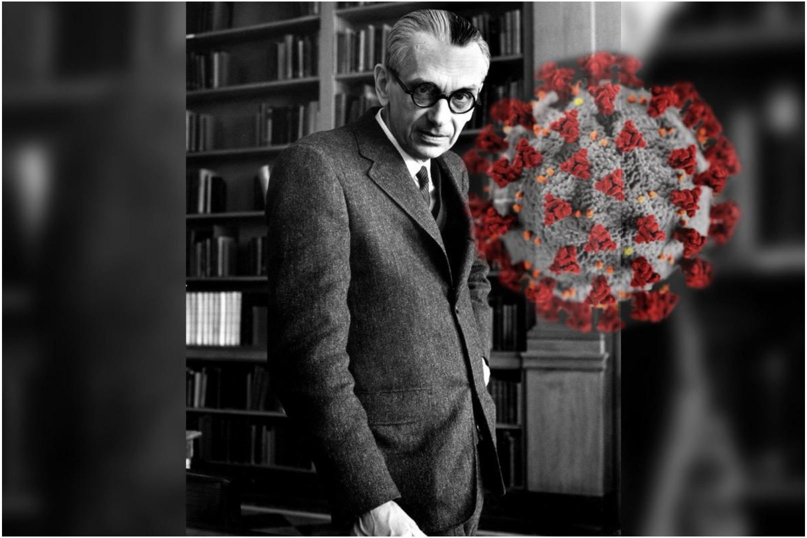


Kurt Gödel

Mathematician, Logician, Philosopher

Anna Frank, Stefan Volz

29.06.2020

University of Applied Sciences Würzburg-Schweinfurt
Faculty of Applied Natural Sciences and Humanities
B.Sc. Industrial Mathematics
English for industrial mathematicians

Structure

1. Who was Kurt Gödel?
2. "We must know. We will know." - Gödel and the Hilbert program
3. Gödel's ontological proof
4. Death and Legacy
5. Interactive Q&A

Who was Kurt Gödel?

Kurt Gödel

Kurt Gödel

- * 28th of April 1906 in Austria-Hungary
- received his doctorate at the age of 24
- regarded as very focused on his work
- fled to the US in the second world war

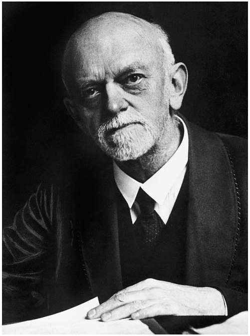
"I don't believe in empirical science. I only believe in a priori truth."

A short anecdote

There's a marvelous story, which this presentation is too short to contain...

”We must know. We will know.”
- Gödel and the Hilbert program

Hilbert's program



David Hilbert

goals:

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of mathematics

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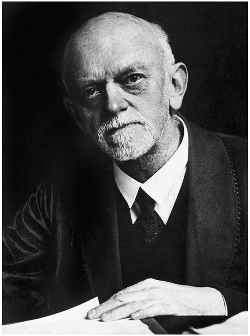

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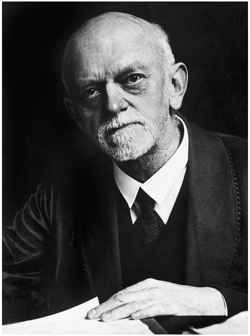

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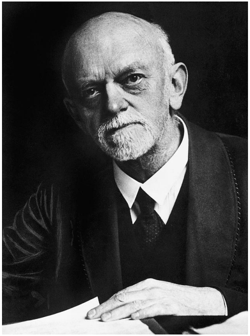

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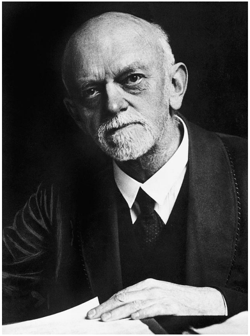

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- $\mathcal{R}$ is a set of inference rules. If w can be infered from x we write $x \vdash w$.

An example

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- The set of axioms is $\mathcal{I} = \{MI\}$.
- Let x, y be metavariables standing in for some strings, then the inference rule are:
  1. Given a string that ends in I you can add an U to the end: $xI \vdash xIU$.
  2. Given a string Mx, where x is some string, you can produce Mxx: $Mx \vdash Mxx$.
  3. Given a string that contains III, you may produce a new string where III is replaced by U: $xIIIy \vdash xUy$.
  4. Given a string that contains UU, you can drop it: $xUUy \vdash xy$.

An example of working inside MIU

Rules of the system:

Axiom		$\vdash MI$
1.	xI	$\vdash xIU$
2.	Mx	$\vdash Mxx$
3.	$xIIIy$	$\vdash xUy$
4.	$xUUy$	$\vdash xy$

Proof of *MUIIU*:

MI	(Axiom)	(1)
MII	(Rule 2)	(2)
MIIII	(Rule 2)	(3)
MIIIIU	(Rule 1)	(4)
MUIU	(Rule 3)	(5)
MUIUUIU	(Rule 2)	(6)
MUIIU	(Rule 4)	(7)

Further examples

- **Propositional logic** - allows to reason about statements: "If it rains, then the street will be wet."

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- **Combinator logic** - allows one to talk about computability without variables: $SII(S(K(SI))(SII))$

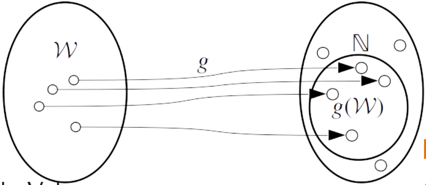
Basic idea:

- encode statements of the formal system as numbers
- use the formal system to reason about those numbers using number-theoretic methods

Gödel numbering

Definition (Gödel numbering)
Let $\mathcal{W}$ be the set of words of a formal system, then we call $g : \mathcal{W} \to \mathbb{N}$ a *Gödel numbering* if:

Informally:
A one to one mapping between words and some numbers.



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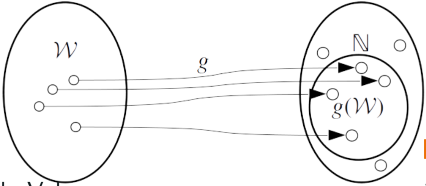

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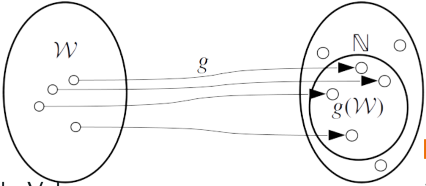

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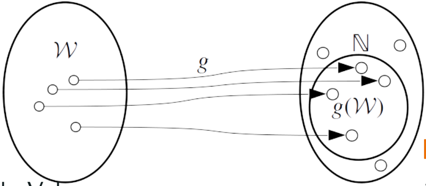

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If g is a Gödel numbering and $w \in \mathcal{W}$, then we call $g(w)$ the *Gödel number of w*.

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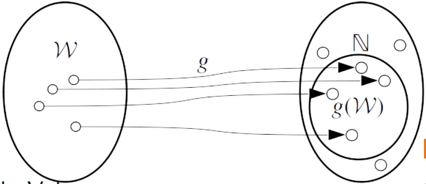

Definition (Gödel encoding)
Let $n \in \mathbb{N}$, $(x_n)_{n\in\{1,\ldots,n\}}$ be an $\mathbb{N}$-valued sequence, and $(p_n)_{n\in\{1,\ldots,n\}}$ be the sequence of the first n primes, then we call $enc(x_1, x_2, \ldots, x_n) := \prod_{i=0}^{n} p_n^{x_n}$ the Gödel encoding of (x_n).

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Example:
$enc(4, 2, 5) = 2^4 \cdot 3^2 \cdot 5^5 = 450,000$

Theorem (First incompleteness theorem)
Any consistent formal system $\mathcal{F}$ within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e. there are statements of the language of $\mathcal{F}$ which can neither be proven nor disproven in $\mathcal{F}$.

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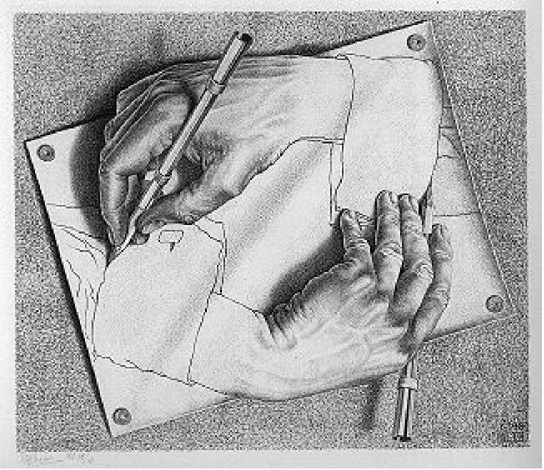
Theorem (Second incompleteness theorem)
Assume $\mathcal{F}$ is a consistent formalized system which contains elementary arithmetic. Then $\mathcal{F} \nvdash \mathit{Cons}(\mathcal{F})$.

The second theorem may be informally stated as:

No system can prove itself consistent.

(not even indirectly)

What's the big deal?

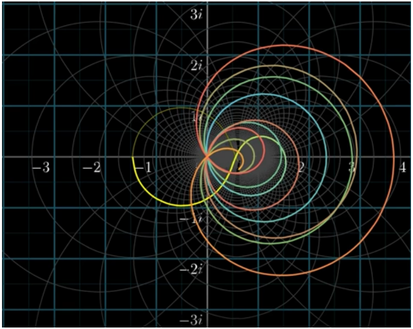


Drawing Hands by M.C. Escher

Instances of Gödel's incompletenes theorems

Examples:

- are mostly quite complicated
- can be found in graph theory, combinatorics (e.g. the Paris-Harrington theorem)



The interested among you can find examples in John Stillwell's *"Roads to Infinity: The Mathematics of Truth and Proof"*.

Gödel's ontological proof

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  - 13. There is a scientific philosophy and theology, which deals with concepts of the highest abstractness and this is also most highly fruitful for science.
  - 14. Religions are for the most part bad - but religion is not.

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Proof

- uses modal logic
- difficult to understand and the notation hard to read
- excerpt of the proof:

$$\text{Ax. } 1 \ (P(\phi) \land \Box\forall x : \phi(x) \Rightarrow \psi(x)) \Rightarrow P(\psi)$$

$$\text{Ax. } 2 \ P(\neg\phi) \iff \neg P(\phi)$$

$$\text{Th. } 1 \ P(\phi) \Rightarrow \Diamond\exists x : \phi(x)$$

$$\vdots$$

$$\text{Df. } 2 \ \phi \text{ ess } x \iff \phi(x) \land \forall\psi : (\psi(x) \Rightarrow \Box\forall y : (\phi(y) \Rightarrow \psi(y)))$$

$$\text{Ax. } 4 \ P(\phi) \Rightarrow \Box P(\phi)$$

$$\vdots$$

$$\text{Th. } 4 \ \Box\exists x : G(x)$$

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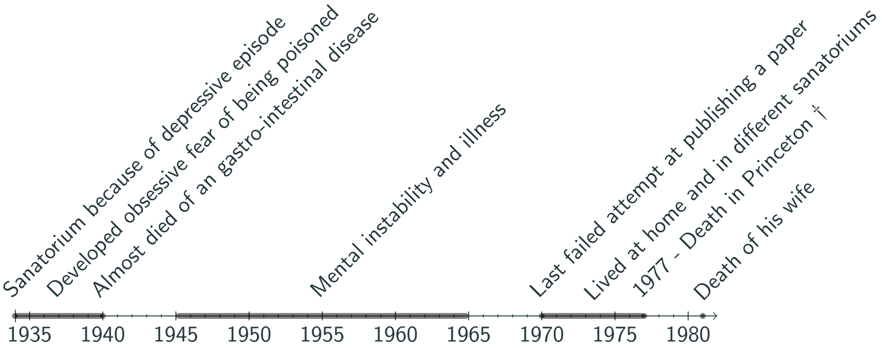
$\Rightarrow$ lots of criticism, but overall ground-breaking and pioneering for all future philosophers and logicians

Death and Legacy

Gödel's last days



Interactive Q&A

After Gödel wrote his ontological proof he couldn't handle the truth and got addicted to cocaine.

True

False

Kurt Gödel had a total of three different citizenships in two different continents.

True

False

Gödel's wife Adele Nimbusky was a stripper by profession.

True

False

In his free time, Gödel was a passionate cineaste and his absolute favourite movie was King Kong.

True

False

It seems clear that the fruitfulness of his ideas will continue to stimulate new work, few mathematicians are granted this kind of immortality.

Rudolf Gödel

Thank’s for your attention

Any questions?