18.445 Introduction to Stochastic Processes

Lecture 10: Hitting times

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Recall

- Consider a network $(G = (V, E), \{c(e) : e \in E\})$. The effective resistance is defined by
$$R(a \leftrightarrow z) = (W(a) - W(z))/||I||.$$
- Consider a random walk on the network, the Green's function is defined by
$$G_\tau(a, x) = \mathbb{E}[\sharp\text{visits to } x \text{ before } \tau].$$
- We have that
$$G_{\tau_z}(a, a) = c(a)R(a \leftrightarrow z).$$

Today's Goal

- hitting time
- commute time
- transitive network

Target time

Suppose that $(X_n)_{n\geq 0}$ is an irreducible Markov chain with transition matrix P and stationary measure π. Let τ_x be the hitting time :

$$\tau_x = \min\{n \geq 0 : X_n = x\}.$$

Lemma

The quantity

$$\sum_x \mathbb{E}_a[\tau_x]\pi(x)$$

does not depend on a ; and we call it target time and denote it by $t_\odot$.

Hitting time

Definition

$$t_{hit} := \max_{x,y} \mathbb{E}_x[\tau_y] \geq t_{\odot}.$$

Lemma

Suppose that the chain is irreducible with stationary measure π. Then

$$t_{hit} \leq 2 \max_{w} \mathbb{E}_{\pi}[\tau_w].$$

Theorem

For an irreducible transitive Markov chain, we have

$$t_{hit} \leq 2 t_{\odot}.$$

Transitive Markov chain

Roughly, a transitive Markov chain “looks the same" from any point in the state space.

Definition

A Markov chain is called transitive if for each pair $(x, y) \in \Omega \times \Omega$, there is a bijection $\varphi : \Omega \to \Omega$ such that

$$\varphi(x) = y; \quad P(\varphi(z), \varphi(w)) = P(z, w), \forall z, w.$$

Example : simple random walk on N-cycle, on hypercube.

Lemma

For a transitive Markov chain on finite state space Ω, the uniform measure is stationary.

Commute time

Definition

Suppose that the Markov chain starts from $X_0 = a$. The commute time between a and b is defined by

$$\tau_{ba} = \min\{n \geq \tau_b : X_n = a\}.$$

Theorem (Commute Time Identity)

Consider a random walk on the network $(G = (V, E), \{c(e) : e \in E\})$*, we have*

$$\mathbb{E}_a[\tau_{ba}] = \mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c_G R(a \leftrightarrow b).$$

Lemma

Suppose that the Markov chain is irreducible with stationary measure π*. Suppose that* τ *is a stopping time satisfying* $\mathbb{P}_a[X_\tau = a] = 1$*. Then*

$$G_\tau(a, x) = \mathbb{E}_a[\tau]\pi(x).$$

Transitive network

Generally, $\mathbb{E}_a[\tau_b]$ and $\mathbb{E}_b[\tau_a]$ can be very different (see Exercise 10.3). However, if the network is transitive, they are equal.

Definition

A network $(G = (V, E), \{c(e) : e \in E\})$ is transitive if for each pair $(x, y) \in V \times V$, there exists a bijection $\varphi : V \to V$ such that

$$\varphi(x) = y; \quad c(\varphi(z), \varphi(w)) = c(z, w), \forall z, w.$$

Remark : The random walk on a transitive network is a transitive Markov chain.

Theorem

For the random walk on a transitive (connected) network, for any vertices a and b, we have

$$\mathbb{E}_a[\tau_b] = \mathbb{E}_b[\tau_a].$$

Summary

For random walk on network

- $t_{\odot} \leq t_{hit} \leq 2 \max_w \mathbb{E}_\pi[\tau_w]$.
- $\mathbb{E}_a[\tau_{ba}] = c_G R(a \leftrightarrow b)$.

For random walk on transitive network

- $t_{\odot} \leq t_{hit} \leq 2t_{\odot}$.
- $\mathbb{E}_a[\tau_b] = \mathbb{E}_b[\tau_a]$.
- $2\mathbb{E}_a[\tau_b] = c_G R(a \leftrightarrow b)$.