Problem 1. P is a set of points in $\mathbb{R}^2$. Prove: if we take a point p from P uniformly at random, the number of Voronoi neighbors of p is $O(1)$ in expectation.

Proof. The number of neighbors of p in Voronoi diagram is the degree of p in the triangulation.

Since there are at most $3n-6$ edges on the Voronoi diagram, every edge corresponds to a Voronoi neighbor pair of p and the number of edges is at most $3n-6$, the total number of neighbors for every possible p is at most $3n-6$.

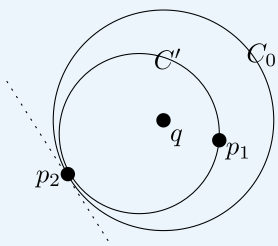
Taking the average between n points, we derive that the number of Voronoi neighbors of p is $O(1)$ in expectation. □

Problem 2. Let P be a set of points in $\mathbb{R}^2$. Consider any point q in $\mathbb{R}^2$ (which may not be in P). Let p_1 be the nearest neighbor of q, and p_2 be the second nearest neighbor of q (i.e., p_2 has the second smallest distance to q among all the points in P). Prove: p_2 must be a Voronoi neighbor of p_1.
Hint: Argue for the existence of a circle that passes p_1, p_2 and contains no points of P in the interior.

Proof. **Lemma 1**: Let C be a circle centered at O and passes through p. Given any point q lying either within the interior of C or on the boundary, there exists a circle C' lying fully within C and passes through p, q. C' can be constructed by an inscribing circle tangented at p and passes through q. It is clear that the circle does not cross over C's region.

Lemma 2: (Property of Voronoi Diagram) A point q lies on the edge in a Voronoi diagram if and only if there is a circle centered at q and passes through at least two points with no points lying in the interior.

Impose a circle C_0 centered at q and passes through p_2. Since p_2 is the second nearest neighbor of q, C_0 has no point belonging to P other than p_1 lying in the interior. By Lemma 1, there is a circle C' which contains no point in P in the interior, and passes through p and q. Therefore, by Lemma 2, there must be some set of points on the Voronoi diagram separating p_1 and p_2, and thus p_2 must be a Voronoi neighbor of p_1 since these points form a line as an edge.



□

Problem 3. Let P be a set of n points in $\mathbb{R}^2$. Describe a data structure of $O(n)$ space that can find the two nearest neighbors of any query point $q \in \mathbb{R}^2$ in $O(\log n)$ time.
Hint: This problem is rather challenging because there are many details to sort out. Consider the Voronoi cell of any point $p \in P$. Think about building a tiny Voronoi diagram inside that Voronoi cell. To build this tiny diagram, it suffices to consider only the Voronoi neighbors of p. The challenging part is to argue that all the tiny Voronoi diagrams have $O(n)$ complexity.
Spoiler: "order-2 Voronoi diagram".

Solution. Facts: The number of vertices in the order-1 Voronoi diagram of a set of n points is at most $V_1 = 2n - 5$ and the number of edges is at most $E_1 = 3n - 6$.

Denote $\mathcal{V}_1(p_1)$ the region with p_1 as the nearest neighbor, $\mathcal{V}_2(p_1, p_2)$ the region with p_1, p_2 as the nearest neighbor, second nearest neighbor, respectively. $\mathcal{V}_2(p_1, p_2)$ is generated by deleting p_1 from P and having p_2 as the nearest neighbor in the order-1 Voronoi diagram of $P - \{p_1\}$.

Consider the number of faces induced by spliting an order-1 Voronoi cell into areas depicting its second nearest neighbor. It is easy to see that $\mathcal{V}_2(p_1, p_2)$ is a piece of continuous region by considering the Voronoi diagram $V_1(P - \{p_1\})$ and the intersection with $\mathcal{V}_1(p_1)$. Since we can find a point's second nearest neighbor by first locating its nearest neighbor p_1 in the Voronoi diagram and iterating through p_1's neighbor, the total number of faces in the order-2 Voronoi diagram is bounded by the number of neighbors for each order-1 Voronoi cell, i.e. $F_2 = 2E_1 \leq 6n$.

We also need to prove that an order-2 Voronoi vertex (after deletion of order-1 Voronoi edges) has degree of 3, assuming general position. By proving this, we have $2E_2 \leq 6(V_2 + 1)$, and by Euler's formula

$$V_2 - E_2 + F_2 + 1 = 2 \Rightarrow E_2 \leq \frac{3F_2}{2} \leq 9n.$$

Suppose that p is a point lying on a order-2 Voronoi edge subdividing $\mathcal{V}_2(p_i, p_j)$ and $\mathcal{V}_2(p_i, p_k)$. Then, this should imply that p_j and p_k are equidistant to p, expand a circle from this point p passing through p_j and p_k. We should have p_i lying inside of this circle since it has distance smaller than $dist(p, p_j)$ and smaller than $dist(p, p_k)$. We can use the same argument in $V_1(P)$ to prove that the order-2 Voronoi vertex bears the similar property as the order-1 vertex and thus giving the degree 3 property. This finishes the proof for $O(n)$ complexity of order-2 Voronoi diagram.

We first build an order-1 voronoi diagram. To build an order-2 Voronoi diagram, compute $V_1'(P - \{p\})$ for each point $p \in P$ and merge it to the order-1 diagram if the face it represents belongs to the neighbor of p. This means any region $\mathcal{V}_2(p_1, p_2)$ is $\mathcal{V}_1(p_1) \cap \mathcal{V}_1'(p_2)$.

Therefore, to locate the face where q is in, locate q in the order-2 diagram to find two nearest points p_i, p_j to q. The point location structure to subdivisions with $O(n)$ complexity allows us to answer any query in $O(\log n)$ time.

Problem 4. Prove: Every triangulation of a set P of n points contains $2n - 2 - h$ triangles and $3n - 3 - h$ edges, where h is the number of points on the convex hull boundary of P.
Hint: Apply Euler's formula.

Proof. Every edge bounds two faces. All faces except the exterior face are triangles. Charge on the number edges by the edge from the T triangles, then all edges is counted twice except those on the hull is counted only once. We have the equality $3T = 2E - h$, where E is the number of edges. The number of faces is the number of triangles plus the outer face, i.e. $T + 1$. Therefore, we substitute $3(F - 1) = 2E + h$ to Euler's formula $V - E + F = 2$, and obtain that $n + \frac{2E-h}{3} + 1 - E = 2$ and thus $E = 3n - 3 - h, T = 2n - 2 - h$. □

Problem 5. The *Gabriel* graph of a set P of points in $\mathbb{R}^2$ is a graph G defined as follows. The vertex set of G is P, i.e., each point of P is a vertex in G. Two points p and q are connected by an edge in G if and only if the circle with segment $\overline{pq}$ as a diameter contains no points of P in the interior. Prove:

- Every edge of G is in the Delaunay triangulation of P.
- Two points p and q are adjacent in G if and only if the segment $\overline{pq}$ intersects with the Voronoi edge shared by the Voronoi cells of p and q.

Proof. **Empty Circle Property:** Two sites p_i and p_j are connected by an edge in the Delaunay triangulation if and only if there is an empty circle passing through p_i and p_j.

Proof of the property: Consider an edge connecting p and q. Exist a circle C with $\overline{pq}$ as diameter which is empty in the interior. The center of C must lie on an edge of the Voronoi diagram since it bisects $\overline{pq}$. Therefore, p and q must be neighbors in the Voronoi diagram, and p and q is by definition connected in the Delaunay triangulation of P.

Every Gabriel edge is a Delaunay edge. The *only-if* direction immediately holds correct with the assistance of the proof shown above.

The *if* direction: The Voronoi edge shared by cells p and q is a part of the perpendicular bisector of $\overline{pq}$. Let the intersection of $\overline{pq}$ and this Voronoi edge be C', then we can create the circle with radius $|C'q|$, which also passes through p due to the aforementioned property (perpendicular, bisecting), now $\overline{pq}$ is the diameter of the circle created. Thus, two points p and q is connected by an edge in G. $\square$

Problem 6. Let P be a set of n points in $\mathbb{R}^2$. Suppose that you are given the Delaunay triangulation of P in the adjacency format (i.e. for each point $p \in P$, you have a linked list containing the neighbors of P in the triangulation). Describe an algorithm to compute the convex hull of P in $O(n)$ time.
Hint: How can you tell whether a point $p \in P$ is on the boundary of the convex hull from its neighbors? Also be reminded that you are supposed to output the vertices of the convex hull in an appropriate order (e.g., clockwise).

Solution. The boundary of $\textsc{Hull}(P)$ must appear on the Delaunay triangulation of P, which is a point set triangulation. Therefore, we can select the lowest point v_1 in P in $O(n)$ time. We can add an auxilary vertex $v_0 = (-\infty, y_{v_1})$ and apply the gift wrapping algorithm. In every iteration, we investigate only the neighbor of the last point p_i and choose the one maximizing the directional angle $\angle v_{i-1}v_i p$, since these vertices must contain the new, undiscovered hull vertex.

Note that the number of edges investigated is bounded by $O(n+h) = O(n)$, since there are $3n-3-h = O(n)$ edges in the triangulation and only those edges connecting to two hull vertices can be visited twice.