18.445 Introduction to Stochastic Processes

Lecture 11: Summary on random walks on network

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Effective Resistance

Consider a network $(G = (V, E), \{c(e) : e \in E\})$.
Suppose that W is a voltage with source $a \in V$ and sink $z \in V$.
Let I be the corresponding current flow :
$I(\overrightarrow{xy}) = (W(x) - W(y))/r(x, y)$.
Define the effective resistance between a and z by

$$R(a \leftrightarrow z) = \frac{W(a) - W(z)}{||I||}.$$

Effective resistance and Escape probability

$$\mathbb{P}_a[\tau_z < \tau_a^+] = \frac{1}{c(a)R(a \leftrightarrow z)}.$$

Effective resistance and Green's function

$$G_{\tau_z}(a, a) = c(a)R(a \leftrightarrow z).$$

Three operations

Define the effective resistance between a and z by

$$R(a \leftrightarrow z) = \frac{W(a) - W(z)}{||I||}.$$

Three operations without changing the effective resistance

Parallel Law : Conductances in parallel add.

Series Law : Resistances in series add.

Gluing : Identify vertices with the same voltage.

Estimates on effective resistance

Effective resistance and energy of flows

$$R(a \leftrightarrow z) = \inf\{\mathcal{E}(\theta) : \theta \text{ unit flow from } a \text{ to } z\}.$$

Corollaries

- If $r(e) \le r'(e)$ for all e, we have
$$R(a \leftrightarrow z; r) \le R(a \leftrightarrow z; r').$$
- **Upper bound** : For any unit flow θ from a to z, we have
$$R(a \leftrightarrow z) \le \mathcal{E}(\theta).$$
- **Lower bound** : **Nash-William Inequality.** $\{\Pi_k\}$ are disjoint edge-cut sets which separate a from z, then
$$R(a \leftrightarrow z) \ge \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Random walk on network

Consider a random walk on network $(G = (V, E), \{c(e) : e \in E\})$.

- Transition matrix : $P(x, y) = c(x, y)/c(x)$
- It is reversible
- The stationary measure : $\pi(x) = c(x)/c_G$.
- The commute time is defined by

$$\tau_{ba} = \min\{n \geq \tau_b : X_n = a\}.$$

- **Commute Time Identity**

$$\mathbb{E}_a[\tau_{ba}] = c_G R(a \leftrightarrow b).$$

- Assume that the network is transitive, then

$$\mathbb{E}_a[\tau_b] = \mathbb{E}_b[\tau_a].$$

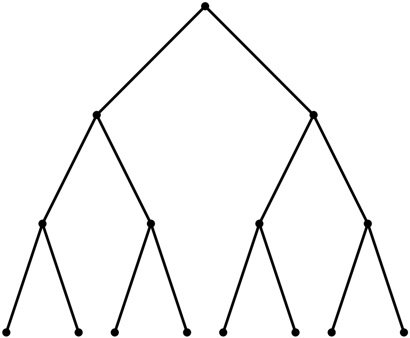
  In particular,

$$2\mathbb{E}_a[\tau_b] = c_G R(a \leftrightarrow b).$$

Random walk on binary tree

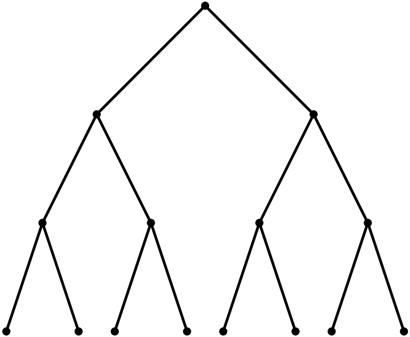
A **tree** is a connected graph with no cycles.
A **rooted tree** has a distinguished vertex v_0, called the root.
The **depth** of a vertex v is its graph distance to the root.
A **leaf** is a vertex with degree one.
A **rooted binary tree** of depth k, denoted by T_2^k, is a tree with a root v_0 such that

- v_0 has degree 2.
- For $1 \leq j \leq k-1$, every vertex at distance j from the root has degree 3.
- The vertices at distance k from the root are leaves (they have degree 1).



Random walk on binary tree

- T_k^2 is a network
- all edges have unit resistance
- there are $N = 2^{k+1} - 1$ vertices
- there are $N - 1$ edges



Theorem

Consider the random walk $(X_n)_n$ on this network. Let B be the set of leaves. Define the commute time

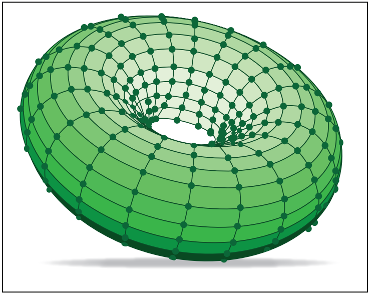
$$\tau_{Bv_0} = \min\{n \geq \tau_B : X_n = v_0.\}$$

Then

Random walk on torus

A 2-dimensional **torus** :
$\mathbb{Z}_N^2 = \mathbb{Z}_N \times \mathbb{Z}_N$.
Two vertices $\overrightarrow{x} = (x^1, x^2)$ and $\overrightarrow{y} = (y^1, y^2)$ are neighbors if,

$$\begin{cases} \text{either} x^1 = y^1, x^2 \equiv y^2 \pm 1 \mod N \\ \text{or} \quad x^2 = y^2, x^1 \equiv y^1 \pm 1 \mod N \end{cases}$$



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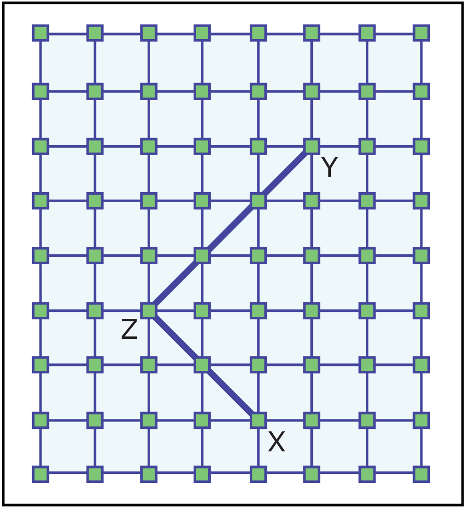
This is a network and assume that all edges have unit resistance.

Theorem

Let $k = |x - y| \geq 2$ on $\mathbb{Z}_N^2$. There exist constants $0 < c < C < \infty$ such that

$$cN^2 \log k \leq \mathbb{E}_x[\tau_y] \leq CN^2 \log k.$$

Random walk on torus



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