Foundations of Abstract Interpretation

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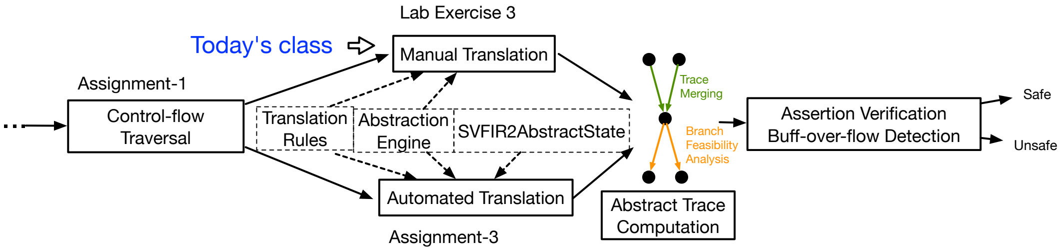
Today’s class



Outline

- The contents of this lecture
  - An Introduction to Abstract Interpretation: What and Why
  - Abstract Interpretation vs Symbolic Execution
  - Definitions: Abstract domains, Abstract State and Abstract Trace.
  - Step-by-Step Motivating Examples.
  - Widening and Narrowing to Improve Analysis Speed and Precision
  - Analysis Order on Control-Flow Graph and Weak Topological Order.

Abstract Interpretation

Abstract interpretation or Abstract Execution [Cousot & Cousot, POPL'77][1], a general framework for static analysis, aims to **soundly approximate** the potential concrete values program variables may take during runtime, based on monotonic functions over ordered sets, particularly **lattices.**

Abstract Interpretation: Levels of Abstractions

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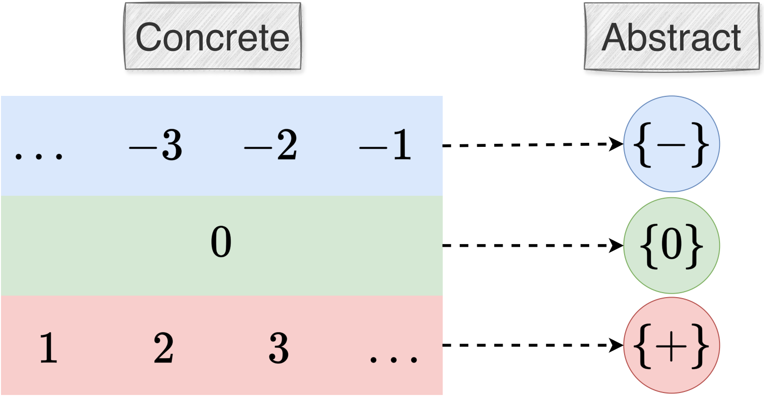

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What is the abstract value?

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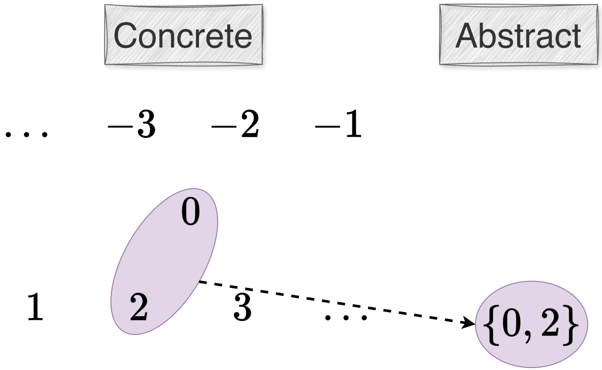

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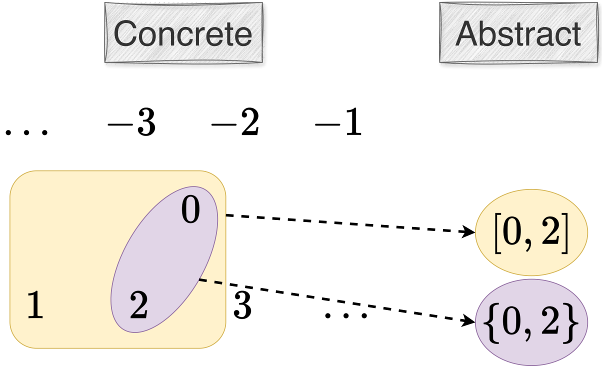

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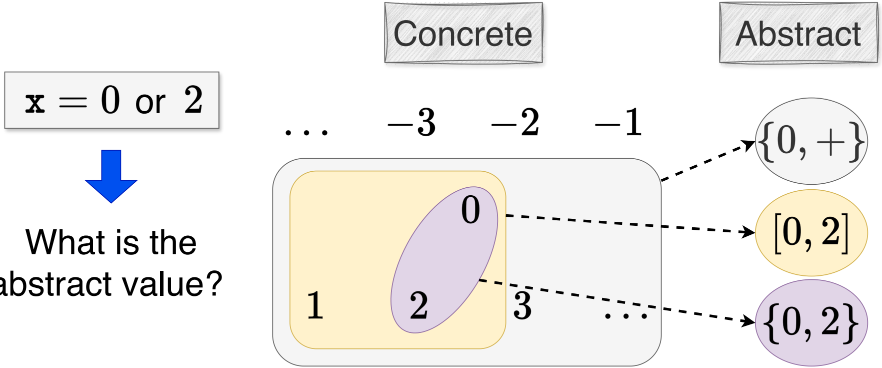

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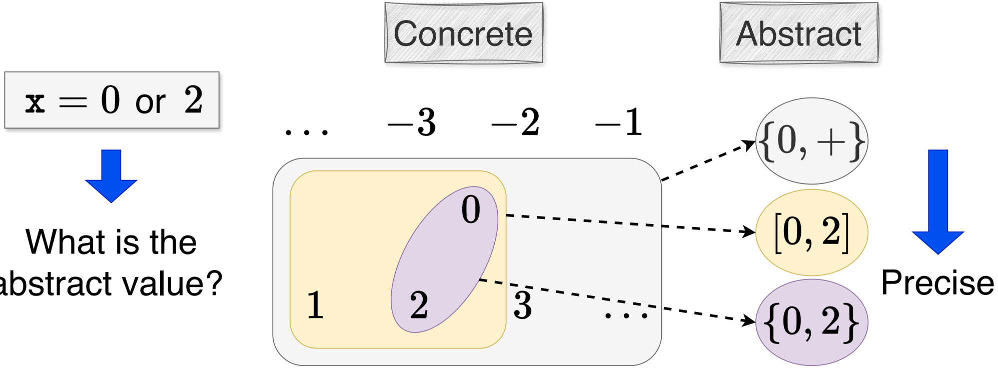

Abstract Interpretation: Applications

- **Program Optimization**: allows compilers to make safe assumptions about a program's behavior, leading to more efficient code generation.
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- **Hardware Design and Analysis:** used to verify that hardware designs meet certain specifications and to optimize the designs for better performance or lower power consumption.
  - **Analyzing Hardware Circuits:** By creating an abstract model of the circuit, it can predict how the circuit will behave under various input conditions.

Abstract Interpretation: Applications

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- **Hardware Design and Analysis:** used to verify that hardware designs meet certain specifications and to optimize the designs for better performance or lower power consumption.
  - **Analyzing Hardware Circuits:** By creating an abstract model of the circuit, it can predict how the circuit will behave under various input conditions.
- **Static Code Analysis (This Subject)**: provides a systematic approach to approximate program behavior without actual execution.
  - **Code Security Analysis:** crucial for early detection of bugs (e.g., assertion errors and buffer overflows), reducing debugging time and enhancing code reliability.

Abstract Interpretation: Tools

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- **Astrée** is used to analyze and ensure the safety of software in modern aircraft, such as the Airbus A380.
- **Polyspace** is highly valued in the automotive and aerospace industries for ensuring software compliance with safety standards such as ISO 26262 for automotive software.
- **Ikos** is specialized in detecting run-time errors and numerical computation issues, making it ideal for space and aeronautics software.
- **SPARK** is used in the aerospace industry for writing and verifying safety-critical avionics software.
- **Infer** is a static analysis tool developed by Facebook to identify bugs in mobile and web applications.
- Other tools: **Frama-C, Julia Static Analyzer, BAP, Soot** and many more . . .

Abstract Interpretation vs. Symbolic Execution

Soundness

- **Abstract interpretation** aims for sound results. It can conservatively approximate all possible execution paths and runtime behaviors.

Abstract Interpretation vs. Symbolic Execution

Soundness

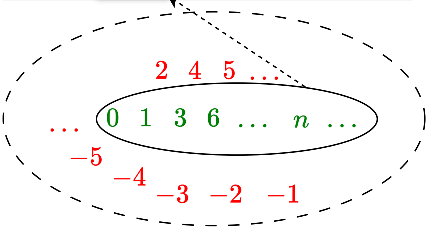
- **Abstract interpretation** aims for sound results. It can conservatively approximate all possible execution paths and runtime behaviors.
- **Symbolic execution** can be unsound. It precisely explores individual yet feasible paths, facing a "path explosion" problem in large programs, and may result in under-approximation of program behaviors.

Abstract Interpretation vs. Symbolic Execution

An Example: Soundness

```
void analyzeThis(int x) {
    int sum = 0;
    for (int i = 0; i < x; ++i) {
        sum += i;
    }
}
```

sum = ?



Abstract Interpretation vs. Symbolic Execution

An Example: Soundness

```
ℓ1 void analyzeThis(int x) {
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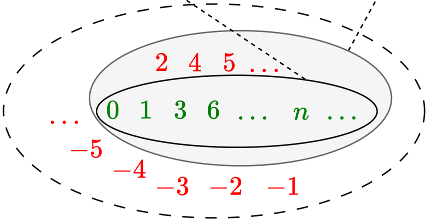
sum = ?



Abstract Interpretation

$\{0, +\}$

Sound (include all non-negative numbers)
imprecise (may include infeasible numbers: 2, 4, 5, ...)

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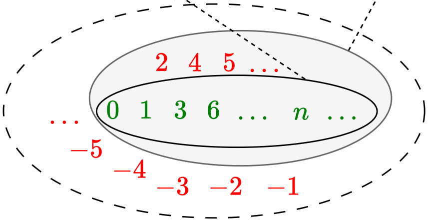

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Symbolic Execution

Path	Answer
$\ell_1 \to \ell_2 \to \ell_3 \to \ell_6$	0

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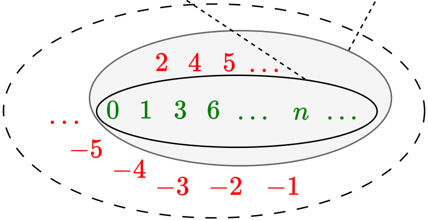

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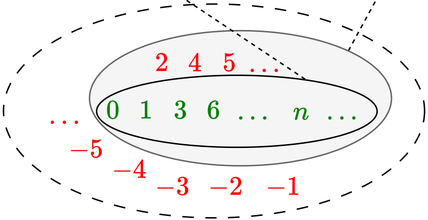

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$\ell_1 \to \ell_2 \to \ell_3 \to \ell_4 \to \ell_5 \to \ell_3 \to \ell_4 \to \ell_5 \to \ell_6$	3

Abstract Interpretation vs. Symbolic Execution

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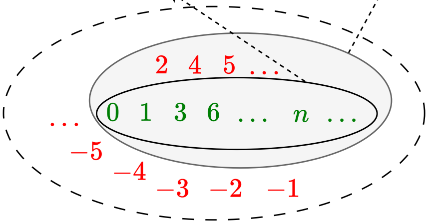

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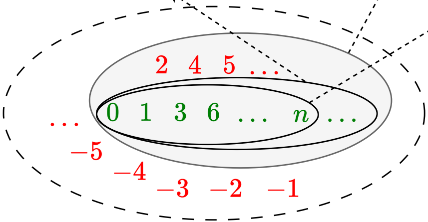

Abstract Interpretation $\{0, +\}$

Sound (include all non-negative numbers)
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Symbolic Execution

Precise (only include feasible numbers: 0, 1, 3, 6, ...)
unsound (cannot cover all possible numbers)

Path	Answer
$\ell_1 \to \ell_2 \to \ell_3 \to \ell_6$	0
$\ell_1 \to \ell_2 \to \ell_3 \to \ell_4 \to \ell_5 \to \ell_6$	1
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Importance of Soundness

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- **Reliability:** Ensures comprehensive coverage of all possible program states, reducing unforeseen behavior in production.
- **Quality Assurance:** Crucial for critical systems where failure can have serious consequences, ensuring software behaves as intended.
- **Confidence in Maintenance:** Provides a safety net for code changes, reducing the risk of introducing new bugs.

Abstract Interpretation vs. Symbolic Execution

Termination

- **Abstract interpretation** is typically guaranteed to terminate within a finite step. Uses an abstracted, and hence more manageable, version of the state space to represent the infinite number of runtime states and paths.

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Termination

- **Abstract interpretation** is typically guaranteed to terminate within a finite step. Uses an abstracted, and hence more manageable, version of the state space to represent the infinite number of runtime states and paths.
- **Symbolic execution** may struggle with termination in complex or large-scale programs. The need to explore numerous paths in detail, especially in programs with loops and recursive calls, can lead to non-termination or impractical analysis times.

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- **Deterministic:** Ensures consistent outcomes and predictable resource use for the same input.
- **Efficiency:** Reduces computational load by using abstracted state spaces, speeding up the analysis process.
- **Coverage:** ensure that all parts of the code are analyzed, avoiding missed sections and ensuring thorough coverage for detecting issues.

Abstract Interpretation: A Code Example

```
if(cond)
   x=1;
else
   x=3;
x = ?
```

Approximation!

x	$\{+\}$
x	$[1,3]$
x	$\{1,3\}$

Abstract Interpretation: A Code Example

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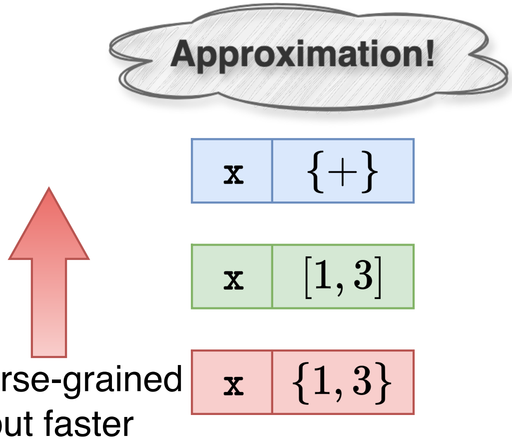

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Fine-grained but slower

Concrete Domain and Abstract Domain: Formal Definition

Concrete Domain

- $\mathbb{S}$ denotes the set of concrete values that a program variable can have.
  - E.g., $\mathbb{S} = \mathbb{Z}$ represents the concrete values that an integer variable can have.
- A **concrete domain** $\mathbb{C}$ is the *powerset* of $\mathbb{S}$, denoted as $\mathbb{C} = \mathcal{P}(\mathbb{S})$.
  - E.g. The *powerset integer domain* is a concrete domain for integer variables.

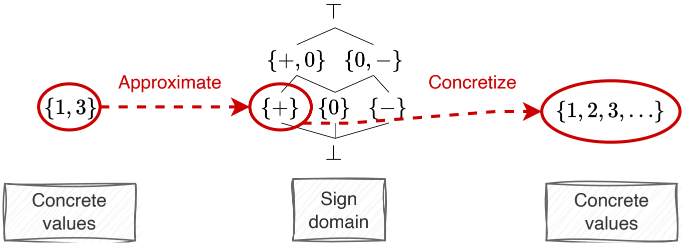
Abstract Domain

- An **abstract domain** $\mathbb{A}$ contains *abstract values* approximating a set of concrete values.
- An **abstract domain** is typically equiped with a lattice $\langle \sqsubseteq, \sqcap, \sqcup, \perp, \top \rangle$, a set of abstract values following a **partial order** $\sqsubseteq$.
  - $\sqsubseteq$ is a binary relation on $\mathbb{A}$ (e.g., $\sqsubseteq$ is the subset relation ($\subseteq$) on a power set).
  - $\sqcap$ and $\sqcup$ are the meet and join operations, and $\perp$ and $\top$ are unique least and greatest elements of $\mathbb{A}$.

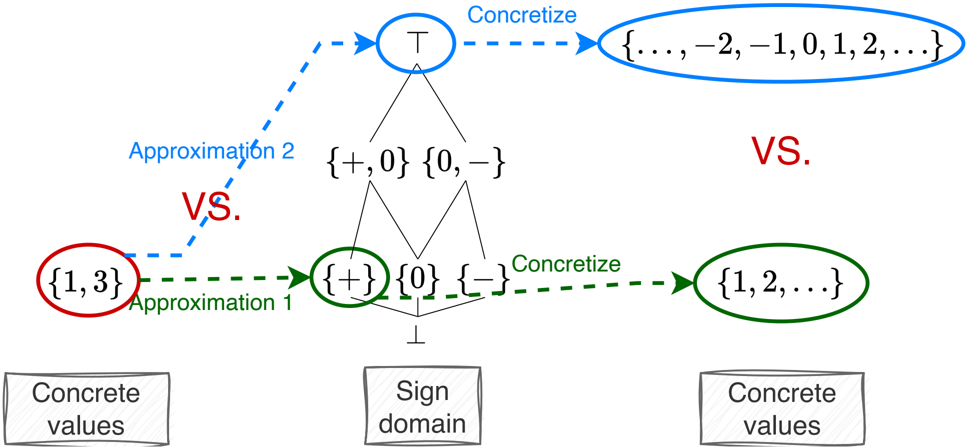
An Example: Abstract Sign Domain

The sign domain is an abstract domain that approximates a set of concrete values with their signs.

- Defined as $\mathbb{A} = \mathcal{P}(\{-, 0, +\})$.
- **Partial order**: $a \sqsubseteq b \Leftrightarrow a \subseteq b$.
  - E.g., $\{+\} \sqsubseteq \{0, +\} \Leftrightarrow \{+\} \subseteq \{0, +\}$.
- **Approximation**: concrete value set $\{1, 3\}$ is over-approximated as $\{+\}$, because after concretization, it is restored as $\{x \in \mathbb{Z} | x > 0\}$, a super set of $\{1, 3\}$.



An Example, the Best Abstraction using Sign Domain



Approximation 1 (more precise than Approximation 2) is the best abstraction!

Galois Connection

When each concrete value has a unique best abstraction, the correspondence is a **Galois connection**, which is a two-way connections between abstract domain and concrete domain using abstraction function and concretization function.

- Abstraction function $\alpha : \mathbb{C} \rightarrow \mathbb{A}$ maps a set of concrete values to its abstract ones;
- Concretization function $\gamma : \mathbb{A} \rightarrow \mathbb{C}$ maps a set of abstract values to concrete ones.

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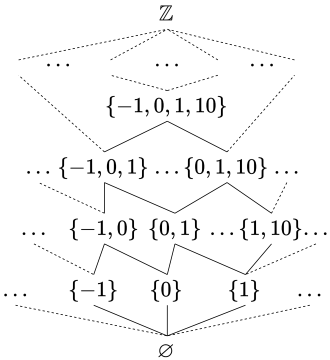
Example: Abstraction/concretization functions on sign domain

$$\gamma_{Sign}(\top) = \mathbb{Z}$$
$$\gamma_{Sign}(\{-\}) = \{x \mid x < 0\}$$
$$\gamma_{Sign}(\{+\}) = \{x \mid x > 0\}$$
$$\ldots$$

$$\alpha_{Sign}(c) = \{+\} \text{ if } c \in \mathbb{Z}_{>0}$$
$$\alpha_{Sign}(c) = \{-\} \text{ if } c \in \mathbb{Z}_{<0}$$
$$\alpha_{Sign}(c) = \{+, 0\} \text{ if } c \in \mathbb{Z}_{\geq 0}$$
$$\ldots$$

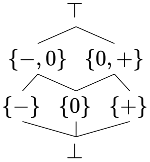
Galois Connection of Sign Domain



(a) Powerset integer domain

$$
\begin{aligned}
\gamma_{Sign}(\top) &= \mathbb{Z}\\
\gamma_{Sign}(\{-\}) &= \{x \mid x < 0\}\\
\gamma_{Sign}(\{+\}) &= \{x \mid x > 0\}\\
\gamma_{Sign}(\{-,0\}) &= \{x \mid x \le 0\}\\
\gamma_{Sign}(\{+,0\}) &= \{x \mid x \ge 0\}\\
\gamma_{Sign}(\{\bot\}) &= \varnothing
\end{aligned}
$$

$$
\alpha_{Sign}(c) = \begin{cases}
\{+\}\ \mathit{if}\, c \in \mathbb{Z}_{>0}\\
\{-\}\ \mathit{if}\, c \in \mathbb{Z}_{<0}\\
\{+,0\}\ \mathit{if}\, c \in \mathbb{Z}_{\ge 0}\\
\{-,0\}\ \mathit{if}\, c \in \mathbb{Z}_{\le 0}\\
\bot\ \ \mathit{if}\, c = \varnothing\\
\top\ \mathit{otherwise}
\end{cases}
$$

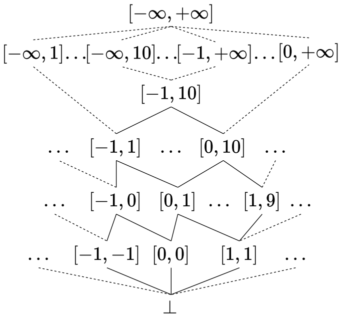


(b) Sign domain

Interval Domain

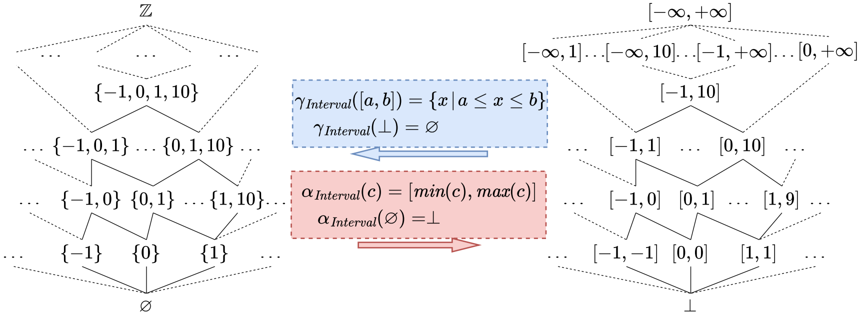
The interval domain is an abstract domain that represents a set of integers that fall between two given endpoints.

- Defined as $\mathbb{A}_{interval} = \{[a, b] \mid a, b \in \mathbb{R} \cup \{-\infty, +\infty\}\} \cup \{\perp\}$.
- Partial order: $[a_1, b_1] \sqsubseteq [a_2, b_2] \Leftrightarrow a_2 \leq a_1 \wedge b_1 \leq b_2$.
  - E.g., $[0, 0], [0, 1] \in \mathbb{A}_{interval}$, satisfying $[0, 0] \sqsubseteq [0, 1]$.



(b) Interval domain

Galois Connection between $\mathbb{C}$ and $\mathbb{A}_{interval}$



(a) Powerset integer domain

(b) Interval domain

Figure: Powerset integer domain $\mathbb{C}$ and its abstraction as the interval domain $\mathbb{A}_{interval}$.

Abstract State and Abstract Trace

- An **abstract state** is defined as a map $AS : \mathcal{V} \rightarrow \mathbb{A}$ associating program variables $\mathcal{V}$ with an abstract value in $\mathbb{A}$, approximating the runtime states of program variables.

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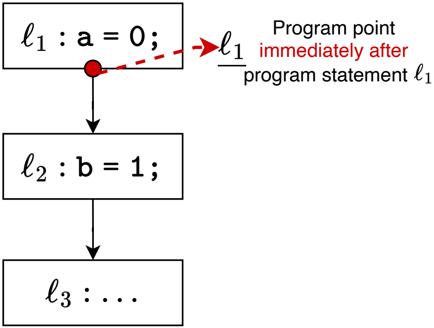
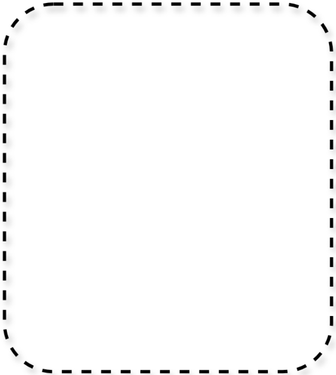
- An **abstract state** is defined as a map $AS : \mathcal{V} \rightarrow \mathbb{A}$ associating program variables $\mathcal{V}$ with an abstract value in $\mathbb{A}$, approximating the runtime states of program variables.
- An **abstract trace** $\sigma \in \mathbb{L} \times \mathcal{V} \rightarrow \mathbb{A}$ represents a list of abstract states before ($\overline{\ell}$) and after ($\underline{\ell}$) each program statement ℓ *following the execution order*.

	Notation	Domain
Abstract trace	σ	$\mathbb{L} \times \mathcal{V} \rightarrow \mathbb{A}_{Interval}$
Abstract state at program point $L \in \mathbb{L}$	σ_L	$\mathcal{V} \rightarrow \mathbb{A}_{Interval}$
Abstract value of x at program point $L \in \mathbb{L}$	$\sigma_L(x)$	$\mathbb{A}_{Interval}$

Abstract Trace : A Simple Example



Control Flow Graph



Abstract Trace σ

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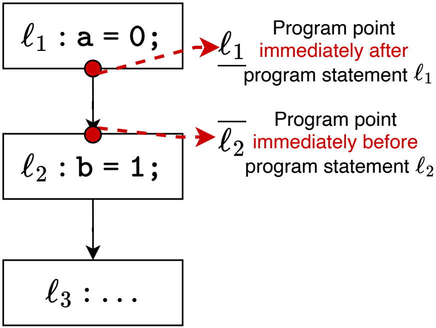

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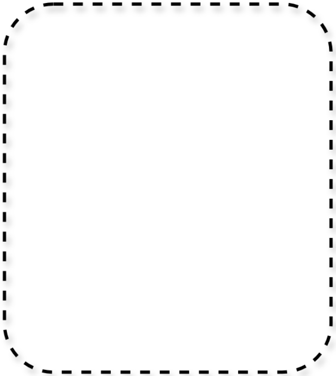

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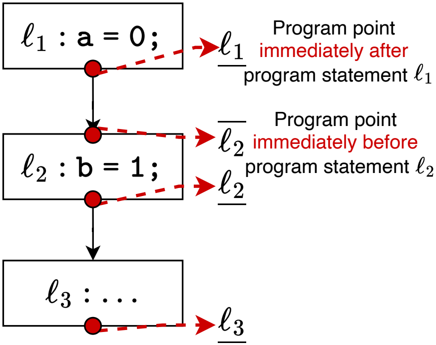

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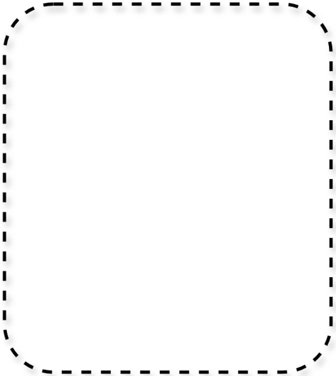

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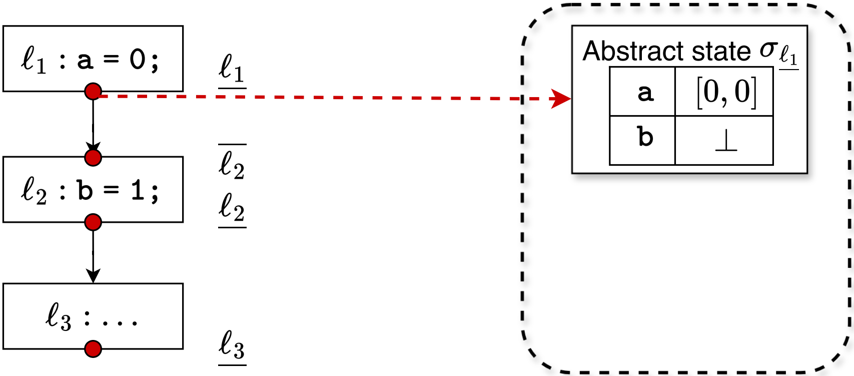
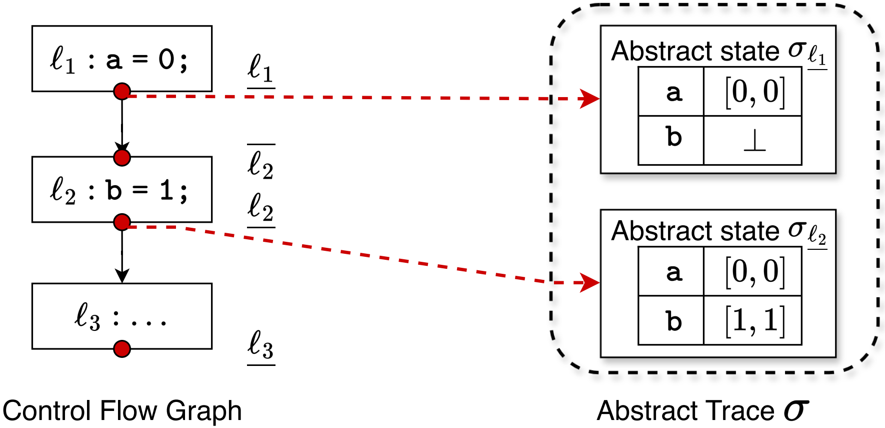


Control Flow Graph

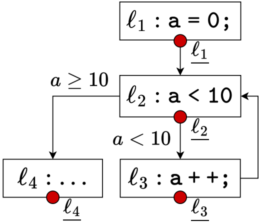
Abstract Trace σ

Abstract Trace : A Simple Example



Abstract Trace: A Loop Example

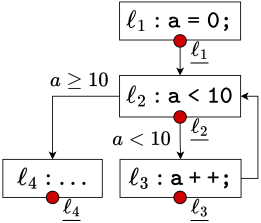
Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

Abstract Trace: A Loop Example

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												

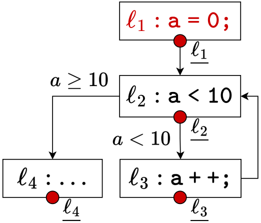


Control Flow Graph

What is the abstract state after analyzing each statement?

Abstract Trace: A Loop Example

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



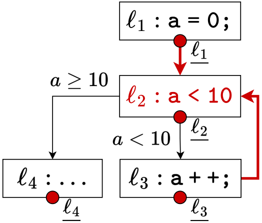
Control Flow Graph

What is the abstract state after analyzing each statement?

$\sigma_{\underline{\ell_1}}(a) := F_1() = [0, 0]$ $\quad$ F_1 is a **transfer function**

Abstract Trace: A Loop Example

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

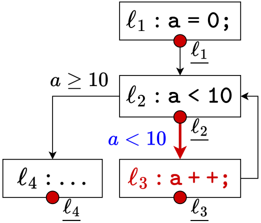
What is the abstract state after analyzing each statement?

$\sigma_{\underline{\ell_1}}(a) := F_1() = [0,0]$ $\quad$ F_1 is a **transfer function**

$\sigma_{\underline{\ell_2}}(a) := F_2(\sigma_{\underline{\ell_1}}, \sigma_{\underline{\ell_3}}) = \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$

Abstract Trace: A Loop Example

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

What is the abstract state after analyzing each statement?

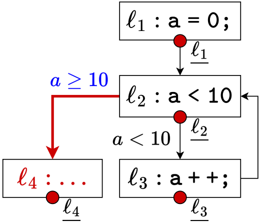
$\sigma_{\underline{\ell_1}}(a) := F_1() = [0, 0]$ $\quad F_1$ is a **transfer function**

$\sigma_{\underline{\ell_2}}(a) := F_2(\sigma_{\underline{\ell_1}}, \sigma_{\underline{\ell_3}}) = \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$

$\sigma_{\underline{\ell_3}}(a) := F_3(\sigma_{\underline{\ell_2}}) = ([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1, 1]$

Abstract Trace: A Loop Example

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

What is the abstract state after analyzing each statement?

$\sigma_{\underline{\ell_1}}(a) := F_1() = [0,0]$ $\quad F_1$ is a **transfer function**

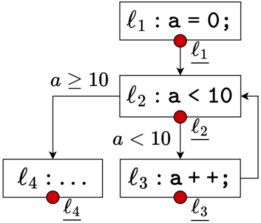
$$\sigma_{\underline{\ell_2}}(a) := F_2(\sigma_{\underline{\ell_1}}, \sigma_{\underline{\ell_3}}) = \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

$$\sigma_{\underline{\ell_3}}(a) := F_3(\sigma_{\underline{\ell_2}}) = ([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$$

$$\sigma_{\underline{\ell_4}}(a) := F_4(\sigma_{\underline{\ell_2}}) = [10, \infty] \sqcap \sigma_{\underline{\ell_2}}(a)$$

Abstract Trace: A Loop Example

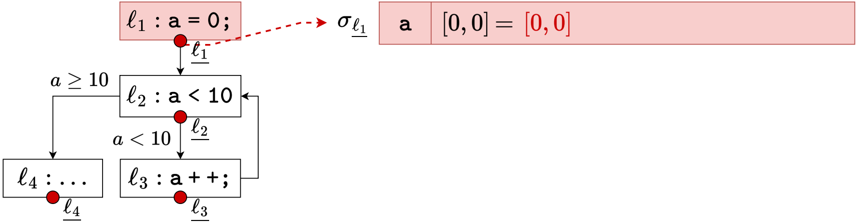
Abstract trace	Init											
$\sigma_{\underline{\ell_1}}(a)$	$\perp$											
$\sigma_{\underline{\ell_2}}(a)$	$\perp$											
$\sigma_{\underline{\ell_3}}(a)$	$\perp$											
$\sigma_{\underline{\ell_4}}(a)$	$\perp$											



Control Flow Graph

Abstract Trace: A Loop Example

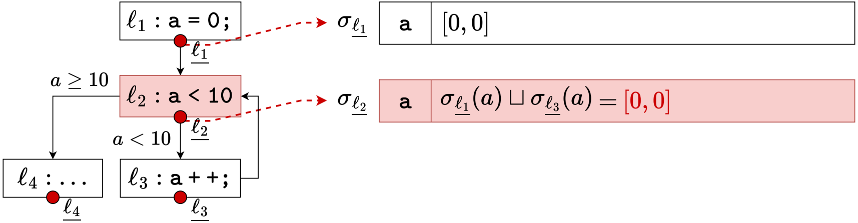
Abstract trace	Init	After analyzing ℓ_1										
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$										
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$										
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$										
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$										



Control Flow Graph

Abstract Trace: A Loop Example

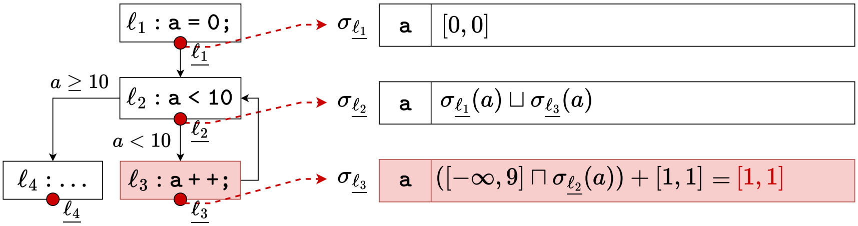
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter									
			After ℓ_2									
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0, 0]$	$[0, 0]$									
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0, 0]$									
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$									
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$									



Control Flow Graph

Abstract Trace: A Loop Example

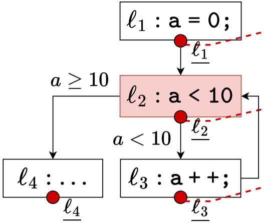
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter										
			After ℓ_2	After ℓ_3									
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$									
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$									
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$									
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$									



Control Flow Graph

Abstract Trace: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter							
			After ℓ_2	After ℓ_3	After ℓ_2							
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$							
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,1]$							
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$							
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$							

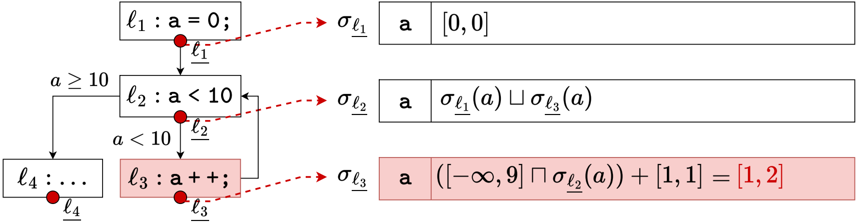


$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a) = [0,1]$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Control Flow Graph

Abstract Trace: A Loop Example

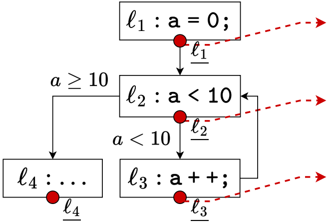
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter							
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3						
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$						
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$						
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$						
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$						



Control Flow Graph

Abstract Trace: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3			
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	...	$[0,0]$	$[0,0]$			
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	...	$[0,10]$	$[0,10]$			
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$	...	$[1,9]$	$[1,10]$			
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$			



$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

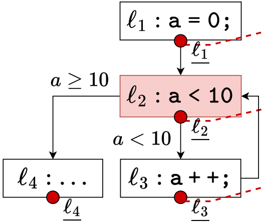
$\sigma_{\underline{\ell_2}}$ | a | $\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$

$\sigma_{\underline{\ell_3}}$ | a | $([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Control Flow Graph

Abstract Trace: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		. . .	11^{th} loop iter		12^{nd} loop iter	
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	. . .	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	. . .	$[0,10]$	$[0,10]$	$[0,10]$	
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,2]$	. . .	$[1,9]$	$[1,10]$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	. . .	$\bot$	$\bot$	$\bot$	

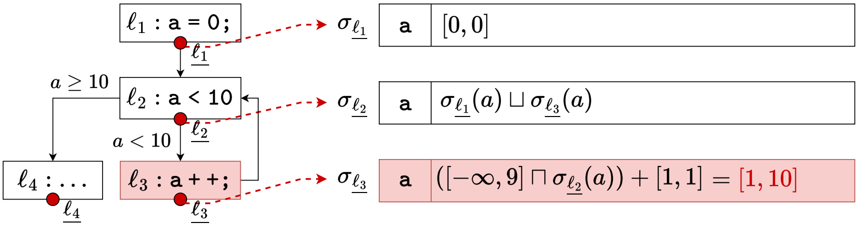


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a) = [0,10]$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Abstract Trace: A Loop Example

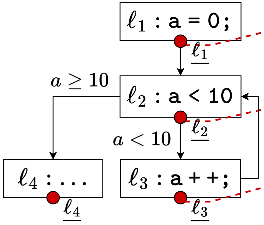
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter		12^{nd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	...	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	...	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$	...	$[1,9]$	$[1,10]$	$[1,10]$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	



Control Flow Graph

Abstract Trace: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		. . .	11^{th} loop iter		12^{nd} loop iter	
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	. . .	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	. . .	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$	. . .	$[1,9]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	. . .	$\perp$	$\perp$	$\perp$	$\perp$



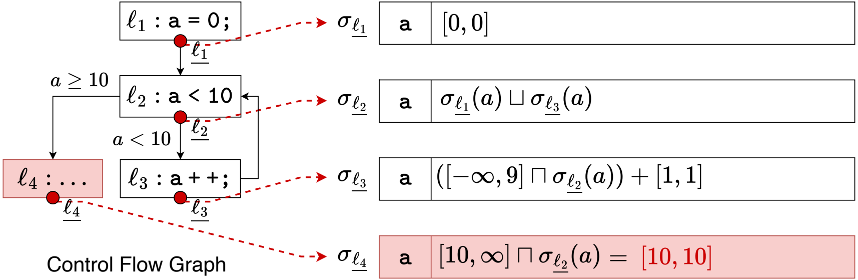
Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Fixpoint is reached!
(Abstract trace after loop round 11 = Abstract trace after loop round 12)

Abstract Trace: A Loop Example

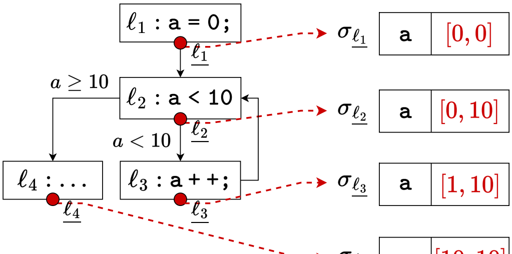
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter		12^{nd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	...	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	...	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$	...	$[1,9]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



Control Flow Graph

Abstract Trace: A Loop Example

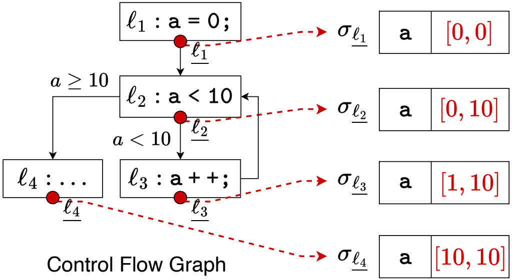
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		. . .	11^{th} loop iter		12^{nd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	. . .	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	. . .	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,2]$	. . .	$[1,9]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	. . .	$\bot$	$\bot$	$\bot$	$\bot$	$[10,10]$



Control Flow Graph

Abstract Trace: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1th loop iter		2nd loop iter			11th loop iter		12nd loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	...	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	[0, 0]	[0, 0]	[0, 1]	[0, 1]	...	[0, 10]	[0, 10]	[0, 10]	[0, 10]	[0, 10]
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	[1, 1]	[1, 1]	[1, 2]	...	[1, 9]	[1, 10]	[1, 10]	[1, 10]	[1, 10]
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	[10, 10]



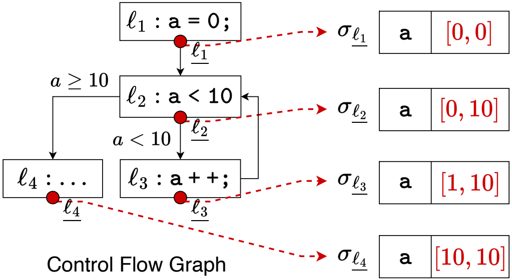
Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,1] \Rightarrow [0,2] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]}$$

12 iterations while analyzing the loop

Abstract Trace: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter			11^{th} loop iter		12^{nd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	...	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	[0, 0]	[0, 0]	[0, 1]	[0, 1]	...	[0, 10]	[0, 10]	[0, 10]	[0, 10]	[0, 10]
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	[1, 1]	[1, 1]	[1, 2]	...	[1, 9]	[1, 10]	[1, 10]	[1, 10]	[1, 10]
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	[10, 10]



Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,1] \Rightarrow [0,2] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]}$$

12 iterations while analyzing the loop

What if $a < 10000$?
More iterations!

Widening: An accelerating approach

Widening technique can accelerate the fixpoint computation of $\sigma_{\underline{\ell_2}}(a)$.

Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$

$$[0,0] \Rightarrow [0,1] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]$$

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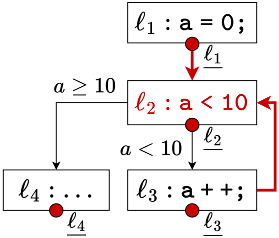
$$[0,0] \Rightarrow [0,1] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]$$

$$[0,0] \xrightarrow[\text{aggressively update } \sigma_{\underline{\ell_2}}(a)]{\text{Widening}} [0,+\infty]$$

Widening: An Accelerating Approach

Widening at the k^{th} iteration in the loop for analyzing ℓ_2 to update $\sigma_{\underline{\ell_2}}$.



Control Flow Graph

$$\sigma_{\underline{\ell_2}}(a) := \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

Apply widening operator ∇

$$\sigma^{k}_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma^{k-1}_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

$\sigma^{k}_{\underline{\ell_2}}$ denotes the value of $\sigma_{\underline{\ell_2}}$ after the k^{th} analysis of ℓ_2

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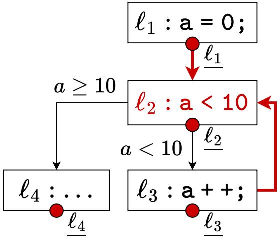
Widening at the k^{th} iteration in the loop for analyzing ℓ_2 to update $\sigma_{\underline{\ell_2}}$.



Control Flow Graph

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$\sigma^k_{\underline{\ell_2}}$ denotes the value of $\sigma_{\underline{\ell_2}}$ after the k^{th} analysis of ℓ_2

What is a **widening operator**?

Widening operator

The widening operator ($\nabla : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$) is formally defined on a poset ($\mathbb{A}, \sqsubseteq$). ∇ on interval domain could be defined as:

$$[\ell_1, h_1] \nabla [\ell_2, h_2] = [\ell_3, h_3]$$

Widening operator

The widening operator ($\nabla : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$) is formally defined on a poset ($\mathbb{A}, \sqsubseteq$). ∇ on interval domain could be defined as:

$$[\ell_1, h_1]\nabla[\ell_2, h_2] = [\ell_3, h_3]$$

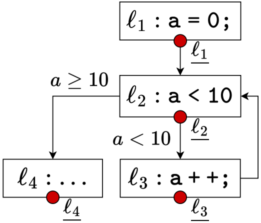
where

$$l_3 = \begin{cases} -\infty & l_2 < l_1 \\ l_1 & l_2 \geq l_1 \end{cases}, h_3 = \begin{cases} +\infty & h_2 > h_1 \\ h_1 & h_2 \leq h_1 \end{cases}$$

As a concrete example, $[0, 0]\nabla[0, 1] = [0, +\infty]$.

Widening: A Loop Example

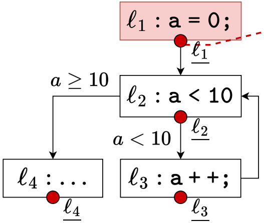
Abstract trace	Init								
$\sigma_{\underline{\ell_1}}(a)$	$\bot$								
$\sigma_{\underline{\ell_2}}(a)$	$\bot$								
$\sigma_{\underline{\ell_3}}(a)$	$\bot$								
$\sigma_{\underline{\ell_4}}(a)$	$\bot$								



Control Flow Graph

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1							
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	[0, 0]							
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$							
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$							
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$							

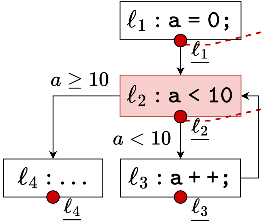


$\sigma_{\underline{\ell_1}}$ | a | $[0, 0] = [0, \infty]$

Control Flow Graph

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter						
			After ℓ_2						
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$						
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$						
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$						
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$						



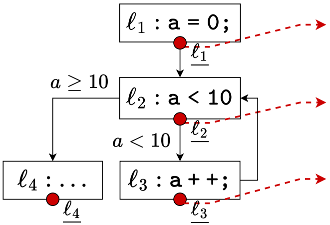
Control Flow Graph

$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

$\sigma_{\underline{\ell_2}}$ | a | $(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)) = [0,0]$

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter						
			After ℓ_2	After ℓ_3					
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$					
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$					
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$					
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$					



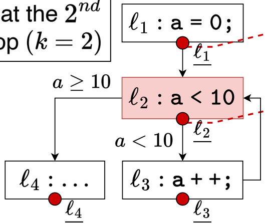
$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a))$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1] = [1,1]$

Control Flow Graph

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2				
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$				
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0, 0]$	$[0, 0]$	$[0, \infty]$				
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1, 1]$	$[1, 1]$				
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$				

Start widening at the 2^{nd} iteration of loop ($k = 2$)

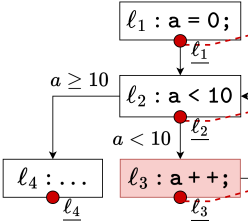


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0, 0]$
$\sigma^2_{\underline{\ell_2}}$	a	$\sigma^1_{\underline{\ell_2}}(a) \nabla (\sigma^1_{\underline{\ell_1}}(a) \sqcup \sigma^1_{\underline{\ell_3}}(a)) = [0, \infty]$

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3			
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$			
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$			
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,10]$			
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$			

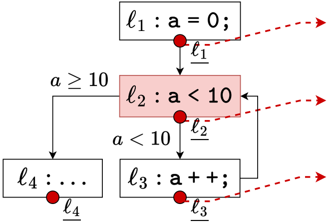


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^2_{\underline{\ell_2}}$	a	$\sigma^1_{\underline{\ell_2}}(a) \nabla (\sigma^1_{\underline{\ell_1}}(a) \sqcup \sigma^1_{\underline{\ell_3}}(a))$
$\sigma^2_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^2_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter After ℓ_2	After ℓ_3	2^{nd} loop iter After ℓ_2	After ℓ_3	3^{rd} loop iter After ℓ_2		
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$		
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$		
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$		
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$		

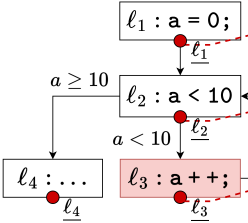


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^3_{\underline{\ell_2}}$	a	$\sigma^2_{\underline{\ell_2}}(a) \,\nabla\, (\sigma^2_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a)) = [0,\infty]$
$\sigma^2_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^2_{\underline{\ell_2}}(a)) + [1,1]$

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	



$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

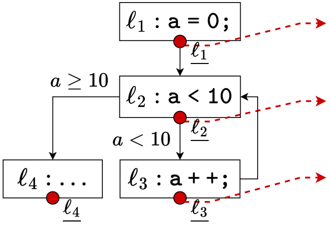
$\sigma^3_{\underline{\ell_2}}$ | a | $\sigma^2_{\underline{\ell_2}}(a) \,\nabla\, (\sigma^2_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a))$

$\sigma^3_{\underline{\ell_3}}$ | a | $([-\infty,9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Control Flow Graph

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	



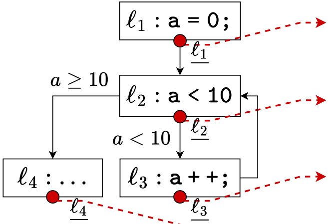
Control Flow Graph

σ_{ℓ_1}	a	$[0,0]$
$\sigma^3_{\ell_2}$	a	$\sigma^2_{\ell_2}(a)\ \nabla\ (\sigma^2_{\ell_1}(a) \sqcup \sigma^2_{\ell_3}(a))$
$\sigma^3_{\ell_3}$	a	$([-\infty,9] \sqcap \sigma^3_{\ell_2}(a)) + [1,1]$

Fixpoint is reached!
(Abstract trace after loop round 2 = Abstract trace after loop round 3)

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$

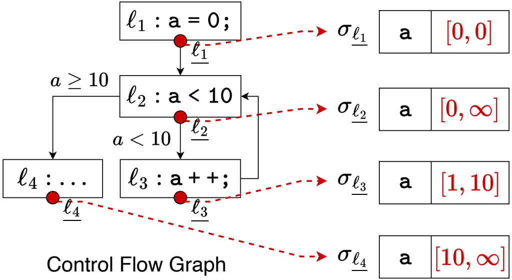


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^3_{\underline{\ell_2}}$	a	$\sigma^2_{\underline{\ell_2}}(a) \nabla (\sigma^2_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a))$
$\sigma^3_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1]$
$\sigma_{\underline{\ell_4}}$	a	$[10,\infty] \sqcap \sigma_{\underline{\ell_2}}(a) = [10,\infty]$

Widening: A Loop Example

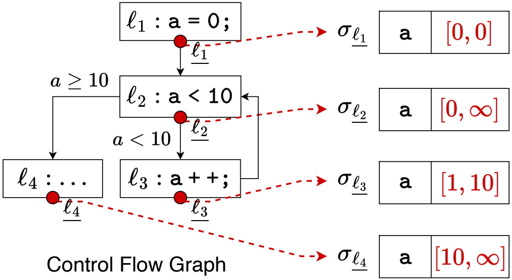
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter After ℓ_2	After ℓ_3	2^{nd} loop iter After ℓ_2	After ℓ_3	3^{rd} loop iter After ℓ_2	After ℓ_3	After analyzing ℓ_4
$\sigma_{\ell_1}(a)$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	$[1, 1]$	$[1, 1]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10, \infty]$



Control Flow Graph

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$



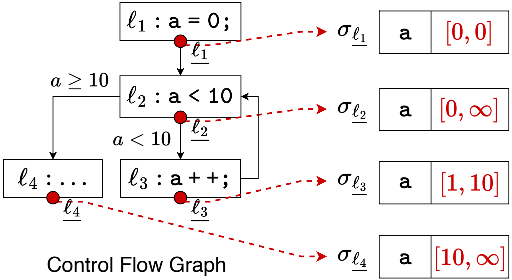
Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty]}$$

3 iterations while analyzing the loop

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$



Control Flow Graph

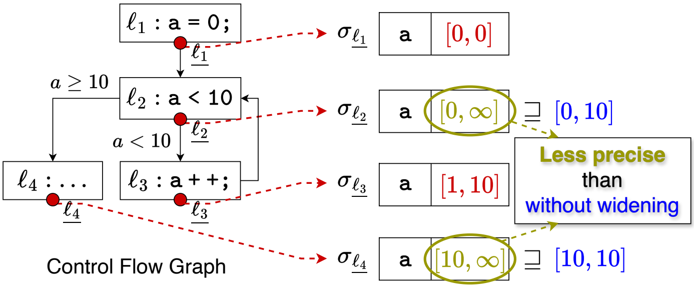
$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty]}$$

3 iterations while analyzing the loop

Faster than naive fixpoint computation (12 iterations)!

Widening: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\ell_2}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\ell_3}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\ell_4}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$[10,\infty]$



$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty]}$$

3 iterations while analyzing the loop

Faster than without widening (12 iterations)!

Narrowing: A precision improving approach

Narrowing technique can eliminate the precision loss after a widening operation (e.g., by improving imprecise $\sigma_{\underline{\ell_2}}$ and $\sigma_{\underline{\ell_4}}$).

Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$

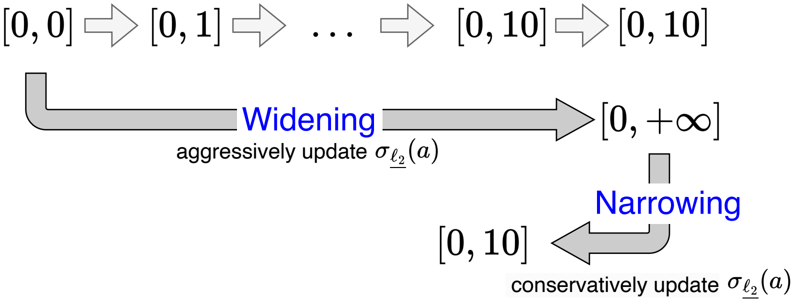
$$[0,0] \Rightarrow [0,1] \Rightarrow \dots \Rightarrow [0,10] \Rightarrow [0,10]$$

$$\xrightarrow[\text{aggressively update } \sigma_{\underline{\ell_2}}(a)]{\text{Widening}} [0,+\infty]$$

Narrowing: A precision improving approach

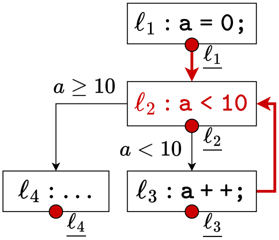
Narrowing technique can eliminate the precision loss after a widening operation (e.g., by improving imprecise $\sigma_{\underline{\ell_2}}$ and $\sigma_{\underline{\ell_4}}$).

Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$



Narrowing: A precision improving approach

After the widening reaches a fixpoint at the k^{th} iteration when analyzing the loop, we start performing narrowing at the $(k+1)^{th}$ to update $\sigma_{\underline{\ell_2}}$.



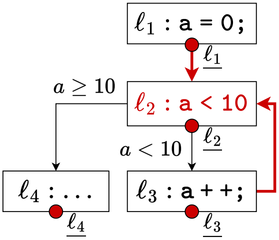
Control Flow Graph

Widening reaches a fixpoint

$$\sigma^k_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma^{k-1}_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

Narrowing: A precision improving approach

After the widening reaches a fixpoint at the k^{th} iteration when analyzing the loop, we start performing narrowing at the $(k+1)^{th}$ to update $\sigma_{\underline{\ell_2}}$.



Control Flow Graph

Widening reaches a fixpoint

$$\sigma^k_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma^{k-1}_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

Apply narrowing operator Δ instead

Start performing narrowing

$$\sigma^{k+1}_{\underline{\ell_2}}(a) := \sigma^k_{\underline{\ell_2}}(a) \Delta (\sigma^k_{\underline{\ell_1}}(a) \sqcup \sigma^k_{\underline{\ell_3}}(a))$$

What is a **narrowing operator**?

Narrowing operator

The narrowing operator ($\Delta : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$) is formally defined on a poset ($\mathbb{A}, \sqsubseteq$). Δ on interval domain could be defined as:

$$[l_1, h_1] \Delta [l_2, h_2] = [l_3, h_3]$$

Narrowing operator

The narrowing operator ($\Delta : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$) is formally defined on a poset ($\mathbb{A}, \sqsubseteq$). Δ on interval domain could be defined as:

$$[l_1, h_1]\Delta[l_2, h_2] = [l_3, h_3]$$

where

$$l_3 = \begin{cases} l_2 & l_1 \equiv -\infty \\ l_1 & l_1 \not\equiv -\infty \end{cases}, h_3 = \begin{cases} h_2 & h_1 \equiv \infty \\ h_1 & h_1 \not\equiv \infty \end{cases}$$

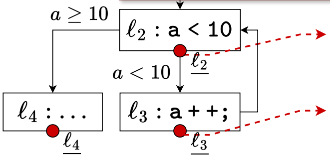
As a concrete example, $[0, \infty]\Delta[0, 10] = [0, 10]$.

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter						
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3					
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$					
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$					
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$					
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$					

Widening reaches a fixpoint at the 3^{rd} loop iteration

$), 0]$



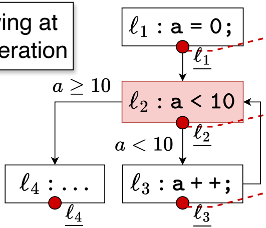
$\sigma^3_{\underline{\ell_2}}$	a	$\sigma^2_{\underline{\ell_2}}(a) \,\nabla\, (\sigma^2_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a))$
$\sigma^3_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1]$

Control Flow Graph

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2				
$\sigma_{\ell_1}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$				
$\sigma_{\ell_2}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$				
$\sigma_{\ell_3}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$				
$\sigma_{\ell_4}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$				

Start narrowing at the 4^{th} loop iteration

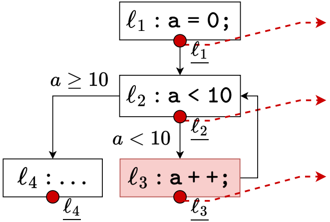


σ_{ℓ_1}	a	$[0,0]$
$\sigma^4_{\ell_2}$	a	$\sigma^3_{\ell_2}(a)\ \Delta(\sigma^3_{\ell_1}(a) \sqcup \sigma^3_{\ell_3}(a)) = [0,10]$
$\sigma^3_{\ell_3}$	a	$([-\infty,9] \sqcap \sigma^3_{\ell_2}(a)) + [1,1]$

Control Flow Graph

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3			
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$			
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$			
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$			
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$			

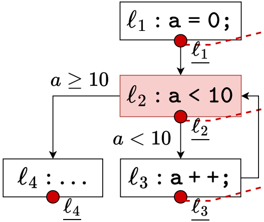


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^4_{\underline{\ell_2}}$	a	$\sigma^3_{\underline{\ell_2}}(a)\,\Delta(\sigma^3_{\underline{\ell_1}}(a) \sqcup \sigma^3_{\underline{\ell_3}}(a))$
$\sigma^4_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^4_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2		
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$		
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$\color{blue}{[0,10]}$	$\color{red}{[0,10]}$		
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$		
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$		



Control Flow Graph

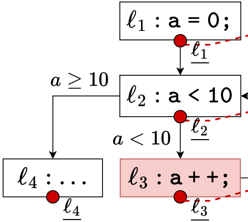
$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

$\sigma^5_{\underline{\ell_2}}$ | a | $\color{blue}{\sigma^4_{\underline{\ell_2}}(a)}\ \Delta(\sigma^4_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a)) = \color{red}{[0,10]}$

$\sigma^4_{\underline{\ell_3}}$ | a | $([-\infty,9] \sqcap \sigma^4_{\underline{\ell_2}}(a)) + [1,1]$

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	

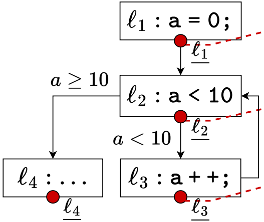


$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^5_{\underline{\ell_2}}$	a	$\sigma^4_{\underline{\ell_2}}(a)\,\Delta(\sigma^4_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a))$
$\sigma^5_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^5_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Control Flow Graph

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1, 1]$	$[1, 1]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	



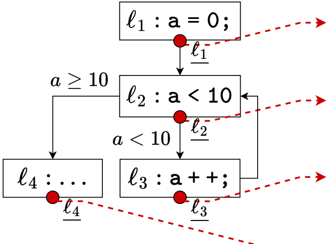
Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0, 0]$
$\sigma^5_{\underline{\ell_2}}$	a	$\sigma^4_{\underline{\ell_2}}(a)\,\Delta(\sigma^4_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a))$
$\sigma^5_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma^5_{\underline{\ell_2}}(a)) + [1, 1]$

Fixpoint is reached!
(Abstract trace after loop round 4 = Abstract trace after loop round 5)

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



Control Flow Graph

σ_{ℓ_1} — a: $[0,0]$

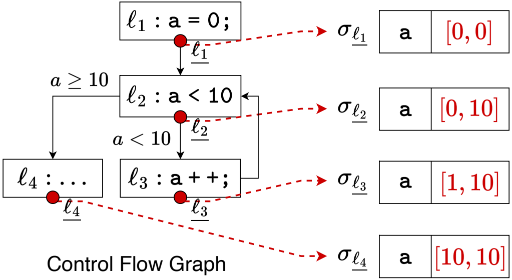
$\sigma^5_{\ell_2}$ — a: $\sigma^4_{\ell_2}(a)\,\Delta\,(\sigma^4_{\ell_1}(a) \sqcup \sigma^4_{\ell_3}(a))$

$\sigma^5_{\ell_3}$ — a: $([-\infty,9] \sqcap \sigma^5_{\ell_2}(a)) + [1,1]$

σ_{ℓ_4} — a: $[10,\infty] \sqcap \sigma_{\ell_2}(a) = [10,10]$

Narrowing: A Loop Example

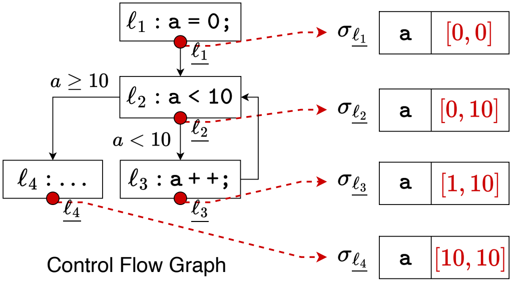
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



Control Flow Graph

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	[0, 0]	[0, 0]	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	[0, 10]	[0, 10]	[0, 10]	[0, 10]	[0, 10]
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	[1, 1]	[1, 1]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	[10, 10]



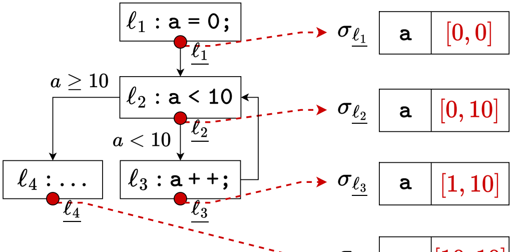
Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty] \Rightarrow [0,10] \Rightarrow [0,10]}$$

5 iterations while analyzing the loop

Narrowing: A Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty] \Rightarrow [0,10] \Rightarrow [0,10]}$$

5 iterations while analyzing the loop

Faster!

Precise!

Analysis Order of Nodes on Control-Flow Graph

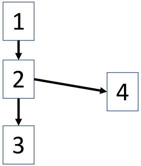
Topological Order

? How to analyze a program **free of loop**?

✓ Analyze each node **once** adhering to the **topological order** on the acyclic control-flow graph of the program.

Definition of topological order: For a direct acyclic graph $G(V, E)$, if a sequence of V satisfies: $a \rightarrow b \in E \Rightarrow a$ is before b in the sequence, then this sequence is a topological order of G.

Example of topological order:



acyclic graph G

Valid/invalid topological order

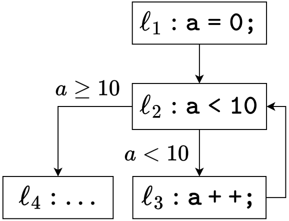
Analysis Order of Nodes on Control-Flow Graph

Weak Topological Order

? How to analyze a program **containing loops**?

✔ We can analyze a program containing loops adhering to the **weak topological order** (WTO) of its control flow graph.

What is the weak topological order?



Control Flow Graph

Analysis Order of Nodes on Control-Flow Graph

Weak Topological Order

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✔ We can analyze a program containing loops adhering to the **weak topological order** (WTO) of its control flow graph.

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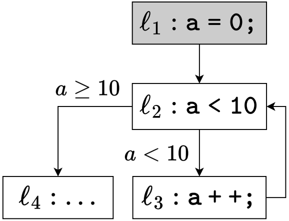
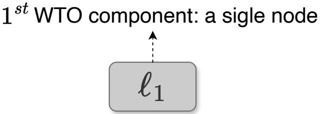


Control Flow Graph



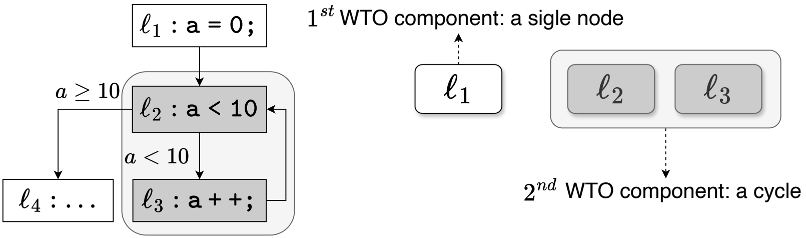
Analysis Order of Nodes on Control-Flow Graph

Weak Topological Order

? How to analyze a program **containing loops**?

✔ We can analyze a program containing loops adhering to the **weak topological order** (WTO) of its control flow graph.

What is the weak topological order?



Control Flow Graph

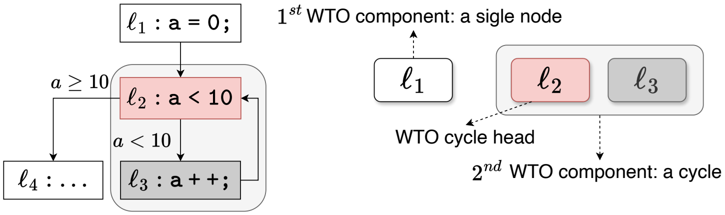
Analysis Order of Nodes on Control-Flow Graph

Weak Topological Order

? How to analyze a program **containing loops**?

✔ We can analyze a program containing loops adhering to the **weak topological order** (WTO) of its control flow graph.

What is the weak topological order?



Control Flow Graph

Analysis Order of Nodes on Control-Flow Graph

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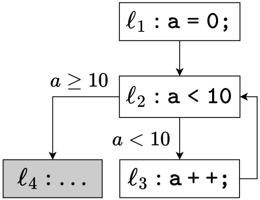
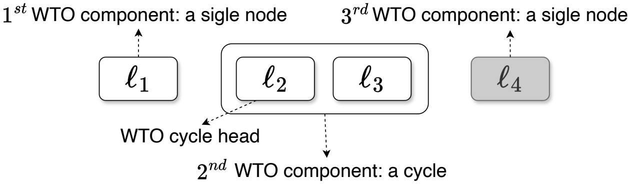


Control Flow Graph



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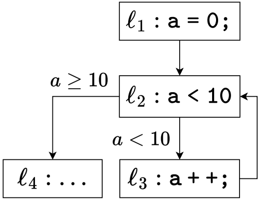
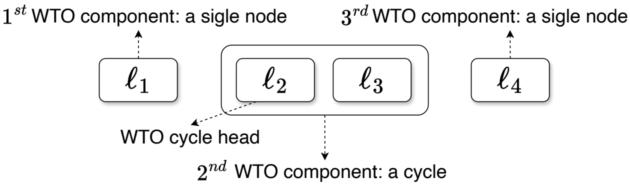


Control Flow Graph



Analyze each node following the WTO

Analyze ℓ_1 ⇨ Repeat: analyze ℓ_2 and ℓ_3 ⇨ Analyze ℓ_4

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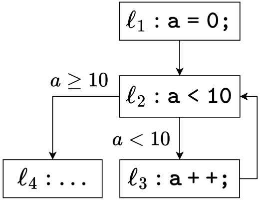
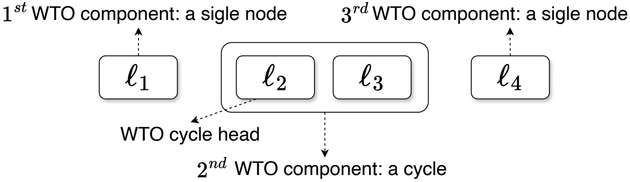


Control Flow Graph



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Perform widening when analyzing the cycle heads (i.e., ℓ_2)