18.445 Introduction to Stochastic Processes

Lecture 8: Random walk on networks 1

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Recall : Reversible Markov chain : there exists a probability measure π such that

$$\pi(x)P(x,y) = \pi(y)P(y,x), \quad \forall x,y \in \Omega.$$

- π is stationary
- $\mathbb{P}_\pi[X_0 = x_0, ..., X_n = x_n] = \mathbb{P}_\pi[X_0 = x_n, ..., X_n = x_0]$.

Today's Goal : Electrical networks

- network, conductance, resistance
- voltage, current flow
- effective resistance

Network

Definition

A network is a finite undirected connected graph $G = (V, E)$ endowed with non-negative numbers $\{c(e) : e \in E\}$.

- $c(e)$: conductance. Write $c(x, y)$ for $c(e)$ where $e = \{x, y\}$. Clearly $c(x, y) = c(y, x)$.
- $r(e) = 1/c(e)$: resistance.

Weighted random walk on network

Definition

Consider the Markov chain on V with transition matrix

$$P(x,y) = \frac{c(x,y)}{c(x)}, \quad \text{where } c(x) = \sum_y c(x,y).$$

This process is called the weighted random walk on G with edge weights $\{c(e) : e \in E\}$.

This Markov chain is reversible with respect to the probability measure π defined by

$$\pi(x) = \frac{c(x)}{c_G}, \quad \text{where } c_G = \sum_x c(x).$$

Therefore π is stationary for P.

Harmonic functions

Ω : state space ; P : the transition matrix, irreducible.
A function $h : \Omega \to \mathbb{R}$ is harmonic at x if $h(x) = \sum_y P(x,y)h(y)$.
Fix $B \subset \Omega$, define the hitting time by

$$\tau_B = \min\{n \geq 0 : X_n \in B\}.$$

Theorem

Let $(X_n)_{n\geq 0}$ be a Markov chain with irreducible transition matrix P. Let $h_B : B \to \mathbb{R}$ be a function defined on B. The function $h : \Omega \to \mathbb{R}$ defined by

$$h(x) = \mathbb{E}_x[h_B(X_{\tau_B})]$$

is the unique extension of h_B such that

$$h(x) = h_B(x), \quad \forall x \in B$$

and that h is harmonic at all $x \in \Omega \setminus B$.

Voltage

Definition

Consider a network $(G = (V, E), \{c(e) : e \in E\})$. We distinguish two vertices a (the source) and z (the sink). A voltage is a function on V which is harmonic on $V \setminus \{a, z\}$.

Remark A voltage is completely determined by its boundary values $W(a)$ and $W(z)$.

Flow

Definition

Consider a function θ defined on oriented edges. The divergence of θ is defined by

$$div\theta(x) = \sum_{y:y\sim x} \theta(\overrightarrow{xy}).$$

Definition

A flow from a to z is a function θ defined on oriented edges satisfying

1. θ is antisymmetric : $\theta(\overrightarrow{xy}) = -\theta(\overrightarrow{yx})$;
2. $div\theta(x) = 0$ for all $x \in V \setminus \{a, z\}$ (Node Law) ;
3. $div\theta(a) \geq 0$.

We define the strength of a flow θ from a to z to be $||\theta|| = div\theta(a)$. A unit flow is a flow with strength 1.

Current flow

Definition

Given a voltage W on the network, the current flow I associated with W is defined by

$$I(\overrightarrow{xy}) = \frac{W(x) - W(y)}{r(x,y)} = c(x,y)(W(x) - W(y)).$$

The current flow satisfies

- Ohm's Law : $r(x,y)I(\overrightarrow{xy}) = W(x) - W(y)$;
- Cycle Law : if the oriented edges $\overrightarrow{e}_1, ..., \overrightarrow{e}_m$ form an oriented cycle, then

$$\sum_{j=1}^{m} r(\overrightarrow{e}_j)I(\overrightarrow{e}_j) = 0.$$

Theorem

If θ is a flow from a to z satisfying Cycle Law for any cycle and $||\theta|| = ||I||$, then $\theta = I$.

Effective resistance

Definition

Given a network, suppose that W is a voltage and I is the corresponding current flow. Define the effective resistance between a and z by

$$R(a \leftrightarrow z) = \frac{W(a) - W(z)}{||I||}.$$

Theorem (Effective resistance and Escape probability)

For any $a, z \in \Omega$, consider the weighted random walk on the network, we have

$$\mathbb{P}_a[\tau_z < \tau_a^+] = \frac{1}{c(a)R(a \leftrightarrow z)}.$$

Effective resistance

Definition

The Green's function for a random walk stopped at a stopping time τ is defined by

$$G_\tau(a, x) = \mathbb{E}_a[\sharp\text{visits to } x \text{ before } \tau] = \sum_{n \geq 0} \mathbb{P}_a[X_n = x, n < \tau].$$

Theorem (Effective resistance and Green's function)

$$G_{\tau_z}(a, a) = c(a) R(a \leftrightarrow z).$$