18.445 Introduction to Stochastic Processes

Lecture 5: Stationary times

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Recall
Suppose that P is irreducible with stationary measure π.

$$d(n) = \max_x ||P^n(x, \cdot) - \pi||_{TV}, \quad t_{mix} = \min\{n : d(n) \leq 1/4\}.$$

Today's Goal Use random times to give upper bound of t_{mix}

- Top-to-Random shuffle
- Stopping time and randomized stopping time
- Stationary time and strong stationary time

Top-to-Random shuffle

Consider the following method of shuffling a deck of N cards :
Take the top card and insert it uniformly at random in the deck.

The successive arrangements of the deck are a random walk $(X_n)_{n\geq 0}$ on the group S_N : $N!$ possible permutations of the N cards starting from $X_0 = (123\cdots N)$.
The uniform measure is the stationary measure.

Question : How long must we shuffle until the orders in the deck is uniform ?
A simpler quenstion : How long must we shuffle until the orginal bottom card become uniform in the deck ?

Answer : Let τ_{top} be the time one move after the first occasion when the original bottom card has moved to the top of the deck. The arrangements of cards at time τ_{top} is uniform in S_N.

Top-to-Random shuffle

Theorem

Let $(X_n)_{n\geq 0}$ be the random walk on S_N corresponding to the top-to-random shuffle. Given at time n there are k cards under the original bottom card, each of the $k!$ possibilities are equally likely. Therefore, $X_{\tau_{top}}$ is uniform in S_N.

Remark : The random time τ_{top} is interesting, since $X_{\tau_{top}}$ has exactly the stationary measure.

Stopping times

Definition

Given a sequence $(X_n)_{n\geq 0}$ of random variables, a number τ, taking values in $\{0, 1, 2, ..., \infty\}$, is a stopping time for $(X_n)_{n\geq 0}$, if for each $n \geq 0$, the event $[\tau = n]$ is measurable with respect to $(X_0, X_1, ..., X_n)$; or equivalently, the indicator function $1_{[\tau=n]}$ is a function of the vector $(X_0, X_1, ..., X_n)$.

Example Fix a subset $A \subset \Omega$, define τ_A to be the first time that $(X_n)_{n\geq 0}$ hits A :

$$\tau_A = \min\{n : X_n \in A\}.$$

Then τ_A is stopping time. (Recall that τ_x and τ_x^+ are stopping times.)

Stopping times

Lemma

Let τ be a random time, then the following four conditions are equivalent.

- $[\tau = n]$ *is measurable w.r.t.* $(X_0, X_1, ..., X_n)$
- $[\tau \leq n]$ *is measurable w.r.t.* $(X_0, X_1, ..., X_n)$
- $[\tau > n]$ *is measurable w.r.t.* $(X_0, X_1, ..., X_n)$
- $[\tau \geq n]$ *is measurable w.r.t.* $(X_0, X_1, ..., X_{n-1})$

Lemma

If τ and τ' are stopping times, then $\tau + \tau'$, $\tau \wedge \tau'$, and $\tau \vee \tau'$ are also stopping times.

Random mapping representation

Definition

A random mapping representation of a transition matrix P on state space Ω is a function $f : \Omega \times \Lambda \to \Omega$, along with a $\Lambda-$valued random variable Z, satisfying

$$\mathbb{P}[f(x, Z) = y] = P(x, y).$$

Question : How is it related to Markov chain ?
Let $(Z_n)_{n\geq 1}$ be i.i.d. with common law the same as Z. Let $X_0 \sim \mu$. Define $X_n = f(X_{n-1}, Z_n)$ for $n \geq 1$. Then $(X_n)_{n\geq 0}$ is a Markov chain with initial distribution μ.
Example : Simple random walk on $N-$cycle. Set $\Lambda = \{-1, +1\}$, let $(Z_n)_{n\geq 1}$ be i.i.d. Bernoulli$(1/2)$. Set

$$f(x, z) \equiv x + z \mod N.$$

Random mapping representation

Definition

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$$\mathbb{P}[f(x, Z) = y] = P(x, y).$$

Theorem

Every transition matrix on a finite state space has a random mapping representation.

Randomized stopping times

Suppose that the transition matrix P has a random mapping representation $f : \Omega \times \Lambda \to \Omega$, along with a random variable Z, such that

$$\mathbb{P}[f(x, Z) = y] = P(x, y).$$

Let $(Z_n)_{n\geq 1}$ be a sequence of i.i.d. with the same law as Z. Define $X_n = f(X_{n-1}, Z_n)$ for $n \geq 1$. Then $(X_n)_{n\geq 0}$ is a Markov chain with transition matrix P.

Definition

A random time τ is called a randomized stopping time if it is a stopping time for the sequence $(Z_n)_{n\geq 1}$.

Remark The sequence $(Z_n)_n$ contains more information than the sequence $(X_n)_n$, therefore the stopping times for $(X_n)_n$ are randomized stopping times, but the reverse does not hold generally.

Stationary time and strong stationary time

Definition

Let $(X_n)_n$ be an irreducible Markov chain with stationary measure π. A stationary time τ for $(X_n)_n$ is a randomized stopping time such that $X_\tau \stackrel{d}{\sim} \pi$:

$$\mathbb{P}[X_\tau = x] = \pi(x), \quad \forall x.$$

A strong stationary time τ for $(X_n)_n$ is a randomized stopping time such that $X_\tau \stackrel{d}{\sim} \pi$ and $X_\tau \perp \tau$:

$$\mathbb{P}[X_\tau = x, \tau = n] = \pi(x)\mathbb{P}[\tau = n], \quad \forall x, n.$$

Example For the top-to-random shuffle, the time τ_{top} is strong stationary.

Strong stationary time

Example Let $(X_n)_n$ be an irreducible Markov chain with state space Ω, stationary measure π, and $X_0 = x$. Let ξ be a $\Omega-$valued random variable with distribution π and it is independent of $(X_n)_n$. Define

$$\tau = \min\{n \geq 0 : X_n = \xi\}.$$

Then

- τ is not a stopping time
- τ is a randomized stopping time
- τ is stationary
- τ is not strong stationary

Strong stationary time

Theorem

Let $(X_n)_{n\geq 0}$ be an irreducible Markov chain with stationary measure π. If τ is a strong stationary time for (X_n), then

$$d(n) := \max_x ||P^n(x,\cdot) - \pi||_{TV} \leq \max_x \mathbb{P}_x[\tau > n].$$

Lemma

For all $n \geq 0$, $\mathbb{P}[\tau \leq n, X_n = y] = \mathbb{P}[\tau \leq n]\pi(y)$.

Lemma

Define the separation distance $S_x(n) = \max_y(1 - P^n(x,y)/\pi(y))$. Then $S_x(n) \leq \mathbb{P}_x[\tau > n]$.

Lemma

$||P^n(x,\cdot) - \pi||_{TV} \leq S_x(n)$.