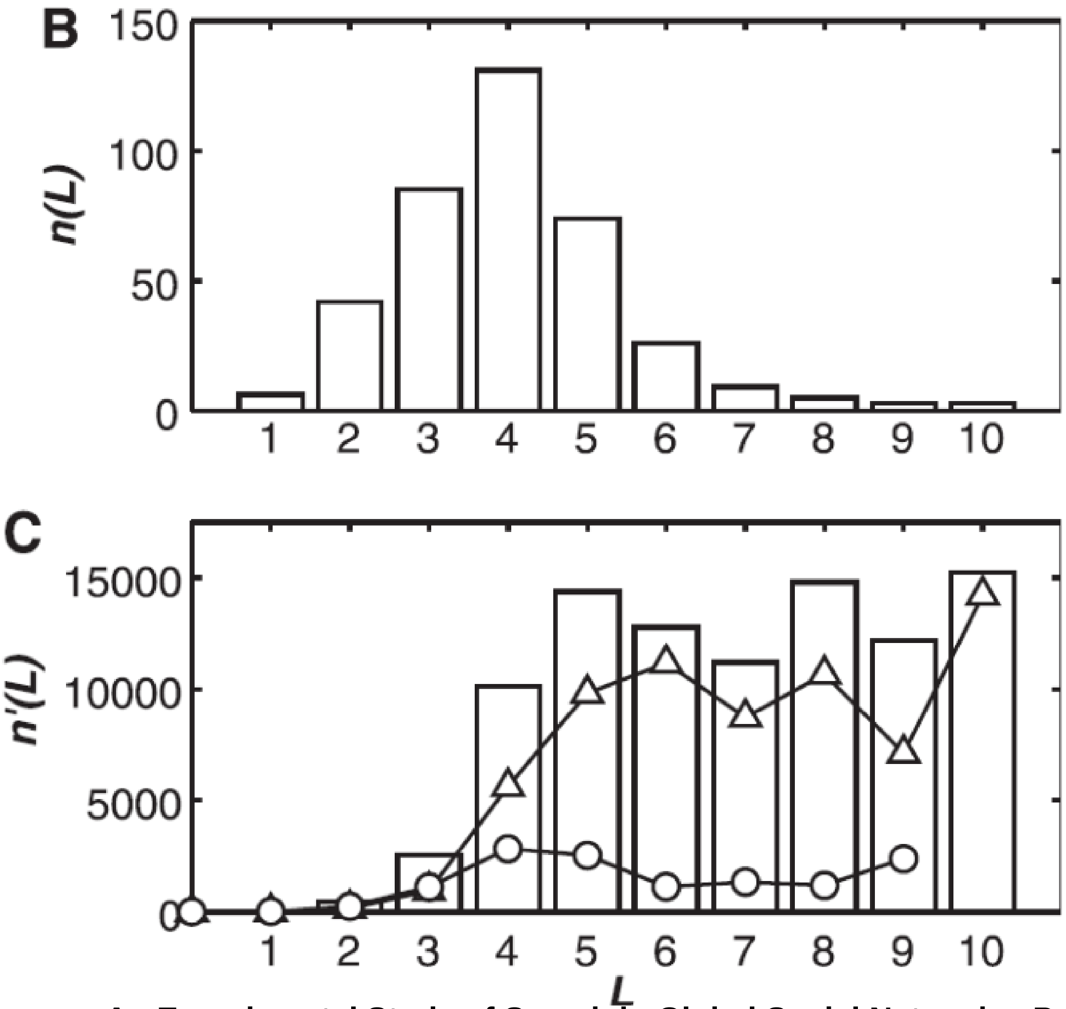


average

95 % confidence interval

Source: An Experimental Study of Search in Global Social Networks: Peter Sheridan Dodds, Roby Muhamad, and Duncan J. Watts (8 August 2003); Science 301 (5634), 827.

Small-world experiment accuracy



- observed chain lengths
- 'recovered' histogram of path lengths

o: inter-country

Δ: intra-country

Source: An Experimental Study of Search in Global Social Networks: Peter Sheridan Dodds, Roby Muhamad, and Duncan J. Watts (8 August 2003); Science 301 (5634), 827.

Small-world experiment accuracy

- Is 6 an **accurate** number?
- Do people find the **shortest** paths?
  - Killworth, McCarty ,Bernard, & House (2005):
  - less than optimal choice for next link in chain is made ½ of the time
- “Social Networking” as a Business:
  - FaceBook, MySpace, Orkut, Friendster (entertainment, keeping and finding friends)
  - LinkedIn (more traditional networking for jobs)
  - Spoke, VisiblePath (helping businesses capitalize on existing client relationships)

Applicable to other networks?

Same pattern:

high clustering $C_{\text{network}} >> C_{\text{random graph}}$

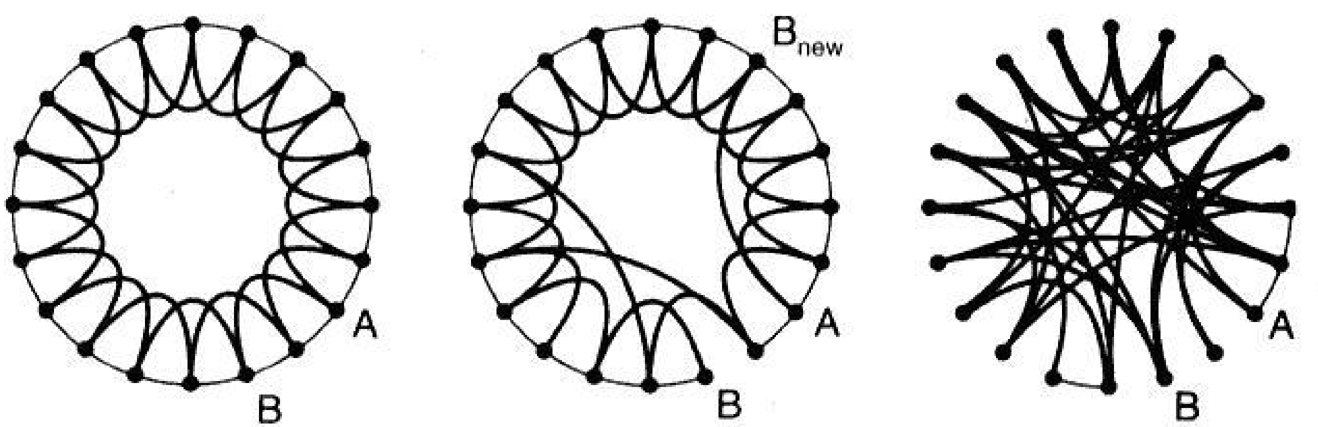
low average shortest path $l_{\text{network}} \approx \ln(N)$

- of course in many social networks
- neural network of C. elegans,
- Human brain
- semantic networks of languages,
- actor collaboration graph
- food webs
- Power grids
- …

Watts-Strogatz model

Reconciling two observations:

- **High clustering:** my friends' friends tend to be my friends
- **Short average paths**



Source: Watts, D.J., Strogatz, S.H.(1998) Collective dynamics of 'small-world' networks. Nature 393:440-442.

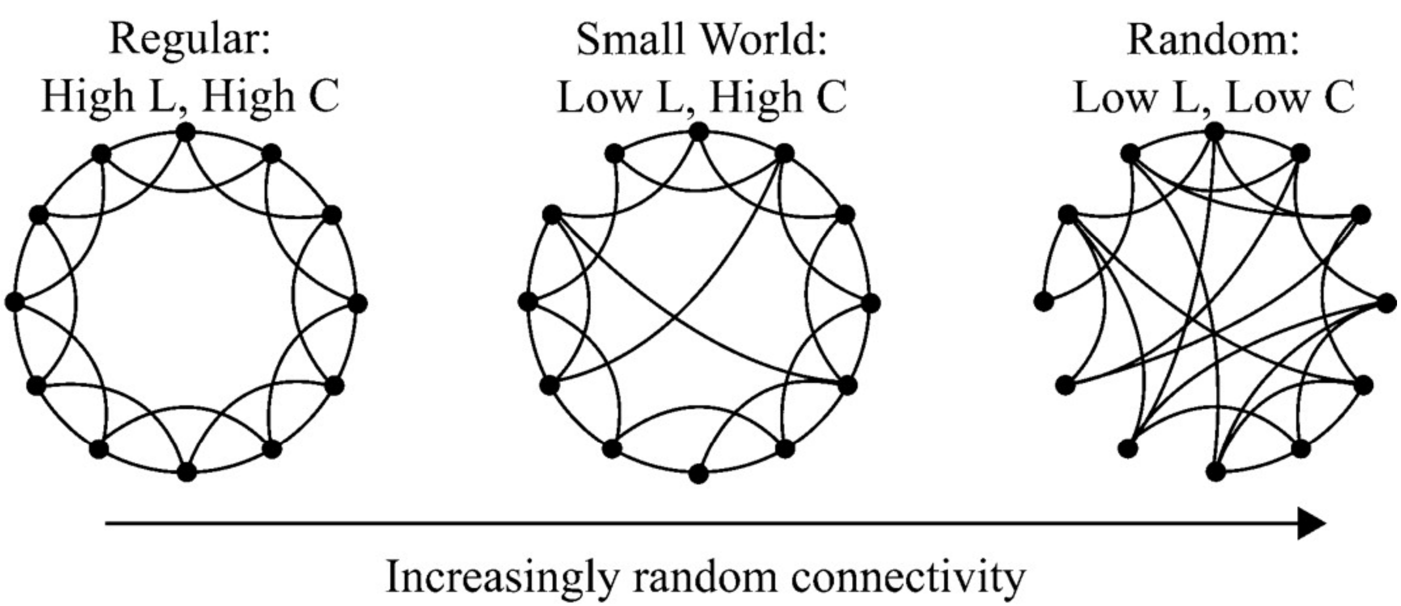
Watts-Strogatz model

The construction algorithm:

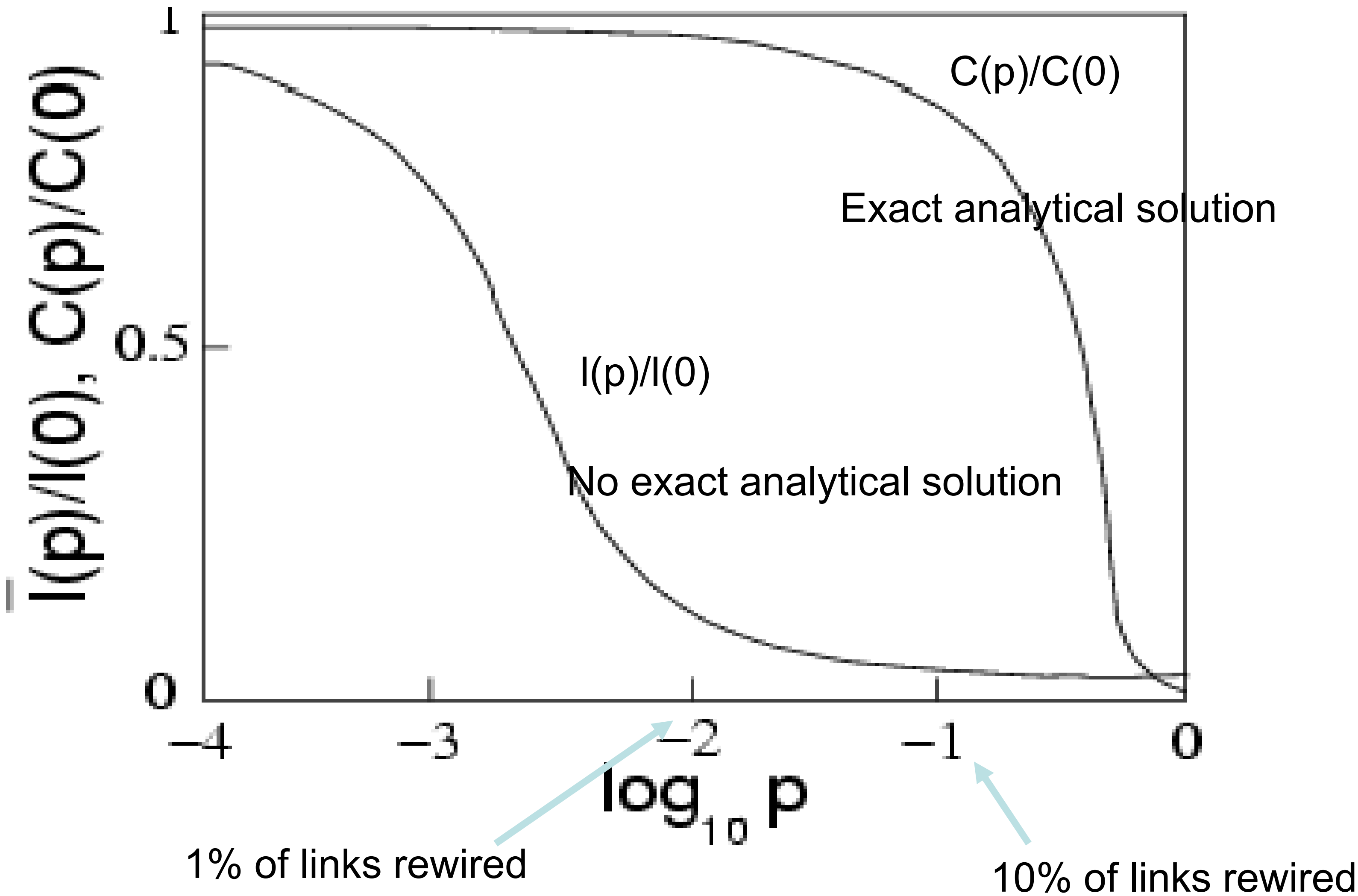
- Consider a ring graph where each node is connected to its m nearest neighbors with undirected edges
- Choose a node and one of the edges that connects it to its nearest neighbors and then with probability P reconnect this edge to a node randomly chosen over the graph
  - provided that the duplication of edges and self-loops are forbidden
- The process is repeated until all nodes and nearest neighbor connecting edges are met
- Next, the edges that connect the nodes to their second-nearest neighbors are reconnected and the rewiring process is performed on them with the same conditions as above
- The same procedure is then repeated for the remaining edges connecting the nodes to their m nearest neighbors

Watts-Strogatz model

- The resulting graph is so that
  - for the value of P = 0 we will have the original ring graph
  - for the value of P = 1 produces a pure random graph
  - For some values of P between these two extremes the resulting network has small characteristics path length ,and at the same time, high clustering coefficient
  - the average degree will be <k> = 2m



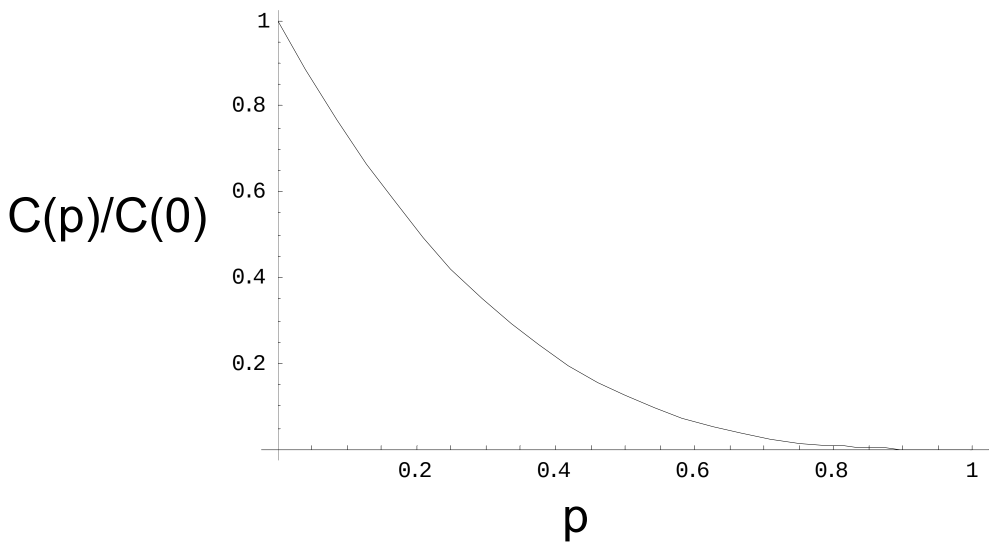
Watts-Strogatz model



Source: Watts, D.J., Strogatz, S.H.(1998) Collective dynamics of 'small-world' networks. Nature 393:440-442.

WS model: clustering coefficient

- The probability that a connected triple stays connected after rewiring
  - probability that none of the 3 edges were rewired $(1\text{-}p)^3$
  - probability that edges were rewired back to each other very small, can ignore
- Clustering coefficient = C(p) = C(p=0)*$(1\text{-}p)^3$

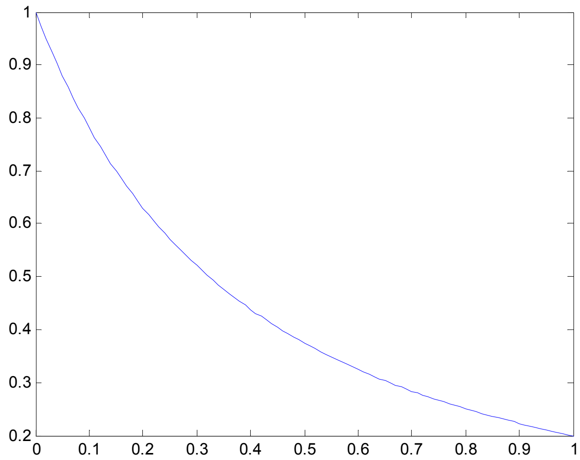


Source: Watts, D.J., Strogatz, S.H.(1998) Collective dynamics of 'small-world' networks. Nature 393:440-442.

WS model: clustering coefficient

- How does C depend on p?
- C(p)= 3xnumber of triangles / number of connected triples
- C(p) computed analytically for the WS model:

$$C(p) = \frac{3(k-1)}{2(2k-1)+4kp(p+2)}$$



Source: Watts, D.J., Strogatz, S.H.(1998) Collective dynamics of 'small-world' networks. Nature 393:440-442.

Real-world networks

Network	size	Characteristic path length	Shortest path in fitted random graph	Clustering coefficient	Clustering in random graph
Film actors	225,226	3.65	2.99	0.79	0.00027
MEDLINE co-authorship	1,520,251	4.6	4.91	0.56	1.8×10^{-4}
E.Coli substrate graph	282	2.9	3.04	0.32	0.026
C.Elegans	282	2.65	2.25	0.28	0.05

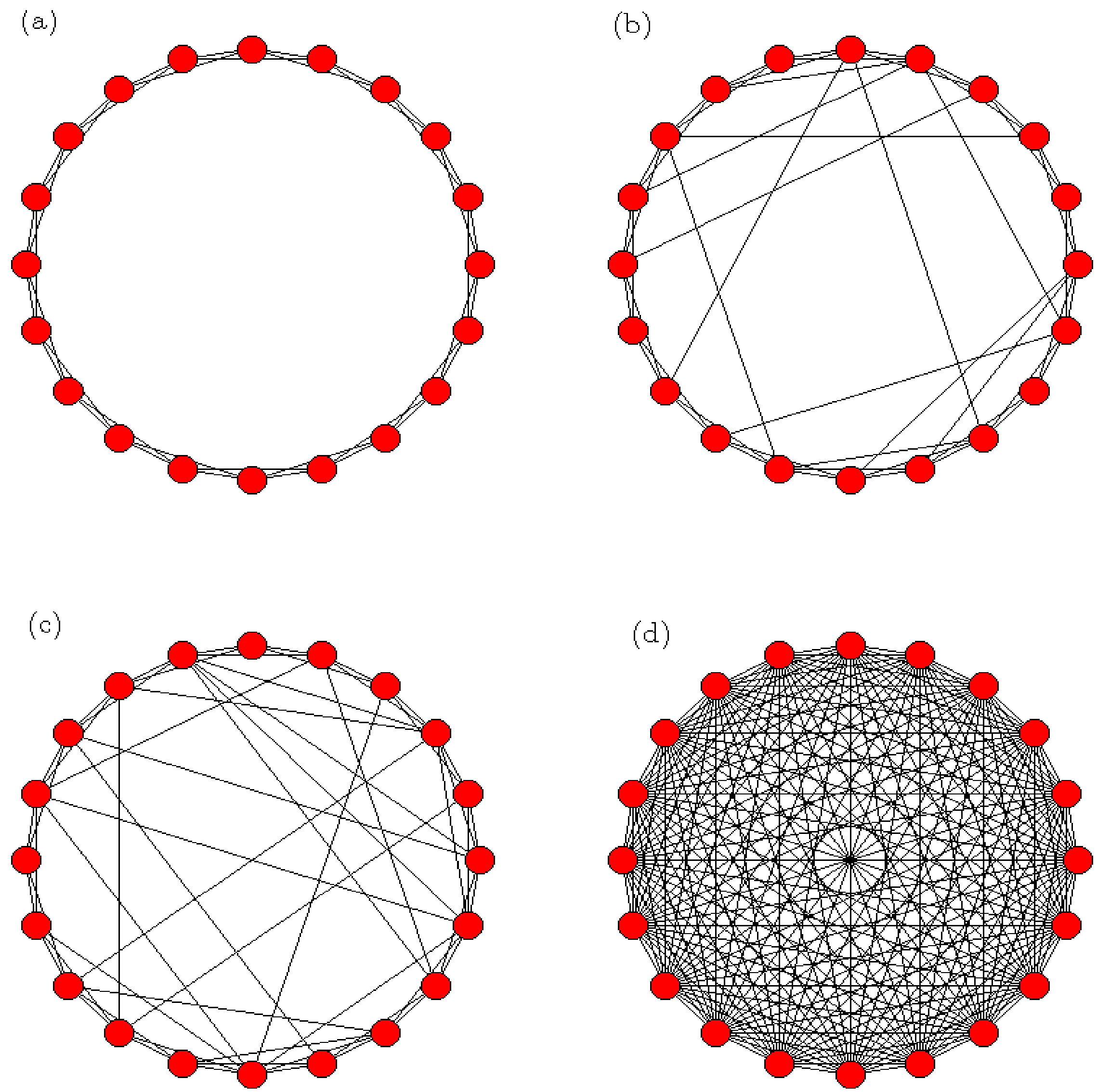
Newman-Watts model

- Starting with a k-ring graph
- *N* nodes
- Non-connected nodes get connected with probability P
- P = 1 results in complete graph
- for some small values of P
  - small-world property
  - high transitivity
- The networks are always connected

Newman-Watts model

20 nodes in a 2-regular ring with

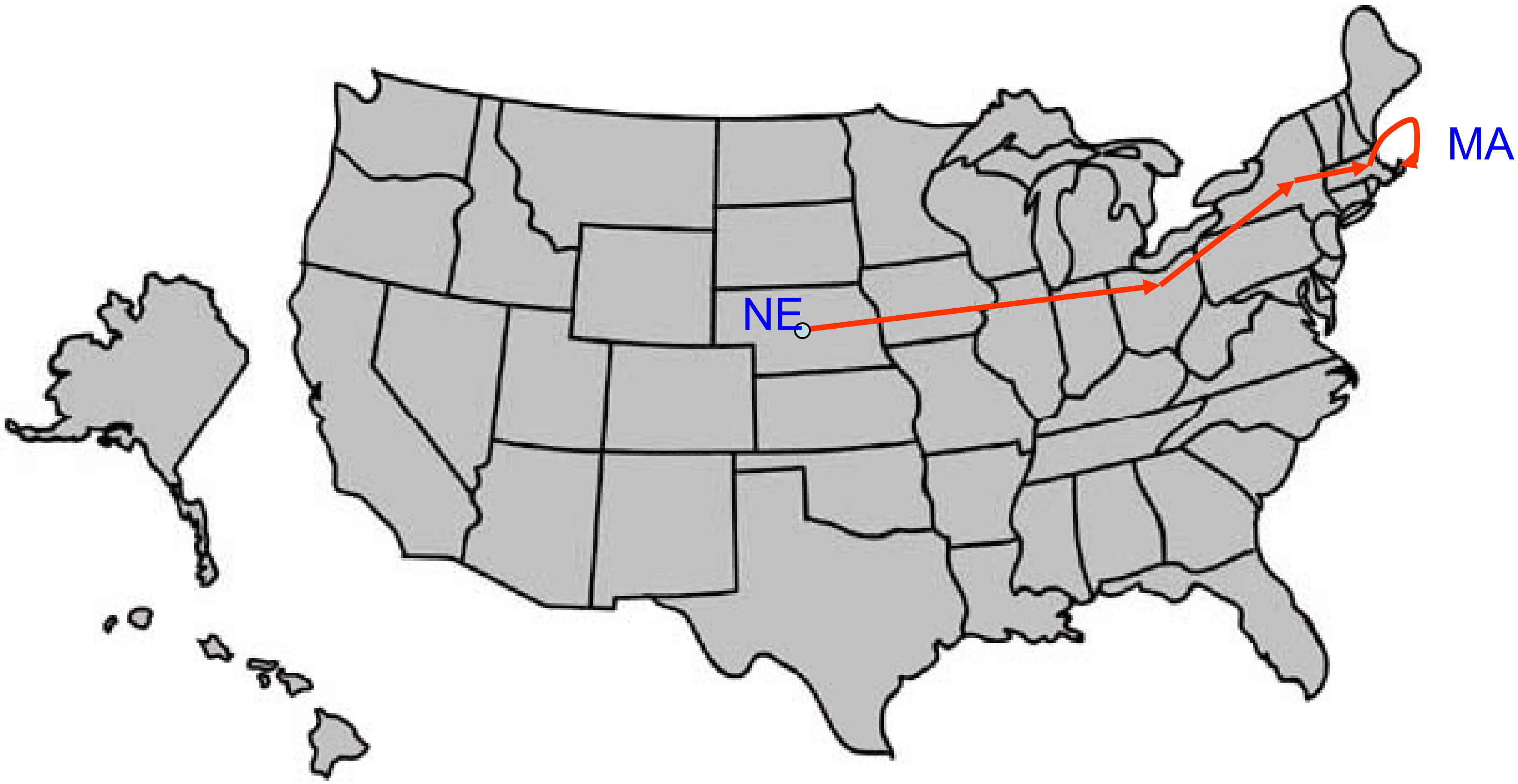
a) $P = 0$
b) $P = 0.05$
c) $P = 0.15$
d) $P = 1$



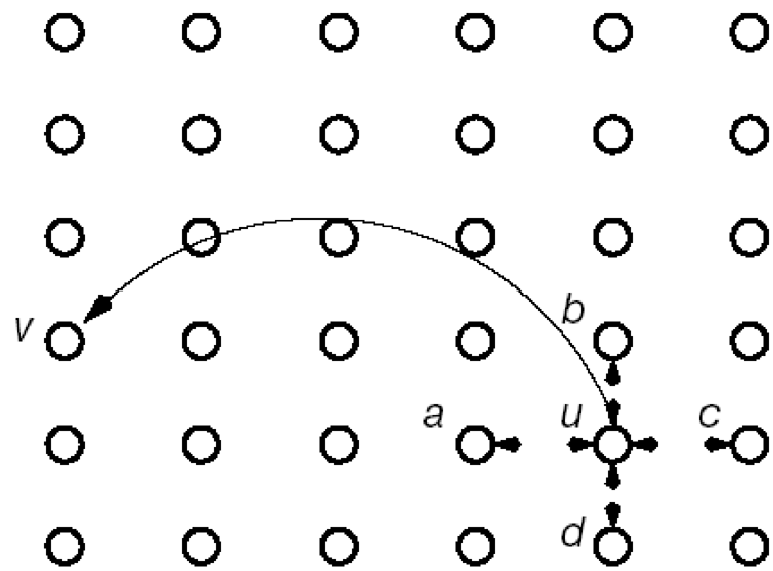
Geographical small-world models

"The geographic movement of the [message] from Nebraska to Massachusetts is striking. There is a progressive closing in on the target area as each new person is added to the chain"

S.Milgram 'The small world problem', Psychology Today 1,61,1967



Kleinberg's geographical model



nodes are placed on a lattice and connect to nearest neighbors

exponent that will determine navigability

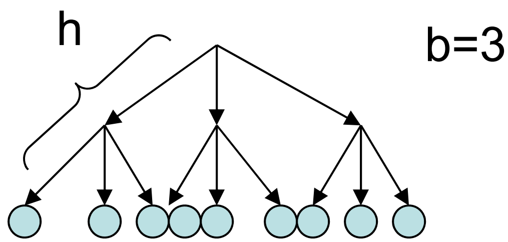
additional links placed with
probability(link between u and v) = $(\text{distance}(u,v))^{-r}$

Source: Kleinberg, 'The Small World Phenomenon, An Algorithmic Perspective' (Nature 2000).

Hierarchal small-world model

Hierarchical network models:

Individuals classified into a hierarchy,
h_{ij} = height of the least common ancestor.



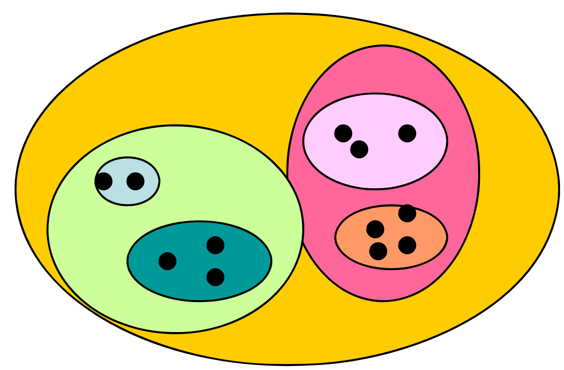
$$p_{ij} \propto b^{-\alpha h_{ij}}$$

e.g. state-county-city-neighborhood
industry-corporation-division-group

Group structure models:

Individuals belong to nested groups
q = size of smallest group that v,w belong to

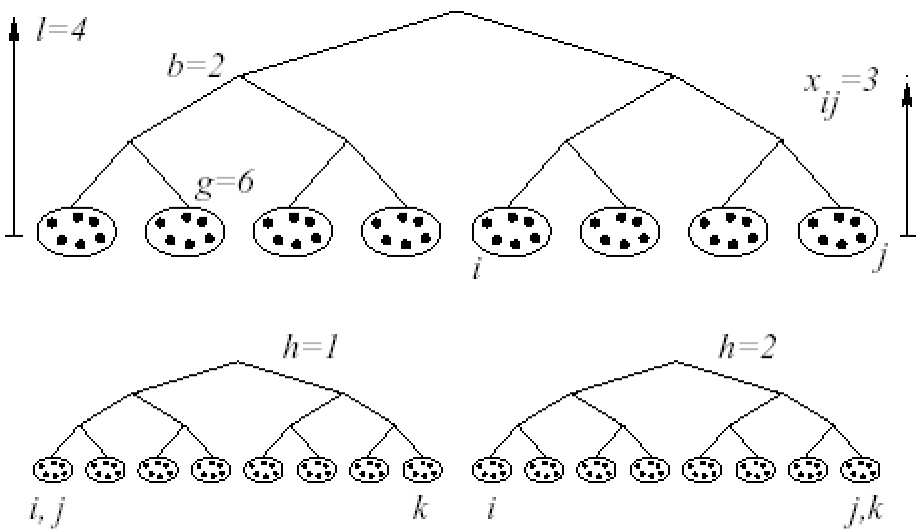
$$f(q) \sim q^{-\alpha}$$



Source: Kleinberg, 'Small-World Phenomena and the Dynamics of Information' NIPS 14, 2001.

Hierarchal small-world model

individuals belong to hierarchically nested groups

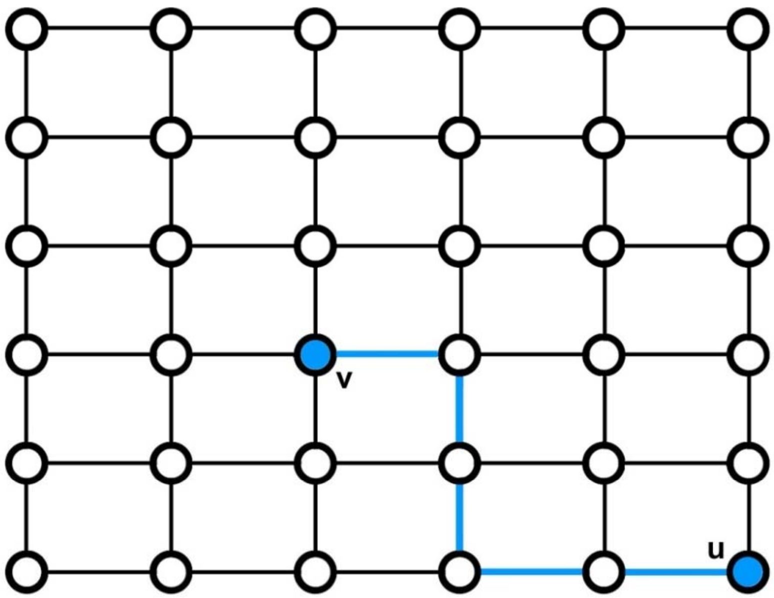


$p_{ij} \sim \exp(-\alpha\ x)$

multiple independent hierarchies h=1,2,..,H coexist corresponding to occupation, geography, hobbies, religion…

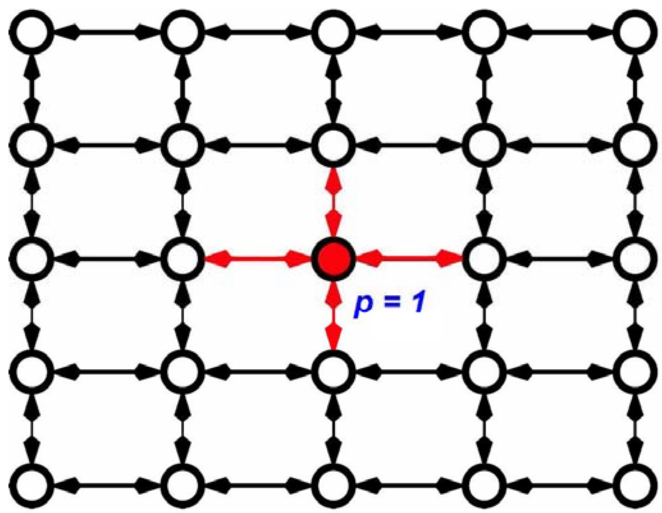
Source: Identity and Search in Social Networks: Duncan J. Watts, Peter Sheridan Dodds, and M. E. J. Newman; Science 17 May 2002 296: 1302-1305. < http://arxiv.org/abs/cond-mat/0205383v1 >

Small-worlds: algorithmic view

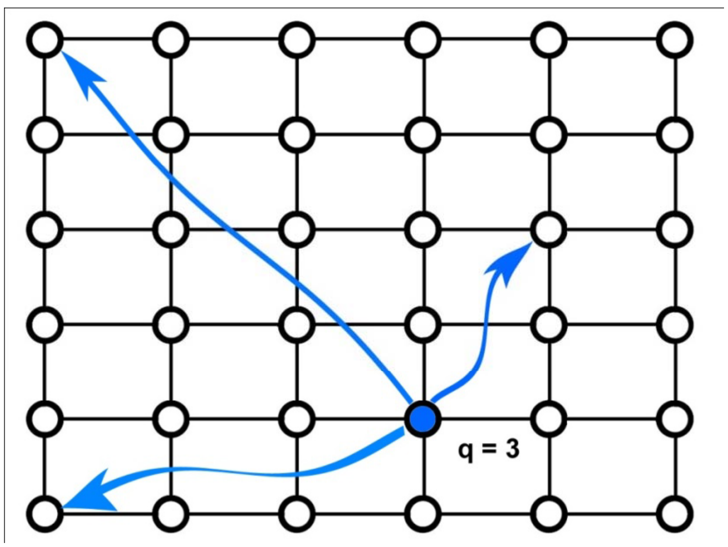


- Imagine everyone lives on an n x n grid
- "lattice distance" – number of lattice steps between two points
- Constants p,q

Small-worlds: algorithmic view



- p: range of local contacts
  - Nodes are connected to all other nodes within distance p



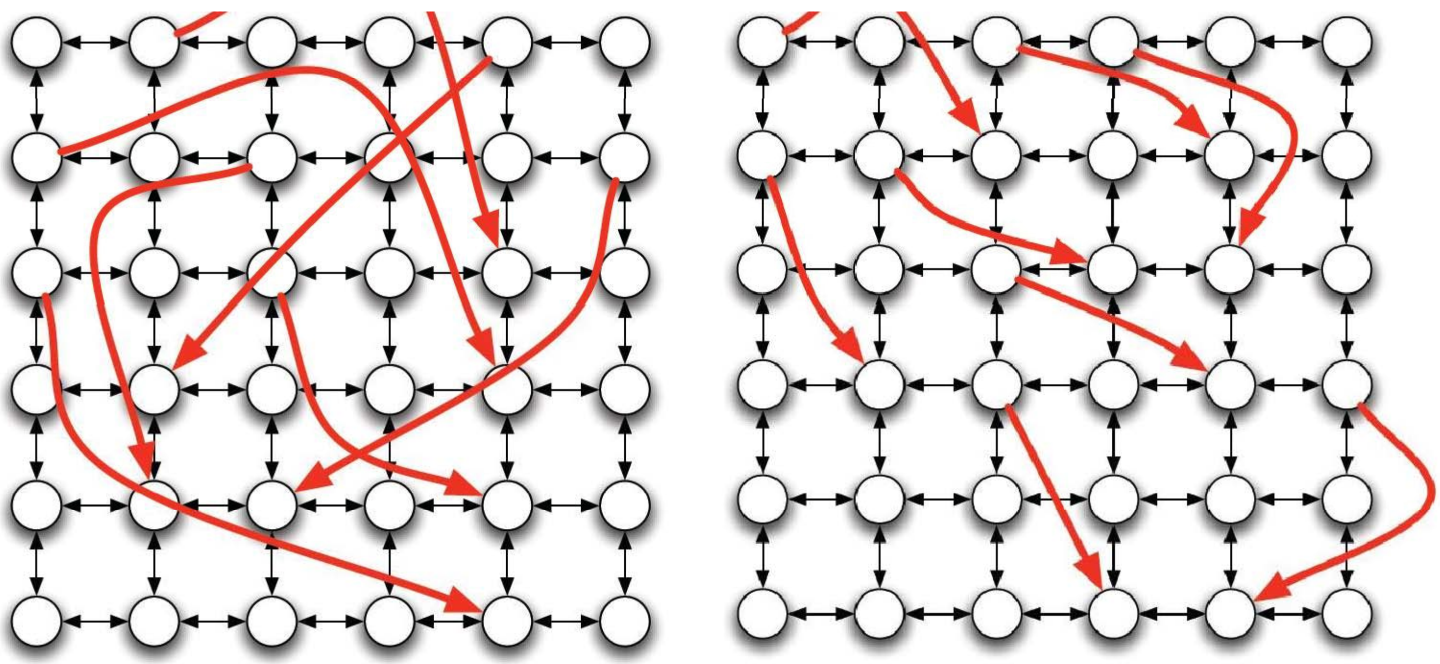
- q: number of long-range contacts
  - add directed edges from node u to q other nodes using independent random trials

Small-worlds: algorithmic view

- Watts – Strogatz (1998)
  - Found that injecting a small amount of randomness (i.e. even q = 1) into the world is enough to make it a small world.
- Kleinberg (2000)
  - Why should arbitrary pairs of strangers, using only locally available information, be able to find short chains of acquaintances that link them together?
  - Does this occur in all small-world networks, or are there properties that must exist for this to happen?

Small-worlds: algorithmic view

- Probability [*u* has *v* as its long range contact] : $\dfrac{[d(u,v)]^{-r}}{\sum_{v:v\neq u}[d(u,v)]^{-r}}$
- Infinite family of networks:
  - r = 0: each node's long-range contacts are chosen independently of its position on the grid
  - As r increases, the long range contacts of a node become clustered in its vicinity on the grid



small 'r' large 'r'

The algorithmic side

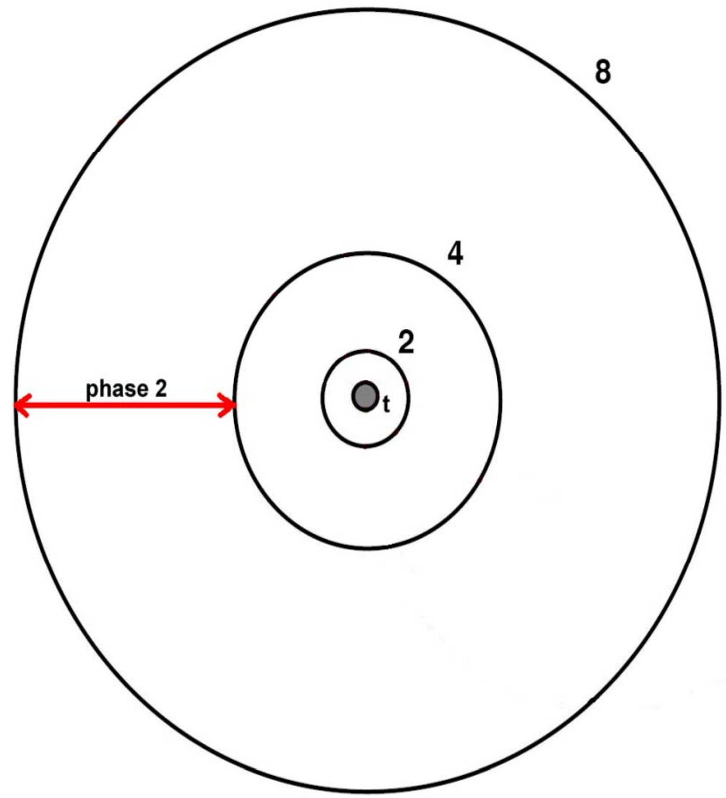
- Input:
  - Grid G = (V,E)
  - arbitrary nodes s, t
- Goal:
  - Transmit a message from s to t in as few steps as possible using only locally available information
- Assumptions:
  - In any step, the message holder u knows
    - The range of local contacts of all nodes
    - The location on the lattice of the target t
    - The locations and long-range contacts of all nodes that have previously touched the message
  - u does not know
    - the long-range contacts of nodes that have not touched the message

Analysis

$$r = 2$$

Analysis

- In each step the current message holder passes the message to the contact that is as close to the target as possible
- Algorithm in phase j:
  - At a given step, $2^j < d(u,t) \leq 2^{j+1}$
  - Algorithm is in phase 0:
    - message is no more than 2 lattice steps away from the target t.
  - $j \leq \log_2 n$.



Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v in the next phase as its long range contact?

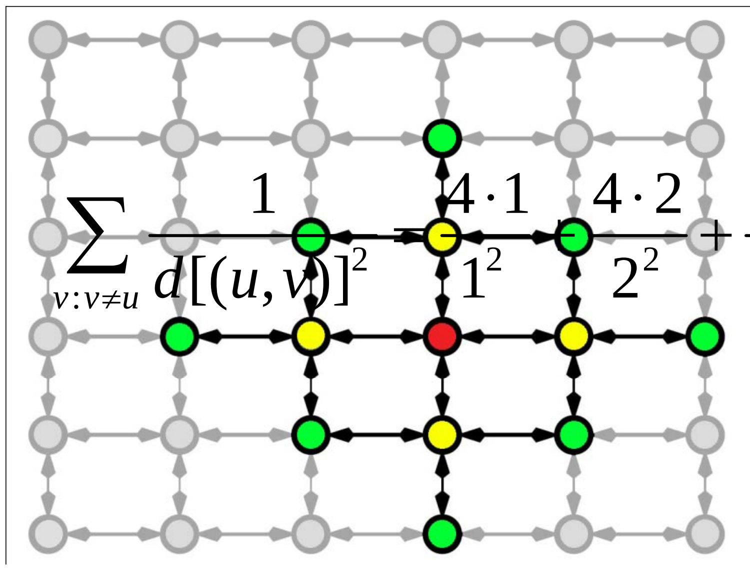
Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v as its long range contact?

- Pr [*u* has *v* as its long range contact] ?

$$= \frac{[d(u,v)]^{-2}}{\sum [d(u,v)]^{-2}}$$



$$\sum_{v:v\neq u} \frac{1}{d[(u,v)]^2} = \frac{4\cdot 1}{1^2} + \frac{4\cdot 2}{2^2} + \frac{4\cdot 3}{3^2} + \ldots \leq \sum_{j=1}^{2n-2} \frac{4j}{j^2}$$

Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v as its long range contact?

- Pr[*u* has *v* as its long range contact]?

$$\boxed{\sum_{v:v\neq u}[d(u,v)]^{-2} \leq \sum_{j=1}^{2n-2}\frac{4j}{j^2} = 4\sum_{j=1}^{2n-2} 1/j \leq 4[1+\ln(2n-2)] \leq 4\ln(6n)}$$

$$\geq \frac{[d(u,v)]^{-2}}{4\ln(6n)}$$

- Thus *u* has *v* as its long-range contact with probability

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

Analysis

Questions:
- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- In any given step, Pr[phase j ends in this step]?
  - Phase j ends in this step if the message enters the set B_j of nodes within distance 2^j of t. Let v_f be the node in B_j that is farthest from u.

$$\Pr[\text{phase } j \text{ ends in this step}] = \sum_{v \in B_j} \Pr\left[u \; is \; friends \; with \; v \in B_j\right]$$

$$\geq |B_j| \cdot \left(\frac{1}{4\ln(6n)\cdot[d(u,v_f)]^2} \right)$$

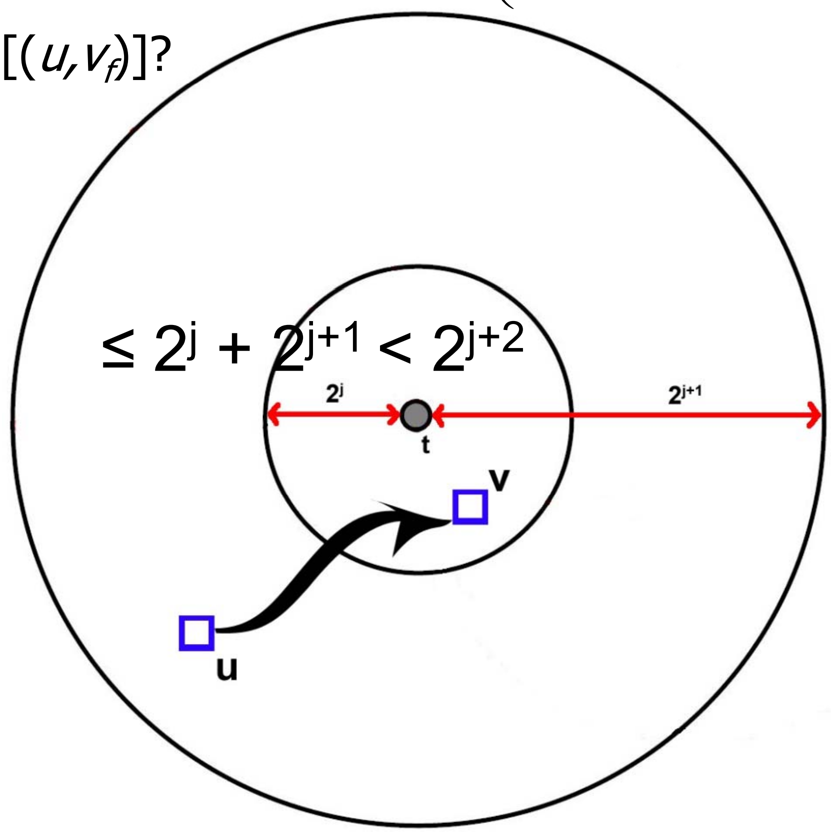
Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- $\Pr[\text{phase } j \text{ ends in this step}] \geq |B_j| \cdot \left(\frac{1}{4\ln(6n)\cdot[d(u,v_f)]^2}\right)$
  - What is d[(u,v_f)]?



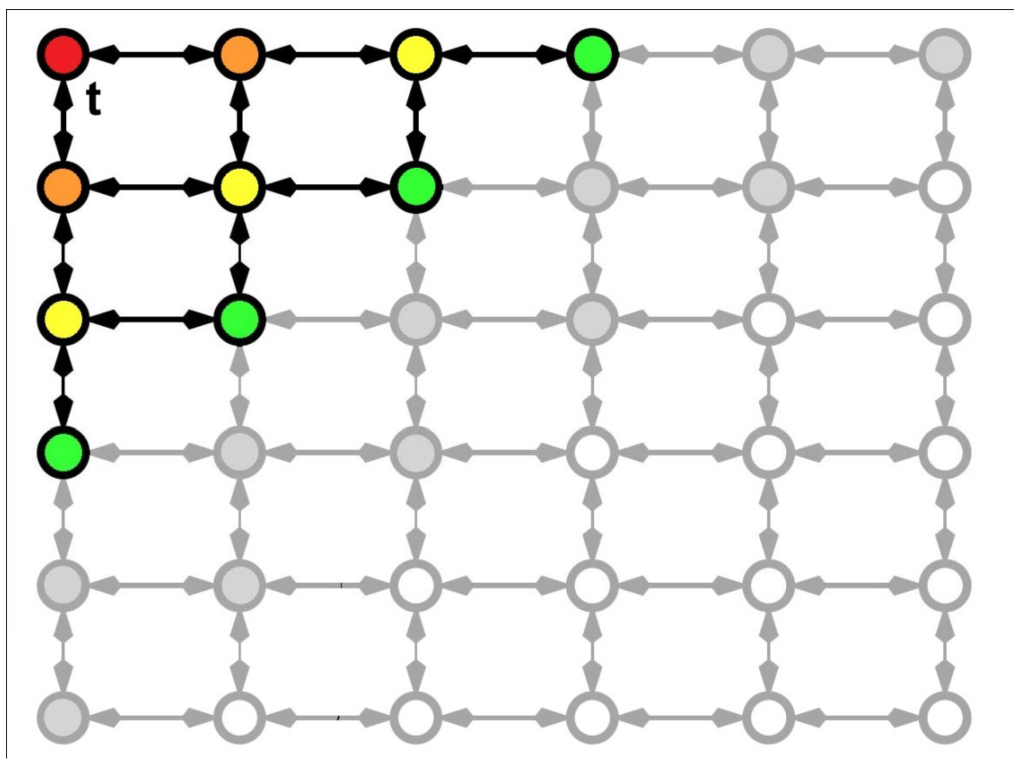
Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- $$\Pr[\text{phase } j \text{ ends in this step}] \geq |B_j| \cdot \left(\frac{1}{4\ln(6n)\cdot 2^{2j+4}}\right)$$
- How many nodes are in B_j?



Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?
- What is the probability that node u has a node v as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- In any given step, Pr[phase j ends in this step]?
  - Pr[*u* has a long-range contact in B_j]?

$$\geq \#\ of\ nodes\ in\ B_j \cdot (probability\ u\ is\ friends\ with\ farthest\ v \in B_j)$$

$$\geq 2^{2j-1}\left(\frac{1}{4\ln(6n)\cdot 2^{2j+4}}\right) = \frac{2^{2j-1}}{4\ln(6n)\cdot 2^{2j+4}} = \frac{1}{128\ln(6n)}$$

Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase *j*?
- In a given step, with what probability will phase *j* end in this step?

$$\geq \frac{1}{128\ln(6n)}$$

- What is the probability that node *u* has a node *v* as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- How many steps will we spend in phase *j*?
  - Let X_j be a random variable denoting the number of steps spent in phase *j*.
    - X_j is a geometric random variable with a probability of success at least

Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?
- In a given step, with what probability will phase j end in this step?

$$\geq \frac{1}{128\ln(6n)}$$

- What is the probability that node u has a node v as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- How many steps will we spend in phase *j*?
  - Since X_j is a geometric random variable, we know that

$$E[X_j] = \frac{1}{p} \leq \frac{1}{1/128\ln(6n)} = 128\ln(6n)$$

Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase *j*?
- In a given step, with what probability will phase *j* end in this step?

$$\geq \frac{1}{128\ln(6n)}$$

- What is the probability that node *u* has a node *v* as its long range contact?

$$\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$$

- How many steps will we spend in phase *j*?
  - Let X_j be a random variable denoting the number of steps spent in phase *j*.

$$E[X_j] = \sum_{i=1}^{\infty} \Pr[X_j \geq i]$$

$$\leq \sum_{i=1}^{\infty} \left(1 - \frac{1}{128\ln(6n)}\right)^{i-1}$$

$$= 128\ln(6n)$$

Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?

  $\leq 128\ln(6n)$

- In a given step, with what probability will phase j end in this step?

  $\geq \frac{1}{128\ln(6n)}$

- What is the probability that node u has a node v as its long range contact?

  $\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$

- How many steps does the algorithm take?
  - Let X be a random variable denoting the number of steps taken by the algorithm.
  - By Linearity of Expectation we have

$$E[X] \leq (1+\log n)(128\ln(6n)) = O(\log n)^2$$

Analysis

Questions:

- How many steps will the algorithm take?
- How many steps will we spend in phase j?

  $\leq 128\ln(6n)$

- In a given step, with what probability will phase j end in this step?

  $\geq \frac{1}{128\ln(6n)}$

- What is the probability that node u has a node v as its long range contact?

  $\geq \frac{1}{4\ln(6n)\cdot[d(u,v)]^2}$

- When r = 2, expected delivery time is

$$O(\log n)^2$$

Analysis

- How about for the cases with $r \neq 2$
- $0 \le r < 2$: The expected delivery time of <u>any</u> decentralized algorithm is $\Omega(n^{(2-r)/3})$.
- $r > 2$: The expected delivery time of <u>any</u> decentralized algorithm is $\Omega(n^{(r-2)/(r-1)})$.

Revisiting Assumptions

- Recall that in each step the message holder *u* knew
  - the locations and long-range contacts of all nodes that have previously touched the message
- Is knowledge of message's history too much info?
- Upper-bound on delivery time in the good case is proven without using this.
- Lower-bound on delivery times for the bad cases still hold even when this knowledge is used.

The Intuition

- For a changing value of *r*
  - *r* = 0 provides no "geographical" clues that will assist in speeding up the delivery of the message.
  - 0 < *r* < 2: provides some clues, but not enough to sufficiently assist the message senders
  - r > 2: as *r* grows, the network becomes more localized. This becomes a prohibitive factor.
  - r = 2: provides a good mix of having relevant "geographical" information without too much localization.

Readings

- "Networks, Crowds, and Markets" by Easley and Kleinberg (Chapters 20)
- MEJ Newman, Random graphs as models of networks, arXiv:cond-mat/0202208v1
- Watts DJ, Strogatz SH (1998) Collective dynamics of 'small-world' networks. Nature 393: 440-442.
- Newman MEJ, Watts DJ (1999) Renormalization group analysis of the small-world network model. Physics Letters A 263: 341-346