Isothermal hydrodynamics

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- Combine mass ρ conservation with momentum $m = \rho v$ conservation

  $\Rightarrow$ assume isothermal gas so that pressure $p = c_i^2 \rho$ with constant c_i^2

  $\Rightarrow$ c_i is the isothermal sound speed

  $\Rightarrow$ 1D isothermal hydrodynamics governed by

$$\begin{cases} \rho_t + (\rho\, v)_x & = \; 0 \\ m_t + (m\, v + p)_x & = \; 0 \end{cases}$$

- 1D dynamics of gas at uniform temperature

- vector of conserved quantities $U = \begin{bmatrix} \rho \\ m \end{bmatrix}$

  $\Rightarrow$ write as $U_t + (F(U))_x = 0$

  $\Rightarrow$ flux vector

$$F(U) = \begin{bmatrix} m \\ \frac{m^2}{\rho} + c_i^2 \rho \end{bmatrix}$$

- Manipulate to $U_t + F_U\, U_x = 0$ with flux Jacobian matrix

$$F_U \equiv \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 \\ c_i^2 - \frac{m^2}{\rho^2} & \frac{2m}{\rho} \end{bmatrix}$$

- Determine (right) eigenvalues and eigenvectors

  $\Rightarrow F_U \mathbf{r} = \lambda \mathbf{r}$

  $\Rightarrow$ find $\lambda_1 = v - c_i$ and $\lambda_2 = v + c_i$

- Eigenvalue $\lambda_1 = v - c_i$ has eigenvector $\mathbf{r}_1 = \begin{bmatrix} 1 \\ v - c_i \end{bmatrix}$

- Eigenvalue $\lambda_2 = v + c_i$ has eigenvector $\mathbf{r}_2 = \begin{bmatrix} 1 \\ v + c_i \end{bmatrix}$

  $\Rightarrow$ introduce matrix R and Λ such that $F_U R = R\Lambda$ as

$$\begin{bmatrix} 0 & 1 \\ c_i^2 - v^2 & 2v \end{bmatrix} \begin{bmatrix} 1 & 1 \\ v - c_i & v + c_i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ v - c_i & v + c_i \end{bmatrix} \begin{bmatrix} v - c_i & 0 \\ 0 & v + c_i \end{bmatrix}$$

  $\Rightarrow$ reminiscent of linear hyperbolic systems

- Note conservation form of equations: $U_t + \nabla \cdot F(U) = 0$

  $\Rightarrow$ discrete equivalent for discontinuity traveling at speed s

  $\Rightarrow$ Rankine-Hugoniot conditions

  $\Rightarrow$ connect left state $U_l = \begin{bmatrix} \rho_l \\ m_l \end{bmatrix}$ and right state $U_r = \begin{bmatrix} \rho_r \\ m_r \end{bmatrix}$

$$F(U_l) - F(U_r) = s\,(U_l - U_r)$$

- given U_r: RH yields system of 2 equations for three unknowns s, ρ_l, m_l

$$\begin{cases} m_l - m_r & = & s\,(\rho_l - \rho_r) \\ \dfrac{m_l^2}{\rho_l} - \dfrac{m_r^2}{\rho_r} + c_i^2\,(\rho_l - \rho_r) & = & s\,(m_l - m_r) \end{cases}$$

  $\Rightarrow$ Note obvious solution $s = 0$ and $U_l = U_r$

- Define Hugoniot locus for state U_r

  $\Rightarrow$ all states U_l that can be connected to U_r in accord with RH

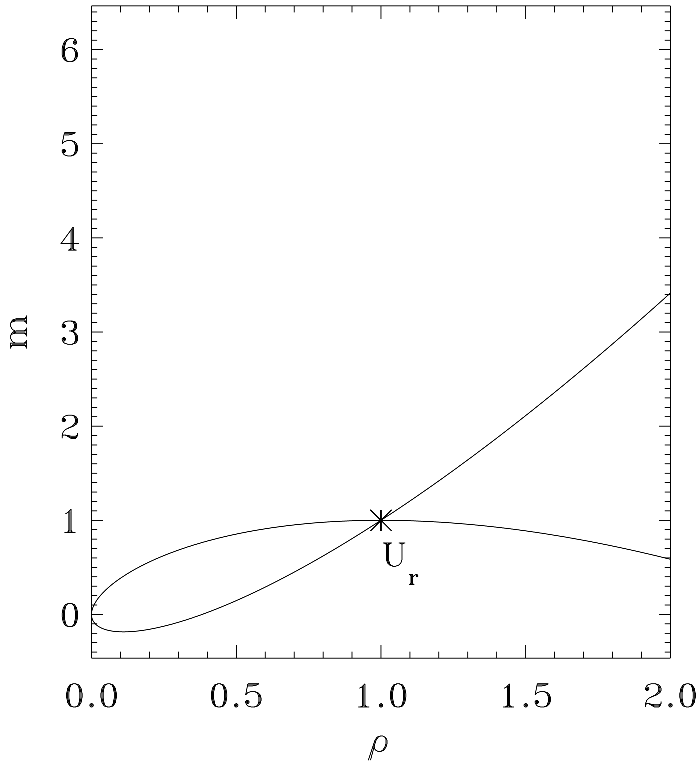
  $\Rightarrow$ manipulate system to

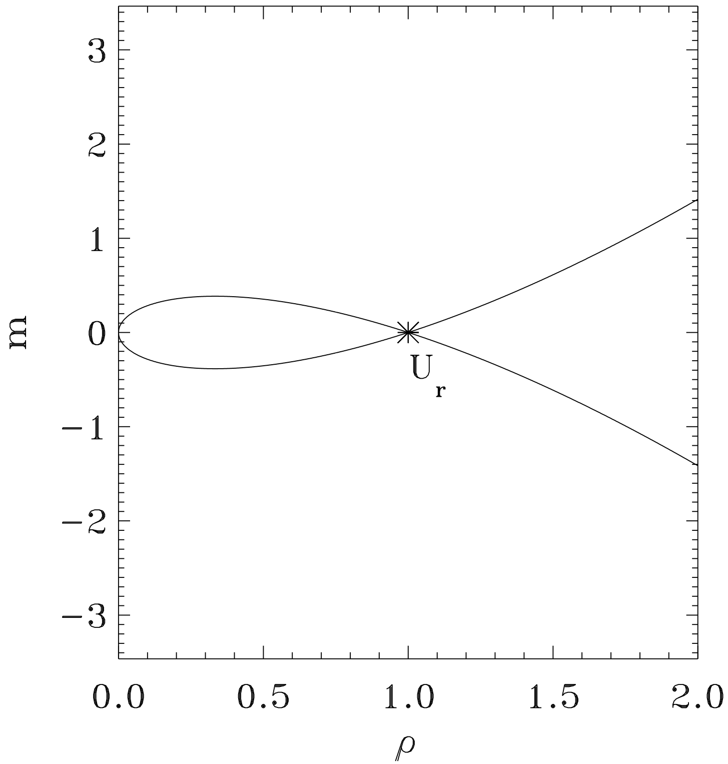
$$\begin{cases} m_l &= \frac{\rho_l}{\rho_r} m_r \pm \sqrt{\frac{\rho_l}{\rho_r}} c_i \left(\rho_l - \rho_r\right) \\ s &= \frac{m_r}{\rho_r} \pm \sqrt{\frac{\rho_l}{\rho_r}} c_i \end{cases}$$

  $\Rightarrow$ parametrize $\rho_l = \rho_r \left(1 + \xi\right)$, vary $\xi \in [-1, +\infty]$

  $\Rightarrow$ plot for given state U_r as curves in $\rho - m$ plane

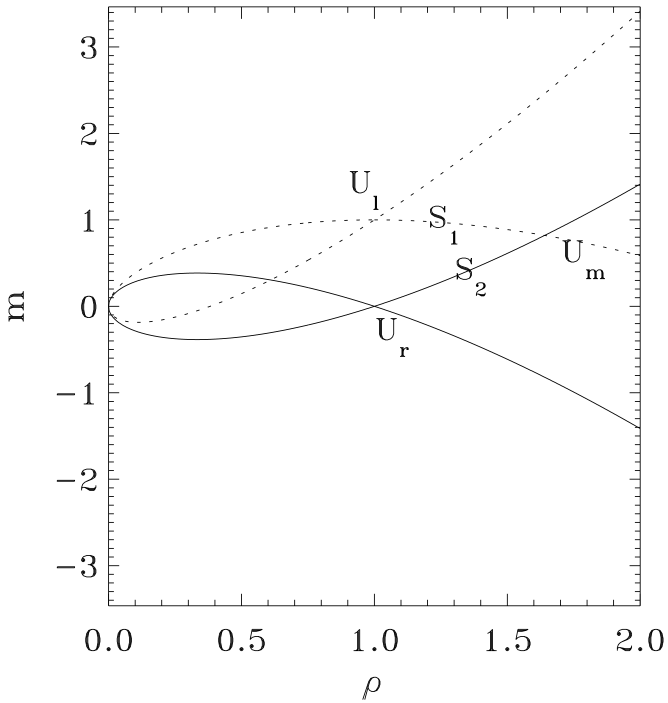
- Example Hugoniot loci for states $U_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

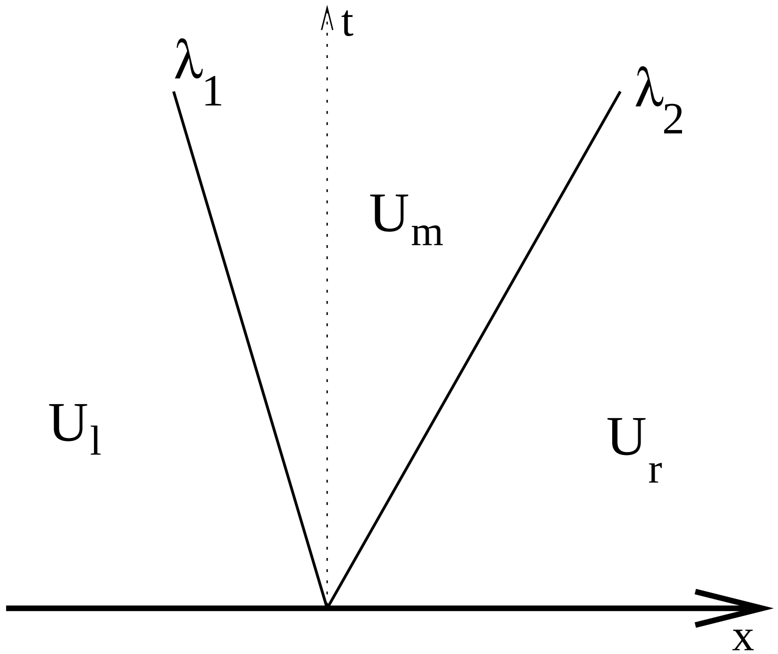




- Graphically, solution to Riemann problem for given states U_l and U_r

  $\Rightarrow$ construct Hugoniot loci: intersect at 2 intermediate states





  $\Rightarrow$ only one intersection physical: due to speed ordering $\lambda_1 < \lambda_2$

  $\Rightarrow$ find state U_m which connects U_l – 1-shock – U_m – 2-shock – U_r

  $\Rightarrow$ counts ok: 4 unknowns $U_m = \begin{bmatrix} \rho_m \\ m_m \end{bmatrix}$, s_1, s_2 for 4 equations (RH)

- unfortunately, we're not done yet . . .

- Return to RH system

$$\begin{cases} m_l - m_r & = & s\left(\rho_l - \rho_r\right) \\ \frac{m_l^2}{\rho_l} - \frac{m_r^2}{\rho_r} + c_i^2\left(\rho_l - \rho_r\right) & = & s\left(m_l - m_r\right) \end{cases}$$

$\Rightarrow$ analyse stationary shock $s = 0 \longrightarrow m_l = m_r$

$\Rightarrow$ second equation can be manipulated to

$$v_l + \frac{c_i^2}{v_l} = v_r + \frac{c_i^2}{v_r} = 2c_i$$

$\Rightarrow$ latter equality holds at sonic point where $v = c_i$

- deduce $c_i^2 = v_r v_l$:**Prandtl-Meyer** relation

$\Rightarrow$ shock separates subsonic from supersonic state

$\Rightarrow$ Mach number $M_l \equiv \frac{v_l}{c_i}$: find $M_l^2 = \frac{v_l}{v_r}$ and $\frac{\rho_l}{\rho_r} = \frac{1}{M_l^2}$

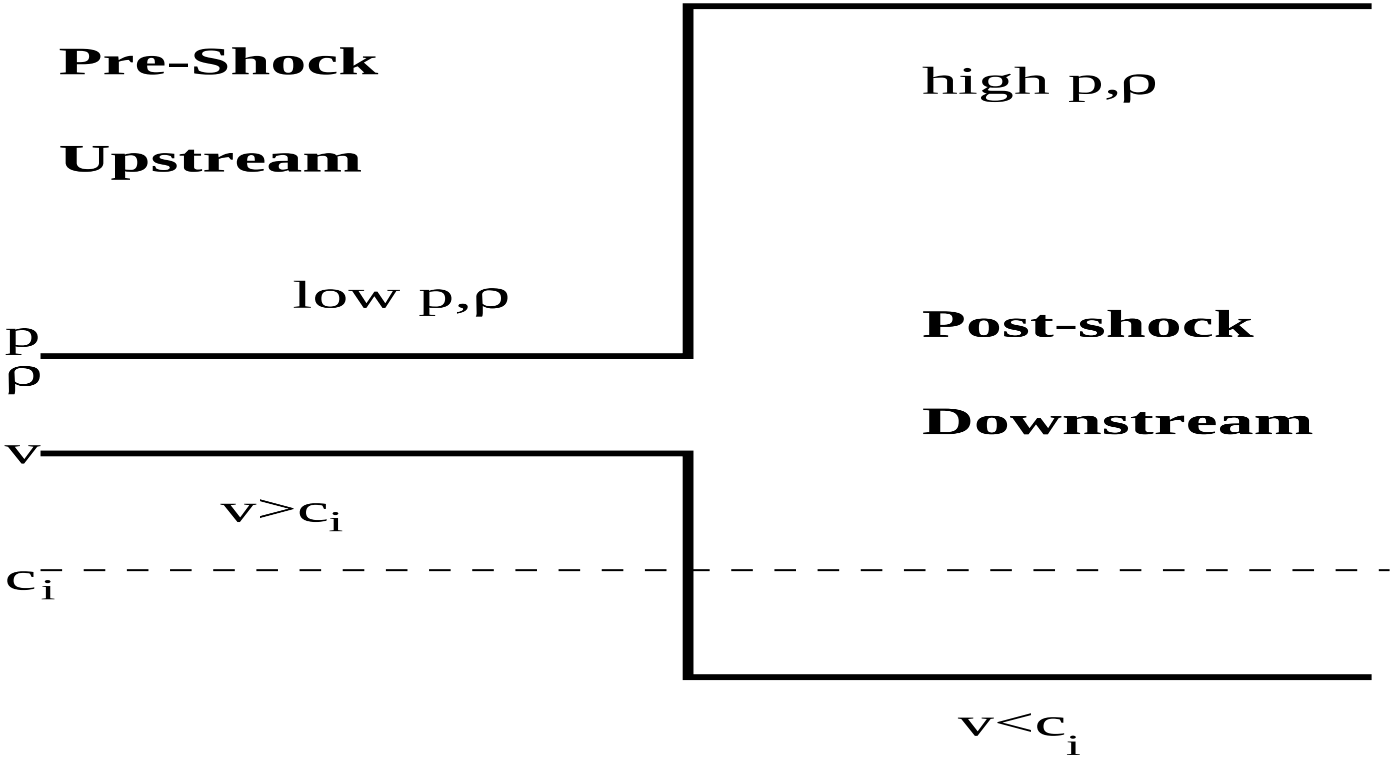
$\Rightarrow$ density is lower in supersonic state than in subsonic state

$\Rightarrow$ high p and ρ subsonic state, low p and ρ supersonic state

- similarly for shock at finite speed s: Galilean transformation

$$M_l^2 = \frac{v_l - s}{v_r - s} = \frac{\rho_r}{\rho_l}$$

$\Rightarrow$ as fluid goes through shock: sees shock coming supersonically, gets compressed and raised in p, v drops

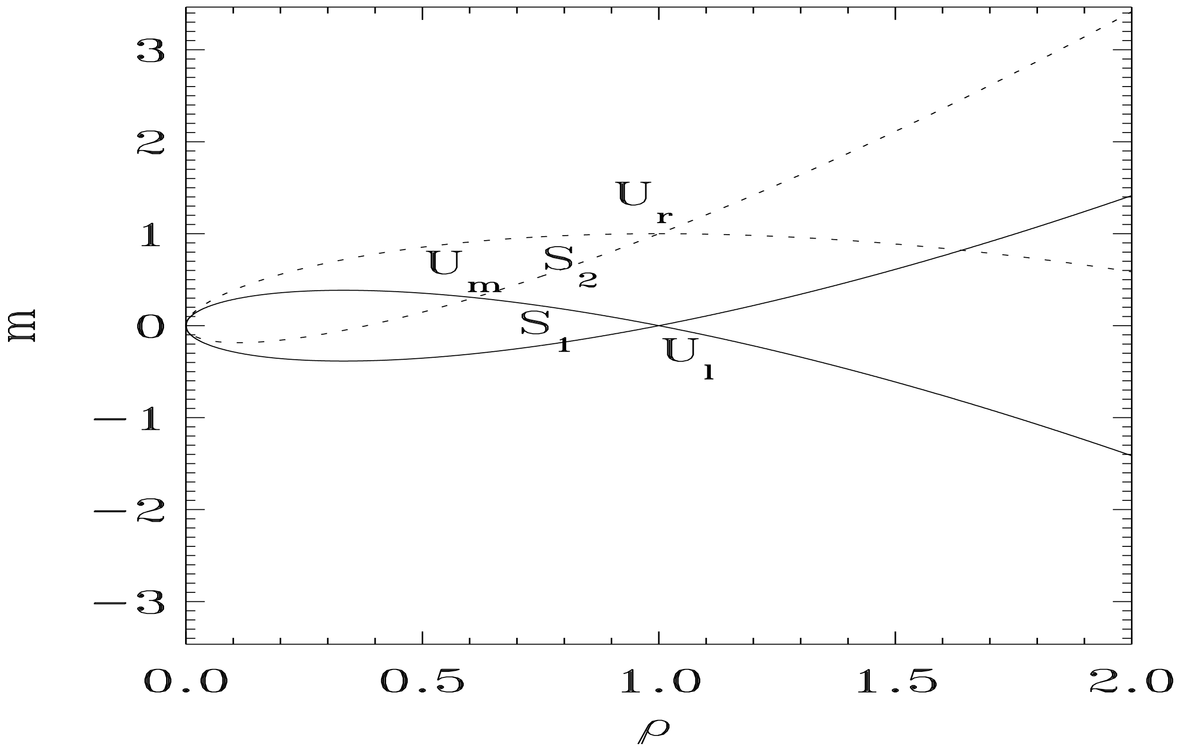


- However: not all shocks from Hugoniot Locus are realizable

  $\Rightarrow$ remember from Burgers example:

  $\Rightarrow$ switch U_l and U_r does not lead to same solution

- in particular for previous case plotted: switch



$\Rightarrow$ will not be realized in practice: something is missing ...

- Remember solution of advection equation:

  $\Rightarrow$ Contact Discontinuity (density jump advected at speed v)

  $\Rightarrow$ is this ingredient missing?

- Turns out that CD is not possible for 1D isothermal HD

  $\Rightarrow$ jump in density while constant $v \longrightarrow$ jump in pressure

  $\Rightarrow$ non-equilibrium: pressure imbalance

  $\Rightarrow$ Does not satisfy RH conditions!

- Remember ‘Rarefaction’ solution from Burgers equation

  $\Rightarrow$ note solution is constant on rays $x = \xi t$

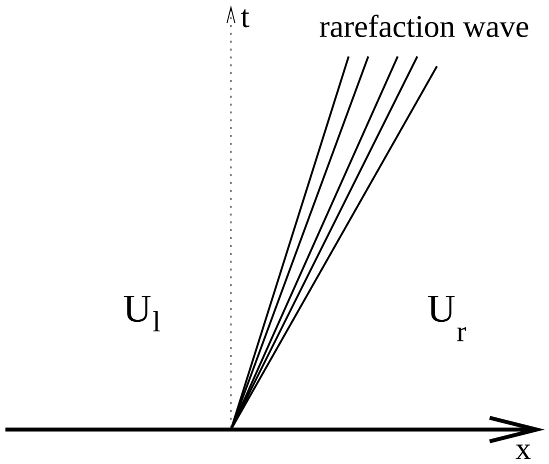
  $\Rightarrow$ generalize to ‘similarity solutions’: function of $x/t = \xi$ alone

- Rarefaction wave is similarity solution of form

$$U(x,t) = U(x/t) = \begin{cases} U_l & x \leq \xi_1 t \\ W(x/t) & \xi_1 t < x < \xi_2 t \\ U_r & \xi_2 t \leq x \end{cases}$$

  $\Rightarrow$ where $W(x/t)$ is smooth function with $\begin{cases} W(\xi_1) = U_l \\ W(\xi_2) = U_r \end{cases}$

- graphically:



- If $U(x,t) = W(x/t)$ then $U_t = -\frac{x}{t^2}W'(x/t)$ and $U_x = \frac{1}{t}W'(x/t)$

  $\Rightarrow$ equation $U_t + F_U U_x = 0$ yields $F_U W' = \frac{x}{t}W'$

  $\Rightarrow$ W' is proportional to eigenvector $\mathbf{r}_p$ of F_U

  $\Rightarrow$ $\frac{x}{t} = \xi$ will be eigenvalue λ_p

  $\Rightarrow$ write $W' = \alpha(\xi)\mathbf{r}_p(W(\xi))$ and $\xi = \lambda_p(W(\xi))$

  $\Rightarrow$ differentiate to ξ and find

$$1 = \nabla\lambda_p \cdot W' = \nabla\lambda_p \alpha \mathbf{r}_p$$

  $\Rightarrow$ hence $W' = \frac{1}{\nabla\lambda_p \cdot \mathbf{r}_p}\mathbf{r}_p$

  $\Rightarrow$ note $\nabla\lambda_p = \begin{pmatrix} \frac{\partial\lambda_p}{\partial\rho} & \frac{\partial\lambda_p}{\partial m} \end{pmatrix}$

- this gives the following system of ODEs for 1-rarefaction:

$$\begin{pmatrix} \rho' \\ m' \end{pmatrix} = \begin{pmatrix} -\frac{\rho}{c_i} \\ -\frac{m}{c_i} + \rho \end{pmatrix}$$

$\Rightarrow$ solution is $\rho = \rho_l e^{-(\xi - \xi_1)/c_i}$

$\Rightarrow$ and momentum $m = \left[m_l + \rho_l \left(\xi - \xi_1\right)\right] e^{-(\xi - \xi_1)/c_i}$

- these equations define the integral curves for 1-rarefactions:

$\Rightarrow$ eliminate ξ to get $m(\rho)$ for given ρ_l, m_l

$\Rightarrow m = m_l \frac{\rho}{\rho_l} - c_i \rho \ln \frac{\rho}{\rho_l}$

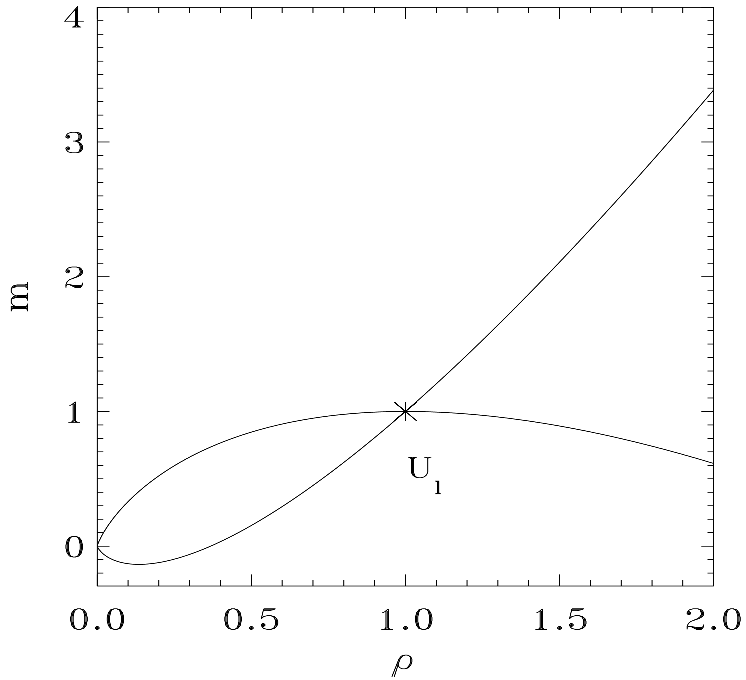
- similarly for integral curves for 2-rarefactions:

$$\Rightarrow m = m_l \frac{\rho}{\rho_l} + c_i \rho \ln \frac{\rho}{\rho_l}$$

- plot integral curves for given ρ_l, m_l

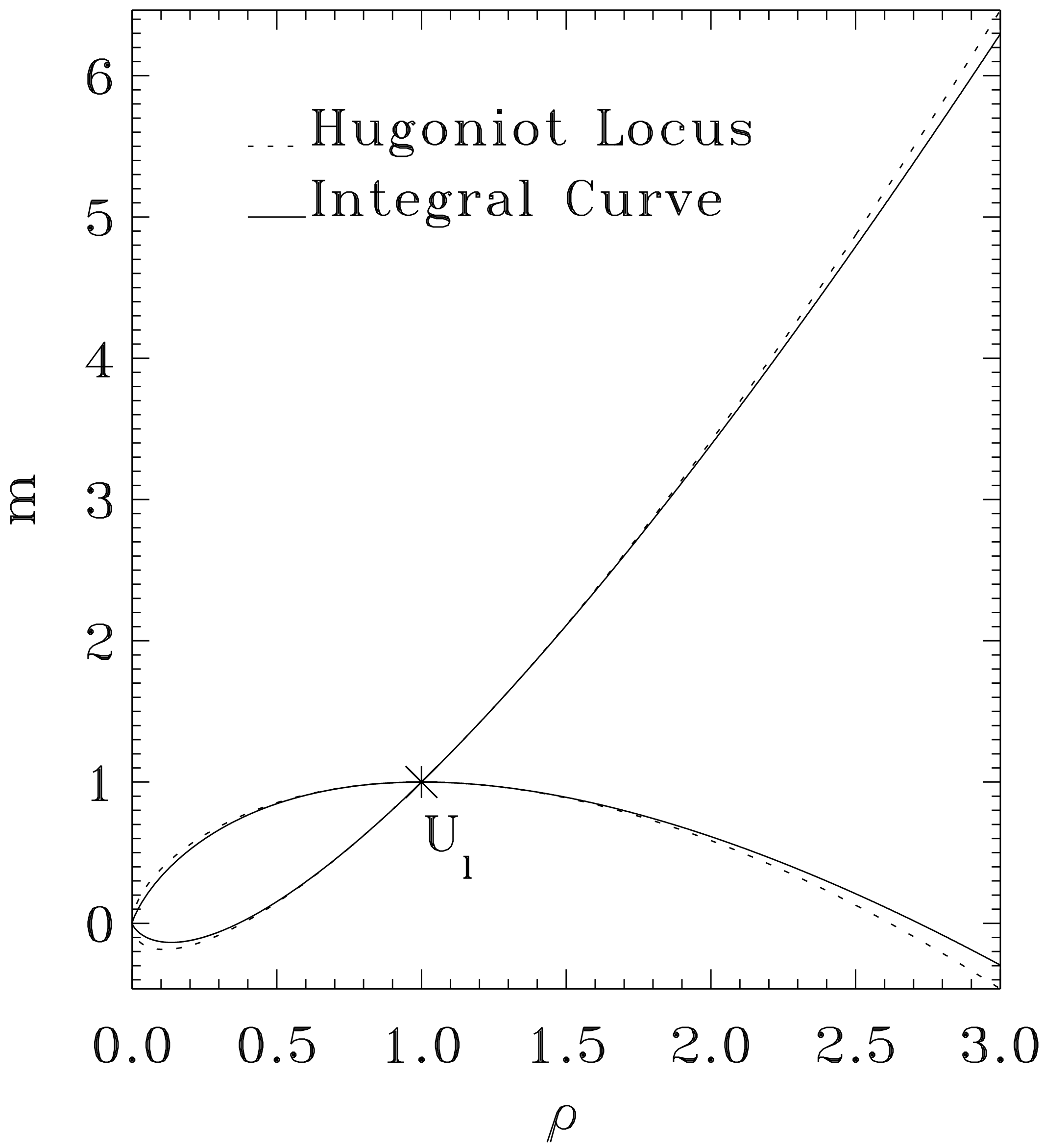
$\Rightarrow$ note obvious solution $m = m_l$ and $\rho = \rho_l$

$\Rightarrow$ parametrize $\rho = \rho_l(1 + \xi)$ and plot in ρ-m plane



- Comparison Hugoniot Loci with Integral curves

  $\Rightarrow$ coincide locally, but different



- For solution to Riemann Problem

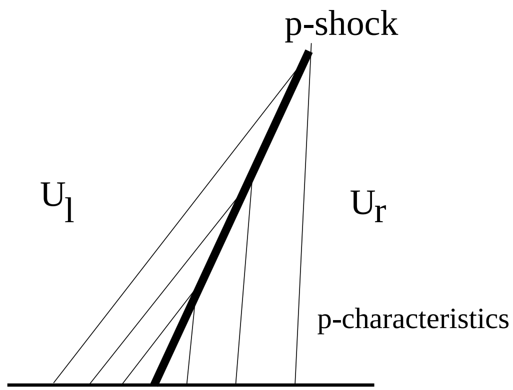
  $\Rightarrow$ which part of Hugoniot Locus and Integral curves permissable?

- **Lax entropy condition:** Jump in p-th field ok if

$$\lambda_p(U_l) > s > \lambda_p(U_r)$$

  $\Rightarrow$ shock speed must lie in between characteristic speed of 2 states

  $\Rightarrow$ Note: asymmetric in left versus right state!



- p-characteristics enter shock

  $\Rightarrow$ from every point on shock: travel along characteristics backward in time, not forward

  $\Rightarrow$ information reaches shock from past, not from future

  $\Rightarrow$ 'causality' and time-irreversibility $\longrightarrow$ 'entropy' condition

- Analyse Hugoniot locus for isothermal case:

$\Rightarrow$ first consider right state $U_r = \begin{pmatrix} \rho_r \\ m_r \end{pmatrix}$ and 1-shock

$\Rightarrow$ recall $\lambda_1 = v - c_i$ and parametrized

$$U_l = \begin{pmatrix} \rho_l \\ m_l \end{pmatrix} = \begin{pmatrix} \rho_r(1+\xi) \\ m_r(1+\xi) \pm \sqrt{1+\xi}c_i\rho_r \mid \xi \mid \end{pmatrix}$$

$\Rightarrow$ associated shock speed $s = \frac{m_r}{\rho_r} \pm \sqrt{1+\xi}c_i\frac{|\xi|}{\xi}$

- entropy condition becomes

$$\underbrace{\frac{m_r}{\rho_r} - c_i}_{\lambda_1(U_r)} < \underbrace{\frac{m_r}{\rho_r} \pm \sqrt{1+\xi}c_i\frac{\mid \xi \mid}{\xi}}_{s} < \underbrace{\frac{m_r(1+\xi) \pm \sqrt{1+\xi}c_i\rho_r \mid \xi \mid}{\rho_r(1+\xi)} - c_i}_{\lambda_1(U_l)}$$

- analyse for higher density left states $\xi > 0$

  $\Rightarrow$ impossible to satisfy simultaneously!

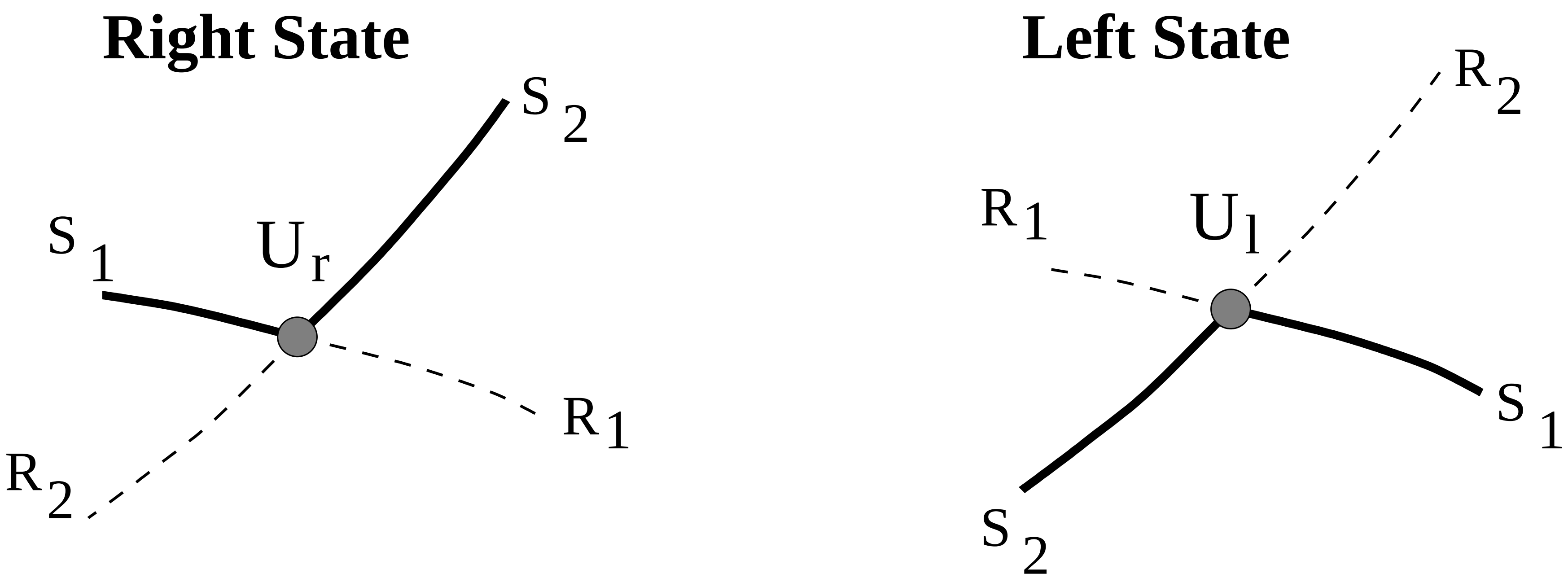
  $\Rightarrow$ right state U_r can not have S_1-shock to higher density states!

- analyse for lower density left states $-1 < \xi < 0$

  $\Rightarrow$ one sign combination allowed

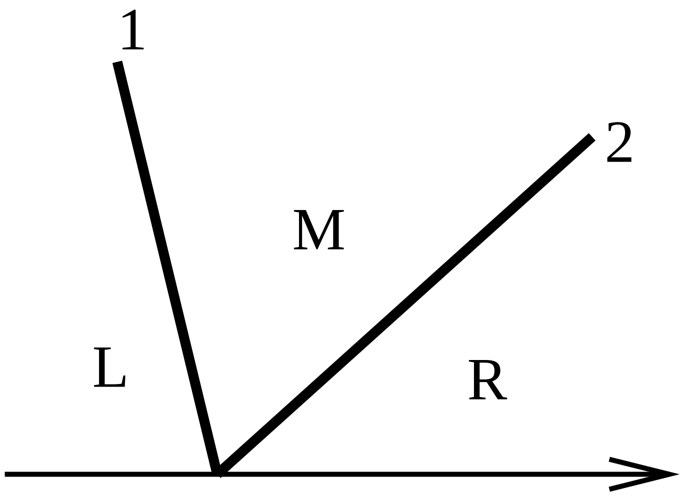
  $\Rightarrow$ can connect right state to lower density left state via S_1-shock

- situation is reversed for left state
- similarly: can connect right state U_r to higher density state via S_2

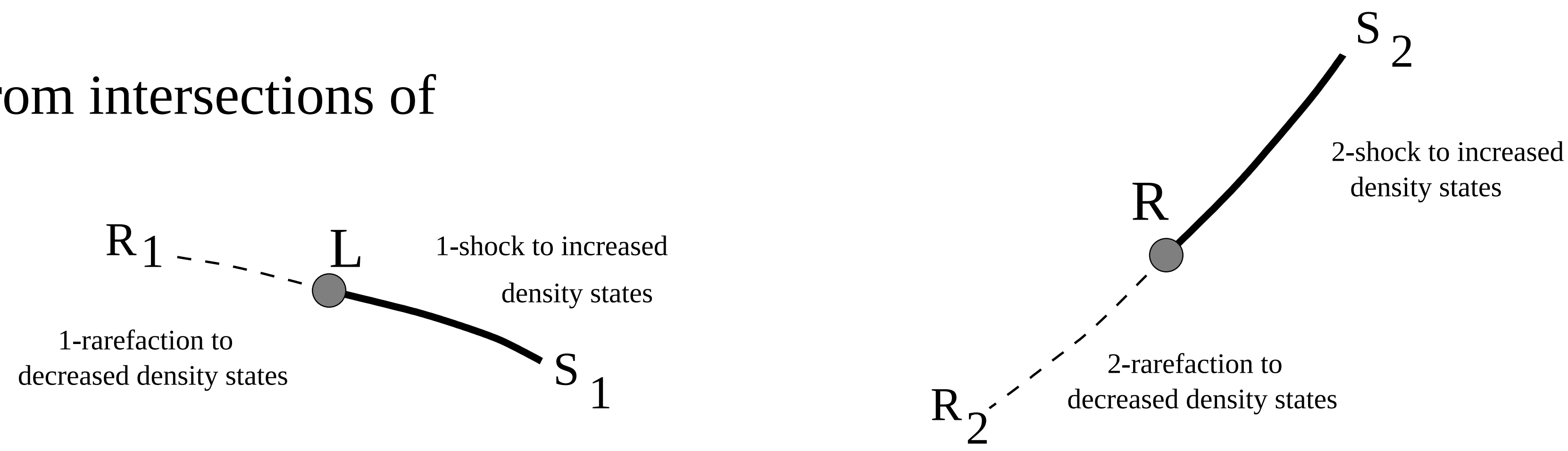


- Solution to RP: graphically

Form L-1-M-2-R

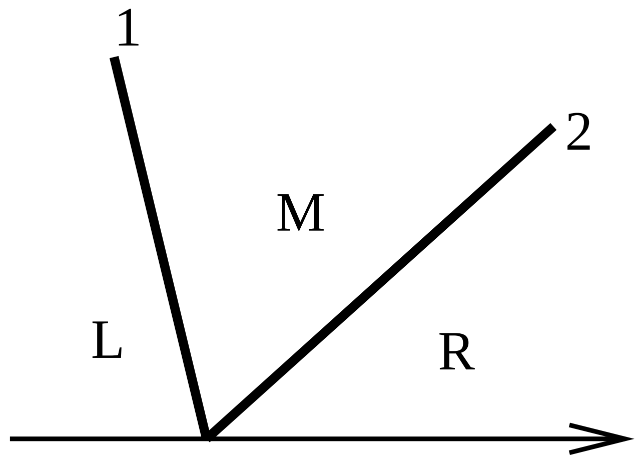


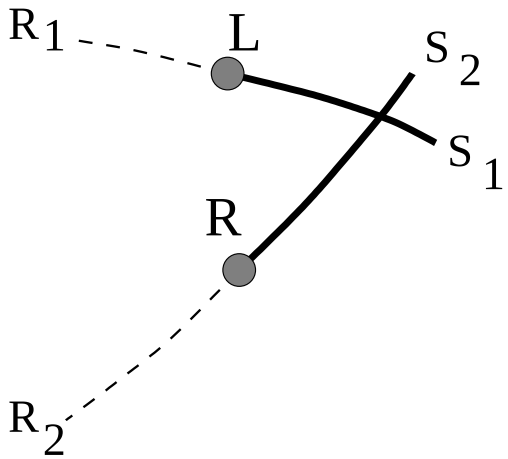
From intersections of

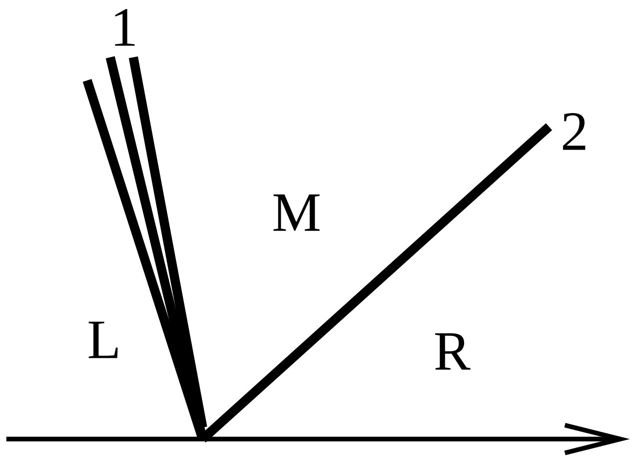


$\Rightarrow$ for shocks: gas gets compressed, ρ increases

$\Rightarrow$ for rarefactions: gas gets rarified, ρ drops


1
2
M
L
R


R1
L
S 2
S 1
R
R2


1
2
M
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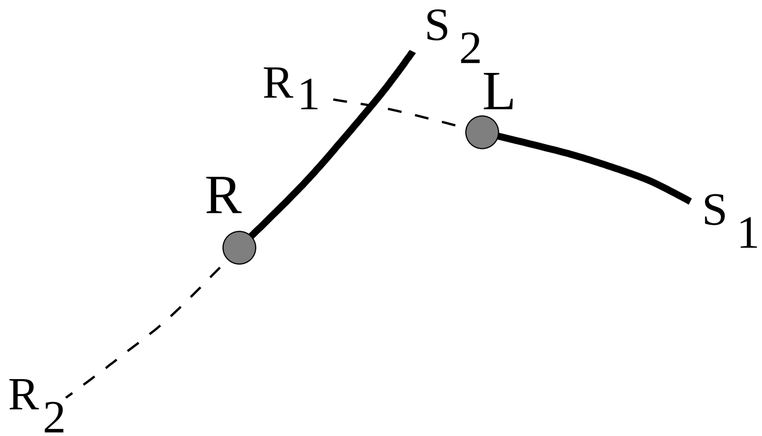
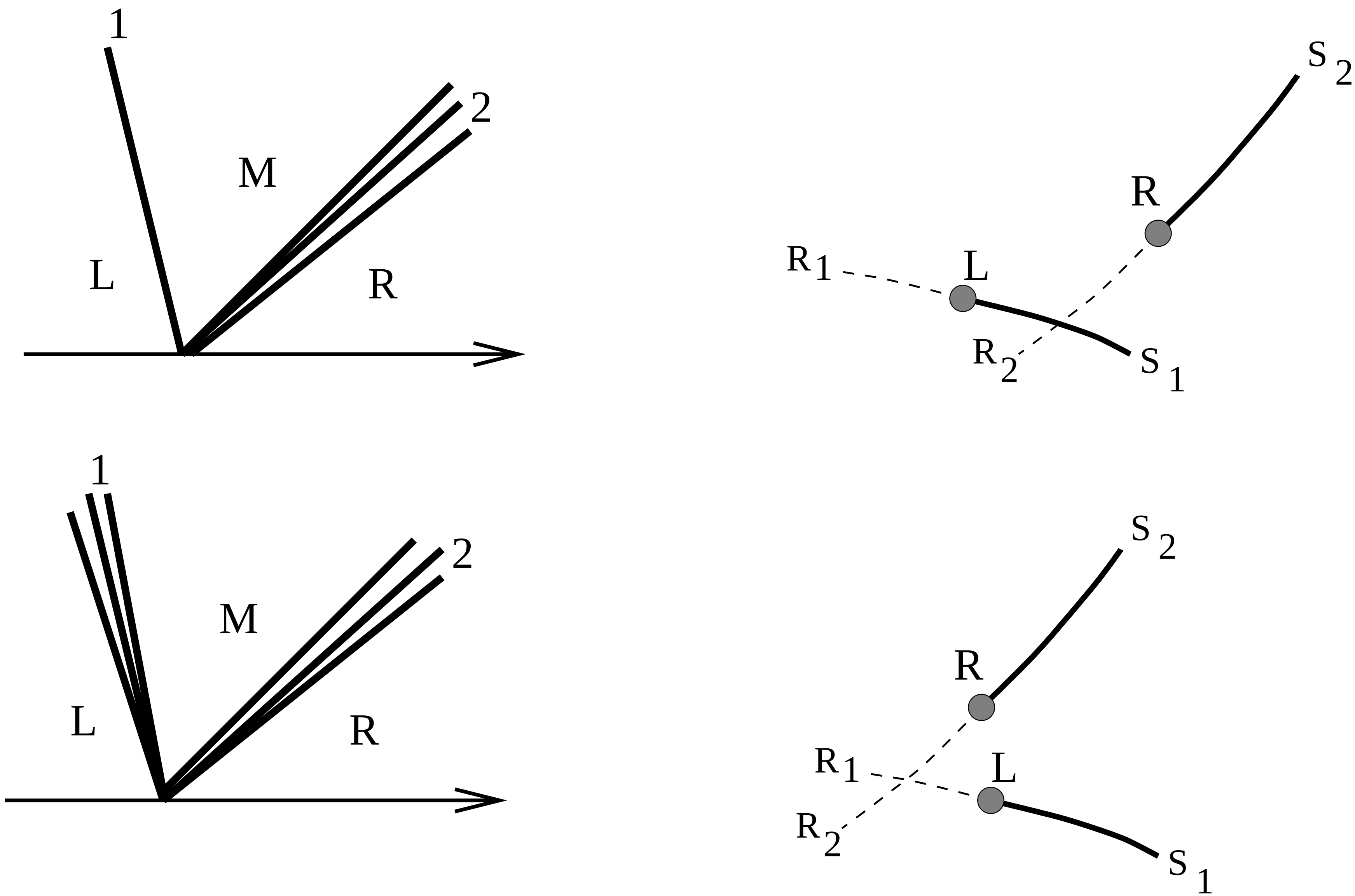
R


S 2
R1
L
R
S 1
R2


1
2
M
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R 2
S 1
1
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M
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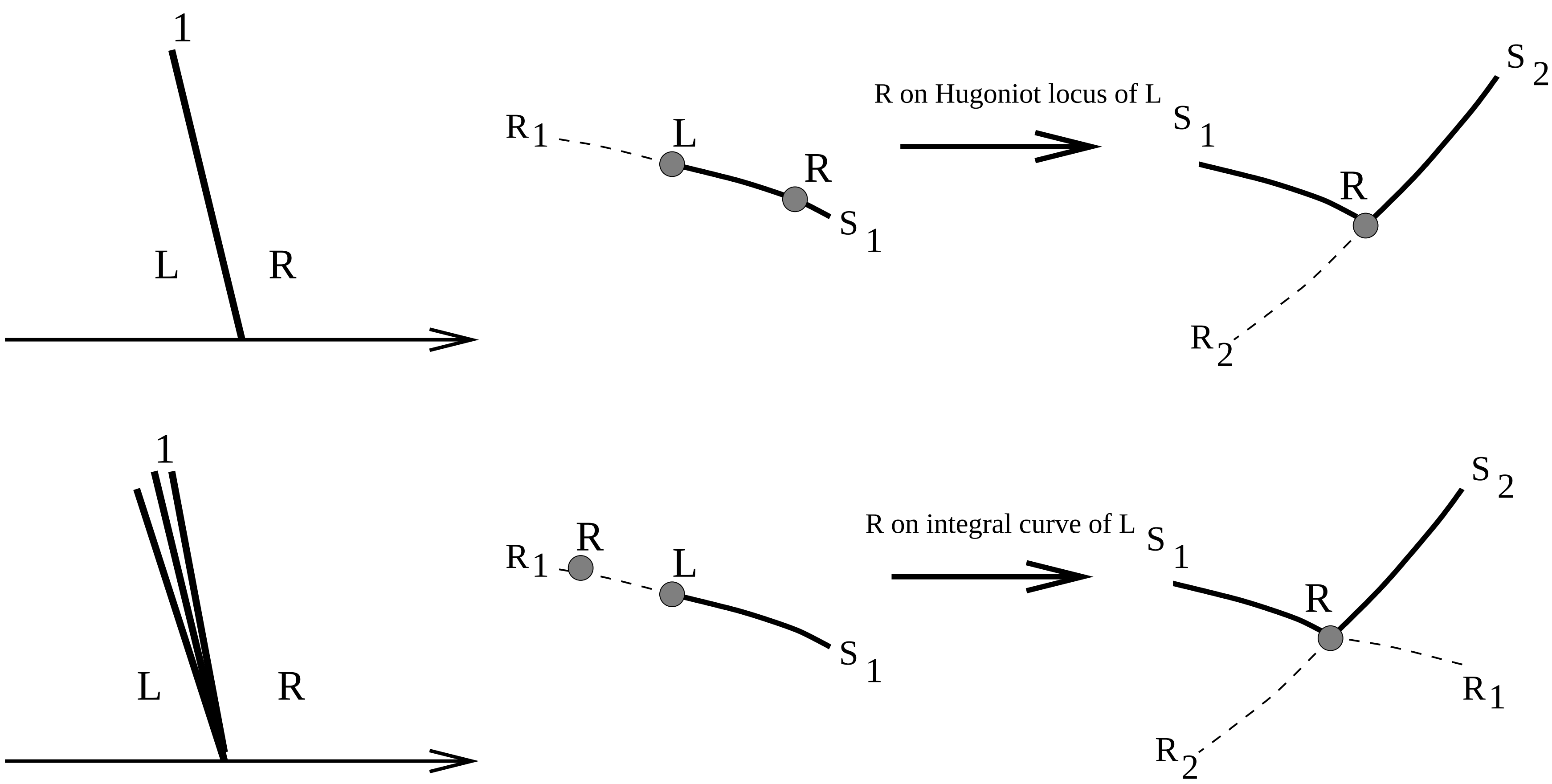
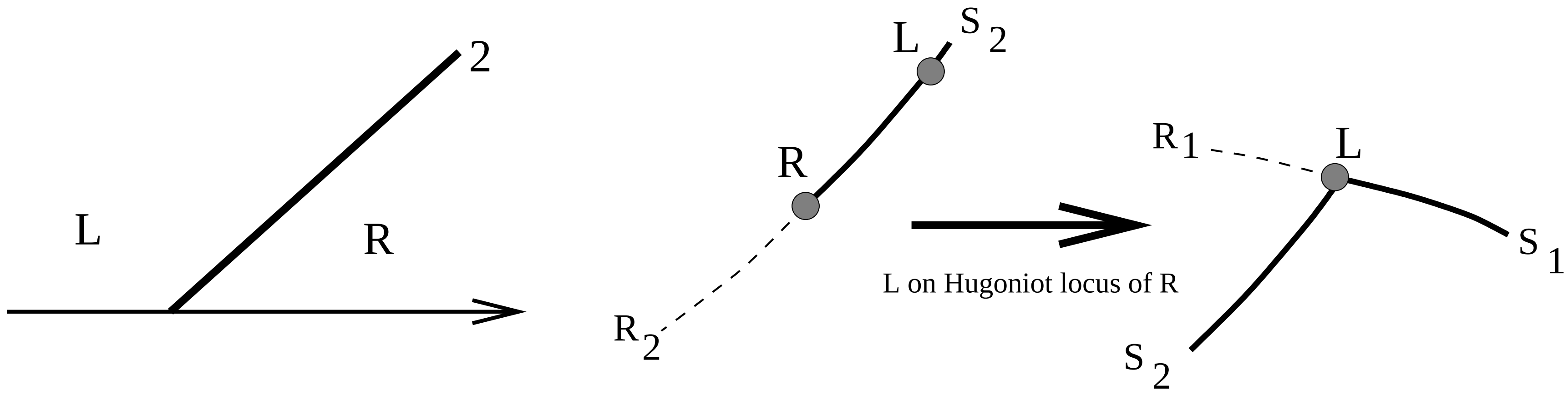
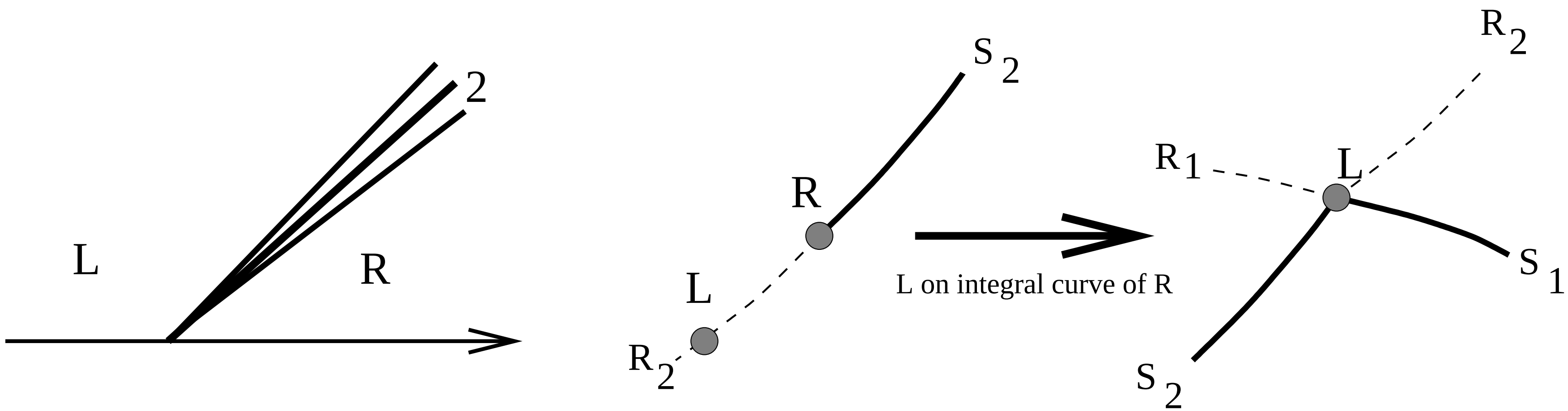
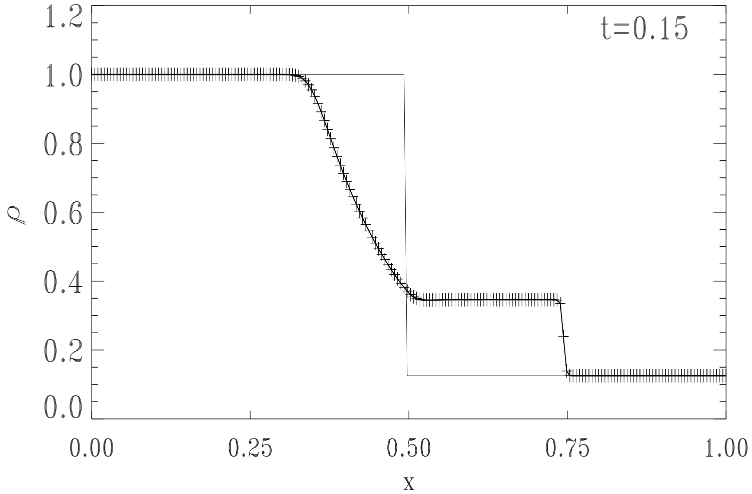
R
R 1
L
R 2
S 1

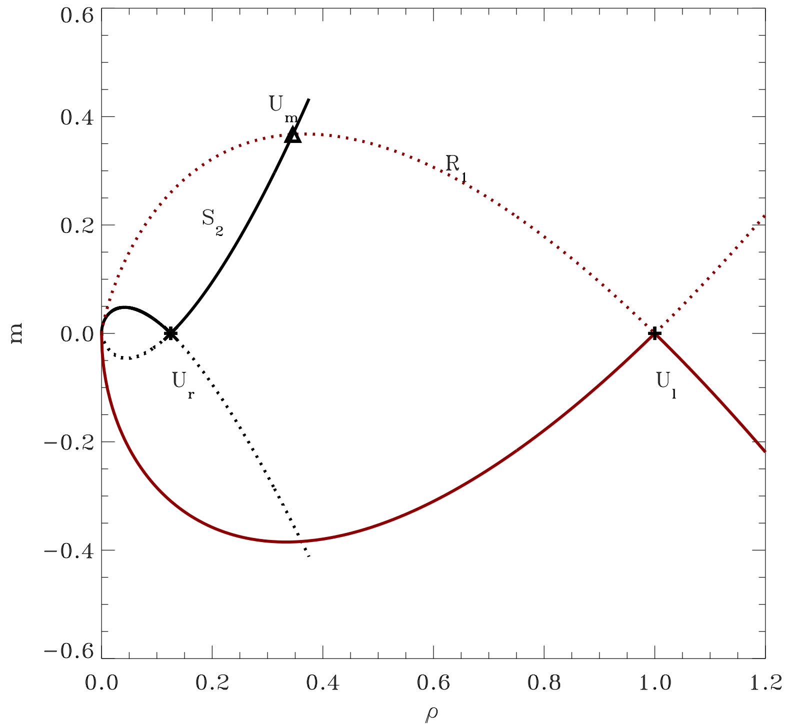

1
L
R
R_1
L
R
S_1
R on Hugoniot locus of L
S_1
S_2
R
R_2
1
L
R
R_1
R
L
S_1
R on integral curve of L
S_1
S_2
R
R_1
R_2


2
L
R
L
S 2
R
R 2
L on Hugoniot locus of R
R 1
L
S 1
S 2


2
L
R
S 2
R
L
R 2
L on integral curve of R
R 2
R 1
L
S 1
S 2

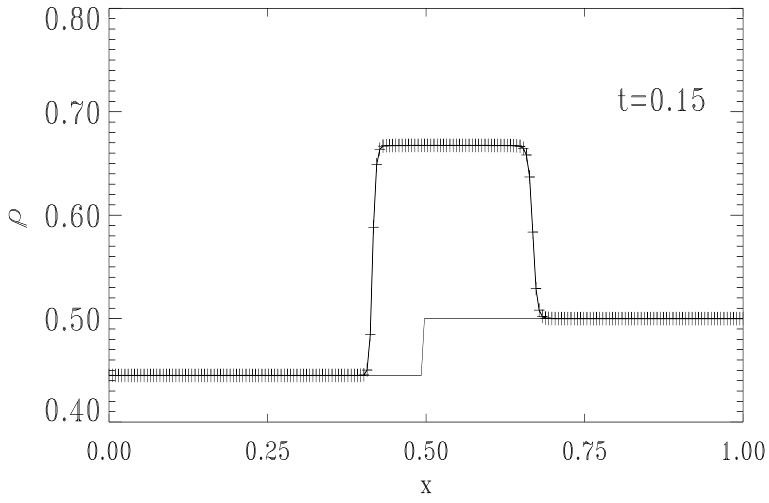
- 8 possibilities: to be verified numerically
- case $U_l = (1, 0)$ and $U_r = (0.125, 0)$, run till $t = 0.15$

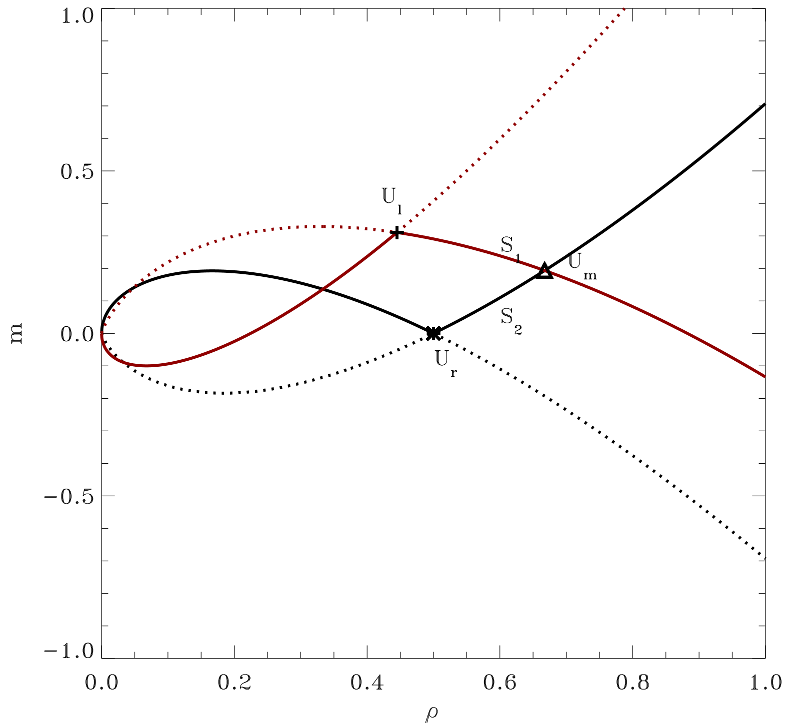




$\Rightarrow$ emerging $U_m = (0.346, 0.367)$ state: agrees with theory!

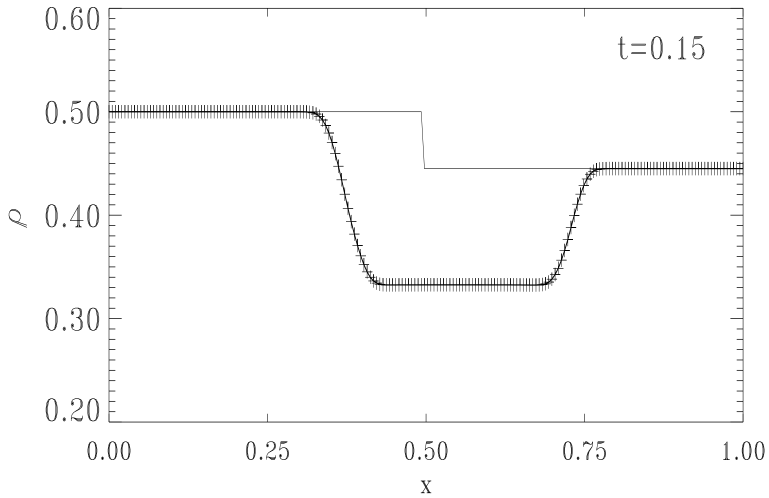
- case $U_l = (0.445, 0.31061)$ and $U_r = (0.5, 0)$, run till $t = 0.15$

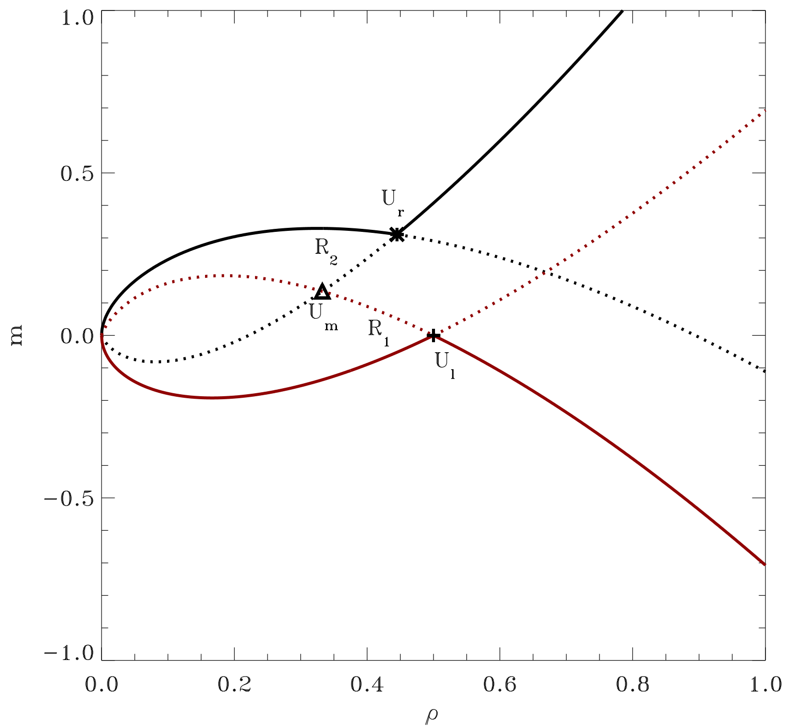




$\Rightarrow$ emerging $U_m = (0.667, 0.193)$ state: agrees with theory!

- case $U_l = (0.5, 0)$ and $U_r = (0.445, 0.31061)$, run till $t = 0.15$





$\Rightarrow$ emerging $U_m = (0.333, 0.136)$ state: agrees with theory!

- nomenclature: primitive variables $\begin{pmatrix} \rho \\ v \end{pmatrix}$ obey

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ \frac{c_i^2}{\rho} & v \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0$$

- can be manipulated to

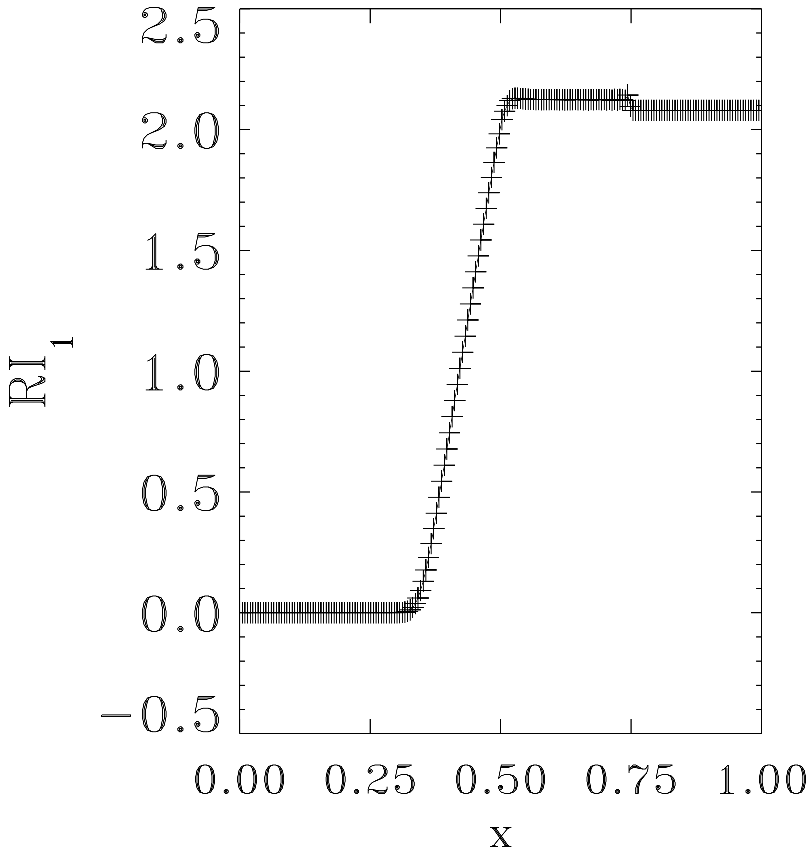
$$\begin{cases} (v - c_i \ln \rho)_t + (v - c_i)\,(v - c_i \ln \rho)_x = 0 \\ (v + c_i \ln \rho)_t + (v + c_i)\,(v + c_i \ln \rho)_x = 0 \end{cases}$$

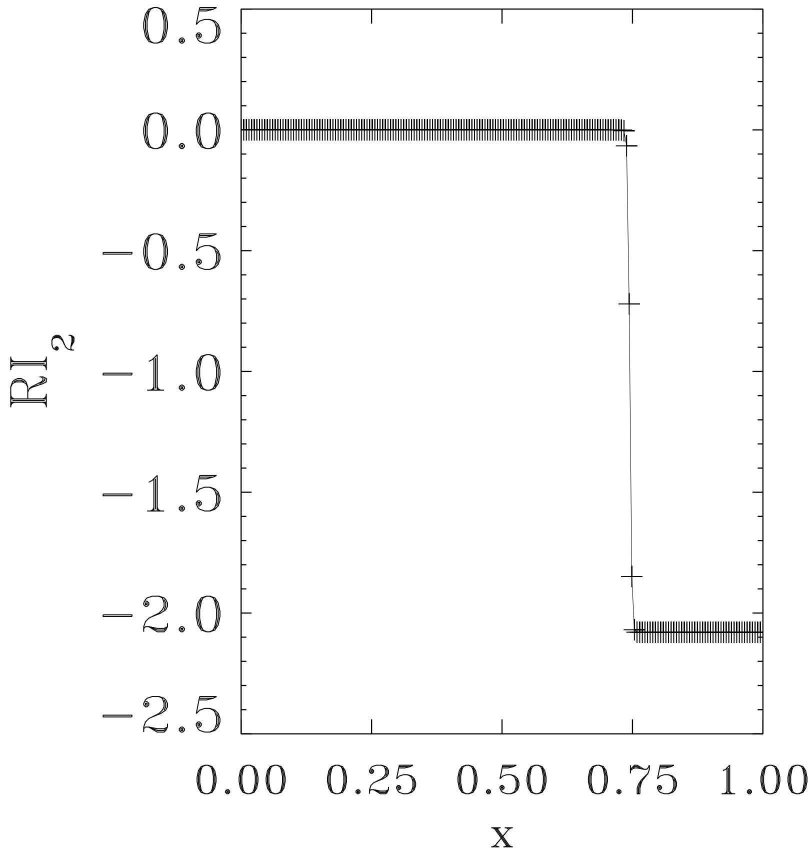
$\Rightarrow$ Riemann Invariants $v \pm c_i \ln \rho$

$\Rightarrow$ constant along characteristics $dx/dt = v \pm c_i$

- Plot RI for case where L – R_1 – M – S_2 – R

  $\Rightarrow$ constant RI_2 through R_1 rarefaction





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- R.J. Leveque *et al.*, *Computational Methods for Astrophysical Fluid Flow*, Saas-Fee Advanced Course 27, 1998, Springer-Verlag, Berlin
- P. Wesseling, *Principles of Computational Fluid Dynamics*, 2001, Spinger-Verlag, Berlin (Chapter 9)