Introduction to the code

This depository contains implementations of all FEM, FDM, and BEM formulations used in "Conditions for numerically accurate TMS electric field simulation," published in Brain Stimulation Journal. This document introduces the reader into the code requirements, as well as, provides some mathematical background on the different simulation methods.

System requirements

FEM, and BEM require the use of windows and Linux operating systems (soon these will be compiled in MacOS). FDM will work on either Windows 64, Linux or MacOS. All codes require a Matlab installation, and FDM and FEM require the PDE toolbox. Finally, these codes should work with most versions of Matlab, however, they have only been tested with Matlab 2010b, and Matlab 2018-2019 versions. If you experience any problems please let me know the operating system and Matlab version, so that I can recompile for these versions.

Required data structures for running code

Each code requires a dipole representation of the coil, a tetrahedral mesh of the head, and observation points where the E-field is to be evaluated $\mathbf{r_o}$.

Dipole data-structure ($\boldsymbol{rs}$, $\boldsymbol{Js}$, ω, and nc): The dipole representation of the coil consists cartesian dipole locations, dipole vector, number of dipoles, and time derivative. The primary E-field is computed as

$$\boldsymbol{E_p}(\boldsymbol{r}) = -\frac{\omega\mu_0}{4\pi}\sum_{i=1}^{nc}\frac{\mathbf{Js}_i}{\|\boldsymbol{r}-\mathbf{rs}_i\|} \quad ,$$

where ω is the time derivative of the current pulse, nc is the number of dipoles, μ_0 is the permeability of free-space, and $\boldsymbol{rs}_i$ and $\boldsymbol{Js}_i$ is the ith dipole cartesian location and dipole weight, respectively.

Head mesh data-structure ($\boldsymbol{te2p}$, $\boldsymbol{p}$, and $\boldsymbol{conductivity}$): The head mesh data structure is in the standard FEM mesh format, which consists of a tetrahedron to point array ($\boldsymbol{te2p}$), point array ($\boldsymbol{p}$), and conductivity array ($\boldsymbol{conductivity}$). The point array has 3 dimensional entries $\boldsymbol{p}_i$ with the cartesian location of one mesh point. Tetrahedron to point array has 4 dimensional entries $\boldsymbol{te2p}_{\mathbf{i}} = (j_1, j_2, j_3, j_4)$, and the i[th] tetrahedron has vertices $\boldsymbol{p}_{j_1}, \boldsymbol{p}_{j_2}, \boldsymbol{p}_{j_3}$, and $\boldsymbol{p}_{j_4}$. The conductivity of tetrahedron made of vertices $\boldsymbol{te2p}_{\mathbf{i}}$ is the i[th] entry of the $\boldsymbol{conductivity}_i$ array. (Note: Because of different algorithmic considerations the specific array dimension ordering are FEM, FDM, and BEM code specific. Each code folder has a file 'testmexcode.m' describing the code specific array ordering.

How to run the codes

Once the above data structures have been generated look for the 'testruncode.m' and modify to pass the correct data structures to runcode. For more advanced users, 'testmexcode.m' and 'runcode.m' are written in a modular fashion to enable users to derive their own modules from them.

The following gives a brief background on each of the methods

FEM solver

The TMS coil is driven by coil currents $\mathbf{J}(\mathbf{r}'; t) = \mathrm{P(t)}\mathbf{J}(\mathbf{r}')$, where $\mathrm{P(t)}$ is the pulse-waveform, and $\mathbf{J}(\mathbf{r}')$ is the spatial distribution of the current. At the low frequencies used for TMS, quasi-stationary assumptions are valid, and the temporal variation and spatial variation of the E-field are separable (i.e. $\mathbf{E}(\mathbf{r}; \mathrm{t}) = \frac{d\mathrm{P(t)}}{dt}\mathbf{E}(\mathbf{r})$), where $\mathbf{E}(\mathbf{r})$ is solely dependent on $\mathbf{J}(\mathbf{r}')$ [1]. The spatial variation of the E-field induced in the head $\mathbf{E}(\mathbf{r})$ is the sum of the incident (primary) field $\mathbf{E_p}(\mathbf{r})$ due to the coil currents and a secondary contribution $\mathbf{E_s}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$ from the scalar potential $\phi(\mathbf{r})$, where

$$\boldsymbol{E_p}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|}d\boldsymbol{r}' \quad , \tag{S.1}$$

$$\nabla\cdot\sigma(\boldsymbol{r})\nabla\phi(\boldsymbol{r}) = \nabla\cdot\sigma(\boldsymbol{r})\boldsymbol{E_p}(\boldsymbol{r}) \quad , \tag{S.2}$$

μ_0 is the electric permeability of free-space, and $\sigma(\mathbf{r})$ is the tissue conductivity. Furthermore, the normal component of the E-field on the surface of the head is zero. To solve for ϕ and $\nabla\phi$, first, the head model is approximated by a mesh consisting of tetrahedrons, and each tetrahedron is assigned a constant tissue conductivity. The scalar potential ϕ is approximated by

$$\phi(\boldsymbol{r}) = \sum_{m=1}^{M} x_m N_m(\boldsymbol{r}) \quad , \tag{S.3}$$

where $N_m(\mathbf{r})$ are piecewise linear, quadratic, and cubic nodal elements for 1st, 2nd, and 3rd order FEM, respectively [2], and $\mathbf{x} = (x_1, \dots, x_M)$ are expansion coefficients. A linear system of equations in terms of $\mathbf{x}$ as unknown is assembled by a standard Galerkin procedure, i.e. each equation is generated by multiplying Eq. S2 by a testing function $N_m(\mathbf{r})$, and integrating it over its support. This results in

$$\begin{aligned} &\boldsymbol{Ax} = \boldsymbol{b} \\ &(\boldsymbol{A})_{m,n} = \int \sigma(\boldsymbol{r})\nabla N_m(\boldsymbol{r})\cdot\nabla N_n(\boldsymbol{r})\, d\boldsymbol{r} \\ &(\boldsymbol{b})_m = \int N_m(\boldsymbol{r})\cdot\boldsymbol{E_p}(\boldsymbol{r})\nabla\sigma(\boldsymbol{r})\, d\boldsymbol{r} \end{aligned} \quad . \tag{S.4}$$

To evaluate $(\mathbf{A})_{m,n}$, exact values of integrals in terms of tetrahedron vertex locations provided in [2] are used. To determine $(\mathbf{b})_m$, a 2th, 3rd, and 4th order accurate quadrature rule is used for 1st, 2nd, and 3rd order FEM, respectively. All codes use quadrature rules [3]. The linear system of Eqs. (S.4) is solved using a minimal residual (MINRES) iterative solver to a relative residual of 10^{-7} [4]. Furthermore, a symmetric diagonal preconditioner is used to improve the convergence of MINRES.

FDM solver

The FDM solver also solves Eq. S.2. The main difference is that the head model is approximated by a regular mesh consisting of identical cuboid elements. Furthermore, first order nodal elements are used to approximate the scalar potential. We use the 27 point finite difference stencil used in [5].

BEM solver

An implementation of the BEM method [6] assumes that the scalar potential arises from charges $\rho(\mathbf{r})$ on tissue interfaces. This charge generates a scalar potential

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|}\,d\boldsymbol{r}' \quad , \tag{S.6}$$

and secondary E-field

$$\boldsymbol{E_s}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3}\,d\boldsymbol{r}' \quad , \tag{S.7}$$

where ϵ_0 is the permittivity of free space. Current continuity across a tissue interface dictates that

$$\sigma_i\left(\boldsymbol{E_{si}}(\boldsymbol{r}) + \boldsymbol{E_p}(\boldsymbol{r})\right)\cdot\hat{\boldsymbol{n}} = \sigma_o\left(\boldsymbol{E_{so}}(\boldsymbol{r}) + \boldsymbol{E_p}(\boldsymbol{r})\right)\cdot\hat{\boldsymbol{n}} \quad , \tag{S.8}$$

where σ_i is the inner and σ_o is the outer tissue conductivity, $\mathbf{E_{si}}(\mathbf{r})$ and $\mathbf{E_{so}}(\mathbf{r})$ is the secondary E-field an infinitesimal distance interior and exterior to the interface, respectively, and $\hat{\boldsymbol{n}}$ is the tissue interface normal pointing towards the outer tissue. Assuming the tissue boundary surface has a continuously varying normal (i.e. does not have corners) and using (S.7) to determine $\boldsymbol{E_{si}}(\boldsymbol{r})$ and $\boldsymbol{E_{so}}(\boldsymbol{r})$ results in

$$4\pi\epsilon_0 \boldsymbol{E_{si}}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \int \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3} d\boldsymbol{r}' = \left(-2\pi\rho(\boldsymbol{r}') + \hat{\boldsymbol{n}} \cdot \int_{p.v.} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3} d\boldsymbol{r}' \right) \quad , \tag{S.9}$$

and

$$4\pi\epsilon_0 \boldsymbol{E_{so}}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \int \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3} d\boldsymbol{r}' = \left(2\pi\rho(\boldsymbol{r}') + \hat{\boldsymbol{n}} \cdot \int_{p.v.} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3} d\boldsymbol{r}' \right) \quad . \tag{S.10}$$

Plugging in (S.9) and (S.10) into (S.8) results in the integral equation

$$-\frac{\sigma_o - \sigma_i}{\sigma_o + \sigma_i} \epsilon_0 \boldsymbol{E_p}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}} = \frac{1}{2}\rho(\boldsymbol{r}) + \frac{\sigma_o - \sigma_i}{\sigma_o + \sigma_i} \frac{1}{4\pi} \int_{p.v.} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3} d\boldsymbol{r}' \quad . \tag{S.11}$$

Here the residue of the integral of Eq. S.7 has been extracted as $\frac{(\sigma_o+\sigma_i)}{2\epsilon_0}\rho(\mathbf{r})$, and p.v. is used to denote the principal value. (For geometries that are not smooth, i.e. have corners or edges, the BEM is still valid, only the factor of ½ has to be changed to a topology dependent value.)

To solve Eq. S.11 for the charge, tissue boundaries are approximated by meshes consisting of triangles. Then, the charge is expanded as

$$P(\boldsymbol{r}) = \sum_{i=1}^{N} x_i B_i(\boldsymbol{r}) \quad , \tag{S.12}$$

where $B_i(\mathbf{r})$ are pulse functions for 0th order BEM and $B_i(\mathbf{r})$ are triangular piecewise linear nodal elements for 1th order BEM [7]. A linear system of equations in terms of $\mathbf{x} = (x_1, \ldots, x_N)$ as unknown is derived by applying a standard Galerkin procedure on Eq. S.11. The resulting system of equations is

$$\begin{aligned} &\boldsymbol{Ax} = \boldsymbol{b} \\ &(\boldsymbol{A})_{m,n} = \int B_m(\boldsymbol{r}) B_n(\boldsymbol{r})\, d\boldsymbol{r} - \frac{(\sigma_i - \sigma_o)}{(\sigma_o + \sigma_i)} \frac{1}{2\pi} \int B_m(\boldsymbol{r}) \hat{\boldsymbol{n}} \cdot \int_{p.v.} B_n(\boldsymbol{r}') \frac{(\boldsymbol{r}-\boldsymbol{r}')}{\|\boldsymbol{r}-\boldsymbol{r}'\|^3} d\boldsymbol{r}'\, d\boldsymbol{r} \quad . \\ &(\boldsymbol{b})_m = 2\epsilon_0 \frac{(\sigma_i - \sigma_o)}{(\sigma_o + \sigma_i)} \int B_m(\boldsymbol{r}) \hat{\boldsymbol{n}} \cdot \boldsymbol{E_p}(\boldsymbol{r})\, d\boldsymbol{r} \end{aligned} \tag{S.13}$$

The first term of $(\mathbf{A})_{m,n}$, is computed exactly. For the second term, the quadrature rule used

depends on the distance between the inner integration and outer integration triangular support center. Specifically, we distinguish between entries corresponding to near triangles and far apart ones. Two triangles are considered to be near to each other if they are a distance less than d_{near} apart, measured from triangle centers, or they share a vertex. Otherwise, they are considered far apart. For near triangle entries, we compute the outer integral using an 8th order accurate rule and the inner integral exactly. For far apart triangle entries, a 2nd order accurate quadrature rule is used [3]. To lower computational costs, the FMM library with precision set to δ is used to compute all far apart triangle entries on-the-fly each time the matrix is applied to a vector. The linear system of Eqs. S.13 is solved using TFQMR [4] iterative solver to a relative residual of relres. Finally, a symmetric diagonal preconditioner is used to improve the convergence of TFQMR.

For consistency we chose $\mathrm{relres} = 10^{-7}$ for all FEM and BEM simulations. The accuracy of the resulting $\mathbf{x}$ is such that $\|\mathbf{x}\| \leq \mathrm{C} \times \mathrm{relres}$ [8], where C is the condition number of $\mathbf{A}$. Double layer potential formulations are known to have a low-condition number relative to FEM for the same discretization because of their identity plus compact operator form [9]. The relres parameter can likely be increased beyond what was used here while maintaining the same accuracy level on the E-field.

To determine the E-field from a charge distribution, if a source distribution was less than d_{near} from the observation point, then its contribution was computed analytically, otherwise we use FMM ($\delta = 5 \times 10^{-10}$) along with a 2th-order accurate quadrature rule.

Analytical E-field in a sphere head model

The total E-field in the spherical head model is computed via a spherical harmonic series expansion using [10]

$$\boldsymbol{E}(r,\theta,\phi) = \sum_{l=0}^{N}\sum_{m=-l}^{l} a_{lm}\left(\frac{r}{R}\right)^{\boldsymbol{l}} \boldsymbol{Y}_{l,l}^{m}(\theta,\phi)$$
$$a_{lm} = \int_0^{\pi}\int_0^{2\pi} \boldsymbol{E_p}(R,\theta,\phi)\cdot \boldsymbol{Y}_{l,l}^{m}(\theta,\phi)\, sin(\theta)\, d\phi d\theta \qquad \text{(S.14)}$$

Here $\boldsymbol{Y}_{l,l}^{m}$ is a vector spherical harmonic [10], R is the head radius chosen to be 8.5 cm, and $N = 48$. The primary E-field due to electric and magnetic sources was computed using standard source field relations [7]. Coefficients a_{lm} are computed using an 800th order accurate Gaussian quadrature rule.

References

[1] Plonsey R, Heppner DB. Considerations of quasi-stationarity in electrophysiological systems. The Bulletin of mathematical biophysics 1967;29:657-64.

[2] Jin J-M. The finite element method in electromagnetics. 3rd Edition ed: John Wiley & Sons; 2014.

[3] Xiao H, Gimbutas Z. A numerical algorithm for the construction of efficient quadrature rules in two and higher dimensions. Computers & mathematics with applications 2010;59:663-76.

[4] Paige CC, Saunders MA. Solution of sparse indefinite systems of linear equations. SIAM journal on numerical analysis 1975;12:617-29.

[5] Laakso I, Hirata A. Fast multigrid-based computation of the induced electric field for transcranial magnetic stimulation. Phys. Med. Biol. 2012;57:7753.

[6] Makarov S, Noetscher G, Raij T, Nummenmaa A. A Quasi-Static Boundary Element Approach with Fast Multipole Acceleration for High-Resolution Bioelectromagnetic Models. IEEE Trans. Biomed. Eng. 2018.

[7] Peterson AF, Ray SL, Mittra R, Electrical Io, Engineers E. Computational methods for electromagnetics. Vol 2: IEEE press New York; 1998.

[8] Trefethen LN, Bau III D. Numerical linear algebra. Vol 50: Siam; 1997.

[9] Colton D, Kress R. Integral equation methods in scattering theory. Vol 72: SIAM; 2013.

[10] Koponen LM, Nieminen JO, Ilmoniemi RJ. Minimum-energy coils for transcranial magnetic stimulation: application to focal stimulation. Brain stimulation 2015;8:124-34.