18.445 Introduction to Stochastic Processes

Lecture 9: Random walk on networks 2

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Recall : A voltage W is a harmonic function on $V \setminus \{a, z\}$. A current flow I associated to the voltage W is defined by

$$I(\overrightarrow{xy}) = (W(x) - W(y))/r(x, y).$$

The effective resistance is defined by

$$R(a \leftrightarrow z) = (W(a) - W(z))/||I||.$$

Relation with escape probability

$$\mathbb{P}_a[\tau_z < \tau_a^+] = 1/\left(c(a)R(a \leftrightarrow z)\right).$$

Today's Goal :

- three operations to simplify a network
- effective resistance and energy of a flow
- Nash-William inequality

Three operations to simplify a network

We introduce three operations that simplify the network without changing quantities of interest : all voltages and currents remain unchanged under the following operations.

Parallel Law : Conductances in parallel add.

Series Law : Resistances in series add.

Gluing : Identify vertices with the same voltage.

Example : Biased nearest-neighbor random walk.
Fix $\alpha > 1$ and consider the path with vertices $\{0, 1, 2, ..., N\}$ and weights $c(k-1, k) = \alpha^k$ for $k = 1, ..., N$. Consider the random walk on this network, then we have

$$\mathbb{P}_k[\tau_N < \tau_0] = \frac{1 - \alpha^{-k}}{1 - \alpha^{-N}}.$$

Energy of a flow

Definition

The energy of a flow θ is defined by

$$\mathcal{E}(\theta) = \sum_e \theta(e)^2 r(e),$$

where the summation is taking over unoriented edges.

Theorem (Effective resistance and Energy of flows)

For any finite connected graph,

$$R(a \leftrightarrow z) = \inf\{\mathcal{E}(\theta) : \theta \text{ unit flow from } a \text{ to } z\}.$$

Moreover, the unique minimizer is the unit current flow.

Application

Theorem

If $\{r(e) : e \in E\}$ and $\{r'(e) : e \in E\}$ are sets of resistances on the edges of the same graph G and if $r(e) \leq r'(e)$ for all $e \in E$, then

$$R(a \leftrightarrow z; r) \leq R(a \leftrightarrow z; r').$$

Corollary

- *Adding an edge decreases the effective resistance, hence increases the escape probability.*
- *Gluing vertices decreases the effective resistance, hence increases the escape probability.*

Nash-William inequality

Definition

We call $\Pi \subset E$ an edge-cutset separating a from z if every path from a to z include some edge in Π. In other words, if we cut all edges in Π, then a can not be connected to z.

Theorem (Nash-William inequality)

If $\{\Pi_k\}$ are disjoint edge-cutsets which separate a from z, then

$$R(a \leftrightarrow z) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Example

B_N : $N \times N$ two-dimensional grid graph.
The four corners are $(1,1), (1,N), (N,1), (N,N)$.

Theorem

Let $a = (1,1)$, $z = (N,N)$. Suppose that each edge has unit resistance. Then the effective resistance satisfies

$$\frac{1}{2}\log(N-1) \leq R(a \leftrightarrow z) \leq 2\log N.$$

Proof

Lower bound : Nash-William inequality
Upper bound : Construct a nice unit flow.

Effective resistance

Effective resistances form a metric space.

Theorem

For any vertices x, y, z, we have

$$R(x \leftrightarrow z) \leq R(x \leftrightarrow y) + R(y \leftrightarrow z).$$