Unit 2 Homework: Characterizing Random Variables

w203: Statistics for Data Science

This homework, like all homework, is due at the time of your live session.

Applied Practice

1. (9 points total) Processing Pasta

A certain manufacturing process creates pieces of pasta that vary by length. Suppose that the length of a particular piece (represented as a random variable L that takes on values l), is a continuous random variable with the following probability density function.

$$f(l) = \begin{cases} 0, & l \leq 0 \\ c \cdot l, & 0 < l \leq 2 \\ 0, & 2 < l \end{cases}$$

where c is a constant that makes $f(l)$ a valid probability distribution function.

a. (3 points) Compute the constant c. What value of c makes $f(l)$ a valid pdf?

$$\int_L f(l)dl = 1 \rightarrow \int_0^2 cldl = 1 \rightarrow c(\frac{l^2}{2}|_0^2) = 1 \rightarrow c(\frac{2^2}{2} - \frac{0^2}{2}) \rightarrow c(\frac{4}{2}) = 1 \rightarrow c(2) = 1 \rightarrow c = \frac{1}{2} = 0.5$$

$$f(l) = \begin{cases} 0, & l \leq 0 \\ \frac{l}{2}, & 0 < l \leq 2 \\ 0, & 2 < l \end{cases}$$

b. (1 point) What is the cumulative probability when $l \leq 0$?

$$F(l) = P(L \leq l) = \int_0^l f(l)dl \rightarrow f(l) = \begin{cases} 0, & l \leq 0 \\ \int_0^l \frac{l}{2}dl, & 0 < l \leq 2 \\ 0, & 2 < l \end{cases}$$

$$\int_0^l f(\frac{l}{2})dl = 1 \rightarrow \int_0^l \frac{l}{2}dl = \frac{l^2}{4}$$

$$f(l) = \begin{cases} 0, & l \leq 0 \\ \frac{l^2}{4}, & 0 < l \leq 2 \\ 1, & 2 < l \end{cases}$$

c. (1 point) What is the cumulative probability function when $0 < l \leq 2$?

$$\frac{l^2}{4} = 0 < l \leq 2$$

d. (1 point) What is the cumulative probability function when $2 \leq l$?

$$2 \leq l = 1$$

e. (3 points) Compute the median value of L. That is, compute l such that $P(L \leq l) = 1/2$.

$$P(L \leq l) = P(L > l) = \frac{1}{2}$$

$$F(l) = \frac{1}{2}$$

$$\frac{l^2}{4} = \frac{1}{2}$$

$$l^2 = 2$$

$$Median = \sqrt{2} = 1.41$$

Proof Practice

2. (12 points total) Broken Rulers

You have a ruler of length 2 and you choose a place to break it using a uniform probability distribution. Let random variable X represent the length of the left piece of the ruler. X is distributed uniformly in $[0, 2]$. You take the left piece of the ruler and once again choose a place to break it using a uniform probability distribution. Let random variable Y be the length of the left piece from the second break.

a. (3 points) Draw a picture of the region in the X-Y plane for which the joint density of X and Y is non-zero.

$$f_{(Y|X)}(y|x) \sim Uniform(0, x)$$

$$f(y|x) = \frac{1}{x}, 0 \leq y \leq x$$

$$f(x) = \frac{1}{2}, 0 \leq x \leq 2$$

The area shaded in gray is where the joint density of X and Y is non-zero

b. (3 points) Compute the joint density function for X and Y. (As always, make sure you write a complete expression.)

$$f_{X,Y}(x, y) = f_X(x) \times f_{Y|X}(y|x) \rightarrow f_{X,Y}(x, y) = \frac{1}{2} \times \frac{1}{x} = \frac{1}{2x}$$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2x}, & 0 \leq y \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

c. (3 points) Compute the marginal probability density for $Y, f_Y(y)$.

$$f_Y(y) = \int_y^2 \frac{1}{2x} dx = \frac{1}{2} \log(x)|_y^2 = \frac{1}{2}(\log(2) - \log(y))$$

$$f_Y(y) = \begin{cases} \frac{1}{2}(\log(2) - \log(y)), & 0 \leq y \leq 2 \\ 0, & otherwise \end{cases}$$

d. (3 points) Compute the conditional probability density of X, conditional on $Y = y, f_{X|Y}(x|y)$. (Make sure you state the values of y for which this exists.)

$$f_{X,Y}(x, y) = \frac{f(x, y)}{f(y)} \rightarrow \frac{\frac{1}{2x}}{\frac{1}{2}(\log(2) - \log(y))} = \frac{1}{x(\log(2) - \log(y))}$$

$$f_Y(y) = \begin{cases} \frac{1}{x(\log(2)-\log(y))}, & 0 < y \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

3. (3 points total) Post-Processing and Independence

(3 points) Suppose X and Y are Bernoulli random variables, and $f : \{0, 1\} \rightarrow \{0, 1\}$ is a function. Prove that if X and Y are independent then $f(X)$ and Y are independent.

If X and Y are independent then:

$$X = 0, 1$$
$$Y = 0, 1$$
$$XY = 0, 1$$
$$f(x) = p^x(1 - p)^{1-x}$$

$$y = 0, 1$$
$$f(y) = p^y(1 - p)^{1-y}$$

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$
$$P(XY = 1) = (p^x)(p^y)$$

Note: In fact, this works for any random variables, not just Bernoulli ones, and any function from $\mathbb{R}$ to $\mathbb{R}$.

Optional Challenge Exercise

Characterizing a Function of a Random Variable

Let X be a continuous random variable with probability density function $f(x)$, and let h be an invertible function where h^{-1} is differentiable. Recall that $Y = h(X)$ is itself a continuous random variable.

(Bonus + 3 points) Prove that the probability density function of Y is

$$g(y) = f(h^{-1}(y)) \cdot \left| \frac{d}{dy} h^{-1}(y) \right|$$

Note: The Homework Maximum is 100%