8a)

Sea

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

Se reemplaza h por 2h

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \dots$$

$$= f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \dots$$

Usando (1)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \quad (1)$$

Se obtiene:

$$4f(x+h) = 4f(x) + 4hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{4h^3}{3!}f'''(x) + \dots$$

Restando (1') a (2)

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \dots \quad (2)$$

Se obtiene

$$f(x+2h) - 4f(x+h) = -3f(x) - 2hf'(x) + \frac{4h^3}{3!}f'''(x) + \dots$$

$$f(x+2h) - 4f(x+h) = -3f(x) - 2hf'(x) + O(h^3)$$

$$f(x+2h) - 4f(x+h) + 3f(x) = -2hf'(x) + O(h^3)$$

$$-3f(x) + 4f(x+h) - f(x+2h) = 2hf'(x) + O(h^3)$$

$$= \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} = f'(x) + O(h^2)$$

$\blacksquare$

8b)

Sea $f'(x) = \frac{1}{2h}\left(-3f(x) + 4f(x+h) - f(x+2h)\right) = g(x)$

$$f''(x) = \frac{1}{2h}\left[-3g(x) + 4g(x+h) - g(x+2h)\right]$$

$$g(x) = \frac{1}{2h}\left[\overset{9}{+\cancel{3}}f(x) \overset{-12}{\cancel{-4}}f(x+h) \overset{+3}{+}f(x+2h)\right]$$

$$g(x+h) = \frac{1}{2h}\left[\overset{12}{-\cancel{3}}f(x+h) + \overset{16}{\cancel{4}}f(x+2h) - \overset{4}{\cancel{f}}(x+3h)\right]$$

$$g(x+2h) = \frac{1}{2h}\left[+3f(x+2h) - 4f(x+3h) + f(x+4h)\right]$$

$$\frac{1}{4h^2}\left[\underline{9f(x)} - 12\underline{f(x+h)} + 3\underline{f(x+2h)} - 12\underline{f(x+h)} + \underline{16f(x+2h)} - 4\underline{f(x+3h)} + 3\underline{f(x+2h)} - 4\underline{f(x+3h)} + f(x+4h)\right]$$

$$\frac{1}{4h^2}\left[9f(x) - 24f(x+h) + 22f(x+2h) - 8f(x+3h) + f(x+4h)\right]$$

Así $f''(x_0) =$

$$\frac{1}{4h^2}\left[9f(x_0) - 24f(x_1) + 22f(x_2) - 8f(x_3) + f(x_4)\right]$$