# 18.445 Introduction to Stochastic Processes

## Lecture 16: Optional stopping theorem

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**Recall**

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
- A filtration $(\mathcal{F}_n)_{n \geq 0}$
- $X = (X_n)_{n \geq 0}$ is adapted to $(\mathcal{F}_n)_{n \geq 0}$ and is integrable
- $X$ is a martingale if $\mathbb{E}[X_n \,|\, \mathcal{F}_m] = X_m$ *a.s.* for all $n \geq m$.
- $X$ is a supermartingale if $\mathbb{E}[X_n \,|\, \mathcal{F}_m] \leq X_m$ *a.s.* for all $n \geq m$.
- $X$ is a submartingale if $\mathbb{E}[X_n \,|\, \mathcal{F}_m] \geq X_m$ *a.s.* for all $n \geq m$.

**Today's Goal**

- stopping time
- Optional stopping theorem : $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ ?

## Examples

**Example 1** Let $(\xi_i)_{i \geq 1}$ be i.i.d with $\mathbb{E}[\xi_1] = 0$. Then $X_n = \sum_1^n \xi_i$ is a martingale.

**Example 2** Let $(\xi_i)_{i \geq 1}$ be i.i.d with $\mathbb{E}[\xi_1] = 1$. Then $X_n = \Pi_1^n \xi_i$ is a martingale.

**Example 3** Consider biased gambler's ruin : at each step, the gambler gains one dollar with probability $p$ and losses one dollar with probability $(1 - p)$. Let $X_n$ be the money in purse at time $n$.

- If $p = 1/2$, then $(X_n)$ is a martingale.
- If $p < 1/2$, then $(X_n)$ is a supermartingale.
- If $p > 1/2$, then $(X_n)$ is a submartingale.

## Basic properties

**About the expectations**

- If $(X_n)_{n \geq 0}$ is a martingale, then $\mathbb{E}[X_n] = \mathbb{E}[X_0]$ for all $n$.
- If $(X_n)_{n \geq 0}$ is a supermartingale, then $\mathbb{E}[X_n]$ is decreasing in $n$.
- If $(X_n)_{n \geq 0}$ is a submartingale, then $\mathbb{E}[X_n]$ is increasing in $n$.

**More**

- If $(X_n)_{n \geq 0}$ is a supermartingale, then $(-X_n)_{n \geq 0}$ is a submartingale.
- If $X$ is both supermartingale and submartingale, then it is a martingale.
- If $X = (X_n)_{n \geq 0}$ is a martingale, then $(|X_n|)_{n \geq 0}$ is a non-negative submartingale.

### Lemma

*If $(X_n)_{n \geq 0}$ is a martingale, and $\varphi$ is a convex function, then $(\varphi(X_n))_{n \geq 0}$ is a submartingale.*

# Examples

Suppose that $(Y_n)$ is a biased random walk on $\mathbb{Z} : p \neq 1/2$,

$$Y_{n+1} - Y_n = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p. \end{cases}$$

### Lemma

*Set $\mu = 2p - 1$, and set*

$$X_n = Y_n - \mu n.$$

*Then $(X_n)$ is a martingale.*

# Stopping time

Suppose that $(\mathcal{F}_n)_{n \geq 0}$ is a filtration.

### Definition

A stopping time $T : \Omega \to \mathbb{N} = 0, 1, 2, ..., \infty$ is a random variable such that

$$[T = n] \in \mathcal{F}_n, \quad \forall n.$$

### Lemma

*The following are equivalent.*

- $[T = n] \in \mathcal{F}_n$ *for all n.*
- $[T \leq n] \in \mathcal{F}_n$ *for all n.*
- $[T > n] \in \mathcal{F}_n$ *for all n.*
- $[T \geq n] \in \mathcal{F}_{n-1}$ *for all n.*

### Lemma

*If $S, T, T_j$ are stopping times. The following are also stopping times.*

- $S \vee T$ *and* $S \wedge T$
- $\inf_j T_j$ *and* $\sup_j T_j$
- $\liminf_j T_j$ *and* $\limsup_j T_j$

# Stopping time

$(\Omega, \mathcal{F}, \mathbb{P})$ : a probability space with a filtration $(\mathcal{F}_n)_{n \geq 0}$.
$X = (X_n)_{n \geq 0}$ : a process adapted to $(\mathcal{F}_n)_{n \geq 0}$

## Definition

Let $T$ be a stopping time. Define the $\sigma$-algebra $\mathcal{F}_T$ by

$$\mathcal{F}_T = \sigma\{A \in \mathcal{F} : A \cap [T \leq n] \in \mathcal{F}_n, \forall n\}.$$

- Intuitively, $\mathcal{F}_T$ is the information available at time $T$.
- If $T = n_0$, then $\mathcal{F}_T = \mathcal{F}_{n_0}$.
- $X_T 1_{[T < \infty]}$ is measurable with respect to $\mathcal{F}_T$.
- Let $S$ and $T$ be stopping times, if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.

If $X = (X_n)_{n \geq 0}$ is a process, define $X^T = (X_n^T)_{n \geq 0}$ by $X_n^T = X_{T \wedge n}$.

- $X^T$ is adapted.
- If $X$ is integrable, then $X^T$ is also integrable.

# Optional Stopping Theorem

**Goal :** $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ ?

### Theorem

*Let $X = (X_n)_{n \geq 0}$ be a martingale.*

1. *If $T$ is a stopping time, then $X^T$ is also a martingale. In particular, $\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0]$.*

2. *If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$, a.s. In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.*

3. *If there exists an integrable random variable $Y$ such that $|X_n| \leq Y$ for all $n$, and $T$ is a stopping time which is finite a.s., then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.*

4. *If $X$ has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all $n$, and $T$ is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.*

# Gambler's ruin

The gambler's situation can be modeled by a Markov chain on the state space $\{0, 1, ..., N\}$ :

- $X_0$ : initial money in purse
- $X_n$ : the gambler's fortune at time $n$
- $\mathbb{P}[X_{n+1} = X_n + 1 \mid X_n] = 1/2$,
- $\mathbb{P}[X_{n+1} = X_n - 1 \mid X_n] = 1/2$.
- The states 0 and $N$ are absorbing.
- $\tau$ : the time that the gambler stops.

## Theorem
*Assume that $X_0 = k$ for some $0 \leq k \leq N$. Then*

$$\mathbb{P}[X_\tau = N] = \frac{k}{N}, \quad \mathbb{E}[\tau] = k(N - k).$$

# Optional Stopping Theorem

## Theorem

*Let $X = (X_n)_{n \geq 0}$ be a martingale.*

1. *If $T$ is a stopping time, then $X^T$ is also a martingale.*
   *In particular, $\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0]$.*

2. *If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$, a.s.*
   *In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.*

3. *If there exists an integrable random variable $Y$ such that $|X_n| \leq Y$ for all $n$, and $T$ is a stopping time which is finite a.s., then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.*

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# Optional Stopping Theorem

## Theorem

*Let $X = (X_n)_{n \geq 0}$ be a supermartingale.*

1. *If $T$ is a stopping time, then $X^T$ is also a supermartingale. In particular, $\mathbb{E}[X_{T \wedge n}] \leq \mathbb{E}[X_0]$.*

2. *If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] \leq X_S$, a.s. In particular, $\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$.*

3. *If there exists an integrable random variable $Y$ such that $|X_n| \leq Y$ for all n, and $T$ is a stopping time which is finite a.s., then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*

4. *If $X$ has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all n, and $T$ is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*