A direct (and simple) method to recover shear modulus using displacement data in incompressible media

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1 Model

1.1 Time-Harmonic model for incompressible media

We consider linear time harmonic elasticity

$$-\rho\omega^2\mathbf{u} - \operatorname{div}(2Ge(\mathbf{u})) + \nabla p = \mathbf{0} \tag{1}$$

with incompressibility constraint

$$\operatorname{div}\mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$ is the displacement field, p the pressure, $G = G(x)$ the shear modulus and $e(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ the linear strain tensor.

1.2 Relaxation of incompressibility constraint

Let us consider the following modification to equation (2):

$$\operatorname{div}\mathbf{u} = -\epsilon p \tag{3}$$

which is usually applied in fluid mechanics. Other choice is $\operatorname{div}\mathbf{u} = -i\omega\epsilon p$ (from $\operatorname{div}\mathbf{u} = -\epsilon\partial p/\partial t$ in transient regime) that could be further explored. Note that the parameter $\epsilon << 1$ play the role of the inverse of the first Lamé coefficient. Hence, (1) yields

$$-\rho\omega^2\mathbf{u} - \operatorname{div}\left(\frac{1}{\epsilon}\operatorname{div}\mathbf{u}I + 2Ge(\mathbf{u})\right) = \mathbf{0} \tag{4}$$

This expression seems quite convenient for our purposes since it completely depends upon $\mathbf{u}$, which will be our input data, and G (and eventually ρ), the parameter to be estimated.

2 The method

2.1 Single frequency

If we integrate equation (4) over one single voxel, namely V with boundary denoted by ∂V and exterior normal vector n, we obtain

$$\int_{\partial V}\left(\frac{1}{\epsilon}\operatorname{div}\mathbf{u}I + 2Ge(\mathbf{u})\right)n\,ds = -\omega^2\int_V \rho\mathbf{u}\,dx$$

As a first approximation, let us assume that G is *piece-wise constant* over each independent voxel. Let us define the following vector quantities:

$$\mathbf{b}_V = 2\int_{\partial V} e(\mathbf{u})n ds, \quad \mathbf{p}_V = \int_V \rho\mathbf{u}\,dx, \quad \mathbf{q}_V = \int_{\partial V}\operatorname{div}\mathbf{u}n\,ds \tag{5}$$

Then,

$$G_\omega(V) := -\frac{1}{\|\mathbf{b}_V\|^2}\left(\omega^2\mathbf{p}_V\cdot\mathbf{b}_V^* + \frac{1}{\epsilon}\mathbf{q}_V\cdot\mathbf{b}_V^*\right) \tag{6}$$

whenever $\|\mathbf{b}_V\| > 0$, which is reasonable since in regions with a very low strain induced by external excitation, it is not possible to recover or just to say any something about the elasticity there (rigid-body motion).

Note that the vectors can be calculated using displacement directly, by taking advantage of the direct integration over derivatives of $\mathbf{u}$, which may be stable under noise effects in $\mathbf{u}$.

2.2 Multi frequency

In some experiments, we may benefit of independent sources of information to reconstruct G. Using the expression in (6), we can formulate the following estimation problem

$$G^* = \arg\min_G \sum_\omega \|G - G_\omega\|^2 + R(G) \tag{7}$$

which is simple to solve.