PRIME INTUIT
Finishing School

Bernoulli Distribution

Experiments with only two outcomes?

Bernoulli Trials

Out come: {Positive, Negative}

Outcome: {Pass, Fail}

Outcome: {Hit, Flop}

Outcome: {Spam, Ham}

Outcome: {Approved, Denied}

Ω : Failure Success

X: Ω : → {0. 1}

Bernoulli Random Variable

Bernoulli Distribution

Bernoulli Distribution?

Bernoulli Trials

Ω : Failure, Success

A: event that the outcome is success

X: Ω : → {0. 1}

Let P(A) = P(success) = P

Bernoulli Random Variable

Px(1) = P

Px(0) = 1 - P

$Px(x) = p^{x} * (1\text{-}P)^{(1-x)}$

Bernoulli distribution

Bernoulli Distribution

Is Bernoulli distribution a valid distribution ?

$\mathrm{P}x(x) = p^{x} * (1\text{-P})^{(1-x)}$ Bernoulli distribution

Ω : Failure Success

X: Ω : → {0. 1}

↑

$Bernoulli\ Random\ Variable$

$Px(x) \geq 0$

$\Sigma_{x\in\{0,1\}}Px(x) = 1$?

$\Sigma_{x\in\{0,1\}}Px(x) = Px(0) + Px(1)$

$= (1\text{-P}) + \text{P} = 1$

Binomial Distribution

Repeat a Bernoulli trial n times

  

….. n number of times

Independent :

(Success / failure in one trail does not affect the outcome of other trails)

I*dentical* :

(Probability of success ‘P’ in each trial is the same)

What is the probability of k successes in n trails? ($k \in [0,n]$)

Binomial Distribution

Binomial Distribution (Examples)

….. n number of times

Each ball bearing produced in a given factory is independently non defective with probability p

If you select n ball bearings what is the probability that k of them will be defective ?

What is the probability of k successes in n trails? ($k \in [0,n]$)

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Binomial Distribution

Binomial Distribution (Examples)

….. n number of times

The probability that a customer purchases something from yoru website is P

Assumption 1: Customers are identical (economic strata, interests, needs, etc)

Assumption 2: Customers are independent (one’s decision does not influence another)

What is the probability of k out of n customers will purchase something ?

Binomial Distribution

Binomial Distribution (Examples)

….. n number of times

Marketing Agency: The probability that a customer opens your email is P

Assumption 1: Customers are identical

Assumption 2: Customers are independent

If you send n emails what is probability that the customer will open at least one of them ?

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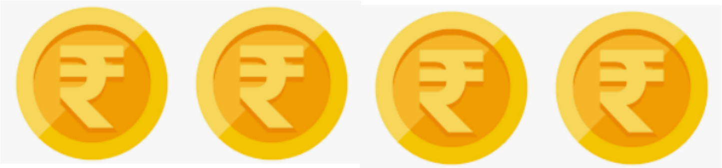
Binomial Distribution

Binomial Distribution



….. n number of times

How many different outcomes can we have if we repeat a Bernoulli trail n times ?

S, F

___ ___ ___ ___ ___ n

(sequence of length n from a given set of 2 objects)

= 2^noutcomes

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Binomial Distribution

Binomial Distribution

….. n number of times

Ω $\qquad\qquad$ X

$T\ T\ T$
T T H
T H T
T H H
H T T
H T H
H H T
H H H

0
1
2
3

Example: n = 3, k = 1

H = Success
T = Fail

____ ____ ____

A = { HTT, THT, TTH}

$P_x(1) = P(A)$

$P_x(1) = P(A)$ = P({HTT}) + P({THT}) + P({TTH})

P({HTT}) = p (1-p) (1-p)

P({THT}) = (1-p) p (1-p)

P({TTH}) = (1-p) (1-p) p

$P_x(1) = P(A) = 3(1\text{-}p)^2\ p$

Binomial Distribution

Binomial Distribution

..... n number of times

Ω X

$T\ T\ T$
T T H
T H T
T H H
H T T
H T H
H H T
H H H

0
1
2
3

Example: n = 3, k = 1

A = { HTT, THT, TTH}

___ ___ ___

3 trials and 1 success

= 3 choose 1 = $\binom{3}{1}$

$P_x(1) = P(A)$ = 3(1-p)^2 p

= 3(1-p)$^{(3-1)}$ p^1

= $\binom{3}{1}$ (1-p)$^{(3-1)}$ p^1

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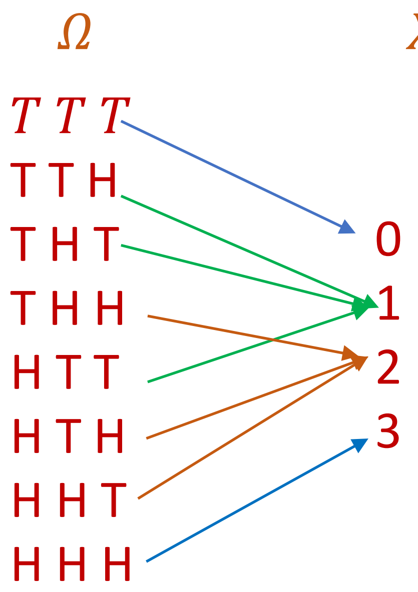
Binomial Distribution

Binomial Distribution

….. n number of times



Example: n = 3, k = 2

B = { HTH, HHT, THH}

____ ____ ____

$P_x(2) = P(B)$ = 3(1-p) p^2

= 3(1-p)$^{(3-2)}p^2$

= $\binom{3}{2}$ (1-p)$^{(3-2)}$ p^2

3 trials and 2 success

= 3 choose 2 = $\binom{3}{2}$

Binomial Distribution

Binomial Distribution

Observations

n trials and k success

$\binom{n}{k}$ terms in the summation

$\binom{n}{k}$ favourable outcomes

each terms will have the factor p^k

$each\ of\ the\ k\ success\ occur\ independently\ with\ a\ probability\ p$

each terms will have the factor (1 – p)$^{(n-k)}$

each of the n – k failures occur independently with a probability 1 - p

$$P_x(\mathrm{k}) = \binom{n}{k}\ p^k\ (1\text{-p})^{(n-k)}$$

Parameters: p, n

The entire distribution is full specified once the value of p and n are known

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Binomial Distribution

Binomial Distribution Example1: Social distancing

$Suppose\ 10\%\ of\ your\ colleagues\ from\ workplace\ are\ infected\ with\ COVID-19\ but\ are\ asymptomatic\ (hence\ come\ to\ office\ as\ usual)$

$Suppose\ you\ come\ in\ close\ proximity\ of\ 50\ of\ your\ colleagues.$
What is the probability of you getting infected

n = 50, p = 0.1

P(getting infected) = P(at least one success)

= 1 – P(0 successes)

= 1 - $P_x(0)$

= 1 - $\binom{50}{0}$ p^0 $(1\text{-}p)^{(50)}$

= $1 - 1 * 1 * 0.9^{(50)} = 0.995$

= $1 - 1 * 1 * 0.9^{(10)} = 0.6513$

= $1 - 1 * 1 * 0.98^{(10)} = 0.1829$ $,P\ change\ to\ 2\%$

Binomial Distribution

Binomial Distribution Example2: Mac Users

$Suppose\ 10\%\ of\ students\ in\ your\ class\ use\ Mac\ book,$
$If\ you\ select\ 25\ students\ at\ random$

$a)\ what\ is\ the\ probability\ that\ exactly\ 3\ of\ them\ are\ using\ Mac\ book\ ?$

b) $what\ is\ the\ probability\ that\ between\ 2\ to\ 6\ of\ them\ are\ using\ Mac\ book\ ?$

C) $How\ would\ the\ above\ probabilities\ change\ if\ instead\ of\ 10\%, 90\%\ were\ using\ Mac\ book\ ?$

a) n = 25, p = 0.1 and k = 3

b) n = 25, p = 0.1 and k ={2,3,4,5,6}

c) n = 25, p = 0.9, 0.5

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Binomial Distribution

Binomial Distribution Example2: Mac Users

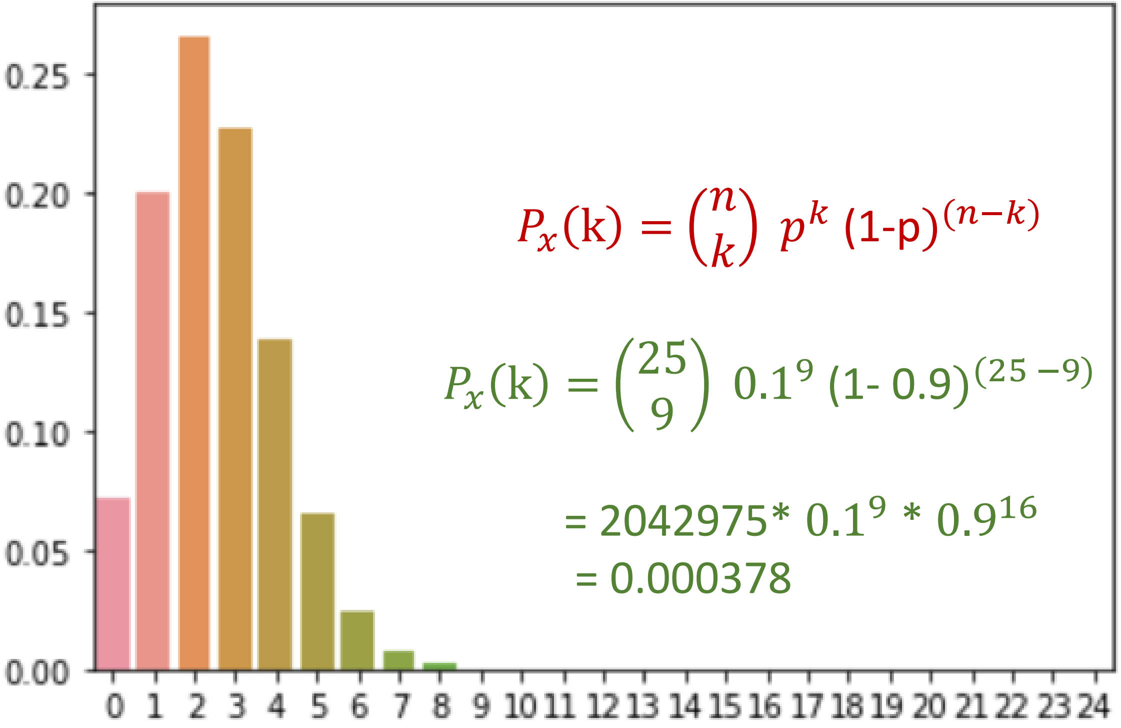
Suppose 10% of students in your class use Mac book,
If you select 25 students at random

a) n = 25, p = 0.1 and k = 3

b) n = 25, p = 0.1 and k ={2,3,4,5,6}

c) n = 25, p = 0.9, p = 0.5

```
import seaborn as sb
import numpy as np
from scipy.stats import binom

x = np.arange(0,25)
n = 25
p = 0.1

dist = binom(n, p)
ax = sb.barplot(x = x, y = dist.pmf(x))
```



$$P_x(\mathrm{k}) = \binom{n}{k}\, p^k\ (1\text{-p})^{(n-k)}$$

$$P_x(\mathrm{k}) = \binom{25}{9}\ 0.1^9\ (1\text{- }0.9)^{(25-9)}$$

$$= 2042975 * 0.1^9 * 0.9^{16}$$
$$= 0.000378$$

Binomial Distribution

Binom

Suppo
If you

a) n = 25, p = 0.1 an

b) n = 25, p = 0.1 an

c) n = 25, p = 0.9, p

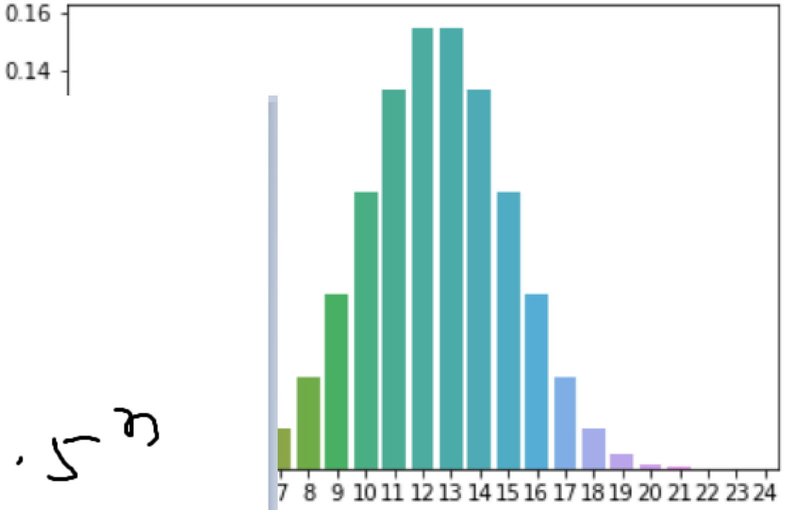
```
import seaborn as
import numpy as r
from scipy.stats

x = np.arange(0,2
n = 25
p = 0.9

dist = binom(n, p)
ax = sb.barplot(x = x, y = dist.pmf(x))
```

$$P_x(k) = \binom{n}{k} p^k (1-p)^{(n-k)}$$

$$(0.5)^k (0.5)^{n-k} = 0.5^n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{k!(n-k)!}$$

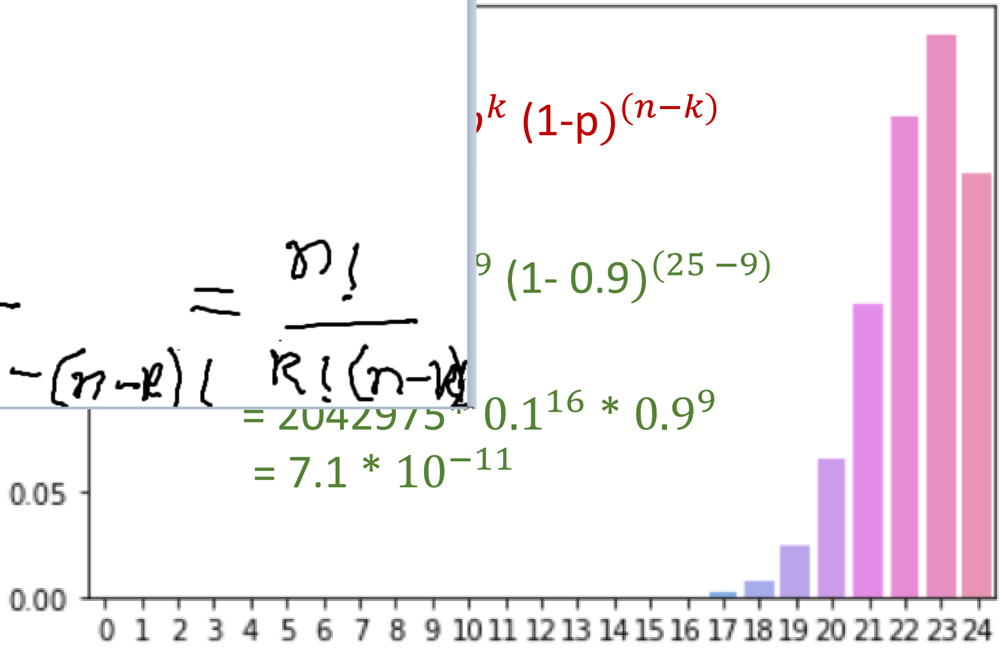


^{k} (1-p)$^{(n-k)}$

^{9} (1- 0.9)$^{(25-9)}$

= 2042975 * 0.1^{16} * 0.9^9

= 7.1 * 10^{-11}



Binomial Distribution

Is Binomial Distribution a valid distribution ?

$p_x(x) \geq 0$ $\qquad$ $P_x(\mathrm{k}) = \binom{n}{k}\, p^k\, (1\text{-p})^{(n-k)}$

$\sum_{i=0}^{n} p_x(i) = 1\ \ ?$

$= p_x(0) + p_x(1) + p_x(2) + \ldots\ldots\ldots\ldots + p_x(n)$

$= \binom{n}{0}\, p^0\, (1\text{-p})^{(n-0)} + \binom{n}{1}\, p^1\, (1-\mathrm{p})^{(n-1)} + \binom{n}{2}\, p^2\, (1-\mathrm{p})^{(n-2)} + \cdots \ldots \ldots \ldots + \binom{n}{n}\, p^n\, (1-\mathrm{p})^{(n-n)}$

$(a+b)^n = \binom{n}{\theta} a^0 b^n + \binom{n}{1} a^1 b^{n-1} + \binom{n}{2} a^2 b^{n-2} \ldots \ldots \ldots + \binom{n}{n} a^{\mathrm{n}} b^0$

$\quad on\ the\ left\ hand\ side\ if\ (a = p, \qquad b = 1 - p)\ then$

$= (\,P + 1\ - P)^n = 1^n = 1$

Binomial Distribution

Relation between Binomial and Bernoulli

Binomial $P_x(\mathrm{k}) = \binom{n}{k} p^k (1\text{-p})^{(n-k)}$

Bernoulli $n = 1, \quad k \in \{0, 1\}$

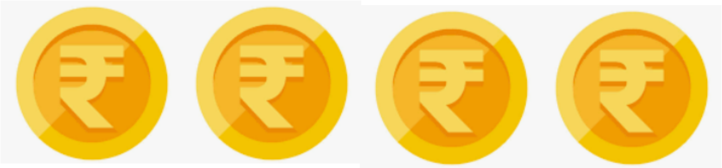
$$p_x(0) = \binom{1}{0} p^0 (1\text{-p})^{(1-0)} = 1 - \mathrm{p}$$

$$p_x(1) = \binom{1}{1} p^1 (1\text{-p})^{(1-1)} = \mathrm{p}$$

Bernoulli distribution is a special case of Binomial distribution

Geometric Distribution

Geometric Distribution



….. ∞ number of times

X : The number of tosses until we see the first heads

$R_x = \{1, 2, 3, 4, 5, \dots.\}$

___ ___ ___ ___ ___ n

Why would we be interested in such a distribution ?

(sequence of length ∞ from a given set of 2 objects)

= 2^{∞}outcomes

$p_x(x)$ = ?

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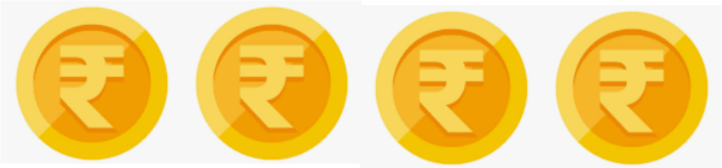
Geometric Distribution

Geometric Distribution



….. ∞ number of times

Why would we be interested in such a distribution ?

Because: It’s useful in any situation involving “waiting times”

Independent trials identical distribution P(success) = p

Street vendor selling vada pav outside a subway station
(Chance that the first vada pav will be sold after k trails)

Salesman handing pamphlets to passersby
(Chance that the k-th person will be the first person to actually read the pamphlet)

A digital marketing agency sending emails
(Chance that the k-th person will be the first person to actually read the email)

Geometric Random variable X, number of trials after which we will get the first success

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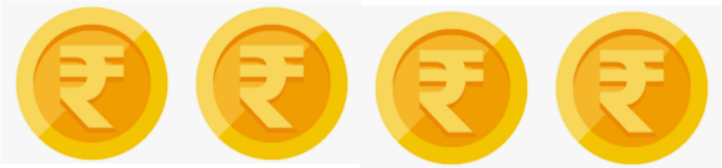
Geometric Distribution

Geometric Distribution



..... ∞ number of times

P(Success) = P

$P_x(\mathrm{x})$

Example: k = 5

$P_x(5)$

F F F F S

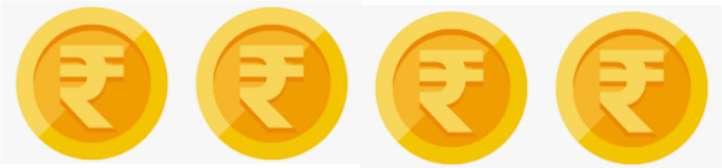
$$\underbrace{(1-p)\,(1-p)\,(1-p)\,\,(1-p)\,\,(1-p)}_{(5\ -1)}\ \underbrace{p}_{1}$$

$P_x(5)$ = (1- p)$^{(5-1)}$ p

$P_x(\mathrm{k})$ = (1- p)$^{(k-1)}$ p

Geometric Distribution

Geometric Distribution



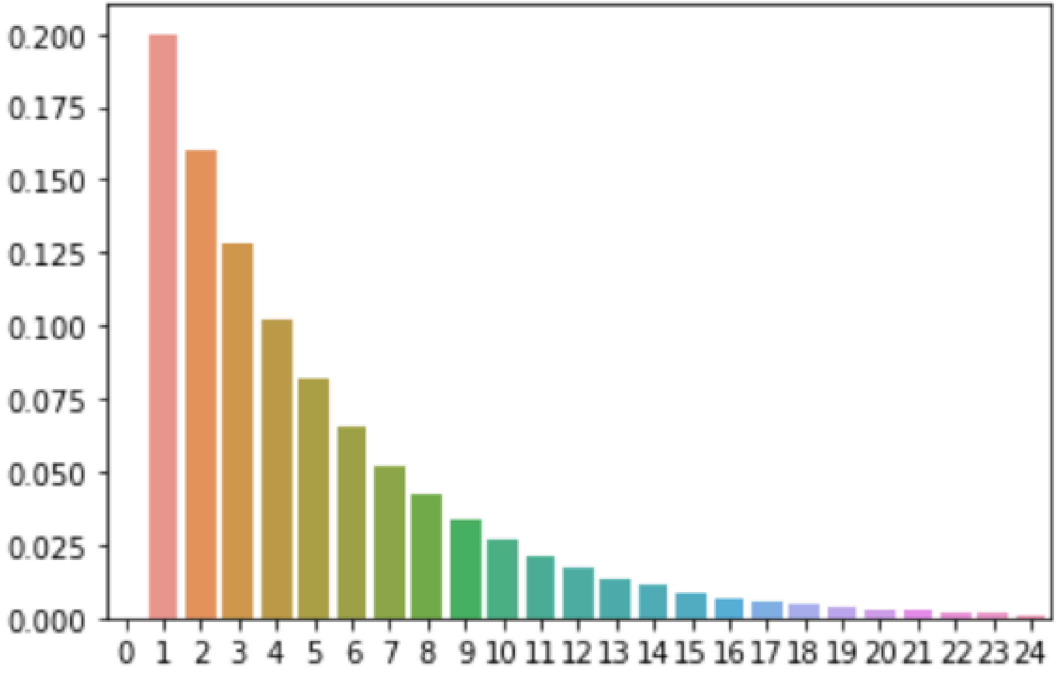
….. ∞ number of times

P (success) = p

```
import seaborn as sb
import numpy as np
from scipy.stats import geom

x = np.arange(0,25)
n = 25
p = 0.2

dist = geom(p)
ax = sb.barplot(x = x, y = dist.pmf(x))
```

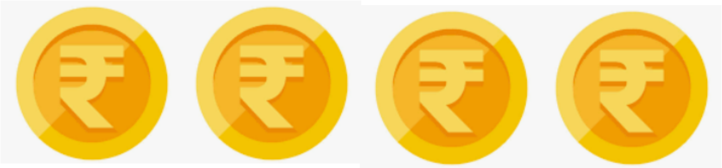
Example: p = 0.2, n = 25

$$P_x(k) = (1-p)^{(k-1)}\, p$$



Geometric Distribution

Geometric Distribution



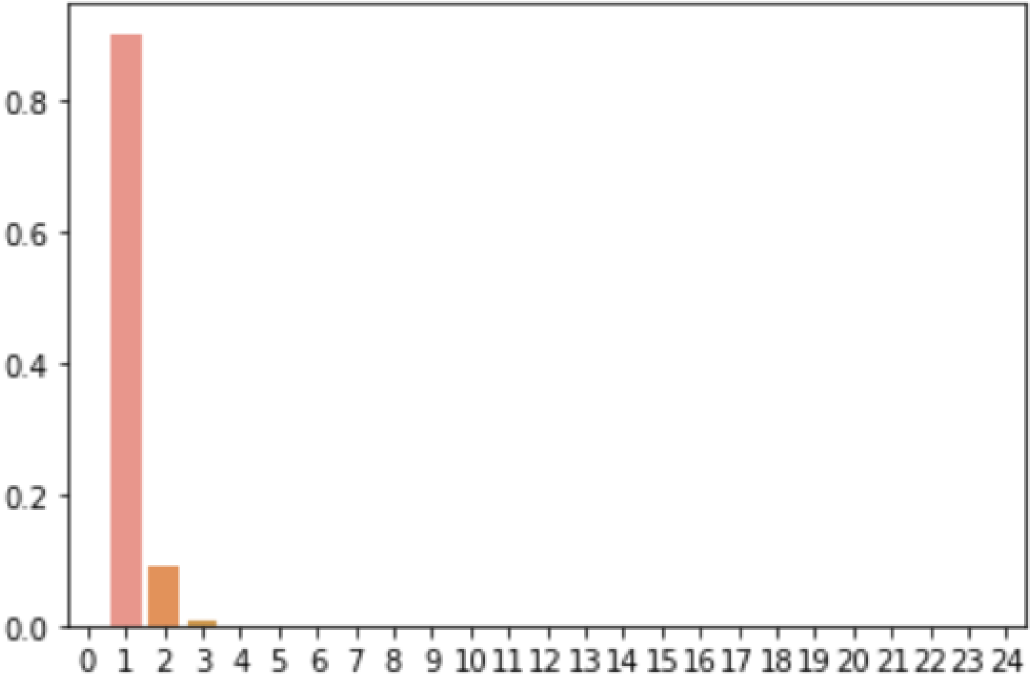
….. ∞ number of times

P (success) = p

```
import seaborn as sb
import numpy as np
from scipy.stats import geom

x = np.arange(0,25)
n = 25
p = 0.9

dist = geom(p)
ax = sb.barplot(x = x, y = dist.pmf(x))
```

Example: p = 0.9, n = 25

$$P_x(k) = (1 - p)^{(k-1)}\, p$$



Geometric Distribution

Geometric Distribution

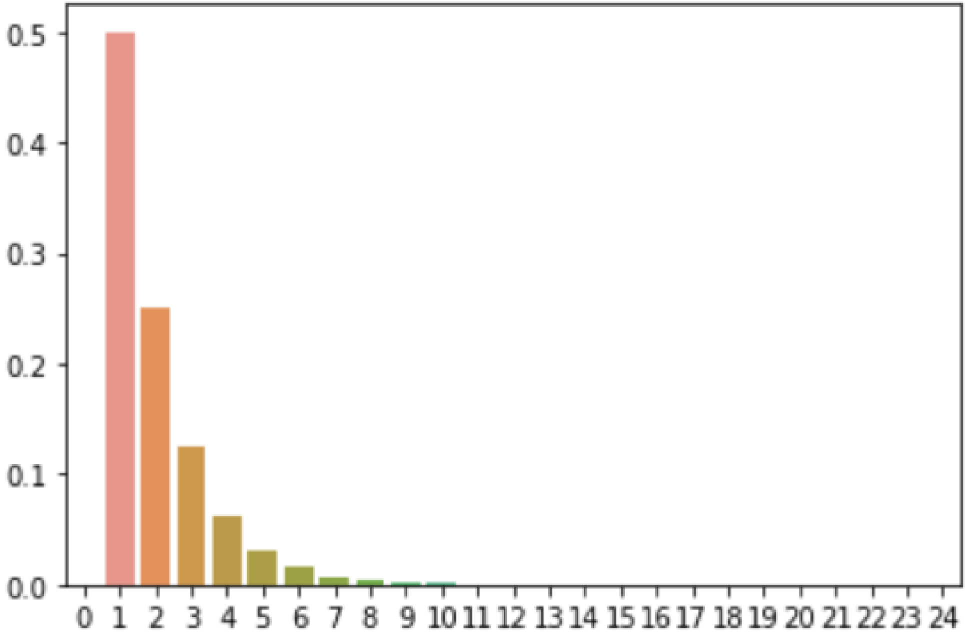
….. ∞ number of times

P (success) = p

```python
import seaborn as sb
import numpy as np
from scipy.stats import geom

x = np.arange(0,25)
n = 25
p = 0.5

dist = geom(p)
ax = sb.barplot(x = x, y = dist.pmf(x))
```

Example: p = 0.5, n = 25

$P_x(\mathrm{k}) = (1-\mathrm{p})^{(k-1)}\ \mathrm{p}$



Geometric Distribution

Is Geometric Distribution a valid distribution ?

$p_x(x) \geq 0$ $P_x(\text{k}) = (1\text{- p})^{(k-1)}\ \text{p}$

$\sum_{\text{k}=1}^{\infty} p_x(k) = 1\ \ ?$

$= p_x(0) + p_x(1) + p_x(2) + \ldots\ldots\ldots\ldots + p_x(\infty)$

$= (1\text{-p})^0\ p + (1-\text{p})^1\ p + (1-\text{p})^2\ p + \cdots \ldots \ldots \ldots + (1-\text{p})^{\infty}\ p$

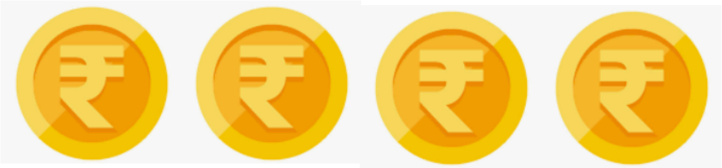
$\sum_{\text{k}=1}^{\infty} (1-\text{p})^k\ p$

$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots \ldots \ldots \ldots \ldots \ldots ar^{\infty}$ $(Geometric\ series)$

$= \frac{a}{1-r} = \frac{p}{1-(1-p)} = 1$

Geometric Distribution

Geometric Distribution



….. ∞ number of times

P (success) = p

A Patient needs a certain blood group which only 9 % of the population has ?

What is the probability that the 7th volunteer that the doctor contacts will be the first one to have a matching blood group ?

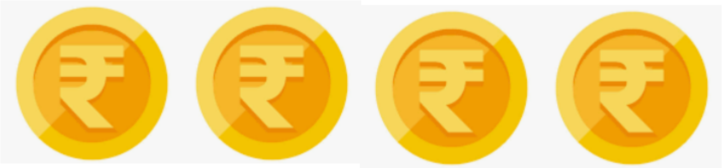
What is the probability that atleast one of the first 10 volunteers will have a matching blood type ?

$$p_x(7) = ?$$

P(x < = 10) = ?

Geometric Distribution

Geometric Distribution



..... ∞ number of times

P (success) = p

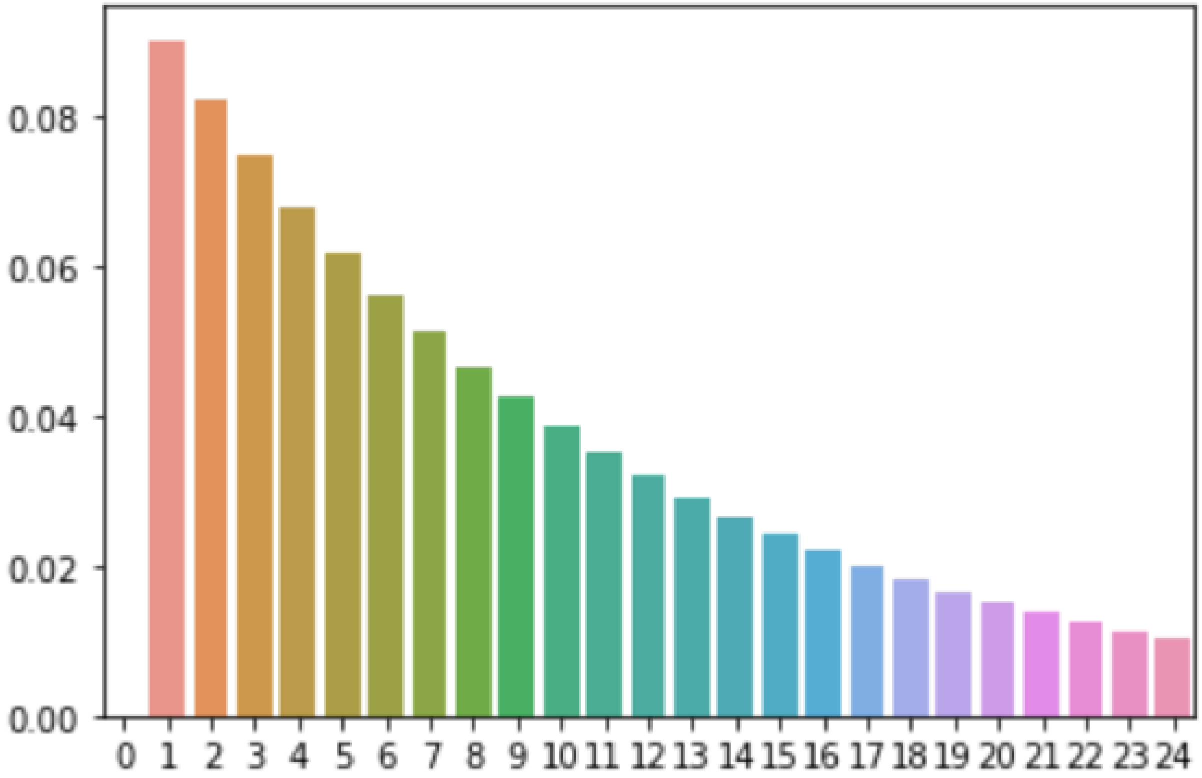
```
import seaborn as sb
import numpy as np
from scipy.stats import geom

x = np.arange(0,25)
n = 25
p = 0.09

dist = geom(p)
ax = sb.barplot(x = x, y = dist.pmf(x))
```

1) $p_x(7) = ?$

Example: p = 0.09, n = 25

2) $P(x <= 10) = 1 - P(x > 10)$

$= 1 - (1 - p)^{10}$



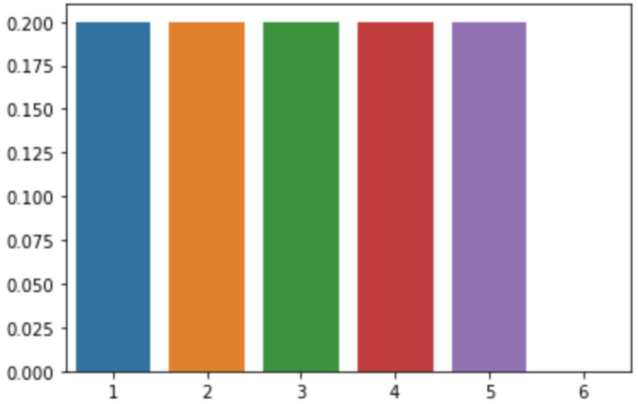
Uniform Distribution

Uniform Distribution

Experiments with equally likely outcomes

X : outcome of a die

$p_x(x)$= 1/6 $\quad \forall x \in \{1, 2, 3, 4, 5, 6\}$



X : outcome of a bingo / housie draw

$p_x(x)$= 1/100 $\quad 1 \leq \text{x} \geq 100\}$

$R_x = \{\, a \leq \text{x} \geq \text{b}\}$

Uniform Distribution

Uniform Distribution

Experiments with equally likely outcomes

X : outcome of a die / bingo / housie draw

$$p_x(x) = \frac{1}{b-a+1} \qquad a \leq x \leq b$$

Special Cases

$a = 1, b = n$

$$p_x(x) = \frac{1}{b-a+1} = \frac{1}{n} \qquad 1 \leq x \leq n$$

$a = c, b = c$

$$p_x(x) = \frac{1}{b-a+1} = \frac{1}{c-c+1} = 1 \qquad x = c$$

Uniform Distribution

Is Uniform Distribution a valid distribution ?

$$p_x(x) \leq 0, \quad p_x(x) = \frac{1}{b-a+1} \qquad a \leq x \leq b$$

$$\sum_{k=1}^{\infty} P_x(i) = 1\ ?$$

$$\sum_a^b P_x(i) = (b-a+1)\frac{1}{b-a+1} = 1$$

Expectation

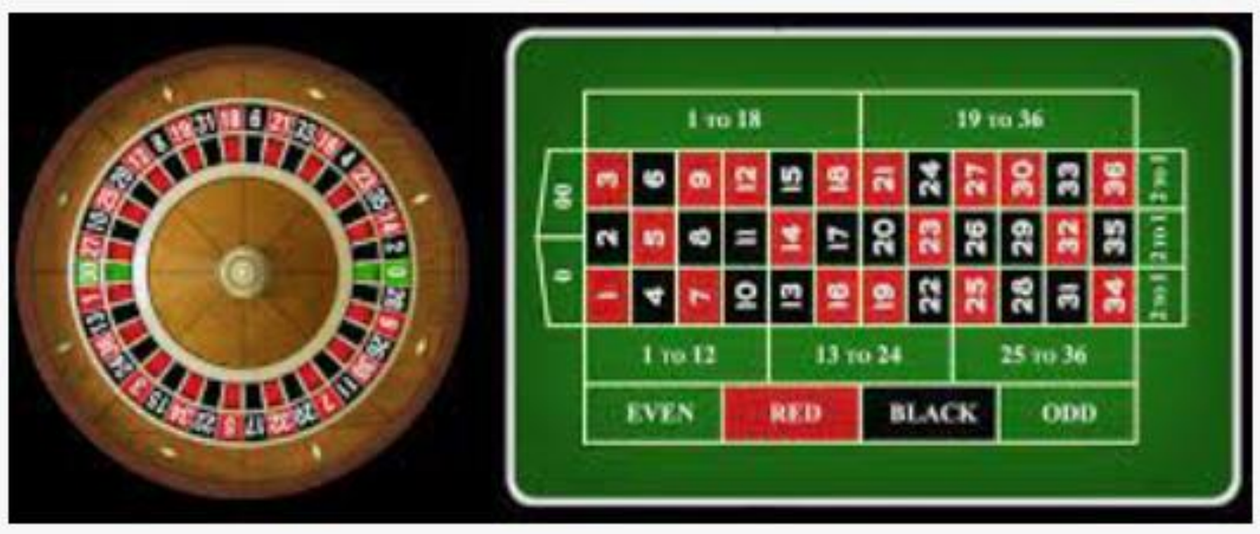
What is Expectation ?

Mathematical **expectation**, also known as the expected value, is the summation or integration of a possible values from a random variable. It is also known as the product of the **probability** of an event occurring, denoted P(x), and the value corresponding with the actual observed occurrence of the event

Expectation

Does gambling pay off ?

Roulette



$0, 0\ , 0, 1, 2, 3\ \dots\dots\dots 36$

$Standard\ pay\ off$
$= \mathbf{35{:}\,1}, if\ the\ ball\ lands$
$on\ your\ number$

$If\ you\ play\ this\ game\ a$
$1000\ times, howmuch\ do$
$you\ expect\ to\ win\ on\ an$
$average$

$if\ \ x\ is\ my\ profit\ random\ variable\ then$ (-1, 34)

$\Omega\ is\ 00, 0, 1, 2, 3, 4, 5, 6, \dots\dots\dots\dots\dots\dots\dots\dots .36$

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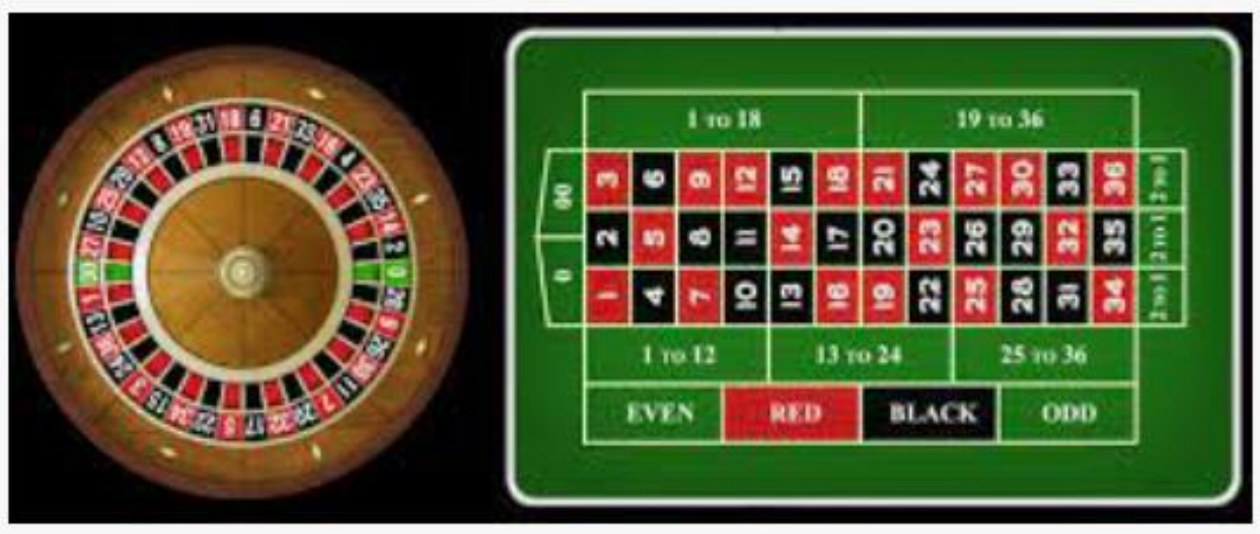
Expectation

Does gambling pay off ?

Roulette



if x is my profit random variable then (-1, 34)

Ω *is* $00, 0, 1, 2, 3, 4, 5, 6, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .36$

$Avg\ Gain = \frac{1}{1000}(26 * 34) + (974 * (-1))$ = - 90

$0, 0, 0, 1, 2, 3 \ldots \ldots \ldots 36$

$p_x(34)$ = 1/38 = 0.026

$p_x(1)$ = 1- 0.026 = 0.974

If you play this game a 1000 times, howmuch do you expect to win on an average

$p_x(win) = \frac{\#\ wins}{\#\ Games}$

$0.026 = \frac{\#\ wins}{1000}$

$\#\ wins = 26$

Expectation

Formula for Expectation:

$$E[X] = \Sigma_{x \in R_x} x * p_x(x)$$

$$E[X] = \frac{1}{1000}(\mathbf{26} * \mathbf{34}) + (\mathbf{974} * (-\mathbf{1}))$$

$$E[X] = \frac{26}{1000} * \mathbf{34} + \frac{974}{1000} * -1$$

$$E[X] = 0.026 * \mathbf{34} + 0.974 * -1$$

$$E[X] = p_x(34) * \mathbf{34} + p_x(-1) * -1$$

$$E[X] = \Sigma_{x \in \{-1,34\}} x * p_x(x)$$

$\frac{1}{2}$ $\frac{1}{2}$

$\frac{34}{2}$ $\frac{-1}{2}$

$\frac{1}{2}$ 34 + $\frac{1}{2}$-1

$p_x(34) * 34 + p_x(-1)$-1

The expected value or expectation of a discrete Random variable X whose possible values are X1, x2, x3, x4, x5, ...x_n is denoted by E[X] and computed as $E[X] = \Sigma_{i=1}^{n} x_i \, P(X = x_i)$

$$= \boldsymbol{\Sigma_{i=1}^{n} x_i * p_x(x_i)}$$

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Expectation

Expectation: Insurance

A person buys a car theft insurance policy of INR 200000 at an annual premium of INR 6000. There is a 2% chance that the car may get stolen

What is the expected gain of the insurance company at the end of 1 year ?

$E[X] = \Sigma_{x \in R_x} x * p_x(x)$

$E[X] = 0.98 * 6000 + 0.02 * (-194000)$

$E[X] = 2000$

X : Profit
X : {6000, - 194000}

$p_x(6000) = 0.98$
$p_x(-194000) = 0.02$

Expectation

Expectation: Insurance

A person buys a car theft insurance policy of INR 200000, Suppose there is a 10% chance that the car may get stolen

What should the premium be so that the expected gain is INR 2000 ?

$E[X] = \Sigma_{x \in R_x} x * p_x(x)$

$E[X] = 0.9 * x + 0.1 * (x - 200000)$

$E[X] = 2000 = 0.9 * x + 0.1 * (x - 200000)$

$X = 22000$

X : Profit

X : {x, -(200000 – x)}

X : {x, (x - 200000)}

$p_x(x) = 0.90$
$p_x(x - 200000) = 0.10$

Expectation

Properties of Expectation:

Linearity of expectation $\qquad$ Y = aX + b

$E[Y] = \Sigma_{x \in R_x} g(x) * p_x(x)$

$E[Y] = \Sigma_{x \in R_x} (ax + b) * p_x(x)$

$E[Y] = \Sigma_{x \in R_x} a * x * p_x(x) + \Sigma_{x \in R_x} b * p_x(x)$

$E[Y] = a * \Sigma_{x \in R_x} x * p_x(x) + b * \Sigma_{x \in R_x} p_x(x)$

$E[Y] = a * E[X] + b * 1 \quad as \ \left(\Sigma_{x \in R_x} p_x(x)\right) = 1$

$E[X] = a\ E[X] + b$

Expectation

Properties of Expectation:

Given a set of random variables X1, X2, X3, X4…….., Xn

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

We will study more about it when we do central limit theorem

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Expectation

Properties of Expectation: Expectation as mean of population

Ω

W: Weights 30 45 60

N Students

$p_w(w_i) = \frac{1}{n}$

E[W] = $\Sigma_{i=1}^{n} p_w(w_i) * w_i$

E[W] = $\frac{1}{n} * w_1 + \frac{1}{n} * w_2 \dots\dots\dots \frac{1}{n} * w_n$

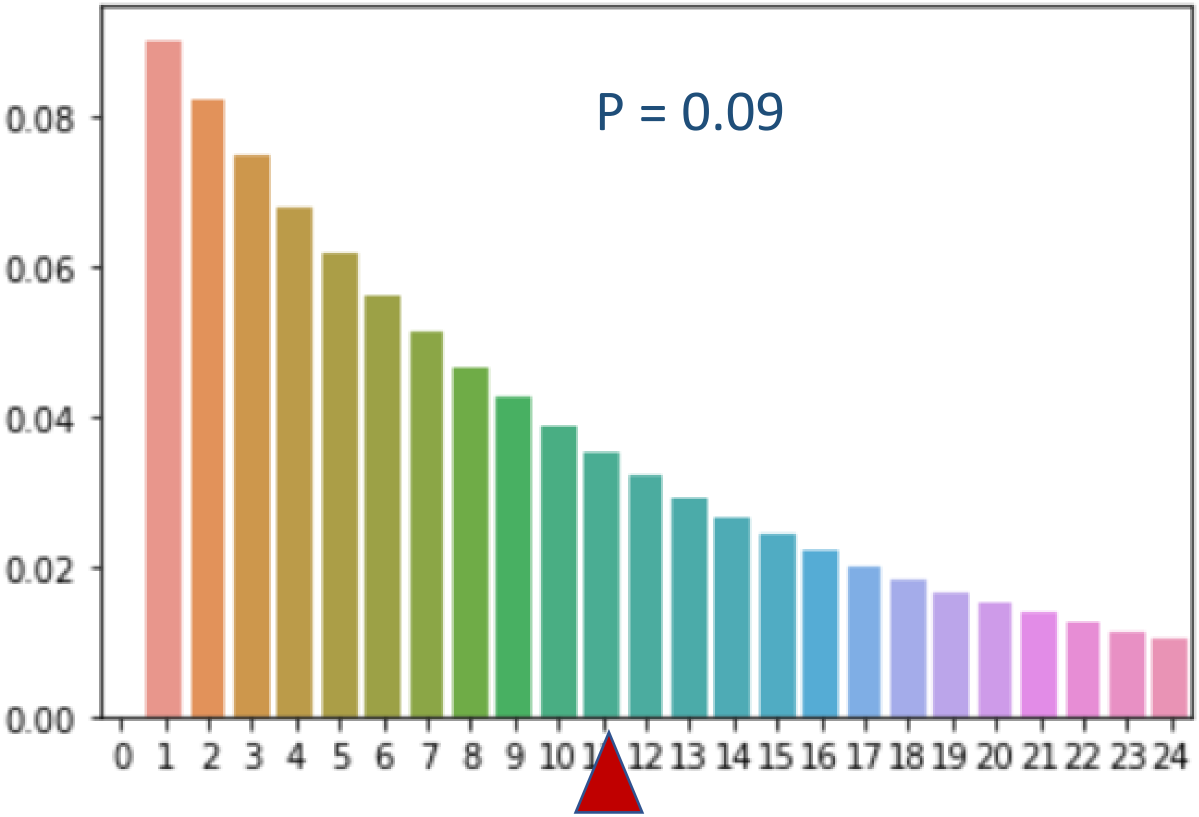
E[W] = $\frac{1}{n}\ \Sigma_{i=1}^{n} w_i$

(Center of gravity)

Expectation

Properties of Expectation: Expectation as mean of population

A patient needs a certain blood group with only 9% of the population has ?



E[X] = ?

$= p + (1-p)^1\, p + (1-p)^2\, p + \cdots \ldots \ldots \ldots + (1-p)^\infty\, p$

$= 1 * 0.09 + 2 * 0.91 * 0.09 + 3 * 0.91^2 * 0.09 + 4 * 0.91^3 * 0.09 + \ldots\ldots\ldots\ldots$

$= 0.09\,(1 + 2 * 0.91 + 3 * 0.91^2 + 4 * 0.91^3 + \ldots\ldots\ldots\ldots$

$= 0.09 \left(\frac{a}{1-r} + \frac{dr}{(1-r)^2}\right) \quad (a = 1, d = 1, r = 0.91)$

$= \frac{1}{P} = \frac{1}{0.09} = 11.11$