

# 18.445 Introduction to Stochastic Processes

## Lecture 6: Lower bounds on mixing times

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**Recall** Suppose that  $P$  is irreducible with stationary measure  $\pi$ .

$$d(n) = \max_x \|P^n(x, \cdot) - \pi\|_{TV}, \quad t_{mix} = \min\{n : d(n) \leq 1/4\}.$$

**Today's Goal** Find lower bounds for the mixing times.

- Bottleneck Ratio
- Distinguishing statistics
- Random walk on hypercube

# Bottleneck ratio

Suppose that  $P$  is an irreducible transition matrix with stationary measure  $\pi$ . Define

$$Q(A, B) = \sum_{x \in A, y \in B} \pi(x) P(x, y).$$

$Q(A, B)$  : the probability of moving from  $A$  to  $B$  within one step when starting from  $\pi$ .

## Definition

For a subset  $S \subset \Omega$ , the bottleneck ratio of  $S$  is defined to be

$$\Phi(S) = Q(S, S^c) / \pi(S).$$

The bottleneck ratio of the whole chain is defined to be

$$\Phi_{\star} = \min\{\Phi(S) : \pi(S) \leq 1/2\}.$$

# Bottleneck ratio

Consider simple random walk on a graph  $G = (V, E)$ .

$$P(x, y) = \frac{1}{\deg(x)} 1_{[x \sim y]}, \quad \pi(x) = \frac{\deg(x)}{2|E|}.$$

Then

$$Q(S, S^c) = \frac{|\partial S|}{2|E|}, \quad \Phi(S) = \frac{|\partial S|}{\sum_{x \in S} \deg(x)}.$$

# Bottleneck ratio

## Theorem

Suppose that  $\Phi_*$  is the bottleneck ratio, then

$$t_{\text{mix}} \geq \frac{1}{4\Phi_*}.$$

## Lemma

For any subset  $S \subset \Omega$ , let  $\mu_S$  be  $\pi$  conditioned on  $S$  :

$$\mu_S(A) = \frac{\pi(A \cap S)}{\pi(S)}.$$

Then

$$\|\mu_S P - \mu_S\|_{TV} = \Phi(S).$$

# Distinguishing statistics

Goal : find a statistic  $f$  (a function on  $\Omega$ ) such that the distance between  $f(X_n)$  and  $f$  can be bounded from below. Recall

$$\mu f = \sum_x \mu(x)f(x), \quad \text{var}_\mu(f) = \mu f^2 - (\mu f)^2.$$

## Lemma

*Let  $\mu$  and  $\nu$  be two probability distributions on  $\Omega$ . Let  $f$  be a real-valued function on  $\Omega$ . If*

$$|\mu f - \nu f| \geq r\sigma, \quad \text{where } \sigma^2 = \frac{1}{2}(\text{var}_\mu(f) + \text{var}_\nu(f)),$$

*then*

$$\|\mu - \nu\|_{TV} \geq \frac{r^2}{4 + r^2}.$$

# Random walk on hypercube

$N$ -dimensional hypercube is a graph with vertex set  $\Omega = \{0, 1\}^N$ ; two vertices are connected by an edge when they differ in exactly one coordinate.

The simple random walk on hypercube moves from one vertex  $(x^1, \dots, x^N)$  by choosing a coordinate  $j \in \{1, \dots, N\}$  uniformly and setting the new state to  $(x^1, \dots, x^{j-1}, 1 - x^j, x^{j+1}, \dots, x^N)$ .

The lazy walk remains at its current position with probability  $1/2$  and moves as above with probability  $1/2$ .

The lazy walk can be constructed using the following random mapping representation :

Uniformly select an element  $(j, B)$  in  $\{1, \dots, N\} \times \{0, 1\}$ , and then update the coordinate  $j$  with  $B$ .

Let  $(Z_n = (j_n, B_n))_{n \geq 1}$  be i.i.d.  $\stackrel{d}{\sim} (j, B)$ . At each step, the coordinate  $j_n$  of  $X_{n-1}$  is updated by  $B_n$ .

# Random walk on hypercube

## Theorem

*For the lazy walk on hypercube, there exists a constant  $c_0 > 0$  such that*

$$t_{\text{mix}} \geq cN \log N.$$

**Proof** Suppose that that lazy walk starts from  $X_0 = (1, \dots, 1)$ . Define

$$W(\vec{x}) = \sum_{j=1}^N x^j.$$

## Lemma

*If  $|\mu f - \nu f| \geq r\sigma$ , where  $\sigma^2 = \frac{1}{2}(\text{var}_{\mu}(f) + \text{var}_{\nu}(f))$ ,*

*then*

$$\|\mu - \nu\|_{TV} \geq \frac{r^2}{4 + r^2}.$$