Vector Calculus

1. For the following, calculate the gradient of f with respect of its input, i.e., compute Df:

   a. $f(x, y, z) = 4x^3y^2 - \mathbf{e}^z y^4 + \frac{z^3}{x^2} + 4y - x^{16}$
   b. $f(u, v, p, t) = 8u^2t^3p - \sqrt{v}\,p^2t^{-5} + 2u^2t + 3p^4 - v$

2. Compute the derivative $\frac{df \circ u(x)}{dx}$ where $u : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ given that $f(z) = (z_1 - z_2)e^{z_1}$, and $u(x) = \begin{bmatrix} x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix}$. **Hint:** The chain rule can make this easier.

   a. Compute $Df = \frac{df(z)}{dz}$
   b. Compute $Du = \frac{du(x)}{dx}$
   c. Use the chain rule and the values of $Df, Du, f, u,$ and x to compute $Df \circ u[x]$

3. Let $A(x) = \begin{bmatrix} 1 & x^3 + 3 \\ x^4 + x^2 + 3 & x^2 + 1 \end{bmatrix}$ be a nonsingular 2×2 matrix for a scalar x. Find $\frac{dA^{-1}(x)}{dx} \in \mathbb{R}^{2\times 2}$

4. if $A(t) = \begin{bmatrix} t^2 & t+1 \\ t^3 + t + 3 & 7 \end{bmatrix}$, calculate the following:

   a. $\frac{dA^2(t)}{dt}$
   b. $\frac{d^2B(t)}{dt^2}$ where $B(t) = ln(det(A(t)))$.

5. What is $\frac{df \circ g \circ h(x)}{dx}$ where $h : \mathbb{R}^3 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, and $f : \mathbb{R}^2 \to \mathbb{R}$ given $f(z) = z_1z_2 + 1$, $g(y) = \begin{bmatrix} y_1 + 2y_2 \\ 4y_1 - y_2 \end{bmatrix}$, and $h(x) = \begin{bmatrix} e^{x_1}\cos x_2 + x_3 \\ e^{x_1}\sin x_2 + x_3 \end{bmatrix}$

   a. Compute $Df = \frac{df(z)}{dz}$
   b. Compute $Dg = \frac{dg(y)}{dy}$
   c. Compute $Dh = \frac{dh(x)}{dx}$
   d. Use the chain rule and the values of Df, Dg, Dh, f, g, h and x to compute $Df \circ g \circ h[x]$