Taller 4 - Métodos computacionales Santiago Vargas & Damián Cortina

Demostración 3, demostrar que

$$\int_a^b f(x)\,dx \cong \int_a^b P_2(x)\,dx = \frac{h}{3}\left(f(a) + 4f(x_m) + f(b)\right)$$

donde

$$f(x) \approx P_2(x) = \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)$$

Por simplicidad, sean $f(a)=h$, $f(x_m)=z$, $f(b)=t$

Así se plantea la siguiente integral

$$\int_a^b \left[\frac{\left((x-b)\left(x-\frac{a+b}{2}\right)\right)h}{(a-b)\left(a-\frac{a+b}{2}\right)} + \frac{((x-a)(x-b))z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} + \frac{\left((x-a)\left(x-\frac{a+b}{2}\right)\right)t}{(b-a)\left(b-\frac{a+b}{2}\right)} \right] dx$$

$$= \frac{h}{\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \int_a^b (x-b)\left(\frac{1}{2}(-a-b)+x\right) dx + \frac{z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \int_a^b (x-a)(x-b)\,dx$$

$$+ \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right) dx$$

$$= \frac{h}{\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \int_a^b \left(\frac{ab}{2} - \frac{ax}{2} + \frac{b^2}{2} - \frac{3bx}{2} + x^2\right) dx + \frac{z}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)}$$

$$\int_a^b (x-a)(x-b)\, dx + \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right) dx$$

$$= \frac{h}{\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \int_a^b x^2\, dx + \left(-\frac{3bh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} - \frac{ah}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right) \int_a^b x\, dx$$

$$+ \left(\frac{b^2h}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} + \frac{abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right) \int_a^b 1\, dx + \frac{z}{\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)} \int_a^b (x-a)(x-b)\, dx$$

$$+ \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)\, x\right) dx$$

Sea $\frac{h}{\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \int_a^b x^2\, dx = \left.\frac{hx^3}{3\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right|_a^b = \frac{b^3h - a^3h}{3\left(\frac{1}{2}(-a-b)+a\right)(a-b)}$

$$= -\frac{2h(a^2+ab+b^2)}{3(a-b)}$$

$$= -\frac{2h(a^2+ab+b^2)}{3(a-b)} + \left[-\frac{3bh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} - \frac{ah}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \right] \int_a^b x\,dx$$

$$+ \left[\frac{b^2h}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} + \frac{abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \right] \int_a^b 1\,dx + \frac{z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \int_a^b (x-a)(x-b)\,dx$$

$$+ \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right)dx$$

Siendo $\int x\,dx = \frac{x^2}{2}$, $\frac{-3bh-ah}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)} \int_a^b x\,dx = \frac{1}{2}x^2\left[\frac{-3bh-ah}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right]\Bigg|_a^b$

$$= \left[\frac{1}{2}b^2\left[\frac{-3bh-ah}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right]\right] - \frac{1}{2}a^2\left[\frac{-3bh-ah}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right] = \frac{h(a+b)(a+3b)}{2(a-b)}$$

$$= \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \left[\frac{b^2h+abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right] \int_a^b 1\,dx +$$

$$\frac{z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \int_a^b (x-a)(x-b)\,dx + \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right)dx$$

Siendo $\int 1\,dx = x$, $\left[\frac{b^2h + abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right]\int_a^b 1\,dx = x\left[\frac{b^2h + abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right]\Bigg|_a^b$

$$= b\left[\frac{b^2h + abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right] - a\left[\frac{b^2h + abh}{2\left(\frac{1}{2}(-a-b)+a\right)(a-b)}\right] = \frac{bh(a+b)}{a-b}$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \frac{z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\int_a^b (x-a)(x-b)\,dx +$$

$$\frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right)dx$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \frac{z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\int_a^b x^2\,dx +$$

$$\left[\frac{-az-bz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]\int_a^b x\,dx + \frac{abz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\int_a^b 1\,dx +$$

$$\frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right)dx$$

Siendo $\int x^2\,dx = \frac{x^3}{3}$, $\frac{z}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\int_a^b x^2\,dx = \left.\frac{x^3 z}{3\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right|_a^b$

$$= \frac{b^3 z - a^3 z}{3\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} = \frac{4z\,(a^2+ab+b^2)}{3(a-b)}$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \frac{4z\,(a^2+ab+b^2)}{3(a-b)} +$$

$$\left[\frac{-az-bz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]\int_a^b x\,dx + \frac{abz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\int_a^b 1\,dx + \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right)dx$$

Siendo $\int x\,dx = \frac{x^2}{2}$, $\left[\frac{-az-bz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]\int_a^b x\,dx = \frac{1}{2}x^2\left[\frac{-az-bz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]$

$$= \left[\frac{1}{2}b^2\left[\frac{-az-bz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]\right] - \left[\frac{1}{2}a^2\left[\frac{-az-bz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]\right] = -\frac{2z(a+b)^2}{a-b}$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} - \frac{2z(a+b)^2}{a-b} + \frac{4z(a^2+ab+b^2)}{3(a-b)}$$

$$+ \frac{abz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \int_a^b 1\,dx + \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right) dx$$

Siendo $\int 1\,dx = x$, $\frac{abz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} \int_a^b 1\,dx = \left[\frac{abzx}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}\right]\Bigg|_a^b$

$$= \frac{abbz - aabz}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)} = \frac{4abz}{a-b}$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \frac{4abz}{a-b} - \frac{2z(a+b)^2}{a-b} + \frac{4z(a^2+ab+b^2)}{3(a-b)}$$

$$+ \frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b (x-a)\left(\frac{1}{2}(-a-b)+x\right) dx$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \frac{4abz}{a-b} - \frac{2z(a+b)^2}{a-b} + \frac{4z(a^2+ab+b^2)}{3(a-b)}$$

$$\frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)} \int_a^b \left(\frac{a^2}{2} + \frac{ab}{2} - \frac{3ax}{2} - \frac{bx}{2} + x^2\right) dx$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} + \frac{4abz}{a-b} - \frac{2z(a+b)^2}{a-b} + \frac{4z(a^2+ab+b^2)}{3(a-b)}$$

$$+\left[\frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b x^2\,dx + \left[\frac{-3at-bt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b x\,dx + \left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b 1\,dx$$

Siendo $\int x^2\,dx = \frac{x^3}{3}$; $\left[\frac{t}{(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b x^2\,dx = \frac{t\,x^3}{3(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\Bigg|_a^b$

$$= \frac{b^3t-a^3t}{3(b-a)\left(\frac{1}{2}(-a-b)+b\right)} = -\frac{2t(a^2+ab+b^2)}{3(a-b)}$$

$$= -\frac{bh(a+b)}{a-b} + \frac{h(a+b)(a+3b)}{2(a-b)} - \frac{2h(a^2+ab+b^2)}{3(a-b)} - \frac{2t(a^2+ab+b^2)}{3(a-b)} + \frac{4abz}{a-b} - \frac{2z(a+b)^2}{a-b}$$

$$+ \frac{4z(a^2+ab+b^2)}{3(a-b)} + \left[\frac{-3at-bt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b x\,dx + \left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b 1\,dx$$

siendo $\int x\,dx = \frac{x^2}{2}$, $\left[\frac{-3at-bt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b x\,dx = \frac{1}{2}x^2\left[\frac{-3at-bt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\Bigg|_a^b$

$$= \left[\frac{1}{2}b^2\left[\frac{-3at-bt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]-\left[\frac{1}{2}a^2\left[\frac{-3at-bt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\right]\right] = \frac{t(a+b)(3a+b)}{2(a-b)}$$

$$= -\frac{bh(a+b)}{a-b}+\frac{h(a+b)(a+3b)}{2(a-b)}-\frac{2h(a^2+ab+b^2)}{3(a-b)}+\frac{t(a+b)(3a+b)}{2(a-b)}-\frac{2t(a^2+ab+b^2)}{3(a-b)}$$

$$+\frac{4abz}{a-b}-\frac{2z(a+b)^2}{a-b}+\frac{4z(a^2+ab+b^2)}{3(a-b)}+\left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b 1\,dx$$

Siendo $\int 1\,dx = x$, $\left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\int_a^b 1\,dx = x\left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]\Bigg|_a^b$

$$= b\left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right]-a\left[\frac{a^2t+abt}{2(b-a)\left(\frac{1}{2}(-a-b)+b\right)}\right] = -\frac{at(a+b)}{a-b}$$

$$= \frac{-2h(a^2+ab+b^2)}{3(a-b)} - \frac{2t(a^2+ab+b^2)}{3(a-b)} + \frac{4z(a^2+ab+b^2)}{3(a-b)} - \frac{bh(a+b)}{a-b} + \frac{h(a+3b)(a+b)}{2(a-b)}$$

$$+ \frac{t(3a+b)(a+b)}{2(a-b)} - \frac{at(a+b)}{a-b} - \frac{2z(a+b)^2}{a-b} + \frac{4abz}{a-b}$$

Siendo,

$$-\frac{2h(a^2+ab+b^2)}{3(a-b)} - \frac{2t(a^2+ab+b^2)}{3(a-b)} + \frac{4z(a^2+ab+b^2)}{3(a-b)} = \frac{(a+b)^2}{3(a-b)}\left(-2h-2t+4z\right)$$

$$-\frac{bh(a+b)}{a-b} - \frac{at(a+b)}{a-b} = -\frac{(a+b)}{a-b}\left(bh+at\right)$$

$$\frac{h(a+3b)(a+b)}{2(a-b)} + \frac{t(3a+b)(a+b)}{2(a-b)} = \frac{1(a+b)}{2(a-b)}\left(h(a+3b)+t(3a+b)\right)$$

$$-\frac{2z(a+b)^2}{a-b} + \frac{4abz}{a-b} = \frac{1}{(a-b)}\left(-2z(a+b)^2+4abz\right)$$

$$= \frac{1}{(a-b)}\left[\frac{(a+b)^2}{3}\left(-2h-2t+4z\right) - (a+b)\left(bh+at\right) + \left(-2z(a+b)^2+4abz\right) + \frac{(a+b)}{2}\left(h(a+3b)+t(3a+b)\right)\right]$$

$$= -\frac{1}{6}(a-b)(h+t+4z)$$

$= \frac{b-a}{6}(h+t+4z)$

Recordando que $h = f(a)$, $t = f(b)$ y $z = f(x_m)$,

$= \frac{b-a}{6}(f(a)+f(b)+4f(x_m))$

Sea $\frac{(b-a)}{2} = h$, se obtiene

$= \frac{h}{3}(f(a)+f(b)+4f(x_m))$