الف) -۱

$$f_X(x;\theta) = \frac{1}{\theta} \mathbb{1}(0 \le x \le \theta)$$

$$\Rightarrow f_{Data}(Data;\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}(0 \le X_i \le \theta)$$

$$= \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}(0 \le X_i \le \theta)$$

$$\underset{\theta \in \mathbb{R}_{>0}}{\operatorname{argmax}} \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}(0 \le X_i \le \theta) \longrightarrow \min_{\theta > 0} \theta \quad \text{s.t.} \quad \theta \ge X_i \quad i=1,\dots,n$$

$$\Rightarrow \boxed{\hat{\theta}_{ML} = \max_{i=1,\dots,n} X_i}$$

2

ب)

$$a\left(1 - \frac{x}{\theta}\right) \qquad 0 \le x \le \theta \longrightarrow \text{PDF}$$

$$\Rightarrow \int_0^\theta a\left(1 - \frac{x}{\theta}\right) dx = a\theta - a\frac{\theta^2/2}{\theta} = 1$$

$$\Rightarrow a\theta - a\frac{\theta}{2} = \frac{a\theta}{2} = 1$$

$$\Rightarrow \boxed{a = \frac{2}{\theta}}$$

0.5

ج)

$$\hat{\theta}_{ML} = \underset{\theta \ge 0}{\operatorname{argmax}} \underbrace{\prod_{i=1}^{n} \frac{2}{\theta}\left(1 - \frac{X_i}{\theta}\right)}_{\left(\frac{2}{\theta}\right)^n \prod_{i=1}^{n}\left(1 - \frac{X_i}{\theta}\right)}$$

$$LLH(\theta) = \log 2^n - n \log\theta + \sum_{i=1}^{n} \log\left(1 - \frac{X_i}{\theta}\right)$$

$$\frac{\partial}{\partial\theta} LLH(\theta) = 0 \Rightarrow \frac{-n}{\theta} + \sum_{i=1}^{n} \frac{1}{1 - \frac{X_i}{\theta}} \frac{X_i}{\theta^2} = 0$$

$$\boxed{\sum_{i=1}^{n} \frac{X_i}{\theta - X_i} = n}$$

2.5

$$g(\theta) \triangleq \sum_{i=1}^{n} \frac{X_i}{\theta - X_i} \qquad M \triangleq \max_i X_i$$

(5

$g(\cdot) \longrightarrow$ Decreasing function

$$\hat{\theta}_m \triangleq M\left(1+\frac{1}{n}\right) \Rightarrow$$

$$g(\hat{\theta}_m) = \cancel{\frac{M}{M(1+\frac{1}{n}) - M}}_{n} + \underbrace{\sum_{\substack{i=1 \\ i \neq i_{max}}}^{n} \frac{X_i}{\hat{\theta}_m - X_i}}_{\geq 0} \geq n$$

$$\Rightarrow \hat{\theta}_{ML} \geq M\left(1+\frac{1}{n}\right)$$

$$\hat{\theta}_M = 2M \Rightarrow$$

$$g(\hat{\theta}_M) = \sum_{i=1}^{n} \frac{X_i}{\hat{\theta}_M - X_i} \leq \sum_{i=1}^{n} \frac{M}{2M - X_i} \leq \sum_{i=1}^{n} \frac{M}{M} = n$$

$$\Rightarrow \hat{\theta}_{ML} \leq \hat{\theta}_M = 2M$$

$$\Rightarrow \boxed{M\left(1+\frac{1}{n}\right) \leq \hat{\theta}_{ML} \leq 2M}$$

*

۲ـ الف)

$$\mathbb{P}(X>a)=\mathbb{P}(e^{tX}>e^{ta})\le\frac{\mathbb{E}(e^{tX})}{e^{ta}}$$

$t>0$

2

$$\Rightarrow \mathbb{P}(X>a)\le e^{-ta}M_X(t)$$

if $M_X(t)$ exists for $\exists\, t>0$

ب) توزیع کوشی $M_X(t)$ ندارد. به ازای هیچ مقداری از $\underline{t>0}$.

1

۳ـ

if $Z\sim \mathrm{Unif}(0,1)$ and $X\sim f_X(x)$

$\hookrightarrow$ CDF $\rightarrow F_X(x)$

Then $F_X^{-1}(Z)\sim f_X$

$$\rightarrow F_X(x)=\int_{-\infty}^{x}\frac{u}{\sigma^2}e^{-u^2/(2\sigma^2)}du$$

$$=\int_{0}^{x}\frac{u}{\sigma^2}e^{-u^2/(2\sigma^2)}du=\int_{0}^{\frac{x^2}{2\sigma^2}}e^{-u'}du'$$

$$u'=\frac{u^2}{2\sigma^2}$$

$$du'=\frac{u\,du}{\sigma^2}$$

$$=\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)\mathbb{1}(x\ge 0)$$

$$F_X^{-1}(Z)=Y\Rightarrow F_X(Y)=Z=1-e^{\frac{-Y^2}{2\sigma^2}}\qquad Y\ge 0$$

$$\Rightarrow \frac{-Y^2}{2\sigma^2}=\log(1-Z)\Rightarrow$$

$$y^2 = -2\sigma^2 \log(1-z)$$

$$\boxed{y = \sigma\sqrt{2\log\frac{1}{1-z}} \sim f_X(y)}$$

4

4-

الف)

1

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

ب)

$$S = \left[\frac{1}{9}(X_1 + \cdots + X_9) - \frac{1}{16}(Y_1 + \cdots + Y_{16})\right]$$

$$\mathbb{E}(S \mid H_0) = 0$$

$$Var(S \mid H_0) = \frac{1}{9^2}\left(\sum_{i=1}^{9} Var(X_i)\right) + \frac{1}{(16)^2}\left(\sum_{i=1}^{16} Var(Y_i)\right)$$

$$= \frac{Var(X_1)}{9} + \frac{Var(Y_1)}{16} = \frac{16+9}{16\times 9} = \frac{25}{9\times 16}$$

$$= \left(\frac{5}{12}\right)^2$$

$$\Rightarrow \quad \bar{S} = \frac{S}{\frac{5}{12}} \rightarrow \begin{array}{l} \mathbb{E}\bar{S} = 0 \\ Var(\bar{S}) = 1 \end{array}$$

قابل استحصال است.

$$\mathbb{P}\left(|\bar{S}| > \frac{13-10.5}{\frac{5}{12}} = \frac{12\times 2.5}{5} = 6\right) \leq \frac{1}{6^2} = \frac{1}{36} > \frac{1}{100}$$

1.5

Chebyshev's Inq.

پس رد می‌شود.

ج)

0.5

$$\bar{S} \sim \mathcal{N}(0,1)$$

$$\mathbb{P}(|\bar{S}| > 6) \ll \frac{1}{100} \longrightarrow$$ قبول می‌شود.

5- الف)

$$Y_i = \mathbb{1}(X_i \geq 0)$$

$$Y_1 \text{—} Y_n \overset{i.i.d.}{\sim} \text{Bern}(p) \qquad p = \mathbb{P}(X \geq 0)$$

$$LH = \prod_{i=1}^{n} p^{Y_i}(1-p)^{1-Y_i}$$

$$\Rightarrow LLH = \sum_{i=1}^{n} Y_i \log p + (1-Y_i)\log(1-p)$$

$$\frac{\partial}{\partial p} LLH(p) = 0 \Rightarrow$$

$$\sum_{i=1}^{n} \frac{Y_i}{p^*} - \frac{1-Y_i}{1-p^*} = 0$$

$$\Rightarrow \frac{N_1}{p^*} = \frac{N_0}{1-p^*} \Rightarrow N_1 - N_1 p^* = N_0 p^*$$

$$\Rightarrow p^* = \frac{N_1}{N_0 + N_1} = \frac{N_1}{n}$$

2

$$\boxed{\hat{P}_{ML} = \frac{N_1}{n} = \frac{\#\{Y_i = 1\}}{n}}$$

ب)

$$\text{Bias}(\hat{P}_{ML}) = \mathbb{E}(\hat{P}_{ML}) - p =$$

$$\frac{\mathbb{E}\, N_1}{n} - p = \frac{np}{n} - p = 0$$

1

$$Var(\hat{P}_{ML}) = Var\left(\frac{N_1}{n}\right) = Var\left(\frac{Y_1 + \text{—} + Y_n}{n}\right) =$$

$$\frac{1}{n^2} \sum_{i=1}^{n} Var(Y_i) = \frac{np(1-p)}{n^2} = \boxed{\frac{p(1-p)}{n}}$$

CamScanner

$$\underbrace{\mathbb{P}(p \mid Data)}_{\text{Posterior}} \propto \underbrace{\mathbb{P}(Data \mid p)}_{\text{Likelihood}} \underbrace{\mathbb{P}(p)}_{\text{Prior}}$$

$$\Rightarrow \mathbb{P}(p \mid Data) = \text{constant} \times \prod_{i=1}^{n} p^{Y_i}(1-p)^{1-Y_i} \, kp^{k-1}$$

$$\Rightarrow \hat{p}_{MAP} = \underset{p \in [0,1]}{\text{argmax}} \underbrace{\sum_{i=1}^{n} Y_i \log p + (1-Y_i)\log(1-p) + (k-1)\log p}_{LP(p)}$$

L ↓ log, P ↓ Posterior

$$\frac{\partial}{\partial p} LP(p) = 0$$

$$\Rightarrow \frac{1}{p^*}\sum_{i=1}^{n} Y_i - \frac{1}{1-p^*}\sum_{i=1}^{n}(1-Y_i) + \frac{k-1}{p^*} = 0$$

$$\Rightarrow \frac{N_1 + k - 1}{p^*} = \frac{N_0}{1-p^*}$$

$$\boxed{\hat{p}_{MAP} = \frac{N_1 + k - 1}{n + k - 1}}$$

2