# 18.445 Introduction to Stochastic Processes

## Lecture 15: Introduction to martingales

Hao Wu

MIT

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**About the midterm** : total=23

$$1 \text{ in } [80, 100], \quad 5 \text{ in } [70, 80), \quad 6 \text{ in } [60, 70)$$

$$4 \text{ in } [40, 60), \quad 7 \text{ in } [10, 40)$$

**Today's Goal** :

- probability space
- conditional expectation
- introduction to martingales

# Probability space

### Definition

$\Omega$ : a set. A collection $\mathcal{F}$ of subsets of $\Omega$ is called a $\sigma$**-algebra** on $\Omega$ if

- $\Omega \in \mathcal{F}$
- $F \in \mathcal{F} \Longrightarrow F^c \in \mathcal{F}$
- $F_1, F_2, ... \in \mathcal{F} \Longrightarrow \cup_n F_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

### Definition

Let $(\Omega, \mathcal{F})$ be a measurable space. A map $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a **probability measure** if

- $\mathbb{P}[\emptyset] = 0$, $\mathbb{P}[\Omega] = 1$
- it is countably additive : whenever $(F_n)_{n \geq 0}$ is a sequence of disjoint sets in $\Omega$, then $\mathbb{P}[\cup_n F_n] = \sum_n \mathbb{P}[F_n]$.

# Probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space

- $\Omega$ : state space
- $\mathcal{F}$ : $\sigma$-algebra
- $\mathbb{P}$ : probability measure

# Conditional expectation—motivation

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
- $X, Z$ two random variables
- elementary conditional probability :

$$\mathbb{P}[X = x \mid Z = z] = \mathbb{P}[X = x, Z = z]/\mathbb{P}[Z = z]$$

- elementary conditional expectation :

$$\mathbb{E}[X \mid Z = z] = \sum_x x\mathbb{P}[X = x \mid Z = z]$$

- $Y = \mathbb{E}[X \mid \sigma(Z)]$ ?
  - $Y$ is measurable with respect to $\sigma(Z)$
  - $\mathbb{E}[Y 1_{Z=z}] = \mathbb{E}[X 1_{Z=z}]$

# Conditional Expectation

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space
- $X$ is a random variable on the probability space with $\mathbb{E}[|X|] < \infty$
- $\mathcal{A} \subset \mathcal{F}$ is a sub $\sigma$-algebra

Then there exists a random variable $Y$ such that

- $Y$ is $\mathcal{A}$-measurable with $\mathbb{E}[|Y|] < \infty$
- for any $A \in \mathcal{A}$, we have $\mathbb{E}[Y1_A] = \mathbb{E}[X1_A]$.

Moreover, if $\tilde{Y}$ also satisfies the above two properties, then $\tilde{Y} = Y$ a.s.
A random variable $Y$ with the above two properties is called the **conditional expectation** of $X$ given $\mathcal{A}$, and we denote it by $\mathbb{E}[X \mid \mathcal{A}]$.

**Remark** :

- If $\mathcal{A} = \{\emptyset, \Omega\}$, then $\mathbb{E}[X \mid \mathcal{A}] = \mathbb{E}[X]$.
- If $X$ is $\mathcal{A}$-measurable, then $\mathbb{E}[X \mid \mathcal{A}] = X$.
- If $Y = \mathbb{E}[X \mid \mathcal{A}]$, then $\mathbb{E}[Y] = \mathbb{E}[X]$

## Conditional Expectation—Basic properties

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that

- $X, X_n$ are random variables on the probability space in $L^1$
- $\mathcal{A} \subset \mathcal{F}$ is a sub $\sigma$-algebra

Then we have the following.

- (Linearity) $\mathbb{E}[a_1 X_1 + a_2 X_2 \,|\, \mathcal{A}] = a_1 \mathbb{E}[X_1 \,|\, \mathcal{A}] + a_2 \mathbb{E}[X_2 \,|\, \mathcal{A}]$ for constants $a_1, a_2$.
- (Positivity) If $X \geq 0$ a.s., then $\mathbb{E}[X \,|\, \mathcal{A}] \geq 0$ a.s.
- (Monotone convergence) If $0 \leq X_n \uparrow X$ a.s. then $\mathbb{E}[X_n \,|\, \mathcal{A}] \uparrow \mathbb{E}[X \,|\, \mathcal{A}]$ a.s.
- (Fatou's Lemma) If $X_n \geq 0$, then $\mathbb{E}[\liminf_n X_n \,|\, \mathcal{A}] \leq \liminf_n \mathbb{E}[X_n \,|\, \mathcal{A}]$ a.s.
- (Dominated convergence) If $|X_n| \leq Z$ with $Z \in L^1$ and $X_n \to X$ a.s., then $\mathbb{E}[X_n \,|\, \mathcal{A}] \to \mathbb{E}[X \,|\, \mathcal{A}]$ a.s.
- (Jensen inequality) If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and $\mathbb{E}[|\varphi(X)|] < \infty$, then $\mathbb{E}[\varphi(X) \,|\, \mathcal{A}] \geq \varphi(\mathbb{E}[X \,|\, \mathcal{A}])$.

# Conditional Expectation—Basic properties

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that

- $X, X_n$ are random variables on the probability space in $L^1$
- $\mathcal{A} \subset \mathcal{F}$ is a sub $\sigma$-algebra

Then we have the following.

- (Tower property) If $\mathcal{B}$ is a sub-$\sigma$-algebra of $\mathcal{A}$, then $\mathbb{E}[\mathbb{E}[X \mid \mathcal{A}] \mid \mathcal{B}] = \mathbb{E}[X \mid \mathcal{B}]$ a.s.
- ("Taking out what is known") If $Z$ is $\mathcal{A}$-measurable and bounded, then $\mathbb{E}[XZ \mid \mathcal{A}] = Z\mathbb{E}[X \mid \mathcal{A}]$ a.s.
- (Independence) If $\mathcal{B}$ is independent of $\sigma(\sigma(X), \mathcal{A})$, then $\mathbb{E}[X \mid \sigma(\mathcal{A}, \mathcal{B})] = \mathbb{E}[X \mid \mathcal{A}]$ a.s. In particular, if $X$ is independent of $\mathcal{B}$, then $\mathbb{E}[X \mid \mathcal{B}] = \mathbb{E}[X]$ a.s.

# Conditional expectation—example

Suppose that $(X_n)_{n\geq 0}$ are i.i.d. with the same distribution as $X$ with $\mathbb{E}[|X|] < \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$, and define

$$\mathcal{A}_n = \sigma(S_n, S_{n+1}, ...) = \sigma(S_n, X_{n+1}, ...).$$

**Question** : $\mathbb{E}[X_1 \mid \mathcal{A}_n]$ ?
**Answer** : $\mathbb{E}[X_1 \mid \mathcal{A}_n] = S_n/n$.

# Martingales

$(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
A filtration $(\mathcal{F}_n)_{n\geq 0}$ is an increasing family of sub $\sigma$-algebras of $\mathcal{F}$.
A sequence of random variables $X = (X_n)_{n\geq 0}$ is adapted to $(\mathcal{F}_n)_{n\geq 0}$ if $X_n$ is measurable with respect to $\mathcal{F}_n$ for all $n$.

Let $(X_n)_{n\geq 0}$ be a sequence of random variables.
The natural filtration $(\mathcal{F}_n)_{n\geq 0}$ associated to $(X_n)_{n\geq 0}$ is given by

$$\mathcal{F}_n = \sigma(X_k, k \leq n).$$

We say that $(X_n)_{n\geq 0}$ is integrable if $X_n$ is integrable for all n.

### Definition

Let $X = (X_n)_{n\geq 0}$ be an integrable process.

- $X$ is a martingale if $\mathbb{E}[X_n \mid \mathcal{F}_m] = X_m$ *a.s.* for all $n \geq m$.
- $X$ is a supermartingale if $\mathbb{E}[X_n \mid \mathcal{F}_m] \leq X_m$ *a.s.* for all $n \geq m$.
- $X$ is a submartingale if $\mathbb{E}[X_n \mid \mathcal{F}_m] \geq X_m$ *a.s.* for all $n \geq m$.

# Examples

**Example 1** Let $(\xi_i)_{i\geq 1}$ be i.i.d with $\mathbb{E}[\xi_1] = 0$. Then $X_n = \sum_1^n \xi_i$ is a martingale.

**Example 2** Let $(\xi_i)_{i\geq 1}$ be i.i.d with $\mathbb{E}[\xi_1] = 1$. Then $X_n = \Pi_1^n \xi_i$ is a martingale.

**Example 3** Consider biased gambler's ruin : at each step, the gambler gains one dollar with probability $p$ and losses one dollar with probability $(1 - p)$. Let $X_n$ be the money in purse at time $n$.

- If $p = 1/2$, then $(X_n)$ is a martingale.
- If $p < 1/2$, then $(X_n)$ is a supermartingale.
- If $p > 1/2$, then $(X_n)$ is a submartingale.