Problem 1. Let $T : \mathbb{R}^d \mapsto \mathbb{R}^d$ be an affine transformation. Given a set P of points in $\mathbb{R}^d$, define $P' = \{T(\mathbf{p}) | \mathbf{p} \in P\}$, namely, P' is the set of points obtained by applying the affine transformation T to P. Prove: if $R \subseteq P$ is an ε-kernel of P, then $T(R)$ is an ε-kernel of $T(P)$.
Hint: Given a directional vector $\mathbf{u}$, the width of P at direction $\mathbf{u}$ can be calculated as

$$W_{\mathbf{u}}(P) = \left(\max_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p}\right) - \left(\min_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p}\right)$$

where $\mathbf{u} \cdot \mathbf{p}$ is the dot product of vectors $\mathbf{u}$ and $\mathbf{p}$. To prove the claim, use your knowledge from linear algebra to figure out how a dot product would change under an affine transformation.

Proof. Every affine transformation can be described in the form of $T(\mathbf{p}) = \mathbf{A}\mathbf{p} + \mathbf{b}$.

Consider the coreset $R \subseteq P$. For any directional vector $\mathbf{u}$, the width of $T(R)$ at direction $\mathbf{u}$ is

$$\begin{aligned}
W_{\mathbf{u}}(T(R)) &= \left(\max_{\mathbf{p} \in R} \mathbf{u} \cdot T(\mathbf{p})\right) - \left(\min_{\mathbf{p} \in R} \mathbf{u} \cdot T(\mathbf{p})\right) \\
&= \left(\max_{\mathbf{p} \in R} \mathbf{u} \cdot (\mathbf{A}\mathbf{p} + \mathbf{b})\right) - \left(\min_{\mathbf{p} \in R} \mathbf{u} \cdot (\mathbf{A}\mathbf{p} + \mathbf{b})\right) \\
&= \left(\max_{\mathbf{p} \in R} \mathbf{u} \cdot (\mathbf{A}\mathbf{p})\right) - \left(\min_{\mathbf{p} \in R} \mathbf{u} \cdot (\mathbf{A}\mathbf{p})\right)
\end{aligned}$$

Let $\mathbf{u} = (\mathbf{u}^* - \mathbf{b})\mathbf{A}^{-1}$. Then,

$$\begin{aligned}
W_{\mathbf{u}}(T(R)) &= \left(\max_{\mathbf{p} \in R} \mathbf{u} \cdot (\mathbf{A}\mathbf{p})\right) - \left(\min_{\mathbf{p} \in R} \mathbf{u} \cdot (\mathbf{A}\mathbf{p})\right) \\
&= \left(\max_{\mathbf{p} \in R} (\mathbf{u}^* - \mathbf{b}) \cdot \mathbf{p}\right) - \left(\min_{\mathbf{p} \in R} (\mathbf{u}^* - \mathbf{b}) \cdot \mathbf{p}\right) \\
&= \left(\max_{\mathbf{p} \in R} \mathbf{u}^* \cdot \mathbf{p}\right) - \left(\min_{\mathbf{p} \in R} \mathbf{u}^* \cdot \mathbf{p}\right) \\
&= \left(\max_{\mathbf{p} \in R} \mathbf{u}^* \cdot \mathbf{p}\right) - \left(\min_{\mathbf{p} \in R} \mathbf{u}^* \cdot \mathbf{p}\right) \\
&= W_{\mathbf{u}^*}(R)
\end{aligned}$$

Apply the same operation to $W_{\mathbf{u}}(T(P))$ and obtain $W_{\mathbf{u}}(T(P)) = W_{\mathbf{u}^*}(P)$.

By definition, $W_{\mathbf{u}^*}(R) \geq (1 - \varepsilon) W_{\mathbf{u}^*}(P)$.

Let $\mathbf{n}_{\mathbf{u}}^*$ be the normal vector of $\mathbf{u}^*$. Then, we know that $W_{\mathbf{n}_{\mathbf{u}^*}}(R) \geq (1 - \varepsilon) W_{\mathbf{n}_{\mathbf{u}^*}}(T(P))$, this is due to the result of dot product under scaling. $\square$

Problem 2. Let P be a set of points in $\mathbb{R}^d$ where each point has a positive coordinate on every dimension. We will view each point $p \in P$ as a d-dimensional vector $\mathbf{p} = (p[1], p[2], \cdots, p[d])$ where $p[i] (1 \le i \le d)$ is the i-th coordinate of p. Given a directional vector $\mathbf{u}$ where $u[i] \ge 0$ for each $i \in [d]$, define

$$\text{top}_{\mathbf{u}}(P) = \max_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p}$$

where $\mathbf{u} \cdot \mathbf{p}$ is the dot product of vectors $\mathbf{u}$ and $\mathbf{p}$.
Given $0 < \varepsilon < 1$, describe an algorithm that computes in $O(n)$ expected time a subset $R \in P$ such that

- $|R| = O(1/\varepsilon^d)$
- for any directional vector $\mathbf{u}$, it holds that $\text{top}_{\mathbf{u}}(R) \ge (1 - \varepsilon) \cdot \text{top}_{\mathbf{u}}(P)$.

Hint: Add the origin to P.

Solution. Let $P' = \{\mathbf{0}\} \cup P$. We construct the ε-kernel $R \subseteq P'$ for P' in $O(n)$ time, and $|R| = O(1/\varepsilon^d)$.

Now, we claim that $\text{top}_{\mathbf{u}}(R) \ge (1 - \varepsilon) \cdot \text{top}_{\mathbf{u}}(P)$ for any $\mathbf{u}$ with $u[i] \ge 0$. Note that from the definition of ε-kernel, we have

$$W_{\mathbf{u}}(R) = \max_{\mathbf{p} \in R} \mathbf{u} \cdot \mathbf{p} - \min_{\mathbf{p} \in R} \mathbf{u} \cdot \mathbf{p} \ge (1 - \varepsilon) W_{\mathbf{u}}(P').$$

Observe that $\min\limits_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p}$ would always return 0, due to the signs of $u[i]$ in $\mathbf{u}$, and $\mathbf{0} \cdot \mathbf{u} \equiv 0$, reaching the minimum attainable.

We see that $W_{\mathbf{u}}(R) = \max\limits_{\mathbf{p} \in R} \mathbf{u} \cdot \mathbf{p} \ge (1 - \varepsilon) W_{\mathbf{u}}(P') = (1 - \varepsilon) \max\limits_{\mathbf{p} \in P'} \mathbf{u} \cdot \mathbf{p}$. Now, noting that the addition of $\mathbf{0}$ to the set P does not affect the result due to the minimality, we can finish the proof of the claim by applying the definition of $\text{top}_{\mathbf{u}}(P) = \max\limits_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p}$, deriving

$$\text{top}_{\mathbf{u}}(R) = \max_{\mathbf{p} \in R} \mathbf{u} \cdot \mathbf{p} \ge (1 - \varepsilon) \cdot \max_{\mathbf{p} \in P'} \mathbf{u} \cdot \mathbf{p} \ge (1 - \varepsilon) \cdot \left(\left(\max_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p} \right) - \left(\min_{\mathbf{p} \in P} \mathbf{u} \cdot \mathbf{p} \right) \right) = (1 - \varepsilon) \cdot \text{top}_{\mathbf{u}}(P).$$

Problem 3. Prove the order-reversal property of dual transformation.

Proof. Consider the point $p = (p_x, p_y)$ and line $\ell : y = ax + b$. Then,

$$p \text{ is on or above } \ell \Leftrightarrow p_y \geq ap_x + b \Leftrightarrow -b \geq ap_x - p_y \Leftrightarrow (a, -b) \text{ is above } \ell' : y = p_x x - p_y,$$

meaning that ℓ^* is above p^* in the dual space. □

Problem 4. Prove the intersection preserving property of dual transformation.

Proof. Consider two lines ℓ_1 and ℓ_2 intersects at the point p. Now, $p \in \ell_1$, $p \in \ell_2$. Therefore, by the incidence preserving property, $\ell_1^* \in p^*$ and $\ell_2^* \in p^*$, implying that the dual line p^* passes through points ℓ_1^* and ℓ_2^*.

□

Problem 5. Let ℓ_1 and ℓ_2 be two parallel non-vertical lines in the primal space $\mathbb{R}^2$. Prove: their vertical distance equals the distance of points ℓ_1^* and ℓ_2^* in the dual space.

Proof. Assume that $\ell_1 : y = ax + b_1, \ell_2 : y = ax + b_2$.

The vertical distance is

$$d = |b_2 - b_1|.$$

In the primal space, the distance of points p_1^*, p_2^*, $dist(p_1^*, p_2^*)$ is $\sqrt{(b_1 - b_2)^2} = |b_2 - b_1|$. □

Problem 6. Let A, B, C, D be four points in the primal space $\mathbb{R}^2$ that have distinct x-coordinates. Suppose that triangle ABC has an area smaller than ABD. Prove: in the dual space, line C^* has smaller vertical distance from the point A^*B^* than D^*.
Note: The vertical distance from a point (a, b) to a line $y = c_1 x - c_2$ equals $|b - (c_1 \cdot a - c_2)|$.

Proof. Denote d_C, d_D the vertical distance from point $C = (x_C, y_C)$, $D = (x_D, y_D)$ to the line formed passing points A and B, $\ell : y = ax + b$, respectively.

The area of $\triangle ABC$ is $\frac{1}{2}|AB| \cdot d_C$, and area of $\triangle ABD$ is $\frac{1}{2}|AB| \cdot d_D$. Therefore, $d_C < d_D$ by $S_{\triangle ABC} < S_{\triangle ABD}$.

Again, this implies

$$|y_C - (a \cdot x_C + b)| < |y_D - (a \cdot x_D + b)| \Leftrightarrow |y_C - a \cdot x_C - b| < |y_D - a \cdot x_D - b|.$$

In the dual space, we verify that the vertical distance from $(a, -b)$ to $C^* : y = x_C x - y_C$ has smaller distance to $D^* : y = x_D x - y_D$.

$$|-b - (x_C a - y_C)| = |-b - x_C a + y_C| < |-b - (x_D a - y_D)| = |-b - x_D a + y_D|,$$

finishing the proof. □