

Rotation Matrices and Homogeneous Transformations

A coordinate frame in an n -dimensional space is defined by n mutually orthogonal unit vectors. In particular, for a two-dimensional (2D) space, i.e., $n = 2$, a coordinate frame is defined by an X axis and a Y axis with these two axes being orthogonal to each other. The intersection between the X axis and the Y axis is the origin of the coordinate frame.

For a three-dimensional (3D) space, i.e., $n = 3$, a coordinate frame is defined by an X axis, a Y axis, and a Z axis, with each pair of these axes being orthogonal to each other. The X , Y , and Z axes meet at a point called the origin of the coordinate frame. Also, we will typically consider only right-handed coordinate frames, i.e., frames wherein the cross-product of the X axis unit vector and the Y axis unit vector is a unit vector in the Z axis direction.

Given two coordinate frames (in either 2D or 3D spaces), there can, in general, be a translation (offset between the origins of the coordinate frames) and a rotation between the coordinate frames.

Rotation matrices in 2D: Given two coordinate frames denoted as the 0 frame and the 1 frame, respectively, the rotation matrix defining the orientation of the 1 frame in terms of the 0 frame is defined as the 2×2 matrix

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix} \quad (1)$$

where x_0 and y_0 denote the X axis and Y axis unit vectors of the 0 frame and x_1 and y_1 denote the X axis and Y axis unit vectors of the 1 frame.

Note that the notation $a \cdot b$ with a and b being any vectors denotes the dot product of the vectors a and b . Also, note that the dot product of two vectors is a scalar number and the numerical value of the dot product is independent of the choice of coordinate frame in which the vectors are written. Hence, for example, the dot product $x_1 \cdot x_0$ can be calculated in either the 0 frame or the 1 frame. Of course, whichever frame is used, the two vectors should both be written in the same frame, i.e., both in 0 frame or both in 1 frame. The numerical value obtained for the dot product would be the same in any choice of coordinate frame used for the calculation.

Note that by the definition of the rotation matrix (1), the columns of the rotation matrix are the unit vectors of the 1 frame when written in terms of the 0 frame. For example, the first column of R_1^0 is the unit vector x_1 written in terms of the coordinate frame 0. The second column of R_1^0 is the unit vector y_1 written in terms of the coordinate frame 0. Equivalently, note that $x_1 = (x_1 \cdot x_0)x_0 + (x_1 \cdot y_0)y_0$ and $y_1 = (y_1 \cdot x_0)x_0 + (y_1 \cdot y_0)y_0$. As physical vector equations, these two equations hold independent of the choice of coordinate frame used to numerically represent the vectors (all the vectors in the equation should be written in the same frame though). When referring to the numerical representation of a vector in a particular coordinate frame, a superscript notation is utilized to show the choice of coordinate frame, e.g., x_1^0 is the numerical representation of the vector x_1 when written in terms of the 0 frame; x_1^1 is the numerical representation of the vector x_1 when written in terms of the 1 frame. Hence, we can write equations such as

$$x_1^0 = (x_1 \cdot x_0)x_0^0 + (x_1 \cdot y_0)y_0^0 \quad ; \quad x_1^1 = (x_1 \cdot x_0)x_0^1 + (x_1 \cdot y_0)y_0^1. \quad (2)$$

Since the dot product is independent of the coordinate frame utilized, the choice of coordinate frame is not explicitly marked for the vectors appearing in the dot products in the equations above. Using the definition of the rotation matrix R_1^0 , we can also write equations such as

$$[x_1^0, y_1^0] = [x_0^0, y_0^0] R_1^0 \quad ; \quad [x_1^1, y_1^1] = [x_0^1, y_0^1] R_1^0. \quad (3)$$

Note that the matrices $[x_1^0, y_1^0]$, $[x_0^0, y_0^0]$, etc., in the equation above, are 2×2 matrices. Also, note that $[x_0^0, y_0^0]$ and $[x_1^1, y_1^1]$ are both 2×2 identity matrices.

By the above definition of the rotation matrix, it can also be written that the rotation matrix defining the orientation of the 0 frame in terms of the 1 frame is given by

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}. \quad (4)$$

Note that the columns of R_0^1 correspond to the unit vectors of the 0 frame as written in terms of the coordinate frame 1.

Since the dot product is commutative (i.e., given two vectors a and b , the dot product satisfies $a \cdot b = b \cdot a$), the rotation matrix R_0^1 is seen to be simply $(R_1^0)^T$. Also, as discussed above, since the columns of the rotation matrix R_1^0 correspond to the unit vectors of the 1 frame written in terms of the 0 frame and the columns of the

rotation matrix R_0^1 correspond to the unit vectors of the 0 frame written in terms of the 1 frame, we can write the equations

$$[x_1^0, y_1^0] = [x_0^0, y_0^0] R_1^0 \quad ; \quad [x_0^0, y_0^0] = [x_1^0, y_1^0] R_0^1. \quad (5)$$

Hence, $[x_0^0, y_0^0] = [x_0^0, y_0^0] R_1^0 R_0^1$. Since $[x_0^0, y_0^0]$ is the identity matrix, this implies that $R_1^0 R_0^1$ is the identity matrix. Also, it can be shown similarly that $R_0^1 R_1^0$ is the identity matrix. Hence, we see that

$$R_0^1 = (R_1^0)^{-1}. \quad (6)$$

But, since we know that $R_0^1 = (R_1^0)^T$ as discussed above, we see that for any rotation matrix R

$$R^{-1} = R^T \quad (7)$$

Hence, for any rotation matrix R , we have the equation $RR^T = R^T R = I$. This property of the rotation matrix is called the orthogonality property.

From the equation $RR^T = I$, we also see that $(\det(R))^2 = 1$, i.e., $\det(R)$ is either +1 or -1. For a rotation matrix, $\det(R)$ has to be +1. The possibility that $\det(R)$ is -1 corresponds to a reflection (e.g., switching direction of one axis while keeping the other axis fixed) and not a simple rotation by an angle. Hence, for rotation matrices, we have the property that $\det(R) = 1$.

From the property that $RR^T = R^T R = I$, we also see the following properties:

1. Each row of R is a unit vector.
2. Each column of R is a unit vector.
3. Any pair of different rows of R is mutually orthogonal.
4. Any pair of different columns of R is mutually orthogonal.

Starting from a coordinate frame 0, if a coordinate frame 1 is obtained by rotating by an angle θ , then it can be easily seen that $x_1^0 = [\cos(\theta), \sin(\theta)]^T$ and $y_1^0 = [-\sin(\theta), \cos(\theta)]^T$. Hence,

$$R_1^0 = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \quad (8)$$

where $c_\theta = \cos(\theta)$ and $s_\theta = \sin(\theta)$.

Rotation matrices in 3D: Rotation matrices in 3D space are 3×3 matrices that have very similar properties to the 2D rotation matrices discussed above. Given two coordinate frames 0 and 1, the rotation matrix defining the orientation of the 1 frame in terms of the 0 frame is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \quad (9)$$

where x_0, y_0 , and z_0 denote the X axis, Y axis, and Z axis unit vectors of the 0 frame and x_1, y_1 , and z_1 denote the X axis, Y axis, and Z axis unit vectors of the 1 frame. Note that the columns of the rotation matrix R_1^0 correspond to the unit vectors of the coordinate frame 1 as written in terms of the coordinate frame 0. The rotation matrix defining the orientation of the 0 frame in terms of the 1 frame can be written similarly as

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 & z_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 & z_0 \cdot y_1 \\ x_0 \cdot z_1 & y_0 \cdot z_1 & z_0 \cdot z_1 \end{bmatrix} \quad (10)$$

As in the case of 2D rotation matrices, it can be shown that any rotation matrix R in 3D space satisfies the following properties:

1. $R^{-1} = R^T$, i.e., $RR^T = I$.
2. $\det(R) = 1$.
3. Each row of R is a unit vector.

4. Each column of R is a unit vector.
5. Any pair of different rows of R is mutually orthogonal.
6. Any pair of different columns of R is mutually orthogonal.

As an example of a rotation matrix in 3D space, consider a rotation around the Z axis by an angle θ , i.e., the 1 frame is obtained from the 0 frame by rotating around the z_0 axis by angle θ . Then, as far as the X and Y axes are concerned, it is simply a 2D rotation, i.e., the relation between the (x_0, y_0) unit vectors and the (x_1, y_1) unit vectors is as in a 2D rotation, i.e., $x_1 = \cos(\theta)x_0 + \sin(\theta)y_0$ and $y_1 = -\sin(\theta)x_0 + \cos(\theta)y_0$. Also, for a rotation around the Z axis, the Z axis would not change, i.e., $z_1 = z_0$. Hence, the 3×3 rotation matrix defining the orientation of the 1 frame in terms of the 0 frame is given by

$$R_1^0 = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

where $c_\theta = \cos(\theta)$ and $s_\theta = \sin(\theta)$. Note that the third row and the third column are both comprised of two zeros and a 1 denoting the properties that x_1 and y_1 vectors do not have a non-zero component along the z_0 vector and that the z_1 vector is identical to the z_0 vector. The rotation matrix given in (11) is called an elementary (or basic) rotation matrix since it is a simple rotation around one of the 0 frame unit vectors (in this case, z_0). The elementary rotation matrix (11) corresponding to a rotation around the Z axis by angle θ is denoted by $R_{z,\theta}$. Similarly, the elementary rotation matrix corresponding to a rotation around the X axis by angle θ is denoted by $R_{x,\theta}$ and the elementary rotation matrix corresponding to a rotation around the Y axis by angle θ is denoted by $R_{y,\theta}$. The expressions for $R_{x,\theta}$ and $R_{y,\theta}$ can be written similar to the expression for $R_{z,\theta}$ above. The elementary rotation matrices in 3D are given by

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix} \quad ; \quad R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \quad ; \quad R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (12)$$

It is easily seen from these definitions of the elementary rotation matrices that for any elementary rotation matrix $R_{k,\theta}$ where k can be any of the axes, the following properties are satisfied:

1. $R_{k,0} = I$
2. $R_{k,\theta}^{-1} = R_{k,\theta}^T = R_{k,-\theta}$
3. $R_{k,\theta_1} R_{k,\theta_2} = R_{k,\theta_1+\theta_2}$ for any angles θ_1 and θ_2 . Hence, the multiplication of the rotation matrices R_{k,θ_1} and R_{k,θ_2} is commutative, i.e., $R_{k,\theta_1} R_{k,\theta_2} = R_{k,\theta_2} R_{k,\theta_1}$. However, in general, multiplication of two arbitrary rotation matrices R_1 and R_2 is **not** commutative.

Transforming the representations of a point between coordinate frames: Given any fixed point p in 3D space, it can be represented in any 3D coordinate frame as a set of three numbers that correspond to the coordinates of the point along the X , Y , and Z axes of the coordinate frame. The representation of a point p in terms of a coordinate frame a is written using the superscript notation as the 3×1 vector p^a .

For example, given a point p written numerically as coordinates (a_1, b_1, c_1) relative to the frame 1, we have the equation

$$p = a_1 x_1 + b_1 y_1 + c_1 z_1 \quad (13)$$

holds. Note that this is a physical vector equation, i.e., holds in any choice of coordinate frame as long as all vector quantities in (13) are written in the same coordinate frame. Therefore, given any frame a , we have the equation

$$p^a = [x_1^a, y_1^a, z_1^a] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}. \quad (14)$$

In particular, taking frame a to be the frame 1, we get, as expected,

$$p^1 = [x_1^1, y_1^1, z_1^1] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad (15)$$

since the 3×3 matrix $[x_1^1, y_1^1, z_1^1]$ is simply the 3×3 identity matrix. Also, taking frame a to be the frame 0, we get

$$p^0 = [x_1^0, y_1^0, z_1^0] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}. \quad (16)$$

Note that the 3×3 matrix $[x_1^0, y_1^0, z_1^0]$ is simply the rotation matrix R_1^0 by definition (from (9)). Hence, from (15) and (16), we get

$$p^0 = R_1^0 p^1. \quad (17)$$

As described above, given a point p , its representation in the frame a is denoted by a 3×1 vector p^a in which the first, second, and third elements correspond to its coordinates with respect to the unit vectors x_a , y_a , and z_a . Denoting the first, second, and third elements of p^a by p_x^a , p_y^a , and p_z^a , we see that (assuming that the origin of the coordinate frame is numerically represented as $[0, 0, 0]^T$) $p_x^a = p \cdot x_a$, $p_y^a = p \cdot y_a$, and $p_z^a = p \cdot z_a$.

Using rotation matrices to denote physical rotation of an object: Consider a coordinate frame 0 attached to a fixed reference (e.g., the ground) and a coordinate frame 1 attached to a rigid body that can rotate relative to the fixed reference frame. Consider a point p on the rigid body (i.e., a fixed point relative to the rigid body). Then, its numerical coordinate representation p^1 relative to the frame 1 is a constant 3×1 vector. If the rigid body rotates relative to the fixed reference frame, then the corresponding orientation of the frame 1 (after rotation of the rigid body) is defined in terms of the rotation matrix R_1^0 . Hence, the numerical coordinate representation p^0 of the point p relative to the frame 0 is given by $p^0 = R_1^0 p^1$ with p^1 remaining a constant as discussed above. In this sense, the rotation of the rigid body is characterized by the rotation matrix R_1^0 and the coordinates of a point on the rigid body (e.g., the center of gravity of the rigid body) can be found relative to the fixed reference frame using the rotation matrix R_1^0 .

Thus, as described above, the rotation matrix has multiple physical interpretations including: unit vectors of one coordinate frame in terms of unit vectors of another coordinate frame; a transformation matrix to transform the numerical coordinate representation of a fixed point from one coordinate frame to another; a transformation matrix that defines the rotation of a rigid body.

Sequences of rotations: Consider three coordinate frames 0, 1, and 2. The rotation matrix defining the orientation of the 1 frame in terms of the 0 frame is given as R_1^0 and the rotation matrix defining the orientation of the 2 frame in terms of the 1 is given as R_2^1 . Then, given a fixed point p with numerical coordinate representations p^0 , p^1 , and p^2 relative to the coordinate frames 0, 1, and 2, respectively, we have the properties that $p^0 = R_1^0 p^1$ and $p^1 = R_2^1 p^2$. Hence, combining these two equations, we see that

$$p^0 = R_1^0 R_2^1 p^2. \quad (18)$$

Therefore, the rotation matrix defining the orientation of the frame 2 in terms of the 0 frame is given by

$$R_2^0 = R_1^0 R_2^1. \quad (19)$$

This equation holds for any set of coordinate frames 0, 1, and 2.

Rotations about current frame or about fixed frame: Given coordinate frames 0 and 1, a new coordinate frame 2 can be defined as a rotation from frame 1 with the rotation defined numerically as a rotation relative to the current frame (i.e., frame 1) or relative to the fixed (or original) frame (i.e., frame 0).

For example, if the rotation from frame 1 to frame 2 is defined as a rotation of angle θ about the z_1 axis, we would say that this rotation is defined relative to the current frame (i.e., frame 1). On the other hand, if the rotation from frame 1 to frame 2 is defined as a rotation of angle θ about the z_0 axis, we would say that this rotation is defined relative to the fixed frame (i.e., frame 0). In both cases, numerically, the rotation matrix is simply the 3×3 matrix $R_{z,\theta}$ defined in (12). If the rotation is defined relative to the frame 1, as given by (19), the rotation matrix R_2^0 is given by $R_1^0 R$, i.e., when the rotation from the 1 frame to the 2 frame is defined numerically relative to the current frame (i.e., the rotation matrix R from frame 1 to frame 2 is defined relative to frame 1), then the composite rotation matrix R_2^0 is calculated by post-multiplying (i.e., R is multiplied to the right of R_1^0).

If the rotation is defined relative to the current frame (i.e., frame 1), then the rotation matrix R_2^1 is simply this numerically defined matrix $R_{z,\theta}$. However, if the rotation is defined relative to the fixed frame (i.e., frame 0), then the rotation matrix R_2^1 is **not**, in general, given by this numerically defined matrix $R_{z,\theta}$ since the rotation is around the z_0 axis and not the z_1 axis. Hence, in terms of the rotation from frame 1 to frame 2, this rotation is defined in terms of a totally different frame 0; hence, R_2^1 is not simply numerically $R_{z,\theta}$. To find the rotation matrix R_2^1 corresponding to a rotation numerically given as R (e.g., $R = R_{z,\theta}$) defined relative to frame 0, one way to physically visualize this is as follows:

Step 1: to apply the rotation defined by R , we need to first transform back to the frame 0 since R is defined in terms of the frame 0

Step 2: then, we can apply rotation R

Step 3: then we need to invert back the additional transformation that we introduced in Step 1.

The rotation matrix to transform back to frame 0 is given by R_0^1 . Hence, by combining the three steps described above, the rotation matrix R_2^1 is given by $R_2^1 = R_0^1 R R_1^0$. Since $R_0^1 = (R_1^0)^T$, we get $R_2^1 = (R_1^0)^T R R_1^0$. Hence, $R_2^0 = R_1^0 (R_1^0)^T R R_1^0 = R R_1^0$, i.e., when the rotation from the 1 frame to the 2 frame is defined numerically relative to the fixed frame 0 (i.e., the rotation matrix R from frame 1 to frame 2 is defined relative to frame 0), then the composite rotation matrix R_2^0 is calculated by pre-multiplying (i.e., R is multiplied to the left of R_1^0).

Another way of deriving the relations discussed above for rotations relative to a current frame or a fixed frame is to use the property that the columns of a rotation matrix are simply the unit vectors of a new coordinate frame written in terms of an old coordinate frame, i.e., as seen from (9), the unit vectors x_1 , y_1 , and z_1 are given in terms of unit vectors x_0 , y_0 , and z_0 as

$$\begin{aligned} x_1 &= (x_1 \cdot x_0)x_0 + (x_1 \cdot y_0)y_0 + (x_1 \cdot z_0)z_0 \\ y_1 &= (y_1 \cdot x_0)x_0 + (y_1 \cdot y_0)y_0 + (y_1 \cdot z_0)z_0 \\ z_1 &= (z_1 \cdot x_0)x_0 + (z_1 \cdot y_0)y_0 + (z_1 \cdot z_0)z_0. \end{aligned} \quad (20)$$

These equations are physical vector equations and hold in any choice of coordinate frame, i.e., x_0 , y_0 , z_0 , x_1 , y_1 , and z_1 in (20) can be written in any coordinate frame; hence, the explicit superscript notation to indicate the choice of coordinate frame for numerical representation of coordinates is omitted in (20) since these equations would hold for any choice of coordinate frame. If a rotation matrix for the transformation from the 1 frame to the 2 frame is given numerically as a 3×3 matrix R defined relative to the frame 1, then the unit vectors x_2 , y_2 , z_2 are given in terms of unit vectors x_1 , y_1 , and z_1 by

$$\begin{aligned} x_2 &= r_{11}x_1 + r_{21}y_1 + r_{31}z_1 \\ y_2 &= r_{12}x_1 + r_{22}y_1 + r_{32}z_1 \\ z_2 &= r_{13}x_1 + r_{23}y_1 + r_{33}z_1 \end{aligned} \quad (21)$$

where r_{ij} denotes the $(i, j)^{th}$ element of R for $i = 1, 2, 3$; $j = 1, 2, 3$, e.g., r_{12} is the element of R on the first row and second column. As with (20), the equations in (21) are physical vector equations that numerically hold in any choice of coordinate frame. Since the equations (20) and (21) hold in any coordinate frame, we have, for example, $[x_1^0, y_1^0, z_1^0] = [x_0^0, y_0^0, z_0^0]R_1^0$ and $[x_2^0, y_2^0, z_2^0] = [x_1^0, y_1^0, z_1^0]R$. Hence, combining these two equations, we get $[x_2^0, y_2^0, z_2^0] = [x_0^0, y_0^0, z_0^0]R_1^0 R$, i.e., the rotation matrix R_2^0 is given as $R_1^0 R$, i.e., when the rotation from the 1 frame to the 2 frame is defined numerically relative to the current frame (i.e., the rotation matrix R from frame 1 to frame 2 is defined relative to frame 1), then the composite rotation matrix R_2^0 is calculated by post-multiplying (i.e., R is multiplied to the right of R_1^0). On the other hand, if the rotation matrix R for the transformation from the 1 frame to the 2 frame is defined numerically in terms of the frame 0, then the geometrical effect is to rotate the vectors x_0 , y_0 , and z_0 used in the definition of the unit vectors x_1 , y_1 , and z_1 in (20), i.e., the unit vectors x_2 , y_2 , and z_2 are given by

$$\begin{aligned} x_2 &= (x_1 \cdot x_0)\bar{x}_0 + (x_1 \cdot y_0)\bar{y}_0 + (x_1 \cdot z_0)\bar{z}_0 \\ y_2 &= (y_1 \cdot x_0)\bar{x}_0 + (y_1 \cdot y_0)\bar{y}_0 + (y_1 \cdot z_0)\bar{z}_0 \\ z_2 &= (z_1 \cdot x_0)\bar{x}_0 + (z_1 \cdot y_0)\bar{y}_0 + (z_1 \cdot z_0)\bar{z}_0 \end{aligned} \quad (22)$$

where \bar{x}_0 , \bar{y}_0 , and \bar{z}_0 are vectors obtained by rotating x_0 , y_0 , and z_0 based on the numerically defined matrix R , i.e.,

$$\begin{aligned} \bar{x}_0 &= r_{11}x_0 + r_{12}y_0 + r_{13}z_0 \\ \bar{y}_0 &= r_{21}x_0 + r_{22}y_0 + r_{23}z_0 \\ \bar{z}_0 &= r_{31}x_0 + r_{32}y_0 + r_{33}z_0 \end{aligned} \quad (23)$$

where r_{ij} denotes the $(i, j)^{th}$ element of R . As in equations (20) and (21), the equations (22) and (23) are physical vector equations that hold in any choice of coordinate frame. Hence, we have, for example, $[x_2^0, y_2^0, z_2^0] = [\bar{x}_0^0, \bar{y}_0^0, \bar{z}_0^0]R_1^0$ and $[\bar{x}_0^0, \bar{y}_0^0, \bar{z}_0^0] = [x_0^0, y_0^0, z_0^0]R$. Hence, combining these two equations, we get $[x_2^0, y_2^0, z_2^0] = [x_0^0, y_0^0, z_0^0]RR_1^0$, i.e., the rotation matrix R_2^0 is given as RR_1^0 , i.e., when the rotation from the 1 frame to the 2 frame is defined numerically relative to the fixed frame (i.e., the rotation matrix R from frame 1 to frame 2 is defined relative to frame 0), then the composite rotation matrix R_2^0 is calculated by pre-multiplying (i.e., R is multiplied to the left of R_1^0). Since R_2^0 is, by (19), given by $R_1^0R_2^1$, this implies that $R_1^0R_2^1 = RR_1^0$, i.e., $R_2^1 = (R_1^0)^T RR_1^0$.

Similar to the above discussion, it can be shown that, in general, if a rotation from a frame 1 to a frame 2 is defined numerically in terms of a frame c , then the rotation matrix R_2^1 is given by

$$R_2^1 = (R_1^c)^T R(R_1^c). \quad (24)$$

Hence, in this general case, $R_2^0 = R_1^0(R_1^c)^T R(R_1^c)$. As a special case, if frame c is the frame 1, then the rotation matrix R_1^c is the identity matrix and hence R_2^1 from (24) is simply R and hence $R_2^0 = R_1^0R$. As another special case, if frame c is the frame 0, then, from (24) $R_2^1 = (R_1^0)^T R(R_1^0)$ and hence $R_2^0 = RR_1^0$. Therefore, the post-multiplication and pre-multiplication rules that we derived above for rotation relative to current frame and fixed frame respectively are simply two special cases of the general rule obtained from (24).

Non-commutativity of rotations: Given two rotation matrices R_1 and R_2 , the multiplication of these two rotation matrices is, in general, not commutative, i.e., in general $R_1R_2 \neq R_2R_1$.

For example, consider $R_1 = R_{y,\phi}$ and $R_2 = R_{z,\theta}$ where $R_{y,\phi}$ and $R_{z,\theta}$ are the elementary rotation matrices given as in (12). It is easy to see by calculating $R_{y,\phi}R_{z,\theta}$ and $R_{z,\theta}R_{y,\phi}$ that these two product matrices have very different elements.

Homogeneous Transformations: Rotation matrices between coordinate frames were discussed above. In general, there can be both a rotation and a translation (offset between the origins) between two coordinate frames, i.e., given two frames 0 and 1, with an associated rotation matrix R_1^0 between these coordinate frames and with an associated translation d_1^0 (coordinates of the origin of frame 1 when written relative to frame 0), the coordinates of a point p relative to frame 0 and to frame 1 are related as

$$p^0 = R_1^0p^1 + d_1^0. \quad (25)$$

Similarly, if frames 1 and 2 are related through a rotation R_2^1 and offset d_2^1 (coordinates of the origin of frame 2 when written relative to frame 1), then the coordinates of a point p relative to frames 1 and 2 are related as

$$p^1 = R_2^1p^2 + d_2^1. \quad (26)$$

Putting (25) and (26) together, we get

$$p^0 = R_1^0R_2^1p^2 + R_1^0d_2^1 + d_1^0. \quad (27)$$

Hence, the equivalent rotation matrix R_2^0 between frames 0 and 2 and the equivalent translation d_2^0 are given by

$$\begin{aligned} R_2^0 &= R_1^0R_2^1 \\ d_2^0 &= R_1^0d_2^1 + d_1^0. \end{aligned} \quad (28)$$

These equations for the rotation and translation between frames 0 and 2 can be derived instead through a combined matrix that includes both rotation and translation, defined as a 4×4 matrix H of the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \quad (29)$$

where R denotes a 3×3 rotation matrix, d denotes a 3×1 translation vector, and the 0 in (29) represents a 1×3 zero vector. Then, we have

$$H_1^0 = \begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \quad ; \quad H_2^1 = \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix}. \quad (30)$$

Therefore,

$$H_1^0 H_2^1 = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}. \quad (31)$$

Comparing (31) and (28), we see that the product $H_1^0 H_2^1$ correctly represents the composite rotation and translation between frames 0 and 2. Thus, the 4×4 matrix H can be used as a combined representation of both rotation and translation and composite transformations between multiple coordinate frames can be calculated through multiplications of such 4×4 matrices. The 4×4 matrix H is called a homogeneous transformation. To represent an equation such as (25) easily in terms of a product with a homogeneous transformation matrix, we also define the homogeneous representation of a 3×1 vector p to be

$$P = \begin{bmatrix} p \\ 1 \end{bmatrix}. \quad (32)$$

Then, denoting the homogeneous representations of a point p in frames 0 and 1 as P^0 and P^1 , respectively, we get

$$P^0 = H_1^0 P^1. \quad (33)$$

Comparing with the equation $p^0 = R_1^0 p^1$, which we would get with a pure rotation (no translation), we see that the homogeneous transformation matrix and homogeneous representation concepts are a generalization (with very similar mathematical behavior) of rotation matrix and 3×1 vector representation concepts. As with rotation matrices, we can infer that since $P^0 = H_1^0 P^1$ and $P^1 = H_2^1 P^2$, we have $H_1^0 H_2^1 = H_2^0$.

To find the inverse of a homogeneous transformation H of form (29), denote the inverse of H to be

$$H^{-1} = \begin{bmatrix} \tilde{R} & \tilde{d} \\ 0 & 1 \end{bmatrix}. \quad (34)$$

Then, computing HH^{-1} , we get

$$HH^{-1} = \begin{bmatrix} R\tilde{R} & R\tilde{d} + d \\ 0 & 1 \end{bmatrix}. \quad (35)$$

Hence, since HH^{-1} should be the 4×4 identity matrix, we get $\tilde{R} = R^{-1} = R^T$ and $\tilde{d} = -R^{-1}d = -R^T d$. Therefore, the inverse of a homogeneous transformation matrix H of form (29) is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}. \quad (36)$$

Similar to the analysis of the composition of rotation matrices, we can show that given a homogeneous transformation matrix H_1^0 between coordinate frames 0 and 1 and given that the transformation from frame 1 to frame 2 is given by a homogeneous transformation matrix H that is numerically defined relative to frame 1 (i.e., the current frame), the composite homogeneous transformation matrix H_2^0 between frames 0 and 2 can be written as $H_1^0 H$, i.e., the composite homogeneous transformation matrix H_2^0 is calculated by post-multiplying (i.e., H is multiplied to the right of H_1^0). If, instead, the homogeneous transformation matrix from frame 1 to frame 2 is given by a homogeneous transformation matrix H that is numerically defined relative to frame 0 (i.e., the fixed or original frame), then the composite homogeneous transformation matrix H_2^0 between frames 0 and 2 can be written as HH_1^0 , i.e., the composite homogeneous transformation matrix H_2^0 is calculated by pre-multiplying (i.e., H is multiplied to the left of H_1^0).

Corresponding to the three rotational degrees of freedom and three translational degrees of freedom, there are six elementary homogeneous transformation matrices, i.e., the three elementary rotations around the X , Y , and Z axes, respectively, and the three elementary translations along the X , Y , and Z axes, respectively. These elementary homogeneous transformations are given by

$$\begin{aligned} \text{Rot}_{x,\theta_x} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_x & -s_x & 0 \\ 0 & s_x & c_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \text{Rot}_{y,\theta_y} &= \begin{bmatrix} c_y & 0 & s_y & 0 \\ 0 & 1 & 0 & 0 \\ -s_y & 0 & c_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \text{Rot}_{z,\theta_z} &= \begin{bmatrix} c_z & -s_z & 0 & 0 \\ s_z & c_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \text{Trans}_{x,d_x} &= \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \text{Trans}_{y,d_y} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \text{Trans}_{z,d_z} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

where $c_x = \cos(\theta_x)$, $s_x = \sin(\theta_x)$, $c_y = \cos(\theta_y)$, $s_y = \sin(\theta_y)$, $c_z = \cos(\theta_z)$, and $s_z = \sin(\theta_z)$.