18.445 2015 Appendix:
Almost Sure Martingale Convergence Theorem

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Theorem 1. *Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space with a filtration $(\mathscr{F}_n)_{n\geq 0}$. Let $X = (X_n)_{n\geq 0}$ be a supermartingale which is bounded in L^1, i.e. $\sup_n \mathbb{E}[|X_n|] < \infty$. Then*

$$X_n \to X_\infty, \quad a.s. \quad as\ n \to \infty$$

where $X_\infty \in L^1(\Omega, \mathscr{F}_\infty, \mathbb{P})$ with $\mathscr{F}_\infty = \sigma(\mathscr{F}_n, n \geq 0)$.

Let $x = (x_n)_{n\geq 0}$ be a sequence of real numbers. Let $a < b$ be two real numbers. We define $T_0(x) = 0$ and inductively, for $k \geq 0$,

$$S_{k+1}(x) = \inf\{n \geq T_k(x) : x_n \leq a\}, \quad T_{k+1}(x) = \inf\{n \geq S_{k+1}(x) : x_n \geq b\},$$

with the usual convention that $\inf \emptyset = \infty$.

Define the number of upcrossings of $[a,b]$ by x by time n to be

$$N_n([a,b],x) = \sup\{k \geq 0 : T_k(x) \leq n\}.$$

As $n \uparrow \infty$, we have

$$N_n([a,b],x) \uparrow N([a,b],x) = \sup\{k \geq 0 : T_k(x) < \infty\},$$

which is the total number of upcrossings of $[a,b]$ by x.

Lemma 2. *A sequence of real numbers x converges in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ if and only if*

$$N([a,b],x) < \infty \quad \text{for all rationals } a < b.$$

Lemma 3. *[Doob's upcrossing inequality] Let X be a supermartingale and $a < b$ be two real numbers. Then, for all $n \geq 0$,*

$$(b-a)\mathbb{E}[N_n([a,b],X)] \leq \mathbb{E}[(a - X_n)^+].$$

Proof. To simplify the notations, we write

$$T_k = T_k(X), \quad S_k = S_k(X), \quad N = N_n([a,b],X).$$

On the one hand, by the definition of (T_k) and (S_k), we have that, for all $k \geq 1$,

$$X_{T_k} - X_{S_k} \geq b - a. \tag{1}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \left(X_{T_k\wedge n} - X_{S_k\wedge n}\right) \\
&\quad = \sum_{k=1}^{N} \left(X_{T_k} - X_{S_k}\right) + \sum_{k=N+1}^{n} \left(X_n - X_{S_k\wedge n}\right) \\
&\quad = \sum_{k=1}^{N} \left(X_{T_k} - X_{S_k}\right) + \left(X_n - X_{S_{N+1}}\right) 1_{[S_{N+1}\le n]}. \quad \text{(Note that } T_N \le n, S_{N+1} < T_{N+1} < S_{N+2}\text{).}
\end{aligned}$$

Since (T_k) and (S_k) are stopping times, we have that $S_k \wedge n \le T_k \wedge n$ are bounded stopping times. Therefore, by *Optional Stopping Theorem*, we have

$$\mathbb{E}[X_{S_k\wedge n}] \ge \mathbb{E}[X_{T_k\wedge n}], \quad \forall k.$$

Combining with Equation (1), we have

$$0 \ge \mathbb{E}\left[\sum_{k=1}^{n} \left(X_{T_k\wedge n} - X_{S_k\wedge n}\right)\right] \ge (b-a)\mathbb{E}[N] - \mathbb{E}[(a - X_n)^+],$$

since $\left(X_n - X_{S_{N+1}}\right) 1_{[S_{N+1}\le n]} \ge -(a - X_n)^+$. This implies the conclusion. □

Proof of Theorem 1. Let $a < b$ be rationals. By Lemma 3, we have that

$$\mathbb{E}[N_n([a,b],X)] \le \frac{\mathbb{E}[(a - X_n)^+]}{b-a} \le \frac{\mathbb{E}[|X_n|] + a}{b-a}.$$

By *Monotone Convergence Theorem*, we have that

$$\mathbb{E}[N([a,b],X)] \le \frac{\sup_n \mathbb{E}[|X_n|] + a}{b-a} < \infty.$$

Therefore, we have almost surely that $N([a,b],X) < \infty$. Write

$$\Omega_0 = \cap_{a<b\in\mathbb{Q}}[N([a,b],X) < \infty].$$

Then $\mathbb{P}[\Omega_0] = 1$. By Lemma 2 on Ω_0, we have that X converges to a possibly infinite limit. Set

$$X_\infty = \begin{cases} \lim_n X_n, & \text{on } \Omega_0, \\ 0 & \text{on } \Omega \setminus \Omega_0. \end{cases}$$

Then X_∞ is $\mathscr{F}_\infty$-measurable and by *Fatou's Lemma*, we have

$$\mathbb{E}[|X_\infty|] \le \mathbb{E}[\liminf_n |X_n|] \le \sup_n \mathbb{E}[|X_n|] < \infty.$$

Therefore $X_\infty \in L^1$. □