Week 3 - Day 1

Application of Gauss's Law

Concept 1: Application of Gauss's Law – Conductor at Equilibrium

Concept 2: Applying Gauss's Law on Symmetrical Charge Distributions

Explanation of Faraday Cages, Screened cables

A simplified way to find $\vec{E}$ under symmetric conditions.

A USB cable with metallic shielding around the wire

Reading:

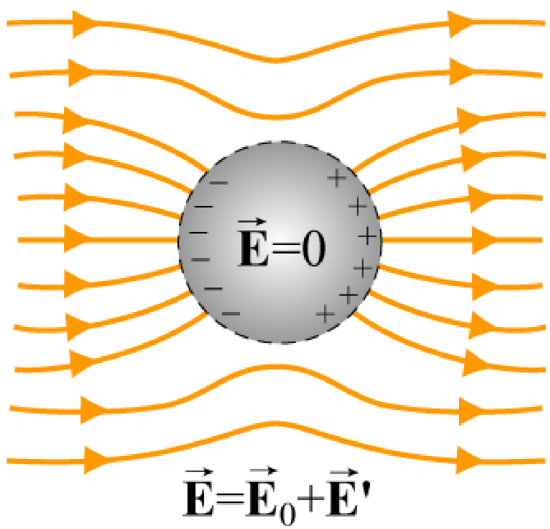
University Physics with Modern Physics – Chapter 22

Introduction to Electricity and Magnetism – Chapter 3

Concept 1: Application of Gauss's Law – Conductor at Equilibrium

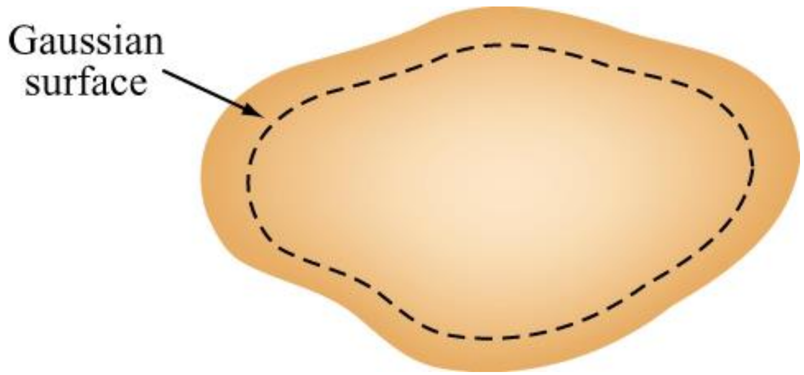
Why so special about conductors?

- If an electric field is present inside a conductor, free charges will move around (not an electrostatic condition).
- The free charges in a conductor will redistribute themselves and very quickly reach electrostatic equilibrium (all charges do not move anymore).
- It implies that at electrostatic equilibrium, the electric field inside a conductor is ZERO!



Placing a conductor in a uniform electric field $\vec{E}_0$

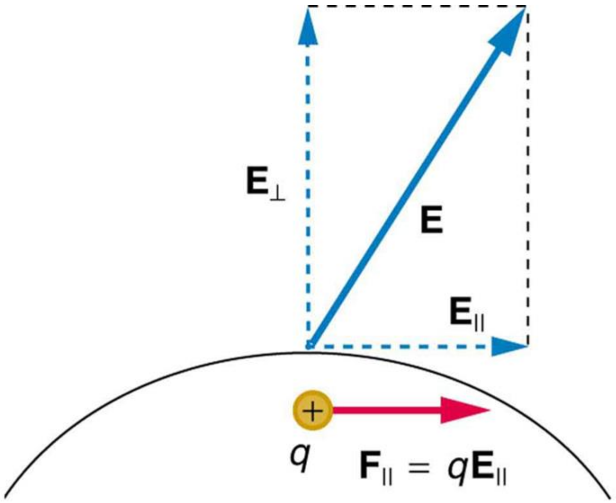
You can have external electric field outside a conductor, but inside the conductor, the $\vec{E}$ is always zero at equilibrium condition.

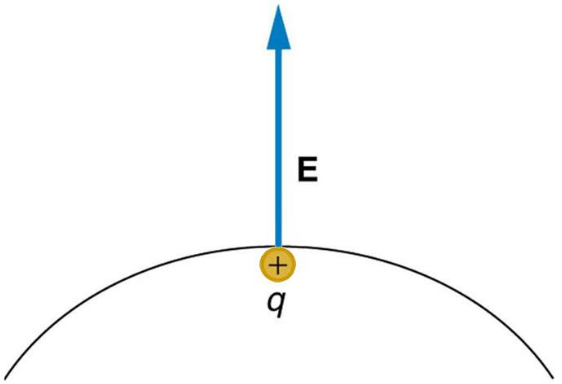


We draw a Gaussian surface inside a conductor. The enclosed charge is zero.

- Now we know E field inside a conductor is always 0 at equilibrium.
- We rightfully draw a Gaussian surface just underneath a conductor surface (or inside a conductor). No electric field inside a conductor means no flux passing through the Gaussian surface.
- According to Gauss's law, the enclosed net charges has to be ZERO! It implies that any excessive charges can only stay on the surface of a conductor in equilibrium, regardless of the shape of the conductor.
- The distribution of the charges on surface may not be necessary uniform. It depends on the shape/ geometry of the conductor.

- In electrostatic equilibrium, the tangential component of $\vec{E}$, $E_{||}$ is zero on the surface of a conductor. (Otherwise, charges will move tangentially along the surface, which is a non-static condition.)
  - A conductor with free charges can reach equilibrium from non-equilibrium state quickly, as free charges can move around in the conductor until $E_{||}$ is zero.
- $\vec{E}$ can only be normal to the surface just outside the conductor.





Summary - Conductors in **Electrostatic Equilibrium**

- The electric field inside a conductor vanishes.
- Any excess charge placed on a conductor resides entirely on the surface of the conductor.
- The electric field can only be perpendicular/ normal to the surface just outside the conductor. (It can never have a tangential component of the electric field just outside the surface of a conductor.)

- FYI: The magnitude of the electric field just above the surface of a conductor is given by $E_{surface} = \frac{\sigma}{\varepsilon_o}$. Note that σ is the surface charge density at the spot; σ may not be uniform throughout the whole surface.

Concept Question 1.1: Spherical Shell

Positive charge Q is distributed uniformly throughout a spherical shell. What is the electric field inside the shell?

A. Zero

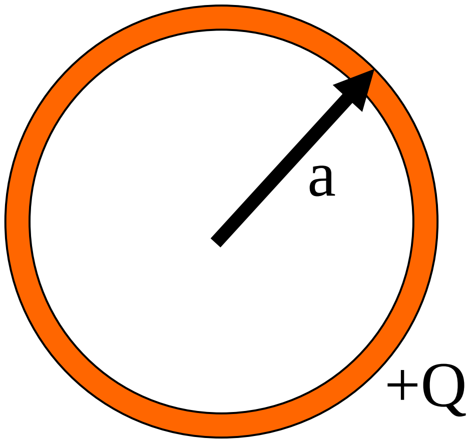
B. Uniform but Non-Zero

C. Still grows linearly

D. Some other functional form (use Gauss' Law)

E. Can't determine with Gauss Law

Multiple Choice



Concept Question 1.1: Solution

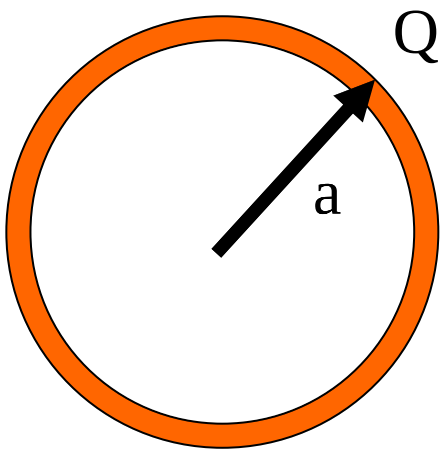
Answer: A. Zero

Spherical symmetry

→ Use Gauss' s Law with spherical surface.

Any surface inside shell contains no charge

→ No flux so E = 0!



Concept Question 1.1: A Conductor with Cavity

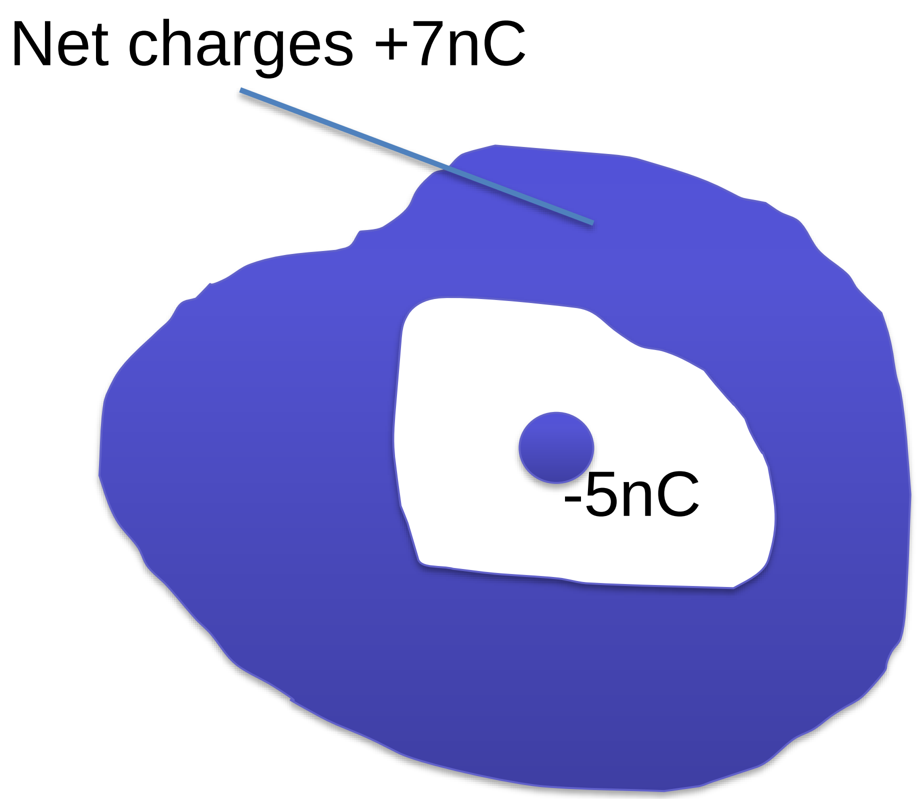
- A conductor with a cavity carries a total charge of +7nC. A point charge of -5nC is inside the conductor.
- How is the charge distributed on the inner surface of cavity wall and outside surface?

A. Inner cavity wall 0C, outside surface +7nC

B. Inner cavity wall +5nC, outside +2nC

C. Not enough information: exact geometry need to be provided

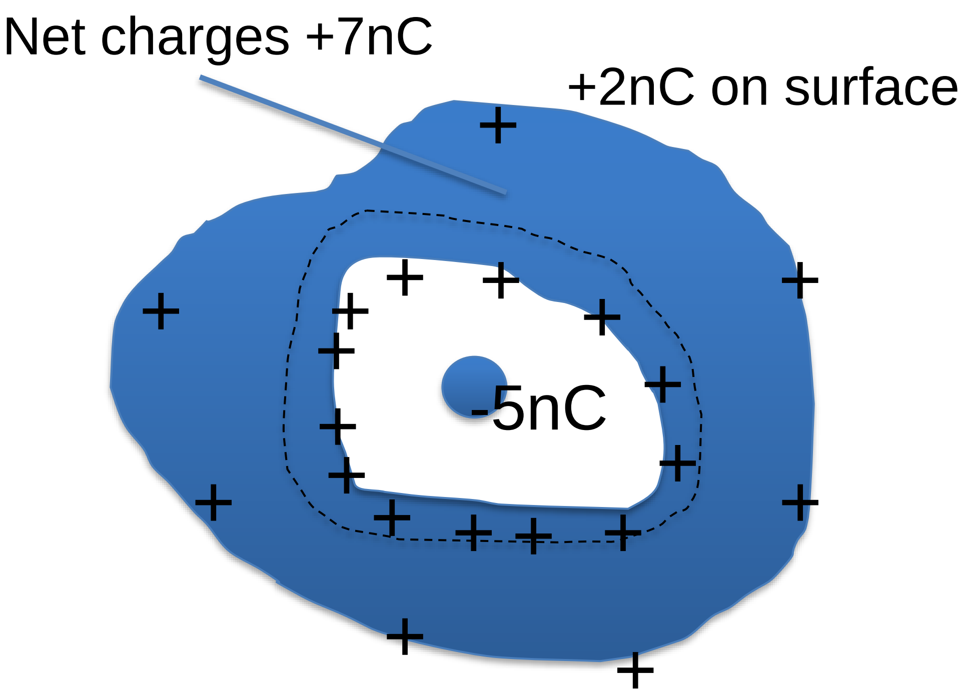
D. Inner cavity wall -5nC, outside +12nC



Multiple Choice

Concept Question 1.1: Solution

- Answer: B. Inner cavity wall +5nC, outside +2nC
- Since there is no electric field in the conductor, the electric flux through the gaussian surface shown must be zero. Therefore the charge on the cavity has to be opposite of the point charge. The remaining charges are distributed on the conductor outside surface.



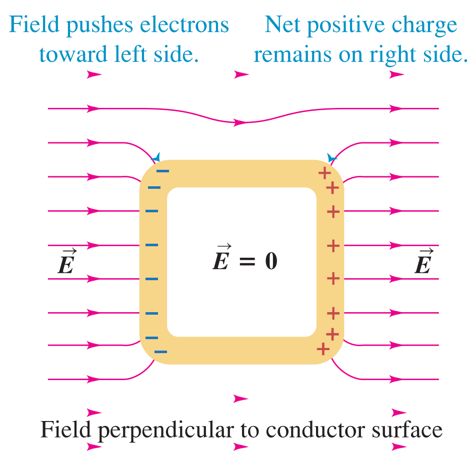
Demo – Faraday Cage

- Turn on the Van de Graaff generator and observe what happen to the tinsel and electroscope. Now cover the tinsel and the electroscope with a metallic cage (Faraday cage). What happen to the tinsel and electroscope then? Why?
- Turn on a radio. Put the Faraday cage over the radio and observe what happen to the sound. Why?
- Put a hand phone in the metallic container, try calling the number of the hand phone. Does the hand phone ring at all? Why?

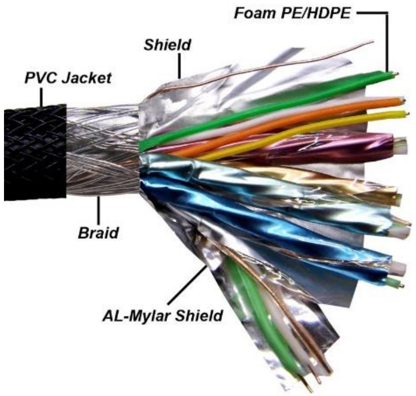
Now you know how the metallic shield works in all signal cables!

Application: Faraday Cage

- Even there is a cavity inside a conductor, $\vec{E}$ inside is also zero. It implies external $\vec{E}$ cannot penetrate in.
- Faraday cage works because the external electrical field causes the electrical charges within the cages conducting material to be disturbed such that they cancel the fields effect inside the cage.
- That is the reason you may lose phone or radio signal in elevators or buildings with metallic conducting frames and walls, which simulate a Faraday cage effect.
- You are safe in cars, trains or even airplane during lighting as the metallic compartments are essentially Faraday cages, protecting passengers from electric charges.

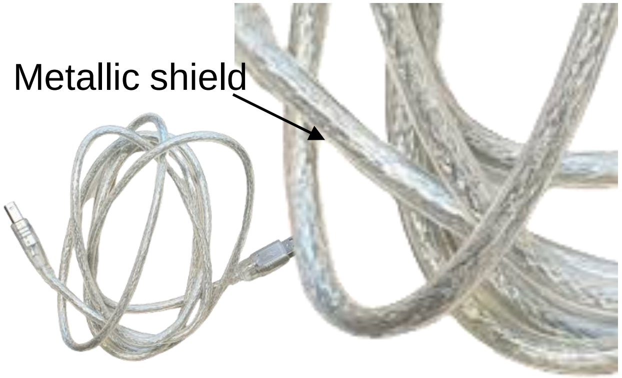


- Microwave ovens are one common product that use a Faraday cage. Instead of keeping microwaves out, the door seals and outer case contain microwaves into a small cooking chamber that cook your food.
- Space Launch Complex (SLC-40) with SpaceX Falcon 9 launch infrastructure has four towers surrounding the rocket as lightning arresters, acting like a giant Faraday cage (from Wikipedia).
- Electronic components in automobiles and aircraft utilize Faraday cages to protect signals from interference.

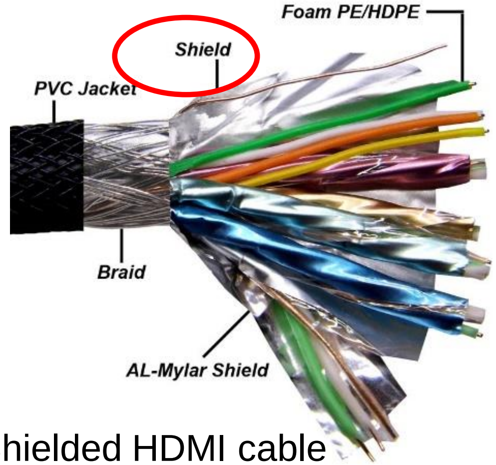


Application: Metallic Shielding in Screened Cables

- You may notice any signal cable (also called screened cable), such as USB, VGA, HDMI, network, coaxial cables for cable television, always have a metallic shielding wrapping around those signal wires.
- Not only it helps to strengthen the mechanical structure, more importantly, the metallic shield protects the internal conductors from external electrical noise and prevents the RF signals from leaking out.
- A conductor has such a property to eliminate external electric field from penetrating into it.
- Gauss's Law explains it!



Shielded USB cable



Shielded HDMI cable

Concept 2: Applying Gauss's Law on Symmetrical Charge Distribution

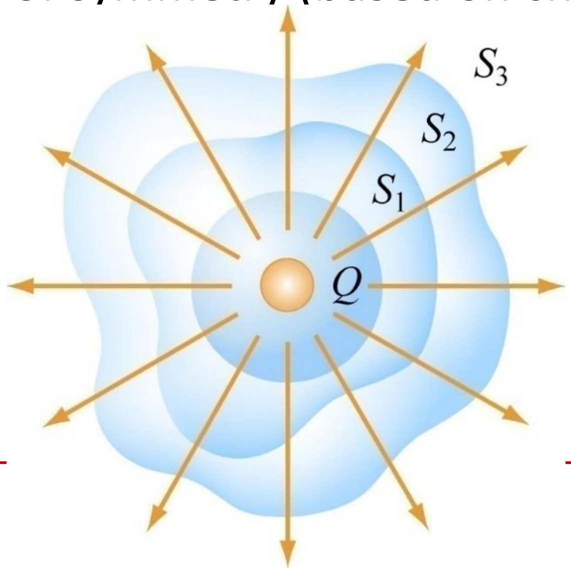
The simplified way to find $\vec{E}$ (under symmetric conditions) helps to understand and analyze other concepts in the subsequent weeks.

Choosing Gaussian Surface

- Desired **E field:** Perpendicular to surface and uniform on surface A. Flux is EA or –EA on these surfaces.
- Other **E field:** Parallel to surface. Flux is zero on these surfaces.

$$\oiint_{\substack{closed\\surface\ S}} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

- Gauss's Law is
  – **True** for all closed surfaces, but only **useful** to calculate electric field for problems with plenty of symmetry (based on charge distribution).



Symmetry & Gaussian Surfaces

- Desired **E field**: perpendicular to surface and constant on surface. So Gauss’s Law useful to calculate electric field from highly symmetric sources.

Source (charge distribution) Symmetry	Gaussian Surface
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian “Pillbox”

Skills to develop:

Describe and picture the electric field pattern and electric field lines for highly symmetrical charge distribution.

Construct an appropriate Gaussian surface and apply Gauss’s Law on highly symmetrical charge distribution to calculate the electric field.

Applying Gauss's Law

1. Based on the highly symmetric charge distribution, perceive the electric field pattern.
2. Identify regions in which to calculate the electric field.
3. Choose a Gaussian surface S, so that the surface is either perpendicular or parallel to the $\vec{E}$. The magnitude of the $\vec{E}$ should be constant on the surface too.
4. Calculate $\Phi_E = \oiint_S \vec{E} \cdot d\vec{A}$ (left hand side of Gauss's Law)
   - You should probably end up with $\Phi_E = E \oiint dA$ for a properly chosen Gaussian surface.
5. Calculate q_{enc}, charge enclosed by surface S (right hand side of Gauss's Law)
6. Apply Gauss's Law (equal LFS to RHS) to calculate the unknown electric field.

$$\oiint_{\substack{closed \\ surface\ S}} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

Applying Gauss's Law

- To apply Gauss's Law to find the electric field, the charge distribution needs to be:
  - Spherical Symmetry
  - Cylindrical Symmetry
  - Planar Symmetry
- Out of these symmetry, Gauss's Law is not useful to calculate electric field (by hand).

- Gauss's Law is always true, but not always useful to calculate electric field! It is only useful for finding electric field in highly symmetric cases!

Concept Question 2.1: Application of Gauss's law

- For which of the following charge distributions would Gauss' s law *not* be useful for calculating the electric field?

A. a uniformly charged sphere of radius *R*

B. a spherical shell of radius *R* with uniform charge distribution over its surface

C. a right circular cylinder of radius *R* and height *h* with charge uniformly distributed over its surface

D. an infinitely long circular cylinder of radius *R* with charge uniformly distributed over its surface

E. Gauss' s law would be useful for finding the electric field in all of these cases.

Multiple Choice

Concept Question 2.1: Solution

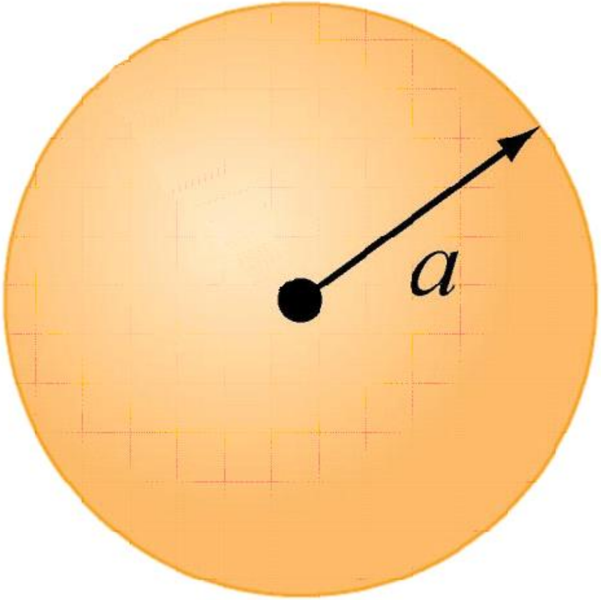
- Answer: C. a right circular cylinder of radius *R* and height *h* with charge uniformly distributed over its surface

- Gauss's law is only useful for finding electric field in highly symmetric cases!

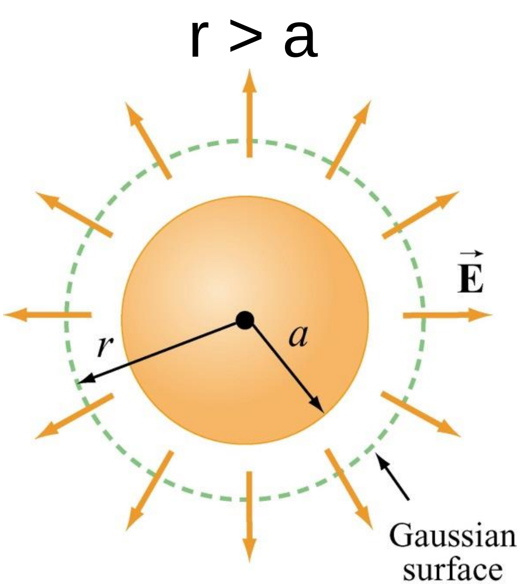
Case Problem 1: Spherical Symmetry

$+Q$ uniformly distributed throughout **non-conducting** solid sphere of radius a. Find $\vec{E}$ everywhere, which are

1. $\vec{E}$ for $r > a$
2. $\vec{E}$ for $r < a$



Case Problem 1: Solution for region $r > a$



- From Gauss's Law,

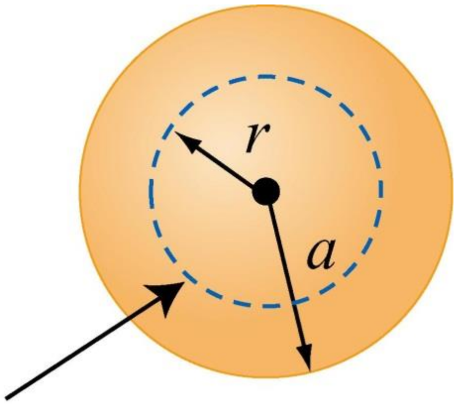
$$\oiint_{\substack{closed\\ surface\ S}} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

- Step 1: We can perceive the electric field $\vec{E} = E\hat{r}$
- Step 2&3: We choose a Gaussian surface that its normal is parallel to $\vec{E}$, which is also a sphere. $d\vec{A} = dA\hat{r}$
- Step 4: $\oiint \vec{E} \cdot d\vec{A} = \oiint E\hat{r} \cdot dA\,\hat{r} = \oiint EdA\,\hat{r} \cdot \hat{r} = \oiint EdA$

- Since E is the same on the Gaussian surface, thus the LHS:
- $\oiint EdA = E \oiint dA = E4\pi r^2$
- From the RHS of Gauss's Law, $q_{enclosed} = +Q$
- So, $E4\pi r^2 = \frac{+Q}{\varepsilon_o} \rightarrow E = \frac{+Q}{4\pi\varepsilon_o r^2}$
- Since we perceive the field is in r̂, thus $\vec{E} = \frac{+Q}{4\pi\varepsilon_o r^2}\,\hat{r}$

Case Problem 1: Solution for region $r < a$

$r < a$



- From Gauss's Law,

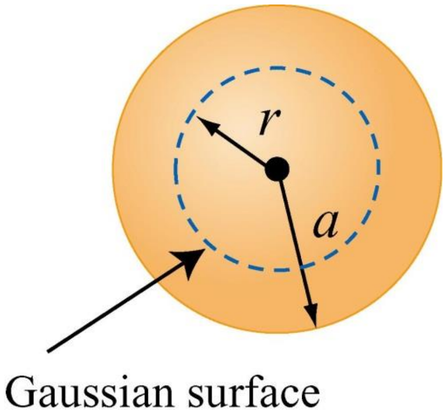
$$\oiint_{\text{closed surface } S} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

- The LHS of Gauss's Law, $\oiint \vec{E} \cdot d\vec{A} = E4\pi r^2$ (same procedure as previous)
- From the RHS of Gauss's Law, $q_{enclosed} = \rho(\frac{4}{3}\pi r^3)$, where $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$. Thus, $q_{enc} = \frac{Qr^3}{R^3}$
- So,

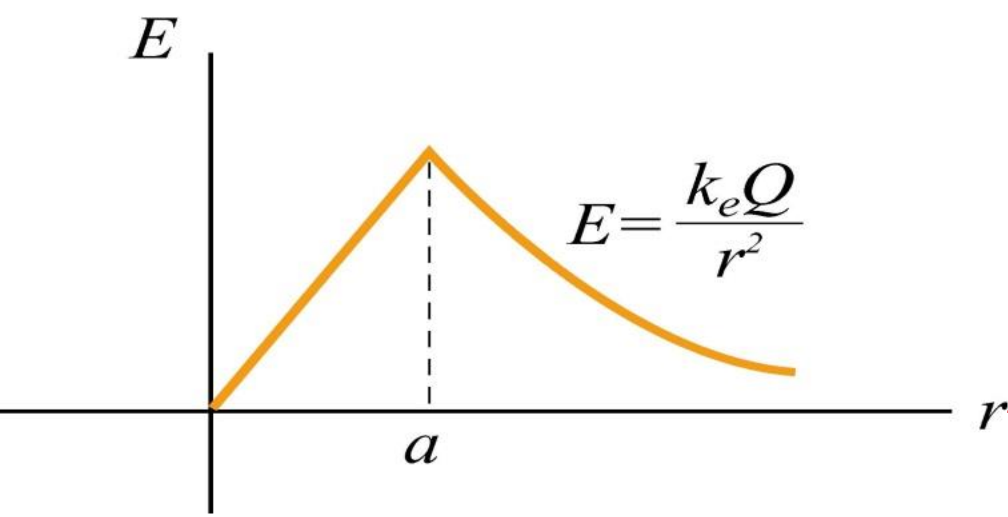
$$E4\pi r^2 = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\varepsilon_o} \rightarrow E = \frac{\rho r}{3\varepsilon_o} = \frac{Qr}{4\pi\varepsilon_o a^3}$$

Case Problem 1: Solution (Summary)

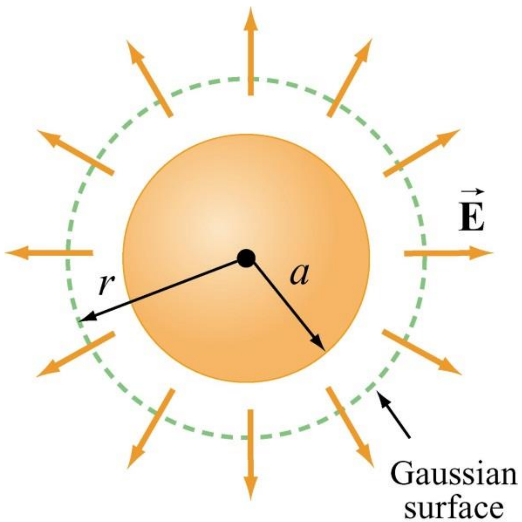
r < a



$$E = \frac{\rho r}{3\varepsilon_o} = \frac{Qr}{4\pi\varepsilon_o a^3}$$



r > a

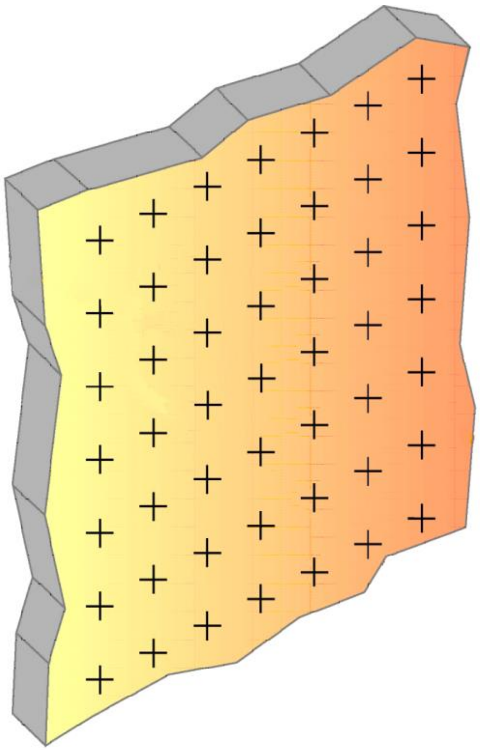


$$E = \frac{Q}{4\pi\varepsilon_o r^2} = k_e\frac{Q}{r^2}$$

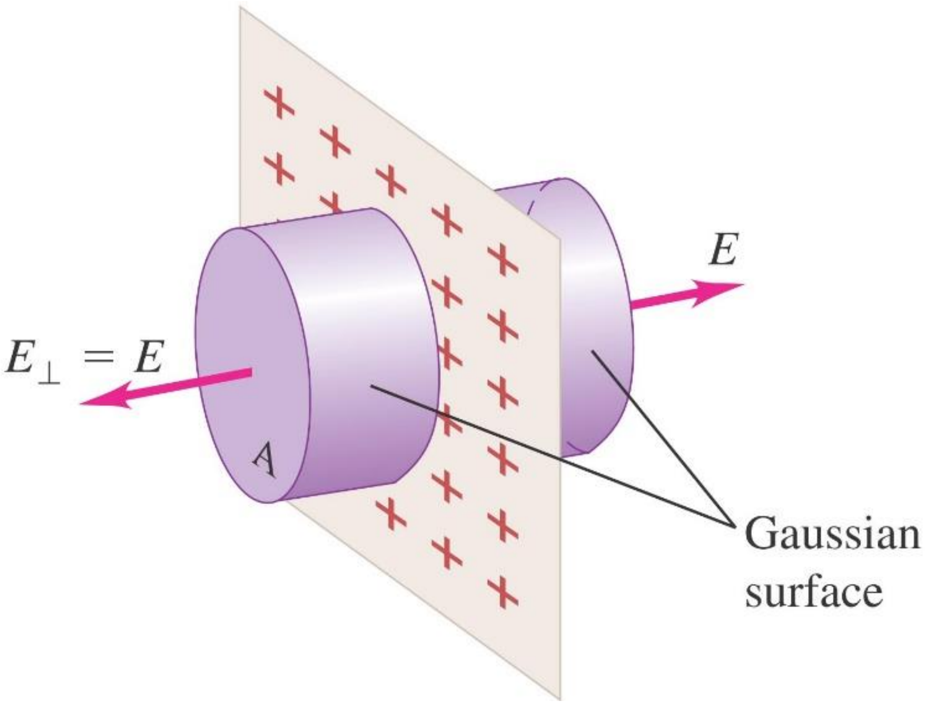
Note: For the electric field outside the sphere, it looks like the field from a point charge located at the center of the sphere.

Case Problem 2: Planar Symmetry

- Consider an infinite thin slab with uniform positive charge density σ. Find a vector expression for the direction and magnitude of the electric field outside the slab. Make sure you show your Gaussian closed surface.

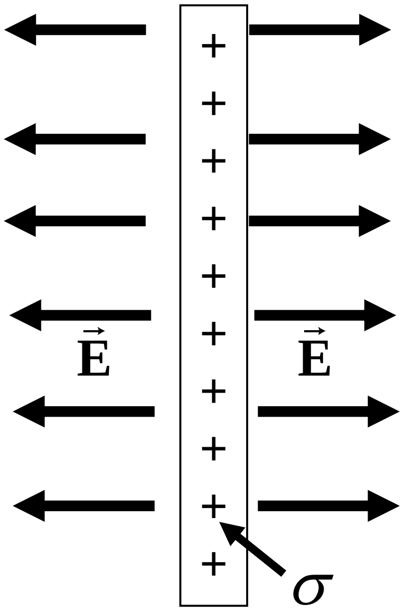


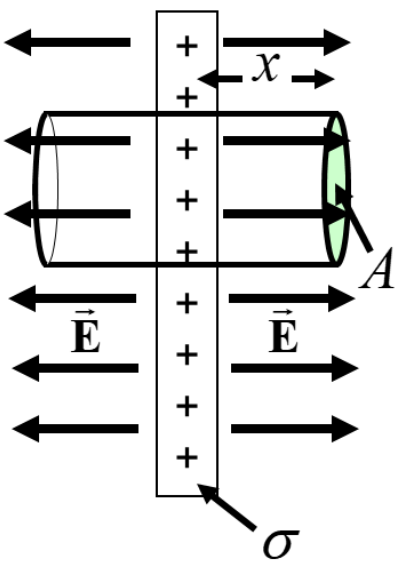
Hint: You may consider the Gaussian surface as shown.

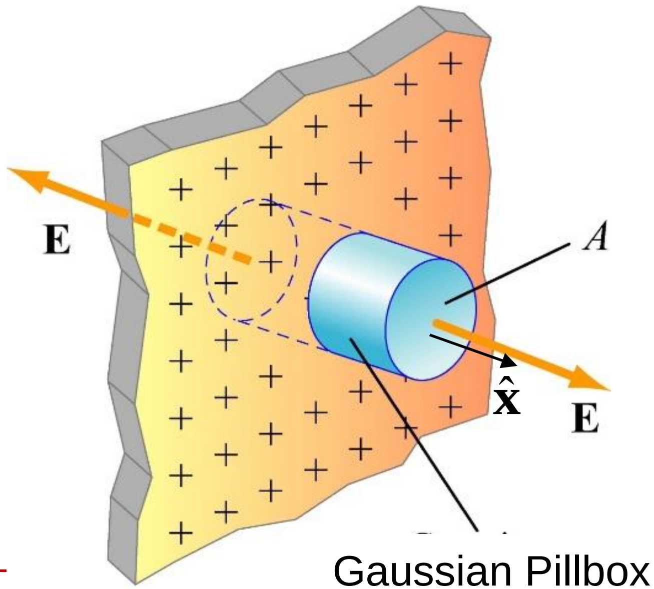


Case Problem 2: Solution

- Because of planar symmetry, the resultant $\vec{E}$ is pointing to +x and –x direction. Thus, $\vec{E} = \pm E\,\hat{x}$
- We draw a Gaussian Pillbox/surface that cuts across the electric field line in the perpendicular and parallel manner.
- Note: A is arbitrary (its size and shape) as it should be divided out later.



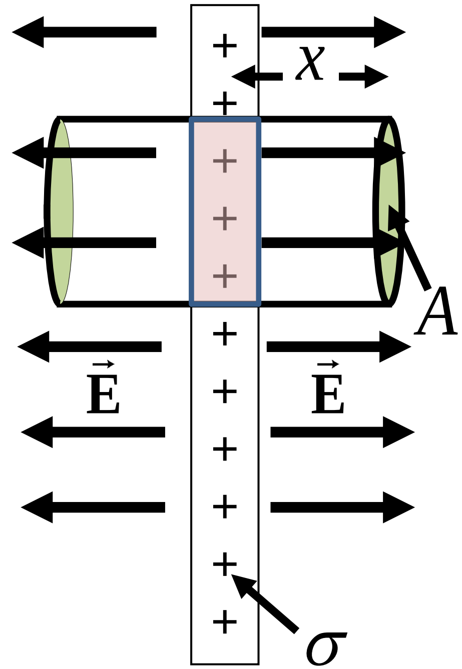




Case Problem 2: Solution

- Total charge enclosed: $q_{enclosed} = \sigma A$
- NOTE: No flux through side of cylinder, only endcaps surface

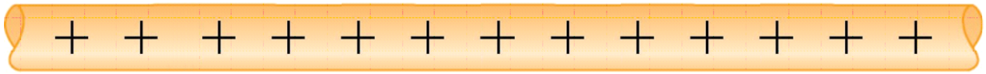
$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{A} = E \oiint_S dA = EA_{Endcaps} = E(2A)$$

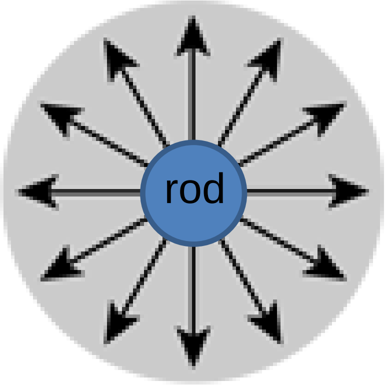


- Thus, using Gauss's Law
- $\oiint_S \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$
- $\mathrm{E(2A)} = \frac{\sigma A}{\varepsilon_0}$
- $\Rightarrow E = \frac{\sigma}{2\varepsilon_o}$
- Thus, $\vec{E} = +\frac{\sigma}{2\varepsilon_o}\hat{\imath}$ (for x>0)
- $\vec{E} = -\frac{\sigma}{2\varepsilon_o}\hat{\imath}$ (for x<0)
- Note: For infinite large charged slab, E field is uniform and independent of the distance x away from the slab.

Case Problem 3: Cylindrical Symmetry

- An infinitely long rod has a uniform positive linear charge density λ. Find the direction and magnitude of the electric field outside the rod. Clearly show your choice of Gaussian closed surface.
- Note that the resultant E field is pointing radial outward.

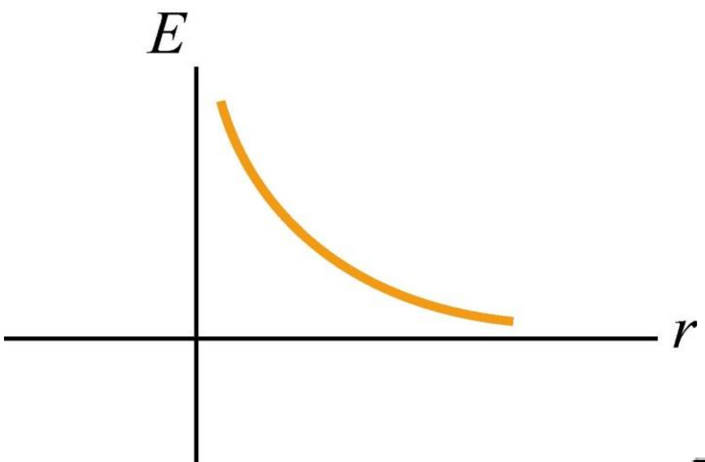


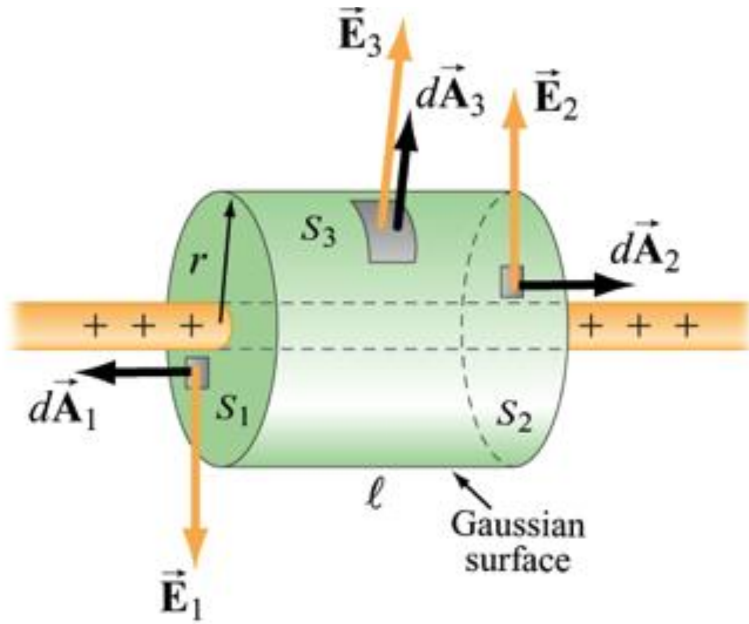


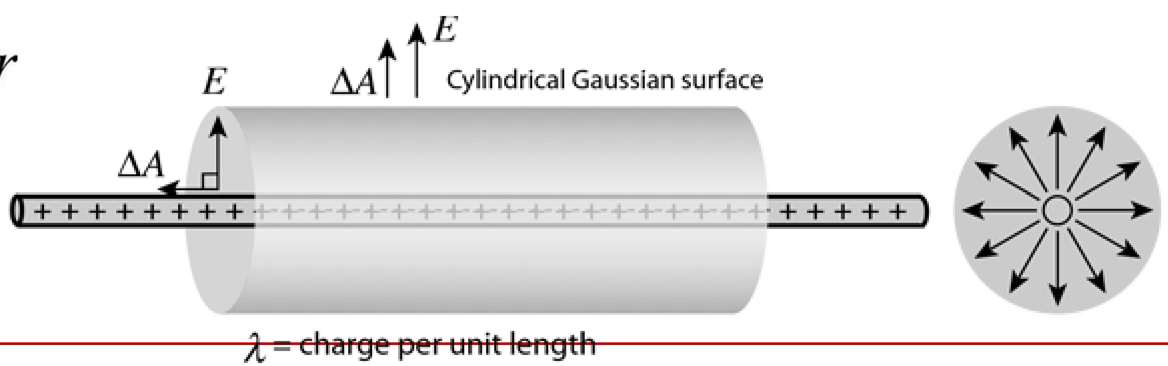
Top view

Case Problem 3: Solution

- $\Phi_E = \iint_S \vec{E} \cdot d\vec{A} = \iint_{S_1} \vec{E}_1 \cdot d\vec{A}_1 + \iint_{S_2} \vec{E}_2 \cdot d\vec{A}_2 + \iint_{S_3} \vec{E}_3 \cdot d\vec{A}_3$
- $\Rightarrow \Phi_E = 0 + 0 + E_3 A_3 = E(2\pi r l)$
- Applying Gauss's Law,
- $E(2\pi r l) = \frac{\lambda l}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}$
- $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$







Reminder: Symmetry & Gaussian Surfaces

- Desired **E field**: perpendicular to surface and constant on surface. So Gauss's Law useful to calculate electric field from highly symmetric sources.

Source (charge distribution) Symmetry	Gaussian Surface
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian "Pillbox"

Skills to develop:

Describe and picture the electric field pattern and electric field lines for highly symmetrical charge distribution.

Construct an appropriate Gaussian surface and apply Gauss's Law on highly symmetrical charge distribution to calculate the electric field.

Summary of $\vec{E}$ for Basic Charged Structures

Charge Distribution	E Field (Outside) Strength
Dipole	Falls off by $1/r^3$
Spherical	$\vec{E} = k_e \frac{Q}{r^2} \hat{r}$ (Falls off by $1/r^2$)
Cylindrical or Line (infinite)	$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r} \hat{r}$ (Falls off by $1/r$)
Plane (infinite)	$\vec{E} = \pm \frac{\sigma}{2\varepsilon_o} \hat{\imath}$ (Uniform on both sides (constant))

Extra Case Problem 1

Positive charge is distributed uniformly over each of two spherical volumes with radius R.

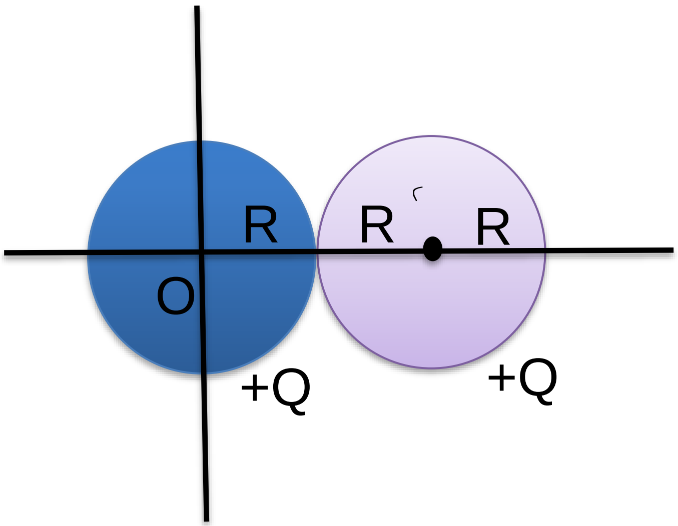
Find the magnitude and direction of Electric field at

a) x=0

b) x=R/2

c) x=R

d) x=3R



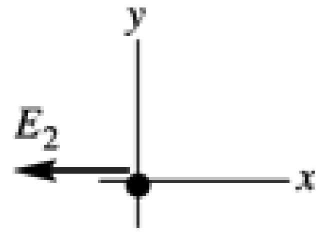
Hint: E inside an uniformly distributed charged sphere $= k_e \frac{Qr}{R^3}\hat{r}$.

E outside an uniformly distributed sphere $= k_e \frac{Q}{r^2}\hat{r}$

Extra Case Problem 1: Solutions

- a)

$$E_1 = \frac{Qr}{4\pi\varepsilon_0 R^3} = 0, \text{ since } r = 0.$$



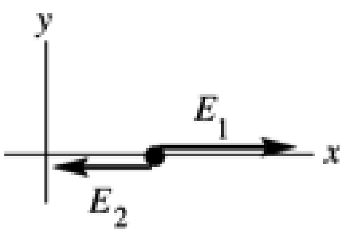
$$E_2 = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ with } r = 2R \text{ so } E_2 = \frac{Q}{16\pi\varepsilon_0 R^2}, \text{ in the } -x\text{-direction.}$$

$$\text{Thus } \vec{E} = \vec{E}_1 + \vec{E}_2 = -\frac{Q}{16\pi\varepsilon_0 R^2}\hat{i}.$$

- b)

$$E_1 = \frac{Qr}{4\pi\varepsilon_0 R^3} \text{ with } r = R/2 \text{ so}$$

$$E_1 = \frac{Q}{8\pi\varepsilon_0 R^2}$$



$$E_2 = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ with } r = 3R/2 \text{ so } E_2 = \frac{Q}{9\pi\varepsilon_0 R^2}$$

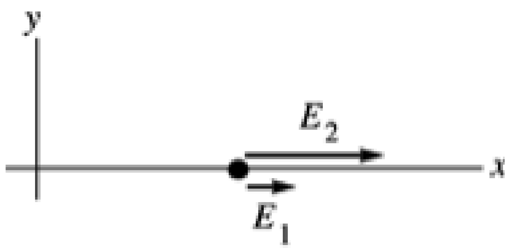
$$E = E_1 - E_2 = \frac{Q}{72\pi\varepsilon_0 R^2}, \text{ in the } +x\text{-direction and } \vec{E} = \frac{Q}{72\pi\varepsilon_0 R^2}\hat{i}$$

- c) $E = 0.$

- d)

$$E_1 = \frac{Q}{4\pi\epsilon_0 r^2} \text{ with } r = 3R \text{ so}$$

$$E_1 = \frac{Q}{36\pi\epsilon_0 R^2}$$



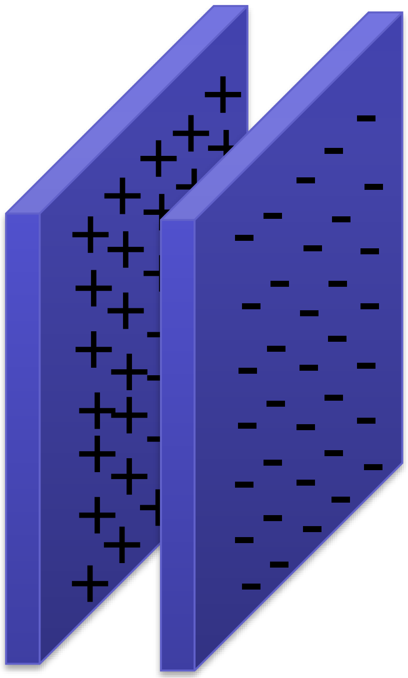
$$E_2 = \frac{Q}{4\pi\epsilon_0 r^2} \text{ with } r = R \text{ so } E_2 = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$E = E_1 + E_2 = \frac{5Q}{18\pi\epsilon_0 R^2}, \text{ in the +x-direction and } \vec{E} = \frac{5Q}{18\pi\epsilon_0 R^2}\hat{i}$$

EVALUATE: The field of each sphere is radially outward from the center of the sphere. We must use the correct expression for $E(r)$ for each sphere, depending on whether the field point is inside or outside that sphere.

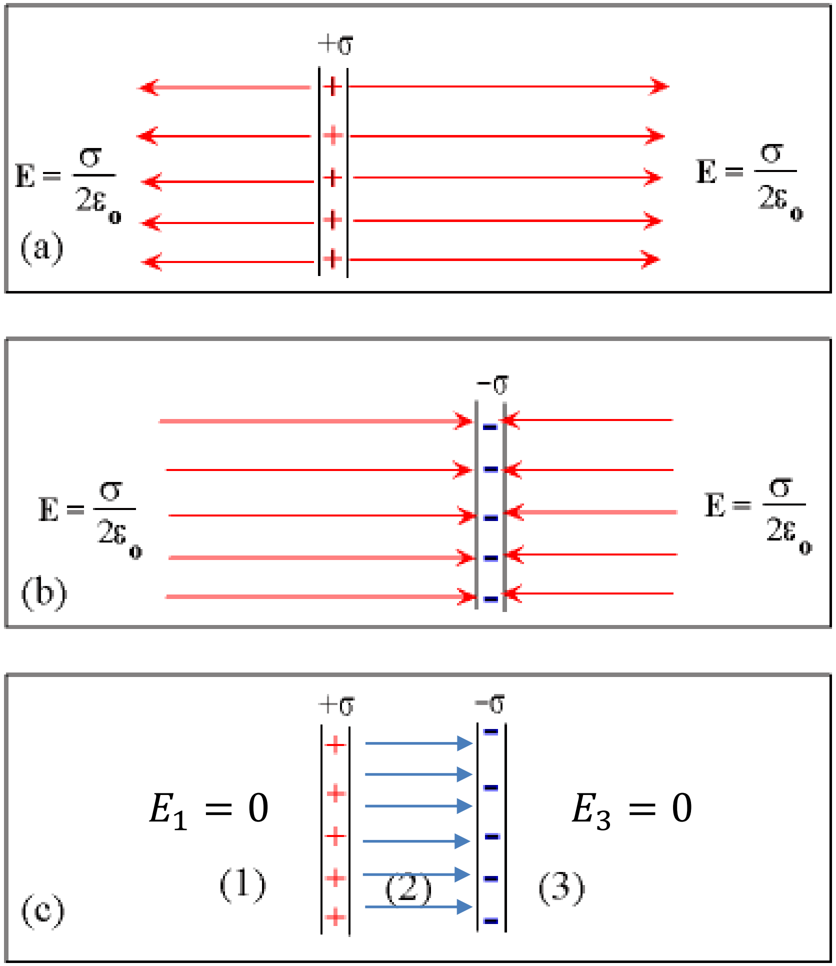
Extra Case Problem 2: Parallel Conducting Plates

Two large parallel conducting plates have equal and opposite charges. The surface charge density are $+\sigma$ and $-\sigma$ respectively. Find the electric field in the regions between and outside the plates.



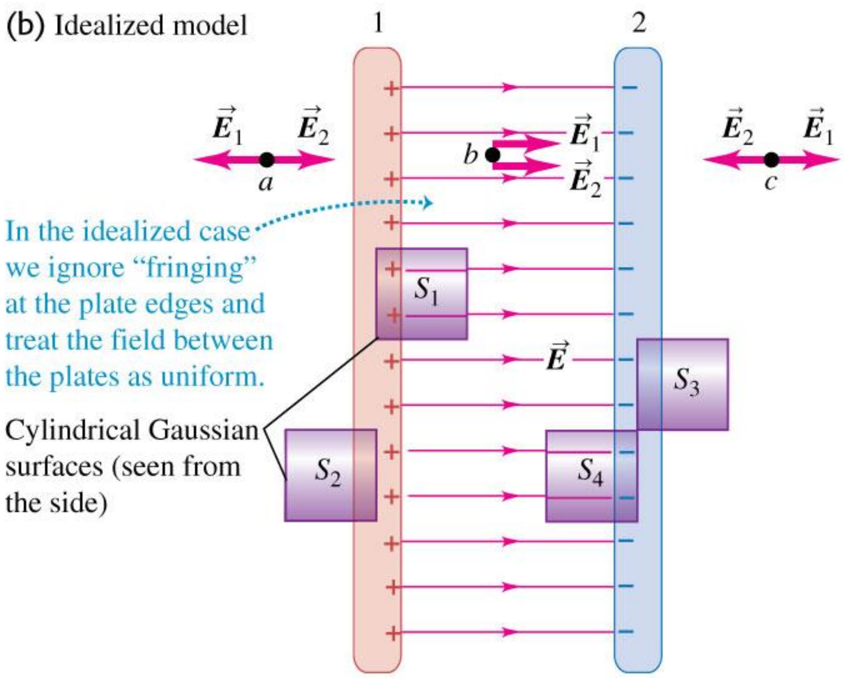
Extra Case Problem 2 Solution:

- Positive charge plate produces $E = \frac{\sigma}{2\varepsilon_o}$ at both sides, similar to the negative charged plate.
- Applying superposition principle, one should find the E field outside the parallel plate is zero due to the cancellation of E field by both plates.
- The E field in between the parallel plates is $E = \frac{\sigma}{2\varepsilon_o} + \frac{\sigma}{2\varepsilon_o} = \frac{\sigma}{\varepsilon_o}$



$$E_2 = \frac{\sigma}{\varepsilon_o}$$

Extra Case Problem 2: Alternative Solution



Note: We draw one end of S1 inside the conductor as we know E inside a conductor is always 0. We draw the other end in between the plates as we want to find out the E field at that region.

We can use the powerful Gauss's Law for this question with 2 highly symmetrical structures.

First, we need to realize that both charges $+\sigma$ and $-\sigma$ will redistribute themselves to the inner surface when both plates come close to each other.

Then, we can use the Gaussian surfaces, S1 or S4 as shown in the figure. Each end of the cylinder surface lies within the conducting plate.

Since the left end of S1 lies within the conductor, the electric field through this end is 0. According to Gauss Law, the total flux is EA and the net flux enclosed by S1 is $\frac{\sigma A}{\varepsilon_o}$. Therefore $E = \frac{\sigma}{\varepsilon_o}$.

We can reach the same result using S4.

Similarly, since S2 and S4 do not enclose any charges, the electric field outside the plates is 0.

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