

EFFICIENCY OF COORDINATE DESCENT METHODS ON HUGE-SCALE OPTIMIZATION PROBLEMS*

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Abstract. In this paper we propose new methods for solving huge-scale optimization problems. For problems of this size, even the simplest full-dimensional vector operations are very expensive. Hence, we propose to apply an optimization technique based on random partial update of decision variables. For these methods, we prove the global estimates for the rate of convergence. Surprisingly, for certain classes of objective functions, our results are better than the standard worst-case bounds for deterministic algorithms. We present constrained and unconstrained versions of the method and its accelerated variant. Our numerical test confirms a high efficiency of this technique on problems of very big size.

Key words. convex optimization, coordinate relaxation, worst-case efficiency estimates, fast gradient schemes, Google problem

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1. Introduction.

1.1. Motivation. Coordinate descent methods were among the first optimization schemes suggested for solving smooth unconstrained minimization problems (see [1, 2] and references therein). The main advantage of these methods is the simplicity of each iteration, both in generating the search direction and in performing the update of variables. However, very soon it became clear that the coordinate descent methods can be criticized in several aspects.

1.2. Theoretical justification. The simplest variant of the coordinate descent method is based on a cyclic coordinate search. However, for this strategy it is difficult to prove convergence and almost impossible to estimate the rate of convergence.¹ Another possibility is to move along the direction corresponding to the component of gradient with maximal absolute value. For this strategy, justification of the convergence rate is trivial. For future reference, we present this result now. Consider the unconstrained minimization problem

$$(1.1) \quad \min_{x \in R^n} f(x),$$

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¹To the best of our knowledge, in the general case the global rate of convergence is not established yet. For the local rate of convergence of the method with almost cyclic and Gauss–Southwell rules see [4].

where the convex objective function f has componentwise Lipschitz continuous gradient,

$$(1.2) \quad |\nabla_i f(x + he_i) - \nabla_i f(x)| \leq M|h|, \quad x \in R^n, \quad h \in R, \quad i = 1, \dots, n,$$

where e_i is the i th coordinate vector in R^n . Consider the following method:

$$(1.3) \quad \boxed{\begin{array}{l} \text{Choose } x_0 \in R^n. \text{ For } k \geq 0 \text{ iterate} \\ 1. \quad \text{Choose } i_k = \arg \max_{1 \leq i \leq n} |\nabla_i f(x_k)|. \\ 2. \quad \text{Update } x_{k+1} = x_k - \frac{1}{M} \nabla_{i_k} f(x_k) e_{i_k}. \end{array}}$$

Then

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\stackrel{(1.2)}{\geq} \frac{1}{2M} |\nabla_{i_k} f(x_k)|^2 \geq \frac{1}{2nM} \|\nabla f(x_k)\|^2 \\ &\geq \frac{1}{2nMR^2} (f(x_k) - f^*)^2, \end{aligned}$$

where $R \geq \|x_0 - x^*\|$, the norm is Euclidean, and f^* is the optimal value of problem (1.1). By standard reasoning (see, for example, the proof of Theorem 1), we conclude that

$$(1.4) \quad f(x_k) - f^* \leq \frac{2nMR^2}{k+4}, \quad k \geq 0.$$

Note that at each test point, method (1.3) requires computation of the *whole* gradient vector. However, if this vector is available, it seems better to apply the usual full-gradient methods. It is also important that for convex functions with Lipschitz-continuous gradient,

$$(1.5) \quad \|\nabla f(x) - \nabla f(y)\| \leq L(f)\|x - y\|, \quad x, y \in R^n,$$

it can happen that $M \geq O(L(f))$. Hence, in general, the rate (1.4) is *worse* than the rate of convergence of the simple gradient method (e.g., section 2.1 in [8]).

1.3. Computational complexity. From the theory of fast differentiation it is known that for all functions defined by an explicit sequence of standard operations, the complexity of computing the whole gradient is proportional to the computational complexity of the value of corresponding function. Moreover, the coefficient in this proportion is a small absolute constant. This observation suggests that for coordinate descent methods the line-search strategies based on the function values are too expensive. Provided that the general directional derivative of the function has the same complexity as the function value, it seems that no room is left for supporting the coordinate descent idea. The versions of this method still appear in the literature. However, they are justified only by local convergence analysis for rather particular problems (e.g., [5, 10]).

At the same time, in recent years we have seen increasing interest in optimization problems of a very big size (Internet applications, telecommunications). In such problems, even computation of a function value can require substantial computational efforts. Moreover, some parts of the problem's data can be distributed in space and

in time. The problem's data may be only partially available at the moment of evaluating the current test point. For problems of this type, we adopt the term *huge-scale* problems.

These applications strongly push us backward to the framework of coordinate minimization. Therefore, let us look again at the above criticism. It appears that there is a small chance for these methods to survive.

1. We can also think about the *random* coordinate search with prespecified probabilities for each coordinate move. As we will see later, the complexity analysis of corresponding methods is quite straightforward. On the other hand, from a technological point of view, this strategy fits very well the problems with distributed or randomly available data.
2. It appears that the computation of a *coordinate directional derivative* can be much simpler than computation of either a function value or a directional derivative along an *arbitrary* direction.

In order to support this claim, let us look at the optimization problem

$$(1.6) \quad \min_{x \in R^n} \left\{ f(x) \stackrel{\text{def}}{=} \sum_{i=1}^n f_i(x^{(i)}) + \frac{1}{2} \|Ax - b\|^2 \right\},$$

where f_i are convex differentiable univariate functions, $A = (a_1, \dots, a_n) \in R^{p \times n}$ is an $p \times n$ -matrix, and $\|\cdot\|$ is the standard Euclidean norm in R^p . Then

$$\begin{aligned} \nabla_i f(x) &= f'_i(x^{(i)}) + \langle a_i, g(x) \rangle, \quad i = 1, \dots, n, \\ g(x) &= Ax - b. \end{aligned}$$

If the residual vector $g(x)$ is already computed, then the computation of i th directional derivative requires $O(p_i)$ operations, where p_i is the number of nonzero elements in vector a_i . On the other hand, the coordinate move $x_+ = x + \alpha e_i$ results in the following change in the residual:

$$g(x_+) = g(x) + \alpha a_i.$$

Therefore, the i th coordinate step for problem (1.6) needs $O(p_i)$ operations. Note that computation of either the function value or the whole gradient or an arbitrary directional derivative requires $O(\sum_{i=1}^n p_i)$ operations. The reader can easily find many other examples of optimization problems with cheap coordinate directional derivatives.

The goal of this paper is to provide the random coordinate descent methods (RCDMs) with the worst-case efficiency estimates. We show that for functions with cheap coordinate derivatives the new methods are always faster than the corresponding full-gradient schemes. A surprising result is that even for functions with expensive coordinate derivatives the new methods can also be faster since their rate of convergence depends on an upper estimate for the *average* diagonal element of the Hessian of the objective function. This value can be much smaller than the maximal eigenvalue of the Hessian, entering the worst-case complexity estimates of the black-box gradient schemes.

1.4. Contents. The paper is organized as follows. In section 2 we analyze the expected rate of convergence of the simplest RCDM. It is shown that it is reasonable to define probabilities for particular coordinate directions using the estimates of Lipschitz constants for partial derivatives. In section 3 we show that for the class of

strongly convex functions RCDM converges with a linear rate. By applying a regularization technique, this allows us to solve the unconstrained minimization problems with arbitrary high confidence level. In section 4 we analyze a modified version of RCDM as applied to constrained optimization problem. In section 5 we show that an unconstrained version of RCDM can be accelerated up to the rate of convergence $O(\frac{1}{k^2})$, where k is the iteration counter. Finally, in section 6 we discuss the implementation issues. In section 6.1 we show that the good estimates for coordinate Lipschitz constants can be efficiently computed. In section 6.2 we present an efficient strategy for generating random coordinate directions. And in section 6.3 we present preliminary computational results.

1.5. Notation. In this paper we work with real coordinate spaces R^n composed by column vectors. For $x, y \in R^n$ denote

$$\langle x, y \rangle = \sum_{i=1}^n x^{(i)} y^{(i)}.$$

We use the same notation $\langle \cdot, \cdot \rangle$ for spaces of different dimension. Thus, its actual sense is defined by the space containing the arguments. If we fix a norm $\| \cdot \|$ in R^n , then the dual norm is defined in the standard way:

$$(1.7) \quad \|s\|^* = \max_{\|x\|=1} \langle s, x \rangle.$$

We denote by $s^\#$ an arbitrary vector from the set

$$(1.8) \quad \text{Arg max}_x \left[\langle s, x \rangle - \frac{1}{2} \|x\|^2 \right].$$

Clearly, $\|s^\#\| = \|s\|^*$.

For function $f(x)$, $x \in R^n$, we denote by $\nabla f(x)$ its *gradient*, which is a vector from R^n composed by partial derivatives. By $\nabla^2 f(x)$ we denote the Hessian of f at x . In what follows we will use the following simple result. It will help us to compare the worst-case complexity bounds for the standard gradient methods, based on the maximal eigenvalues of the Hessians, with the complexity bounds of the coordinate descent schemes.

LEMMA 1. *Let us fix a decomposition of R^n on k subspaces. For positive semidefinite symmetric matrix $A \in R^{n \times n}$, denote by $A_{i,i}$ the corresponding diagonal blocks. If*

$$A_{i,i} \preceq L_i B_i, \quad i = 1, \dots, k,$$

where all $B_i \succeq 0$, then

$$A \preceq \left(\sum_{i=1}^k L_i \right) \cdot \text{diag} \{B_i\}_{i=1}^k,$$

where $\text{diag} \{B_i\}_{i=1}^k$ is a block-diagonal $(n \times N)$ -matrix with diagonal blocks B_i . In particular,

$$A \preceq n \cdot \text{diag} \{A_{i,i}\}_{i=1}^n.$$

Proof. Denote by $x^{(i)}$ the part of vector $x \in R^n$ belonging to i th subspace. Since A is positive semidefinite, we have

$$\begin{aligned} \langle Ax, x \rangle &= \sum_{i=1}^k \sum_{j=1}^k \langle A_{i,j} x^{(i)}, x^{(j)} \rangle \leq \sum_{i=1}^k \sum_{j=1}^k \langle A_{i,i} x^{(i)}, x^{(i)} \rangle^{1/2} \langle A_{j,j} x^{(j)}, x^{(j)} \rangle^{1/2} \\ &= \left(\sum_{i=1}^k \langle A_{i,i} x^{(i)}, x^{(i)} \rangle^{1/2} \right)^2 \leq \left(\sum_{i=1}^k L_i^{1/2} \langle B_{i,i} x^{(i)}, x^{(i)} \rangle^{1/2} \right)^2 \\ &\leq \sum_{i=1}^k L_i \cdot \sum_{i=1}^k \langle B_{i,i} x^{(i)}, x^{(i)} \rangle. \quad \square \end{aligned}$$

2. Coordinate relaxation for unconstrained minimization. Consider the unconstrained minimization problem

$$(2.1) \quad \min_{x \in R^N} f(x),$$

where the objective function f is convex and differentiable on R^N . We assume that the optimal set X^* of this problem is nonempty and bounded.

Let us fix a decomposition of R^N on n subspaces:

$$R^N = \bigotimes_{i=1}^n R^{n_i}, \quad N = \sum_{i=1}^n n_i.$$

Then we can define the corresponding partition of the unit matrix

$$I_N = (U_1, \dots, U_n) \in R^{N \times N}, \quad U_i \in R^{N \times n_i}, \quad i = 1, \dots, n.$$

Thus, any $x = (x^{(1)}, \dots, x^{(n)})^T \in R^N$ can be represented as

$$x = \sum_{i=1}^n U_i x^{(i)}, \quad x^{(i)} \in R^{n_i}, \quad i = 1, \dots, n.$$

Then the *partial gradient* of $f(x)$ in $x^{(i)}$ is defined as

$$f'_i(x) = U_i^T \nabla f(x) \in R^{n_i}, \quad x \in R^N.$$

For spaces R^{n_i} , let us fix some norms $\|\cdot\|_{(i)}$, $i = 1, \dots, n$. We assume that the gradient of function f is coordinatewise Lipschitz continuous with constants $L_i = L_i(f)$:

$$(2.2) \quad \|f'_i(x + U_i h_i) - f'_i(x)\|_{(i)}^* \leq L_i \|h_i\|_{(i)}, \quad h_i \in R^{n_i}, \quad i = 1, \dots, n, \quad x \in R^N.$$

For the sake of simplicity, let us assume that these constants are known.

By the standard reasoning (e.g., section 2.1 in [8]), we can prove that

$$(2.3) \quad f(x + U_i h_i) \leq f(x) + \langle f'_i(x), h_i \rangle + \frac{L_i}{2} \|h_i\|_{(i)}^2, \quad x \in R^N, \quad h_i \in R^{n_i}, \quad i = 1, \dots, n.$$

Let us define the optimal coordinate steps:

$$T_i(x) \stackrel{\text{def}}{=} x - \frac{1}{L_i} U_i f'_i(x)^\#, \quad i = 1, \dots, n.$$

Then, in view of the bound (2.3), we get

$$(2.4) \quad f(x) - f(T_i(x)) \geq \frac{1}{2L_i} \left(\|f'_i(x)\|_{(i)}^* \right)^2, \quad i = 1, \dots, n.$$

In our random algorithm, we need a special random counter \mathcal{A}_α , $\alpha \in R$, which generates an integer number $i \in \{1, \dots, n\}$ with probability

$$(2.5) \quad p_\alpha^{(i)} = L_i^\alpha \cdot \left[\sum_{j=1}^n L_j^\alpha \right]^{-1}, \quad i = 1, \dots, n.$$

Thus, the operation $i = \mathcal{A}_\alpha$ means that an integer random value, chosen from the set $\{1, \dots, n\}$ in accordance to probabilities (2.5), is assigned to variable i . Note that \mathcal{A}_0 generates a uniform distribution.²

Now we can present the scheme of the RCDM. It needs a starting point $x_0 \in R^N$ and value $\alpha \in R$ as input parameters.

$$(2.6) \quad \boxed{\begin{array}{l} \text{Method RCDM}(\alpha, x_0) \\ \text{For } k \geq 0 \text{ iterate:} \\ (1) \text{ Choose } i_k = \mathcal{A}_\alpha. \\ (2) \text{ Update } x_{k+1} = T_{i_k}(x_k). \end{array}}$$

For estimating the rate of convergence of RCDM, we introduce the following norms:

$$(2.7) \quad \begin{aligned} \|x\|_{[\alpha]} &= \left[\sum_{i=1}^n L_i^\alpha \|x^{(i)}\|_{(i)}^2 \right]^{1/2}, \quad x \in R^N, \\ \|g\|_{[\alpha]}^* &= \left[\sum_{i=1}^n L_i^{-\alpha} \left(\|g^{(i)}\|_{(i)}^* \right)^2 \right]^{1/2}, \quad g \in R^N. \end{aligned}$$

Clearly, these norms satisfy the Cauchy–Schwarz inequality:

$$(2.8) \quad \|g\|_{[\alpha]}^* \cdot \|x\|_{[\alpha]} \geq \langle g, x \rangle, \quad x, g \in R^N.$$

In what follows, we use the notation $S_\alpha = S_\alpha(f) = \sum_{i=1}^n L_i^\alpha(f)$ with $\alpha \geq 0$. Note that

$$S_0(f) \equiv n.$$

Let us link our main assumption (2.2) with a full-dimensional Lipschitz condition for the gradient of the objective function.

LEMMA 2. *Let f satisfy condition (2.2). Then for any $\alpha \in R$ we have*

$$(2.9) \quad \|\nabla f(x) - \nabla f(y)\|_{[1-\alpha]}^* \leq S_\alpha \|x - y\|_{[1-\alpha]}, \quad x, y \in R^N.$$

²The only delicate feature of this counter is its complexity. A straightforward transformation of a standard uniform counter to a nonuniform one needs $O(n)$ operations. In section 6.2 we will show that this transformation can be done in $O(\ln n)$ operations.

Therefore,

$$(2.10) \quad f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{1}{2} S_\alpha \|x - y\|_{[1-\alpha]}^2, \quad x, y \in R^N.$$

Proof. Indeed,

$$\begin{aligned} f(x) - f^* &\stackrel{(2.4)}{\geq} \max_{1 \leq i \leq n} \frac{1}{2L_i} \left(\|f'_i(x)\|_{(i)}^* \right)^2 \\ &\geq \frac{1}{2S_\alpha} \sum_{i=1}^n \frac{L_i^\alpha}{L_i} \left(\|f'_i(x)\|_{(i)}^* \right)^2 = \frac{1}{2S_\alpha} \left(\|\nabla f(x)\|_{[1-\alpha]}^* \right)^2. \end{aligned}$$

Applying this inequality to function $\psi(x) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle$, we obtain

$$\begin{aligned} (2.11) \quad f(x) - f(y) - \langle \nabla f(y), x - y \rangle &= \psi(x) - \min_{v \in R^n} \psi(v) \\ &\geq \frac{1}{2S_\alpha} \left(\|\nabla \psi(x)\|_{[1-\alpha]}^* \right)^2 \\ &= \frac{1}{2S_\alpha} \left(\|\nabla f(x) - \nabla f(y)\|_{[1-\alpha]}^* \right)^2, \quad x, y \in R^N. \end{aligned}$$

Adding two variants of (2.11) with x and y interchanged, we get

$$\begin{aligned} \frac{1}{S_\alpha} \left(\|\nabla f(x) - \nabla f(y)\|_{[1-\alpha]}^* \right)^2 &\leq \langle \nabla f(x) - \nabla f(y), x - y \rangle \\ &\stackrel{(2.8)}{\leq} \|\nabla f(x) - \nabla f(y)\|_{[1-\alpha]}^* \cdot \|x - y\|_{1-\alpha}. \end{aligned}$$

Inequality (2.10) can be derived from (2.9) by simple integration. \square

After k iterations, RCDM(α, x_0) generates a random output $(x_k, f(x_k))$, which depends on the observed implementation of random variable

$$\xi_k = \{i_0, \dots, i_k\}.$$

Let us show that the expected value

$$\phi_k \stackrel{\text{def}}{=} E_{\xi_{k-1}} f(x_k)$$

converges to the optimal value f^* of problem (2.1).

THEOREM 1. *For any $k \geq 0$ we have*

$$(2.12) \quad \phi_k - f^* \leq \frac{2}{k+4} \cdot \left[\sum_{j=1}^n L_j^\alpha \right] \cdot R_{1-\alpha}^2(x_0),$$

where $R_\beta(x_0) = \max_x \{ \max_{x_* \in X^*} \|x - x_*\|_{[\beta]} : f(x) \leq f(x_0) \}$.

Proof. Let RCDM generate implementation x_k of a corresponding random variable. Then

$$\begin{aligned} (2.13) \quad f(x_k) - E_{i_k}(f(x_{k+1})) &= \sum_{i=1}^n p_\alpha^{(i)} \cdot [f(x_k) - f(T_i(x_k))] \\ &\stackrel{(2.4)}{\geq} \sum_{i=1}^n \frac{p_\alpha^{(i)}}{2L_i} \left(\|f'_i(x_k)\|_{(i)}^* \right)^2 \stackrel{(2.5)}{=} \frac{1}{2S_\alpha} (\|\nabla f(x_k)\|_{[1-\alpha]}^*)^2. \end{aligned}$$

Note that $f(x_k) \leq f(x_0)$. Therefore,

$$f(x_k) - f^* \leq \min_{x_* \in X^*} \langle \nabla f(x_k), x_k - x_* \rangle \stackrel{(2.8)}{\leq} \|\nabla f(x_k)\|_{[1-\alpha]}^* R_{1-\alpha}(x_0).$$

Hence,

$$f(x_k) - E_{i_k}(f(x_{k+1})) \geq \frac{1}{C}(f(x_k) - f^*)^2,$$

where $C \stackrel{\text{def}}{=} 2S_\alpha R_{1-\alpha}^2(x_0)$. Taking the expectation of both sides of this inequality in ξ_{k-1} , we obtain

$$\phi_k - \phi_{k+1} \geq \frac{1}{C} E_{\xi_{k-1}}[(f(x_k) - f^*)^2] \geq \frac{1}{C}(\phi_k - f^*)^2.$$

Thus,

$$\frac{1}{\phi_{k+1} - f^*} - \frac{1}{\phi_k - f^*} = \frac{\phi_k - \phi_{k+1}}{(\phi_{k+1} - f^*)(\phi_k - f^*)} \geq \frac{\phi_k - \phi_{k+1}}{(\phi_k - f^*)^2} \geq \frac{1}{C}.$$

Summing up the first k inequalities of this type, we obtain $\frac{1}{\phi_k - f^*} \geq \frac{1}{\phi_0 - f^*} + \frac{k}{C} \stackrel{(2.10)}{\geq} \frac{k+4}{C}$ for any $k \geq 0$. \square

Let us look at the most important variants of the estimate (2.12).

- $\alpha = 0$. Then $S_0 = n$, and we get

$$(2.14) \quad \phi_k - f^* \leq \frac{2n}{k+4} \cdot R_1^2(x_0).$$

Note that problem (2.1) can be solved by the standard full-gradient method endowed with the metric $\|\cdot\|_1$. Then its rate of convergence can be estimated as

$$f(x_k) - f^* \leq \frac{\gamma}{k} R_1^2(x_0),$$

where the constant γ is big enough to ensure

$$f''(x) \preceq \gamma \cdot \text{diag}\{L_i \cdot I_{n_i}\}_{i=1}^n, \quad x \in R^N,$$

and I_k is a unit matrix in R^k . (Assume for a moment that f is twice differentiable.) However, since the constants L_i are the upper bounds for the block-diagonal elements of the Hessian, in the worst case we have $\gamma = n$. Hence, the worst-case rate of convergence of this variant of the gradient method is proportional to that of RCDM. However, the iteration of the latter method is usually much cheaper.

- $\alpha = \frac{1}{2}$. Consider the case $n_i = 1$, $i = 1, \dots, n$. Denote by $D_\infty(x_0)$ the size of the initial level set measured in the infinity norm:

$$D_\infty(x_0) = \max_x \left\{ \max_{y \in X^*} \max_{1 \leq i \leq n} |x^{(i)} - y^{(i)}| : f(x) \leq f(x_0) \right\}.$$

Then, $R_{1/2}^2(x_0) \leq S_{1/2} D_\infty^2(x_0)$, and we obtain

$$(2.15) \quad \phi_k - f^* \leq \frac{2}{k+4} \cdot \left[\sum_{i=1}^n L_i^{1/2} \right]^2 \cdot D_\infty^2(x_0).$$

Note that for the first order methods, the worst-case dimension-independent complexity of minimizing the convex functions over an n -dimensional box is infinite [6]. Since for some problems the value $S_{1/2}$ can be bounded even for very big (or even infinite) dimension, the estimate (2.15) shows that RCDM can work in situations where the usual gradient methods have no theoretical justification.

- $\alpha = 1$. Consider the case when all norms $\|\cdot\|_{(i)}$ are the standard Euclidean norms of R^{n_i} , $i = 1, \dots, n$. Then $R_0(x_0)$ is the size of the initial level set in the standard Euclidean norm of R^N , and the rate of convergence of RCDM(1, x_0) is as follows:

$$(2.16) \quad \phi_k - f^* \leq \frac{2}{k+4} \cdot \left[\sum_{i=1}^n L_i \right] \cdot R_0^2(x_0) \equiv \frac{2n}{k+4} \cdot \left[\frac{1}{n} \sum_{i=1}^n L_i \right] \cdot R_0^2(x_0).$$

At the same time, the rate of convergence of the standard gradient method can be estimated as

$$f(x_k) - f^* \leq \frac{\gamma}{k} R_0^2(x_0),$$

where γ satisfies condition

$$f''(x) \preceq \gamma \cdot I_N, \quad x \in R^N.$$

Note that the maximal eigenvalue of a symmetric matrix can reach its trace. Hence, in the worst case, the rate of convergence of the gradient method is the same as the rate of RCDM. However, the latter method has many more chances to accelerate.³

3. Minimizing strongly convex functions. Let us estimate now the performance of RCDM on strongly convex functions. Recall that f is called strongly convex on R^N with convexity parameter $\sigma = \sigma(f) > 0$ if for any x and y from R^N we have

$$(3.1) \quad f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \sigma(f) \|y - x\|^2.$$

Minimizing both sides of this inequality in y , we obtain a useful bound,

$$(3.2) \quad f(x) - f^* \leq \frac{1}{2\sigma(f)} (\|\nabla f(x)\|^*)^2.$$

THEOREM 2. *Let function $f(x)$ be strongly convex with respect to the norm $\|\cdot\|_{[1-\alpha]}$ with convexity parameter $\sigma_{1-\alpha} = \sigma_{1-\alpha}(f) > 0$. Then, for the sequence $\{x_k\}$ generated by RCDM(α, x_0) we have*

$$(3.3) \quad \phi_k - \phi^* \leq \left(1 - \frac{\sigma_{1-\alpha}(f)}{S_\alpha(f)} \right)^k (f(x_0) - f^*).$$

Proof. In view of inequality (2.13), we have

$$f(x_k) - E_{i_k}(f(x_{k+1})) \geq \frac{1}{2S_\alpha} (\|\nabla f(x_k)\|_{[1-\alpha]}^*)^2 \stackrel{(3.2)}{\geq} \frac{\sigma_{1-\alpha}}{S_\alpha} (f(x_k) - f^*).$$

It remains to compute the expectation in ξ_{k-1} . \square

³After this paper was finished, the author learned that method (2.6) with $\alpha = 1$ was suggested in [3] for minimizing the strongly convex quadratic function. In [3], the authors prove the linear rate of convergence of this scheme. However, they do not address the complexity details related to large-scale sparse problems.

In the remaining part of this section, we show how to use this result for generating reliable solutions to problem (2.1), where the objective function is *not* strongly convex. At this moment, we are able to prove only that the *expected quality* of the output of RCDM is good. However, in practice we are not going to run this method many times on the same problem. What is the probability that our single run can also give us a good result?

In order to answer this question, we need to apply RCDM to a regularized objective of the initial problem. For the sake of simplicity, let us endow the spaces R^{n_i} with some Euclidean norms:

$$(3.4) \quad \|h^{(i)}\|_{(i)}^2 = \langle B_i h^{(i)}, h^{(i)} \rangle, \quad h^{(i)} \in R^{n_i},$$

where $B_i \succ 0$, $i = 1, \dots, n$. We present the complexity results for two ways of measuring distances in R^N .

1. Let us use the norm $\|\cdot\|_{[1]}$:

$$(3.5) \quad \|h\|_{[1]}^2 = \sum_{i=1}^n L_i \langle B_i h^{(i)}, h^{(i)} \rangle \stackrel{\text{def}}{=} \langle B_L h, h \rangle, \quad h \in R^N.$$

Since this norm is Euclidean, the regularized objective function

$$f_\mu(x) = f(x) + \frac{\mu}{2} \|x - x_0\|_{[1]}^2$$

is strongly convex with respect to $\|\cdot\|_{[1]}$ with convexity parameter μ . Moreover,

$$S_0(f_\mu) = n$$

for any value of $\mu > 0$. Hence, by Theorem 2, method RCDM(0, x_0) can quickly approach the value $f_\mu^* = \min_{x \in R^N} f_\mu(x)$. The following result will be useful.

LEMMA 3. *Let random point x_k be generated by RCDM(0, x_0) as applied to function f_μ . Then*

$$(3.6) \quad E_{\xi_k}(\|\nabla f_\mu(x_k)\|_{[1]}^*) \leq \left[2(n + \mu) \cdot (f(x_0) - f^*) \cdot \left(1 - \frac{\mu}{n}\right)^k \right]^{1/2}.$$

Proof. In view of Lemma 2, for any $h = (h^{(1)}, \dots, h^{(n)}) \in R^N$ we have

$$\begin{aligned} f_\mu(x + h) &\leq f(x) + \langle \nabla f(x), h \rangle + \frac{n}{2} \|h\|_{[1]}^2 + \frac{\mu}{2} \|x + h - x_0\|_{[1]}^2 \\ &\stackrel{(3.5)}{=} f_\mu(x) + \langle \nabla f_\mu(x), h \rangle + \frac{n + \mu}{2} \|h\|_{[1]}^2. \end{aligned}$$

Thus, function f_μ has a Lipschitz-continuous gradient (e.g., [8, equation (2.1.6) in Theorem 2.1.5]) with constant $L(f_\mu) \stackrel{\text{def}}{=} n + \mu$. Therefore (see, for example, [8, equation (2.1.7) in Theorem 2.1.5]),

$$\frac{1}{2L(f_\mu)} \left(\|\nabla f_\mu(x)\|_{[1]}^* \right)^2 \leq f_\mu(x) - f_\mu^*, \quad x \in R^N.$$

Hence,

$$\begin{aligned} E_{\xi_k}(\|\nabla f_\mu(x_k)\|_{[1]}^*) &\leq E_{\xi_k} \left([2(n + \mu) \cdot (f_\mu(x_k) - f_\mu^*)]^{1/2} \right) \\ &\leq [2(n + \mu) \cdot (E_{\xi_k}(f_\mu(x_k)) - f_\mu^*)]^{1/2} \\ &\stackrel{(3.3)}{\leq} \left[2(n + \mu) \cdot \left(1 - \frac{\mu}{n}\right)^k (f(x_0) - f_\mu^*) \right]^{1/2}. \end{aligned}$$

It remains to note that $f_\mu^* \geq f^*$. \square

Now we can estimate the quality of the random point x_k generated by $\text{RCDM}(0, x_0)$, taken as an approximate solution to problem (2.1). Let us fix the desired accuracy of the solution $\epsilon > 0$ and the confidence level $\beta \in (0, 1)$.

THEOREM 3. *Let us define $\mu = \frac{\epsilon}{4R_1^2(x_0)}$ and choose*

$$(3.7) \quad k \geq 1 + \frac{8nR_1^2(x_0)}{\epsilon} \ln \frac{2nR_1^2(x_0)}{\epsilon(1-\beta)}.$$

If the random point x_k is generated by $\text{RCDM}(0, x_0)$ as applied to function f_μ , then

$$\mathbf{Prob}(f(x_k) - f^* \leq \epsilon) \geq \beta.$$

Proof. Note that $f(x_k) \leq f_\mu(x_k) \leq f_\mu(x_0) = f(x_0)$. Therefore, there exists $x_* \in X^*$ such that $\|x_k - x_*\|_{[1]} \leq R_1(x_0) \stackrel{\text{def}}{=} R$. Hence, $\|x_k - x_0\|_{[1]} \leq \|x_k - x_*\|_{[1]} + \|x_* - x_0\|_{[1]} \leq 2R$. Since

$$(3.8) \quad \nabla f_\mu(x) = \nabla f(x) + \mu \cdot B_L(x - x_0),$$

we conclude that

$$\begin{aligned} \mathbf{Prob}(f(x_k) - f^* \geq \epsilon) &\leq \mathbf{Prob}(\|\nabla f(x_k)\|_{[1]}^* \cdot R \geq \epsilon) \\ &\stackrel{(3.8)}{\leq} \mathbf{Prob}(\|\nabla f_\mu(x_k)\|_{[1]}^* \cdot R + 2\mu R^2 \geq \epsilon) \\ &= \mathbf{Prob}\left(\|\nabla f_\mu(x_k)\|_{[1]}^* \geq \frac{\epsilon}{2R}\right). \end{aligned}$$

Now, using the Chebyshev inequality, we obtain

$$\begin{aligned} \mathbf{Prob}\left(\|\nabla f_\mu(x_k)\|_{[1]}^* \geq \frac{\epsilon}{2R}\right) &\leq \frac{2R}{\epsilon} \cdot E_{\xi_k}\left(\|\nabla f_\mu(x_k)\|_{[1]}^*\right) \\ &\stackrel{(3.6)}{\leq} \frac{2R}{\epsilon} \cdot \left[2(n + \mu) \cdot (f(x_0) - f^*) \cdot \left(1 - \frac{\mu}{n}\right)^k\right]^{1/2}. \end{aligned}$$

Since the gradient of function f is Lipschitz continuous with constant n , we get

$$f(x_0) - f^* \leq \frac{n}{2} R^2.$$

Taking into account that $(n + \mu)(1 - \frac{\mu}{n})^k \leq n(1 - \frac{\mu}{n})^{k-1}$, we obtain the following bound:

$$\mathbf{Prob}\left(\|\nabla f_\mu(x_k)\|_{[1]}^* \geq \frac{\epsilon}{2R}\right) \leq \frac{2nR^2}{\epsilon} \cdot \left(1 - \frac{\mu}{n}\right)^{\frac{k-1}{2}} \leq \frac{2nR^2}{\epsilon} \cdot e^{-\frac{\mu(k-1)}{2n}} \leq 1 - \beta.$$

This proves the statement of the theorem. \square

For the current choice of norm (3.5), we can guarantee only $L(f) \leq n$. Therefore, the standard gradient method as applied to the problem (2.1) has the worst-case complexity bound of

$$O\left(\frac{nR_1^2(x_0)}{\epsilon}\right)$$

full-dimensional gradient iterations. Up to a logarithmic factor, this bound coincides with (3.7).

2. Consider now the following norm:

$$(3.9) \quad \|h\|_{[0]}^2 = \sum_{i=1}^n \langle B_i h^{(i)}, h^{(i)} \rangle \stackrel{\text{def}}{=} \langle Bh, h \rangle, \quad h \in R^N.$$

Again, the regularized objective function

$$f_\mu(x) = f(x) + \frac{\mu}{2} \|x - x_0\|_{[0]}^2$$

is strongly convex with respect to $\|\cdot\|_{[0]}$ with convexity parameter μ . However, now

$$S_1(f_\mu) = \sum_{i=1}^n [L_i(f) + \mu] = S_1(f) + n\mu.$$

Since $L(f_\mu) = S_1(f) + \mu$, using the same arguments as in the proof of Lemma 3, we get the following bound.

LEMMA 4. *Let random point x_k be generated by $RCDM(1, x_0)$ as applied to function f_μ . Then*

$$(3.10) \quad E_{\xi_k}(\|\nabla f_\mu(x_k)\|_{[0]}^*) \leq \left[2(S_1(f) + \mu) \cdot (f(x_0) - f^*) \cdot \left(1 - \frac{\mu}{S_1(f) + n\mu}\right)^k \right]^{1/2}.$$

Using this lemma, we can prove the following theorem.

THEOREM 4. *Let us define $\mu = \frac{\epsilon}{4R_0^2(x_0)}$ and choose*

$$(3.11) \quad k \geq 2 \left[n + \frac{4S_1(f)R_0^2(x_0)}{\epsilon} \right] \cdot \left(\ln \frac{1}{1-\beta} + \ln \left(\frac{1}{2} + \frac{2S_1(f)R_0^2(x_0)}{\epsilon} \right) \right).$$

If the random point x_k is generated by $RCDM(1, x_0)$ as applied to function f_μ , then

$$\mathbf{Prob}(f(x_k) - f^* \leq \epsilon) \geq \beta.$$

Proof. As in the proof of Theorem 3, we can prove that $\|x_k - x_0\|_{[0]} \leq 2R \equiv 2R_0(x_0)$. Since

$$(3.12) \quad \nabla f_\mu(x) = \nabla f(x) + \mu \cdot B(x - x_0),$$

we conclude that

$$\begin{aligned} \mathbf{Prob}(f(x_k) - f^* \geq \epsilon) &\leq \mathbf{Prob}(\|\nabla f(x_k)\|_{[0]}^* \cdot R \geq \epsilon) \\ &\stackrel{(3.12)}{\leq} \mathbf{Prob}(\|\nabla f_\mu(x_k)\|_{[0]}^* \cdot R + 2\mu R^2 \geq \epsilon) \\ &= \mathbf{Prob}\left(\|\nabla f_\mu(x_k)\|_{[0]}^* \geq \frac{\epsilon}{2R}\right). \end{aligned}$$

Now, using the Chebyshev inequality, we obtain

$$\begin{aligned} &\mathbf{Prob}\left(\|\nabla f_\mu(x_k)\|_{[0]}^* \geq \frac{\epsilon}{2R}\right) \\ &\leq \frac{2R}{\epsilon} \cdot E_{\xi_k}\left(\|\nabla f_\mu(x_k)\|_{[0]}^*\right) \\ &\stackrel{(3.10)}{\leq} \frac{2R}{\epsilon} \cdot \left[2(S_1(f) + \mu) \cdot (f(x_0) - f^*) \cdot \left(1 - \frac{\mu}{S_1(f) + n\mu}\right)^k \right]^{1/2}. \end{aligned}$$

Since the gradient of function f is Lipschitz continuous with constant $S_1(f)$, we get

$$f(x_0) - f^* \leq \frac{1}{2} S_1(f) R^2.$$

Thus, we obtain the following bound:

$$\begin{aligned} \mathbf{Prob} \left(\|\nabla f_\mu(x_k)\|_{[1]}^* \geq \frac{\epsilon}{2R} \right) &\leq \frac{2R^2}{\epsilon} \cdot (S_1(f) + \mu) \cdot \left(1 - \frac{\mu}{S_1(f) + n\mu} \right)^{\frac{k}{2}} \\ &\leq \frac{2R^2}{\epsilon} \cdot (S_1(f) + \mu) \cdot e^{-\frac{\mu k}{2(S_1(f) + n\mu)}} \leq 1 - \beta. \end{aligned}$$

This proves the statement of the theorem. \square

We conclude the section with some remarks.

- The dependence of complexity bounds (3.7), (3.11) in the confidence level β is very moderate. Hence, even a very high confidence level is easily achievable.
- The standard gradient method has the complexity bound of $O(\frac{L(f)R_0^2(x_0)}{\epsilon})$ full-dimensional gradient iterations. Note that in the worst case $L(f)$ can reach $S_1(f)$. Hence, for the class of objective functions treated in Theorem 4, the worst-case complexity bounds of RCDM(1, x_0) and the standard gradient method are essentially the same. Note that RCDM needs a certain number of full cycles. But this number grows proportionally to the logarithms of accuracy and of the confidence level. Note that the computational cost of a single iteration of RCDM is very often much smaller than that of the standard gradient method.
- Consider very sparse problems with the cost of n coordinate iterations being proportional to a single full-dimensional gradient iteration. Then the complexity bound of RCDM(1, x_0) in terms of the groups of iterations of size n becomes $O^*(1 + \frac{S_1}{n} \cdot \frac{R_0^2(x_0)}{\epsilon})$. As compared with the gradient method, in this complexity bound the largest eigenvalue of the Hessian is replaced by an estimate for its *average* eigenvalue.

4. Constrained minimization. Consider now the constrained minimization problem

$$(4.1) \quad \min_{x \in Q} f(x),$$

where $Q = \bigotimes_{i=1}^n Q_i$, and the sets $Q_i \subseteq R^{n_i}$, $i = 1, \dots, n$, are closed and convex. Let us endow the spaces R^{n_i} with some Euclidean norms (3.4) and assume that the objective function f of problem (4.1) is convex and satisfies our main smoothness assumption (2.2).

We can define now the constrained coordinate update as follows:

$$(4.2) \quad \begin{aligned} u^{(i)}(x) &= \arg \min_{u^{(i)} \in Q_i} \left[\langle f'_i(x), u^{(i)} - x^{(i)} \rangle + \frac{L_i}{2} \|u^{(i)} - x^{(i)}\|_{(i)}^2 \right], \\ V_i(x) &= x + U_i^T (u^{(i)}(x) - x^{(i)}), \quad i = 1, \dots, n. \end{aligned}$$

The optimality conditions for these optimization problems can be written in the following form:

$$(4.3) \quad \langle f'_i(x) + L_i B_i(u^{(i)}(x) - x^{(i)}), u^i - u^{(i)}(x) \rangle \geq 0 \quad \forall u^{(i)} \in Q_i, \quad i = 1, \dots, n.$$

Using this inequality for $u^{(i)} = x^{(i)}$, we obtain

$$\begin{aligned} f(V_i(x)) &\stackrel{(2.3)}{\leq} f(x) + \langle f'_i(x), u^{(i)}(x) - x^{(i)} \rangle + \frac{L_i}{2} \|u^{(i)}(x) - x^{(i)}\|_{(i)}^2 \\ &\stackrel{(4.3)}{\leq} f(x) + \langle L_i B_i(u^{(i)}(x) - x^{(i)}), x^i - u^{(i)}(x) \rangle + \frac{L_i}{2} \|u^{(i)}(x) - x^{(i)}\|_{(i)}^2. \end{aligned}$$

Thus,

$$(4.4) \quad f(x) - f(V_i(x)) \geq \frac{L_i}{2} \|u^{(i)}(x) - x^{(i)}\|_{(i)}^2, \quad i = 1, \dots, n.$$

Let us apply to problem (4.1) the *uniform* coordinate decent method (UCDM).

Method UCDM(x_0)

For $k \geq 0$ iterate:

(1) Choose randomly i_k by uniform distribution on $\{1 \dots n\}$.

(2) Update $x_{k+1} = V_{i_k}(x_k)$.

(4.5)

THEOREM 5. For any $k \geq 0$ we have

$$\phi_k - f^* \leq \frac{n}{n+k} \cdot \left[\frac{1}{2} R_1^2(x_0) + f(x_0) - f^* \right].$$

If f is strongly convex in $\|\cdot\|_{[1]}$ with constant σ , then

$$(4.6) \quad \phi_k - f^* \leq \left(1 - \frac{2\sigma}{n(1+\sigma)} \right)^k \cdot \left(\frac{1}{2} R_1^2(x_0) + f(x_0) - f^* \right).$$

Proof. We will use the notation of Theorem 1. Let UCDM generate an implementation x_k of corresponding random variable. Denote

$$r_k^2 = \|x_k - x_*\|_1^2 = \sum_{i=1}^n L_i \left\langle B_i \left(x_k^{(i)} - x_*^{(i)} \right), x_k^{(i)} - x_*^{(i)} \right\rangle.$$

Then

$$\begin{aligned} r_{k+1}^2 &= r_k^2 + 2L_{i_k} \left\langle B_{i_k} \left(u^{(i_k)}(x_k) - x_k^{(i_k)} \right), x_k^{(i_k)} - x_*^{(i_k)} \right\rangle + L_{i_k} \|u^{(i_k)}(x_k) - x_k^{(i_k)}\|_{(i_k)}^2 \\ &= r_k^2 + 2L_{i_k} \left\langle B_{i_k} \left(u^{(i_k)}(x_k) - x_k^{(i_k)} \right), u^{(i_k)}(x_k) - x_*^{(i_k)} \right\rangle \\ &\quad - L_{i_k} \|u^{(i_k)}(x_k) - x_k^{(i_k)}\|_{(i_k)}^2 \\ &\stackrel{(4.3)}{\leq} r_k^2 + 2 \langle f'_{i_k}(x_k), x_*^{(i_k)} - u^{(i_k)}(x_k) \rangle - L_{i_k} \|u^{(i_k)}(x_k) - x_k^{(i_k)}\|_{(i_k)}^2 \\ &= r_k^2 + 2 \langle f'_{i_k}(x_k), x_*^{(i_k)} - x_k^{(i_k)} \rangle \\ &\quad - 2 \left[\langle f'_{i_k}(x_k), u^{(i_k)}(x_k) - x_k^{(i_k)} \rangle + \frac{1}{2} L_{i_k} \|u^{(i_k)}(x_k) - x_k^{(i_k)}\|_{(i_k)}^2 \right] \\ &\stackrel{(2.3)}{\leq} r_k^2 + 2 \langle f'_{i_k}(x_k), x_*^{(i_k)} - x_k^{(i_k)} \rangle + 2 [f(x_k) - f(V_{i_k}(x_k))]. \end{aligned}$$

Taking the expectation in i_k , we obtain

$$(4.7) \quad E_{i_k} \left(\frac{1}{2} r_{k+1}^2 + f(x_{k+1}) - f^* \right) \leq \frac{1}{2} r_k^2 + f(x_k) - f^* + \frac{1}{n} \langle \nabla f(x_k), x_* - x_k \rangle.$$

Thus, for any $k \geq 0$ we have

$$\begin{aligned} & \frac{1}{2} r_0^2 + f(x_0) - f^* \\ & \geq \phi_{k+1} - f^* + \frac{1}{n} \left[\langle \nabla f(x_0), x_0 - x_* \rangle + \sum_{i=1}^k E_{\xi_{i-1}} (\langle \nabla f(x_i), x_i - x_* \rangle) \right] \\ & \geq \phi_{k+1} - f^* + \frac{1}{n} \sum_{i=0}^k (\phi_i - f^*) \stackrel{(4.4)}{\geq} \left(1 + \frac{k+1}{n} \right) (\phi_{k+1} - f^*). \end{aligned}$$

Finally, let f be strongly convex in $\|\cdot\|_{[1]}$ with convexity parameter σ .⁴ Then (e.g., section 2.1 in [8]),

$$\langle \nabla f(x), x - x_* \rangle \geq f(x) - f^* + \frac{\sigma}{2} \|x - x_*\|_{[1]}^2 \geq \sigma \|x - x_*\|_{[1]}^2.$$

Define $\beta = \frac{2\sigma}{1+\sigma} \in [0, 1]$. Substituting now the above inequalities with weights β and $1 - \beta$ in inequality (4.7), we have

$$\begin{aligned} E_{i_k} \left(\frac{1}{2} r_{k+1}^2 + f(x_{k+1}) - f^* \right) & \leq \frac{1}{2} r_k^2 + f(x_k) - f^* \\ & \quad - \frac{1}{n} \left[\beta \left(f(x_k) - f^* + \frac{\sigma}{2} r_k^2 \right) + (1 - \beta) \sigma r_k^2 \right] \\ & = \left(1 - \frac{2\sigma}{n(1+\sigma)} \right) \cdot \left(\frac{1}{2} r_k^2 + f(x_k) - f^* \right). \end{aligned}$$

It remains to take expectation in ξ_{k-1} . \square

5. Accelerated coordinate descent. It is well known that the usual gradient method can be transformed in a faster scheme by applying an appropriate multistep strategy [7]. Let us show that this can be done also for RCDMs. Consider the following accelerated scheme as applied to the unconstrained minimization problem (2.1) with strongly convex objective function. We assume that the convexity parameter $\sigma = \sigma_1(f) \geq 0$ is known.

⁴In view of assumption (2.2), we always have $\sigma \leq 1$.

Method ACDM(x_0)

1. Define $v_0 = x_0$, $a_0 = \frac{1}{n}$, $b_0 = 2$.
2. For $k \geq 0$ iterate:
 - (1) Compute $\gamma_k \geq \frac{1}{n}$ from equation $\gamma_k^2 - \frac{\gamma_k}{n} = \left(1 - \frac{\gamma_k \sigma}{n}\right) \frac{a_k^2}{b_k^2}$.
Set $\alpha_k = \frac{n - \gamma_k \sigma}{\gamma_k(n^2 - \sigma)}$, and $\beta_k = 1 - \frac{1}{n} \gamma_k \sigma$.
 - (2) Select $y_k = \alpha_k v_k + (1 - \alpha_k)x_k$.
 - (3) Choose $i_k = \mathcal{A}_0$ and update
 $x_{k+1} = T_{i_k}(y_k)$, $v_{k+1} = \beta_k v_k + (1 - \beta_k)y_k - \frac{\gamma_k}{L_{i_k}} U_{i_k} f'_i(y_k)^\#$.
 - (4) Set $b_{k+1} = \frac{b_k}{\sqrt{\beta_k}}$, and $a_{k+1} = \gamma_k b_{k+1}$.

(5.1)

Note that steps (2) and (3) of this scheme need $O(n)$ operations. Therefore, this method should be applied to functions with the cost of directional derivative of the order $O(n)$.

For $n = 1$ and $\sigma = 0$, method (5.1) coincides with method [7]. Its parameters satisfy the following identity:

$$(5.2) \quad \gamma_k^2 - \frac{\gamma_k}{n} = \beta_k \frac{a_k^2}{b_k^2} = \frac{\beta_k \gamma_k}{n} \cdot \frac{1 - \alpha_k}{\alpha_k}.$$

THEOREM 6. For any $k \geq 0$ we have

$$(5.3) \quad \begin{aligned} \phi_k - f^* &\leq \sigma \left[2\|x_0 - x^*\|_{[1]}^2 + \frac{1}{n^2}(f(x_0) - f^*) \right] \cdot \left[\left(1 + \frac{\sqrt{\sigma}}{2n}\right)^{k+1} - \left(1 - \frac{\sqrt{\sigma}}{2n}\right)^{k+1} \right]^{-2} \\ &\leq \left(\frac{n}{k+1} \right)^2 \cdot \left[2\|x_0 - x^*\|_{[1]}^2 + \frac{1}{n^2}(f(x_0) - f^*) \right]. \end{aligned}$$

Proof. Let x_k and v_k be the implementations of corresponding random variables generated by ACDM(x_0) after k iterations. Denote $r_k^2 = \|v_k - x^*\|_{[1]}^2$. Then using representation

$$v_k = y_k + \frac{1 - \alpha_k}{\alpha_k}(y_k - x_k),$$

we obtain

$$\begin{aligned} r_{k+1}^2 &= \|\beta_k v_k + (1 - \beta_k)y_k - x^*\|_{[1]}^2 + \frac{\gamma_k^2}{L_{i_k}^2} \left(\|f'_{i_k}(y_k)\|_{(i_k)}^* \right)^2 \\ &\quad + 2\gamma_k \langle f'_{i_k}(y_k), (x^* - \beta_k v_k - (1 - \beta_k)y_k)^{(i_k)} \rangle \\ &\leq \|\beta_k v_k + (1 - \beta_k)y_k - x^*\|_{[1]}^2 + 2\gamma_k^2 (f(y_k) - f(T_{i_k}(y_k))) \\ &\quad + 2\gamma_k \left\langle f'_{i_k}(y_k), \left(x^* - y_k + \frac{\beta_k(1 - \alpha_k)}{\alpha_k}(x_k - y_k) \right)^{(i_k)} \right\rangle. \end{aligned}$$

Taking the expectation of both sides in i_k , we obtain

$$\begin{aligned}
 E_{i_k}(r_{k+1}^2) &\leq \beta_k r_k^2 + (1 - \beta_k) \|y_k - x^*\|_{[1]}^2 + 2\gamma_k^2 [f(y_k) - E_{i_k}(f(x_{k+1}))] \\
 &\quad + 2\frac{\gamma_k}{n} \left\langle \nabla f(y_k), x^* - y_k + \frac{\beta_k(1 - \alpha_k)}{\alpha_k} (x_k - y_k) \right\rangle \\
 &\leq \beta_k r_k^2 + (1 - \beta_k) \|y_k - x^*\|_{[1]}^2 + 2\gamma_k^2 [f(y_k) - E_{i_k}(f(x_{k+1}))] \\
 &\quad + 2\frac{\gamma_k}{n} \left[f^* - f(y_k) - \frac{1}{2} \sigma \|y_k - x^*\|_{[1]}^2 + \frac{\beta_k(1 - \alpha_k)}{\alpha_k} (f(x_k) - f(y_k)) \right] \\
 &\stackrel{(5.2)}{=} \beta_k r_k^2 - 2\gamma_k^2 E_{i_k}(f(x_{k+1})) + 2\frac{\gamma_k}{n} \left[f^* + \frac{\beta_k(1 - \alpha_k)}{\alpha_k} f(x_k) \right].
 \end{aligned}$$

Note that

$$b_{k+1}^2 = \frac{1}{\beta_k} b_k^2, \quad a_{k+1}^2 = \gamma_k^2 b_{k+1}^2, \quad \gamma_k \frac{\beta_k(1 - \alpha_k)}{n\alpha_k} = \frac{a_k^2}{b_{k+1}^2}.$$

Therefore, multiplying the last inequality by b_{k+1}^2 , we obtain

$$b_{k+1}^2 E_{i_k}(r_{k+1}^2) \leq b_k^2 r_k^2 - 2a_{k+1}^2 (E_{i_k}(f(x_{k+1})) - f^*) + 2a_k^2 (f(x_k) - f^*).$$

Taking now the expectation of both sides of this inequality in ξ_{k-1} , we get

$$\begin{aligned}
 2a_{k+1}^2 (\phi_{k+1} - f^*) + b_{k+1}^2 E_{\xi_k}(r_{k+1}^2) &\leq 2a_k^2 (\phi_k - f^*) + b_k^2 r_k^2 \\
 &\leq 2a_0^2 (f(x_0) - f^*) + b_0^2 \|x_0 - x^*\|_{[1]}^2.
 \end{aligned}$$

It remains to estimate the growth of coefficients a_k and b_k . We have

$$b_k^2 = \beta_k b_{k+1}^2 = \left(1 - \frac{\sigma}{n} \gamma_k\right) b_{k+1}^2 = \left(1 - \frac{\sigma}{n} \frac{a_{k+1}}{b_{k+1}}\right) b_{k+1}^2.$$

Thus, $\frac{\sigma}{n} a_{k+1} b_{k+1} \leq b_{k+1}^2 - b_k^2 \leq 2b_{k+1}(b_{k+1} - b_k)$, and we conclude that

$$(5.4) \quad b_{k+1} \geq b_k + \frac{\sigma}{2n} a_k.$$

On the other hand, $\frac{a_{k+1}^2}{b_{k+1}^2} - \frac{a_{k+1}}{nb_{k+1}} \stackrel{(5.2)}{=} \frac{\beta_k a_k^2}{b_k^2} = \frac{a_k^2}{b_{k+1}^2}$. Therefore,

$$\frac{1}{n} a_{k+1} b_{k+1} \leq a_{k+1}^2 - a_k^2 \leq 2a_{k+1}(a_{k+1} - a_k),$$

and we obtain

$$(5.5) \quad a_{k+1} \geq a_k + \frac{1}{2n} b_k.$$

Further, denoting $Q_1 = 1 + \frac{\sqrt{\sigma}}{2n}$ and $Q_2 = 1 - \frac{\sqrt{\sigma}}{2n}$ and using inequalities (5.4) and (5.5), it is easy to prove by induction that

$$a_k \geq \frac{1}{\sqrt{\sigma}} [Q_1^{k+1} - Q_2^{k+1}], \quad b_k \geq Q_1^{k+1} + Q_2^{k+1}.$$

Finally, using the trivial inequality $(1+t)^k - (1-t)^k \geq 2kt$, $t \geq 0$, we obtain

$$Q_1^{k+1} - Q_2^{k+1} \geq \frac{k+1}{n} \sqrt{\sigma}. \quad \square$$

The rate of convergence (5.3) of ACDM is much better than the rate (2.14). However, for some applications (e.g., section 6.3), the complexity of one iteration of the accelerated scheme is rather high since for computing y_k it needs to operate with full-dimensional vectors.

6. Implementation details and numerical test.

6.1. Dynamic adjustment of Lipschitz constants. In RCDM (2.6) we use the valid upper bounds for Lipschitz constants $\{L_i\}_{i=1}^n$ of the directional derivatives. For some applications (e.g., section 6.3) this information is easily available. However, for more complicated functions we need to apply a dynamic adjustment procedure for finding appropriate bounds. Let us estimate the efficiency of a simple backtracking strategy with restore (e.g., [9]) inserted in RCDM(0, x_0). As already discussed, such a strategy *should not* be based on computation of the function values.

Consider the random adaptive coordinate decent method. For the sake of notation, we assume that $N = n$.

Method RCDM ₊ (x_0)	
Setup: lower bounds $\hat{L}_i := L_i^0 \in (0, L_i]$, $i = 1, \dots, n$.	
(6.1)	For $k \geq 0$ iterate:
	(1) Choose $i_k = \mathcal{A}_0$.
	(2) Set $x_{k+1} := x_k - \hat{L}_{i_k}^{-1} \cdot \nabla_{i_k} f(x_k) \cdot e_{i_k}$.
	while $\nabla_{i_k} f(x_k) \cdot \nabla_{i_k} f(x_{k+1}) < 0$ do
	$\left\{ \hat{L}_{i_k} := 2\hat{L}_{i_k}, \quad x_{k+1} := x_k - \hat{L}_{i_k}^{-1} \cdot \nabla_{i_k} f(x_k) \cdot e_{i_k} \right\}$
	(3) Set $L_{i_k} := \frac{1}{2}L_{i_k}$.

THEOREM 7.

1. At the beginning of each iteration, we have

$$\hat{L}_i \leq L_i, \quad i = 1, \dots, n.$$

2. RCDM₊ has the following rate of convergence:

$$(6.2) \quad \phi_k - f^* \leq \frac{8nR_1^2(x_0)}{16 + 3k}, \quad k \geq 0.$$

3. After iteration k , the total number N_k of computations of directional derivatives in method (6.1) satisfies inequality

$$(6.3) \quad N_k \leq 2(k+1) + \sum_{i=1}^n \log_2 \frac{L_i}{\hat{L}_i^0}.$$

Proof. For proving the first statement, we assume that at the entrance to the internal cycle in method (6.1), we have $\hat{L}_{i_k} \leq L_{i_k}$. If during this cycle we get $\hat{L}_{i_k} > L_{i_k}$, then immediately

$$|\nabla_{i_k} f(x_{k+1}) - \nabla_{i_k} f(x_k)| \stackrel{(2.2)}{\leq} \hat{L}_{i_k}^{-1} L_{i_k} |\nabla_{i_k} f(x_k)| < |\nabla_{i_k} f(x_k)|.$$

In this case, the termination criterion is satisfied. Thus, during the internal cycle

$$(6.4) \quad \hat{L}_{i_k} \leq 2L_{i_k}.$$

Therefore, after execution of step (3), we have again $\hat{L}_{i_k} \leq L_{i_k}$.

Further, the internal cycle is terminated with \hat{L}_{i_k} satisfying inequality (6.4). Therefore, using inequality (2.1.17) in [8], we have

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \nabla_{i_k} f(x_{k+1}) \cdot \left(x_k^{(i_k)} - x_{k+1}^{(i_k)} \right) + \frac{1}{2L_{i_k}} (\nabla_{i_k} f(x_{k+1}) - \nabla_{i_k} f(x_k))^2 \\ &= \hat{L}_{i_k}^{-1} \cdot \nabla_{i_k} f(x_{k+1}) \cdot \nabla_{i_k} f(x_k) + \frac{1}{2L_{i_k}} (\nabla_{i_k} f(x_{k+1}) - \nabla_{i_k} f(x_k))^2 \\ &\stackrel{(6.4)}{\geq} \frac{1}{2L_{i_k}} \left[(\nabla_{i_k} f(x_{k+1}))^2 + (\nabla_{i_k} f(x_k))^2 - \nabla_{i_k} f(x_{k+1}) \cdot \nabla_{i_k} f(x_k) \right] \\ &\geq \frac{3}{8L_{i_k}} (\nabla_{i_k} f(x_k))^2. \end{aligned}$$

Thus, we obtain the following bound:

$$f(x_k) - E_{i_k}(f(x_{k+1})) \geq \frac{3}{8n} \left(\|\nabla f(x_k)\|_{[1]}^* \right)^2$$

(compare with (2.13)). Now, using the same arguments as in the proof of Theorem 1, we can prove that

$$\phi_k - \phi_{k+1} \geq \frac{1}{C} (\phi_k - f^*)^2$$

with $C = \frac{8n}{3} R_1^2(x_0)$. And we conclude that

$$\frac{1}{\phi_k - f^*} \geq \frac{1}{f(x_0) - f^*} + \frac{k}{C} \geq \frac{2}{nR_1^2(x_0)} + \frac{3k}{8nR_1^2(x_0)}.$$

Finally, let us find an upper estimate for N_k . Denote by K_i a subset of iteration numbers $\{0, \dots, k\}$, at which the index i was active. Denote by $p_i(j)$ the number of computations of i th partial derivative at iteration $j \in K_i$. If \hat{L}_i is the current estimate of the Lipschitz constant L_i in the beginning of iteration j , and \hat{L}'_i is the value of this estimate in the end of this iteration, then these values are related as follows:

$$\hat{L}'_i = \frac{1}{2} \cdot 2^{p_i(j)-1} \cdot \hat{L}_i.$$

In other words, $p_i(j) = 2 + \log_2[\hat{L}'_i/\hat{L}_i]$. Taking into account the statement of Theorem 7, item 1, we obtain the following estimate for M_i , the total number of computations of i th partial in the first k iterations:

$$M_i \leq 2 \cdot |K_i| + \log_2 \frac{L_i}{L_i^0}.$$

It remains to note that $\sum_{i=1}^n |K_i| = k + 1$. \square

Note that a similar technique can be used in the accelerated version (6.1). For that, we need to choose its parameters from the equations

$$(6.5) \quad \begin{aligned} \frac{4}{3}\gamma_k^2 - \frac{\gamma_k}{n} &= \left(1 - \frac{\gamma_k\sigma}{n}\right) \frac{a_k^2}{b_k^2}, \\ \beta_k &= 1 - \frac{\gamma_k\sigma}{n}, \\ \frac{1-\alpha_k}{\alpha_k} &= \frac{n}{\beta_k} \cdot \left(\frac{4}{3}\gamma_k - \frac{1}{n}\right). \end{aligned}$$

This results in a minor change in the rate of convergence (5.3).

6.2. Random counters. For problems of very big size, we should treat carefully any operation with multidimensional vectors. In RCDMs, there is such an operation, which must be fulfilled at each step of the algorithms. This is the random generation of active coordinates. In a straightforward way, this operation can be implemented in $O(n)$ operations. However, for huge-scale problems this complexity can be prohibitive. Therefore, before presenting our computational results, we describe a strategy for generating random coordinates with complexity $O(\ln n)$ operations.

Given the values L_i^α , $i = 1, \dots, n$, we need to generate efficiently random integer numbers $i \in \{1, \dots, n\}$ with probabilities

$$\text{Prob}[i = k] = L_k^\alpha / \left[\sum_{j=1}^n L_j^\alpha \right], \quad k = 1, \dots, n.$$

Without loss of generality, we assume that $n = 2^m$. Define $m+1$ vectors $P_k \in R^{2^{m-k}}$, $k = 0, \dots, m$, as follows:

$$(6.6) \quad \begin{aligned} P_0^{(i)} &= L_i^\alpha, \quad i = 1, \dots, n. \\ P_k^{(i)} &= P_{k-1}^{(2i)} + P_{k-1}^{(2i-1)}, \quad i = 1, \dots, 2^{m-k}, \quad k = 1, \dots, m. \end{aligned}$$

Clearly, $P_m^{(1)} = S_\alpha$. Note that the preliminary computations (6.6) need $\frac{n}{2}$ additions.

Let us describe now our random generator.

$$(6.7) \quad \begin{aligned} &1. \quad \text{Choose } i = 1 \text{ in } P_m. \\ &2. \quad \text{For } k = m \text{ down to } 1 \text{ do:} \\ &\quad \text{Let the element } i \text{ of } P_k \text{ be chosen. Choose in } P_{k-1} \\ &\quad \text{either } 2i \text{ or } 2i - 1 \text{ with probabilities } \frac{P_{k-1}^{(2i)}}{P_k^{(i)}} \text{ or } \frac{P_{k-1}^{(2i-1)}}{P_k^{(i)}}. \end{aligned}$$

Clearly, this procedure implements correctly the random counter \mathcal{A}_α . Note that its complexity is $O(\ln n)$ operations. In the above form, its execution requires generating m random numbers. However, a simple modification can reduce this amount up to one. Note also that corrections of vectors P_k , $k = 0, \dots, m$, due to the change of a single entry in the initial data needs $O(\ln n)$ operations.

6.3. Numerical test. Let us describe now our test problem (sometimes called the *Google problem*). Let $E \in R^{n \times n}$ be an incidence matrix of a graph. Denote

$e = (1, \dots, 1)^T$ and

$$\bar{E} = E \cdot \text{diag}(E^T e)^{-1}.$$

Since, $\bar{E}^T e = e$, it is a stochastic matrix. Our problem consists in finding a right maximal eigenvector of the matrix \bar{E} :

$$\text{Find } x^* \geq 0 : \quad \bar{E}x^* = x^*, \quad \langle e, x^* \rangle = 1.$$

Clearly, this problem can be rewritten in an optimization form:

$$(6.8) \quad f(x) \stackrel{\text{def}}{=} \frac{1}{2} \|\bar{E}x - x\|^2 + \frac{\gamma}{2} [\langle e, x \rangle - 1]^2 \rightarrow \min_{x \in R^n},$$

where $\gamma > 0$ is a penalty parameter for the equality constraint and the norm $\|\cdot\|$ is Euclidean. Note that an increase of parameter γ has a moderate influence on the diagonal elements of the quadratic form. However, its impact to the maximal eigenvalue of corresponding matrix is much higher.

If the degree of each node in the graph is small, then the computation of partial derivatives of function f is cheap. Hence, we can apply RCDM to (6.8) even if the size of the matrix \bar{E} is very big.

In our numerical experiments we applied RCDM(1,0) to a randomly generated graph with average node degree p . The termination criterion was

$$\|\bar{E}x - x\| \leq \epsilon \cdot \|x\|$$

with $\epsilon = 0.01$. In the table below, k denotes the total number of groups by n coordinate iterations. The computations were performed on a standard Pentium 4 computer with frequency 1.6 GHz.

n	p	γ	k	Time(s)
65536	10	$\frac{1}{n}$	47	7.41
	20	$\frac{1}{n}$	30	5.97
	10	$\frac{1}{\sqrt{n}}$	65	10.5
	20	$\frac{1}{\sqrt{n}}$	39	7.84
262144	10	$\frac{1}{n}$	47	42.7
	20	$\frac{1}{n}$	32	39.1
	10	$\frac{1}{\sqrt{n}}$	72	76.5
	20	$\frac{1}{\sqrt{n}}$	45	62.0
1048576	10	$\frac{1}{n}$	49	247
	20	$\frac{1}{n}$	31	240
	10	$\frac{1}{\sqrt{n}}$	82	486
	20	$\frac{1}{\sqrt{n}}$	64	493

We can see that the number of n -iteration groups grows very moderately with the dimension of the problem. The increase of parameter γ also does not create significant difficulties for RCDM. Note that for the standard Euclidean norm we have $L(f) \approx \gamma n$. Thus, for the black-box gradient methods the problems with $\gamma = \frac{1}{\sqrt{n}}$ and very big n are very difficult ($L(f) \approx n^{1/2}$). Note also that the accelerated scheme (6.1) is not efficient on problems of this size. Indeed, each coordinate iteration of this method needs an update of n -dimensional vector. Therefore, one group of n iterations takes at least $n^2 \approx 10^{12}$ operations (this is for the maximal dimension in the above table). For our computer, this amount of computations takes at least 20 minutes.

Note that the dominance of RCDM on sparse problems can be supported by comparison of the efficiency estimates. Let us put in one table the complexity results related to RCDM (2.6), accelerated scheme ACDM (5.1), and the fast gradient method (FGM) working with the full gradient (e.g., section 2.2 in [8]). Denote by T the complexity of computation of a single directional derivative and by F the complexity of computation of the function value. We assume that $F = nT$. Note that this can be true even for dense problems (e.g., quadratic functions). We will measure distances in $\|\cdot\|_{[1]}$. For this metric, denote by γ the Lipschitz constant for the gradient of the objective function. Note that

$$1 \leq \gamma \leq n.$$

In the table below, we compare the cost of iteration, cost of the oracle, and iteration complexity for these three methods.

(6.9)

	RCDM	ACDM	FGM
Iteration	1	n	n
Oracle	T	T	F
Complexity	$n \frac{R^2}{\epsilon}$	$n \frac{R}{\sqrt{\epsilon}}$	$\frac{\sqrt{\gamma} R}{\sqrt{\epsilon}}$
Total	$(F + n) \frac{R^2}{\epsilon}$	$(F + n^2) \frac{R}{\sqrt{\epsilon}}$	$(F + n) \frac{\sqrt{\gamma} R}{\sqrt{\epsilon}}$

We can see that RCDM is better than FGM if $\frac{R}{\sqrt{\epsilon}} < \sqrt{\gamma}$. On the other hand, FGM is better than ACDM if $F < \frac{n^2}{\sqrt{\gamma}}$. For our test problem, both conditions are satisfied.

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