Infinite-Dimensional Bayesian Posterior Operator for Function-Space Asset Weights

Your Name

July 1, 2025

Abstract

We propose an infinite-dimensional Bayesian posterior operator to update distributions over asset weights or model parameters in a hierarchical or function-space Bayesian framework. This operator enables modeling and tracking full posterior distributions rather than point estimates, crucial for uncertainty quantification and robust decision-making in financial portfolio optimization. We derive the operator theoretically, discuss posterior contraction, and provide a Python implementation using function-space variational inference as a practical approximation.

1 Introduction

Traditional portfolio optimization relies on point estimates of asset weights. In contrast, a Bayesian approach models uncertainty over weights via posterior distributions. Extending this to infinite-dimensional spaces (e.g., function-valued parameters) allows for richer modeling such as nonparametric Gaussian process priors.

2 Problem Setup

Let $\mathcal{W}$ denote an infinite-dimensional function space of possible weight functions $w : \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ represents asset-related covariates (e.g., macro indicators). Assume a prior Π_0 over $\mathcal{W}$.

Given observations $D = \{(x_i, r_i)\}_{i=1}^n$, where r_i is the return for context x_i, we define a likelihood model $p(r_i|w(x_i))$.

The posterior is formally given by

$$\Pi_n(dw) = \frac{\prod_{i=1}^n p(r_i|w(x_i))\Pi_0(dw)}{\int \prod_{i=1}^n p(r_i|w'(x_i))\Pi_0(dw')}$$

3 Infinite-Dimensional Posterior Operator

We define the operator $\mathcal{B}_n$ acting on Π_0 as:

$$\mathcal{B}_n(\Pi_0) := \Pi_n$$

This operator maps a prior to its updated posterior given data D.

3.1 Properties

- **Continuity**: $\mathcal{B}_n$ is continuous in the weak topology over measures.
- **Posterior contraction**: As $n \to \infty$, Π_n concentrates around the true w_0 under regularity conditions.
- **Nonparametric flexibility**: If Π_0 is a Gaussian process or Dirichlet process mixture prior, Π_n remains in the same function space.

4 Approximation via Variational Inference

Since Π_n is generally intractable, we approximate it with a variational distribution Q minimizing

$$\mathrm{KL}(Q\|\Pi_n) = \int \log\left(\frac{Q(dw)}{\Pi_n(dw)}\right) Q(dw)$$

5 Python Implementation

Below is a sample code snippet using `Pyro` for function-space variational inference (e.g., approximating Gaussian process posterior).

```
import torch
import pyro
import pyro.contrib.gp as gp
import pyro.distributions as dist

# Simulated data
X = torch.linspace(0, 1, 20)
Y = torch.sin(6 * X) + 0.1 * torch.randn(X.size())

# GP prior
kernel = gp.kernels.RBF(input_dim=1)
gpr = gp.models.GPRegression(X.unsqueeze(-1), Y, kernel, noise=torch.tensor(0.1))

# Variational inference approximation
optimizer = pyro.optim.Adam({"lr": 0.01})
loss_fn = pyro.infer.TraceELBO().differentiable_loss

for i in range(1000):
    optimizer.step(lambda: loss_fn(gpr.model, gpr.guide))
    if i % 100 == 0:
```

```
    print(f"Step␣{i},␣loss␣=␣{loss_fn(gpr.model,␣gpr.guide):.4f}")

# Predictive posterior
X_test = torch.linspace(0, 1, 100).unsqueeze(-1)
mean, cov = gpr(X_test, full_cov=True)
```

6 Visualization

posterior_evolution.png

Figure 1: Example evolution of function-space posterior mean and uncertainty bands.

7 Applications

- Robust portfolio optimization under full parameter uncertainty.
- Hierarchical multi-level allocations: sector-level functions feeding into asset-level.
- Adaptive asset strategies continuously refined by market data.

8 Conclusion

We presented an operator $\mathcal{B}_n$ for infinite-dimensional Bayesian updating, theoretically grounded and practically approximated via variational inference. Future work includes exhaustive stress tests and convergence diagnostics to further validate and refine the operator.

Next Steps

1. Design and run large-scale posterior contraction tests.
2. Conduct stress simulations with synthetic heavy-tailed returns.
3. Build empirical coverage plots and KL divergence traces.
4. Expand to dynamic time-evolving function-space priors.