18.445 Introduction to Stochastic Processes

Lecture 23: Irreducibility and recurrence

Hao Wu

MIT

11 May 2015

Recall : $(X_t)_{t\geq 0}$ is a continuous time Markov chain on countable state space with the following requirements

- (Homogeneity) $\mathbb{P}[X_{t+s} = y \,|\, X_s = x] = P_t(x, y)$
- (Right-continuity for the chain) For any $t \geq 0$, there exists $\epsilon > 0$, such that $X_{t+s} = X_t$ for all $s \in [0, \epsilon]$
- (Non explosion) The explosion time $\xi = \infty$
- (Right-continuity in the semigroup) $P_\epsilon \to P_0 = I$ as $\epsilon \to 0$, pointwise for each entry.

Consider the transition semigroup $(P_t)_{t\geq 0}$

- the infinitesimal generator $A = \lim_{\epsilon\to 0}(P_\epsilon - I)$. We write $A = P_0'$.
- Since $P_{t+s} = P_t P_s$, we have $P_t' = AP_t$

Today's goal :

- the infinitesimal generator A characterizes the chain
- irreducible, recurrent

Infinitesimal generator characterizes the transition semigroup

Theorem

Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain with generator A. Then the semigroup $(P_t)_{t\geq 0}$ is the minimal nonnegative solution to the backward equation

$$P_t' = AP_t, \quad P_0 = I.$$

Recall

- the limits $q_x = \lim_\epsilon (1 - P_\epsilon(x,x))/\epsilon, q_{xy} = \lim_{\epsilon\to 0} P_\epsilon(x,y)/\epsilon$ exist.
- the holding time $J_1 : \mathbb{P}_x[J_1 > t] = e^{-q_x t}$
- the jump process : $\mathbb{P}_x[X_{J_1} = y] = q_{xy}/q_x$
- J_1 and X_{J_1} are independent

Irreducible

Suppose that $X = (X_t)_{t\geq 0}$ is a continuous time Markov chain.

- the jump times $J_0, J_1, J_2, ... : J_0 = 0, J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}$.
- the jump chain $Y_0, Y_1, Y_2, ... : Y_n = X_{J_n}$
- the limits $q_x = \lim_{\epsilon}(1 - P_{\epsilon}(x,x))/\epsilon, q_{xy} = \lim_{\epsilon\to 0} P_{\epsilon}(x,y)/\epsilon$ exist.
- the holding time S_x : exponential with parameter q_x
- the jump process : $\mathbb{P}_x[X_{J_1} = y] = q_{xy}/q_x$

Definition

A continuous time Markov chain is irreducible if and only if its jump chain is irreducible.

Lemma

For $x, y \in \Omega$, the following statements are equivalent

- $\exists n \geq 1$ *such that* $\mathbb{P}_x[Y_n = y] > 0$*.*
- $\exists x_0 = x, x_1, ..., x_n = y$ *such that* $q_{x_0x_1}q_{x_1x_2}\cdots q_{x_{n-1}x_n} > 0$*.*
- $P_t(x,y) > 0$ *for all* $t > 0$

Recurrence

Suppose that X is a continuous time Markov chain and that Y is its jump chain.

Definition

A state x is recurrent if $\mathbb{P}_x[\{t : X_t = x\}\text{is unbounded}] = 1$.
A state x is transient if $\mathbb{P}_x[\{t : X_t = x\}\text{is unbounded}] < 1$.

Theorem

Let X be an irreducible continuous time Markov chain.

- *If x is recurrent for Y, then x is recurrent for X.*
- *If x is transient for Y, then x is transient for X.*
- *Either all states are recurrent, or all states are transient.*

Remark A state x is recurrent for X if and only if $\int_0^\infty P_t(x,x)dt = \infty$.
A state x is transient for X if and only if $\int_0^\infty P_t(x,x)dt < \infty$.