Multivariate q

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$$LL = 0.5 * (\mathbf{I} - log(q))^T \Sigma^{-1} (\mathbf{I} - log(q))$$

$$\mathbf{I} = \mathbf{Index}/\hat{\mathbf{N}}$$

$$LL = 0.5 * \begin{bmatrix} I_1 - log(q) \\ I_2 - log(q) \end{bmatrix}^T \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} I_1 - log(q) \\ I_2 - log(q) \end{bmatrix}$$

Invert

$$LL = 0.5 * \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} I_1 - log(q) \\ I_2 - log(q) \end{bmatrix}^T \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} I_1 - log(q) \\ I_2 - log(q) \end{bmatrix}$$

Multiply out

$$LL = 0.5 * \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} I_1 - log(q) \\ I_2 - log(q) \end{bmatrix}^T \begin{bmatrix} \sigma_2^2(I_1 - log(q)) - \rho\sigma_1\sigma_2(I_2 - log(q)) \\ -\rho\sigma_1\sigma_2(I_1 - log(q)) + \sigma_1^2(I_2 - log(q)) \end{bmatrix}$$

$$LL = 0.5 * \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} I_1 - log(q) \\ I_2 - log(q) \end{bmatrix}^T \begin{bmatrix} \sigma_2^2(I_1 - log(q)) - \rho\sigma_1\sigma_2(I_2 - log(q)) \\ -\rho\sigma_1\sigma_2(I_1 - log(q)) + \sigma_1^2(I_2 - log(q)) \end{bmatrix}$$

$$LL = \frac{\sigma_1^2 \left(I_2 - log(q)\right)^2 + -2\varphi\sigma_1\sigma_2 \left(I_1 - log(q)\right)\left(I_2 - log(q)\right) + \sigma_2^2 \left(I_1 - log(q)\right)^2}{2\left(1-\varphi^2\right)\sigma_1^2\sigma_2^2}$$

Take derivative

$$\frac{dLL}{dq} = -\frac{\left(\sigma_2^2 - 2\varphi\sigma_1\sigma_2 + \sigma_1^2\right) log(q) - I_1\sigma_2^2 + \left(I_2 + I_1\right)\varphi\sigma_1\sigma_2 - I_2\sigma_1^2}{\left(\varphi^2 - 1\right)\sigma_1^2\sigma_2^2 x}$$

Set to zero and solve for q

$$0 = -\frac{\left(\sigma_2^2 - 2\varphi\sigma_1\sigma_2 + \sigma_1^2\right) log(q) - I_1\sigma_2^2 + \left(I_2 + I_1\right)\varphi\sigma_1\sigma_2 - I_2\sigma_1^2}{\left(\varphi^2 - 1\right)\sigma_1^2\sigma_2^2 x}$$

$$0 = \left(\sigma_2^2 - 2\varphi\sigma_1\sigma_2 + \sigma_1^2\right) log(q) - I_1\sigma_2^2 + \left(I_2 + I_1\right)\varphi\sigma_1\sigma_2 - I_2\sigma_1^2$$

$$-\left(\sigma_2^2 - 2\varphi\sigma_1\sigma_2 + \sigma_1^2\right) log(q) = -I_1\sigma_2^2 + \left(I_2 + I_1\right)\varphi\sigma_1\sigma_2 - I_2\sigma_1^2$$

$$log(q) = \frac{-I_1\sigma_2^2 + \left(I_2 + I_1\right)\varphi\sigma_1\sigma_2 - I_2\sigma_1^2}{-(\sigma_2^2 - 2\varphi\sigma_1\sigma_2 + \sigma_1^2)}$$

$$q = exp\left(\frac{I_1\sigma_2^2 - \left(I_2 + I_1\right)\varphi\sigma_1\sigma_2 + I_2\sigma_1^2}{(\sigma_2^2 - 2\varphi\sigma_1\sigma_2 + \sigma_1^2)}\right)$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

Numerator (sum hessian times index)

$$\Sigma^{-1}\mathbf{I} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} I_1\sigma_2^2 - I_2\rho\sigma_1\sigma_2 \\ -I_1\rho\sigma_1\sigma_2 + I_2\sigma_1^2 \end{bmatrix}$$

$$\sum \left(\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 I_1 - \rho\sigma_1\sigma_2 I_2 \\ -\rho\sigma_1\sigma_2 I_1 + \sigma_1^2 I_2 \end{bmatrix} \right) = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \sum \begin{bmatrix} \sigma_2^2 I_1 - \rho\sigma_1\sigma_2 I_2 \\ -\rho\sigma_1\sigma_2 I_1 + \sigma_1^2 I_2 \end{bmatrix} = \frac{\sigma_2^2 I_1 - \rho\sigma_1\sigma_2(I_1+I_2) + \sigma_1^2 I_2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

Denominator (sum hessian)

$$\sum \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \sum \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} = \frac{\sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

$$\frac{\sum \Sigma^{-1}\mathbf{I}}{\sum \Sigma^{-1}} = \frac{\frac{\sigma_2^2 I_1 - \rho\sigma_1\sigma_2(I_1+I_2) + \sigma_1^2 I_2}{\sigma_1^2\sigma_2^2(1-\rho^2)}}{\frac{\sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}} = \frac{\sigma_2^2 I_1 - \rho\sigma_1\sigma_2(I_1+I_2) + \sigma_1^2 I_2}{\sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2} = log(q)$$

Sim analytical q

```
library(MASS)
q <- 0.5
Nvec <- seq(50, 200, length.out = 4)
varcov <- matrix(c(0.2,0.1,0,0,
                   0.1,0.2,0.1,0,
                   0,0.1,0.2,0.1,
                   0,0,0.1,0.2), 4, 4)
hess <- solve(varcov)
simddata <- exp(mvrnorm(10000, mu = log(Nvec * q) - diag(varcov)/2, Sigma = varcov))

q_est <- (apply(simddata,1, function(x) exp(sum((hess) %*% log(x/Nvec))/(sum(hess)))))
mean(q_est)
```

```
## [1] 0.4723102
```