18.445 Introduction to Stochastic Processes

Lecture 1: Introduction to finite Markov chains

Hao Wu

MIT

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Today's goal

1. Definitions
2. Gambler's ruin
3. coupon collecting
4. stationary distribution

Ω : finite state space
P : transition matrix $|\Omega| \times |\Omega|$

Definition

A sequence of random variables $(X_0, X_1, X_2, ...)$ is a Markov chain with state space Ω and transition matrix P if
for all $n \geq 0$, and all sequences $(x_0, x_1, ..., x_n, x_{n+1})$, we have that

$$\mathbb{P}[X_{n+1} = x_{n+1} \,|\, X_0 = x_0, ..., X_n = x_n]$$

$$= \mathbb{P}[X_{n+1} = x_{n+1} \,|\, X_n = x_n] = P(x_n, x_{n+1}).$$

Gambler's ruin

Consider a gambler betting on the outcome of a sequence of independent fair coin tosses.
If head, he gains one dollar.
If tail, he loses one dollar.
If he reaches a fortune of N dollars, he stops.
If his purse is ever empty, he stops.

Questions :

- What are the probabilities of the two possible fates ?
- How long will it take for the gambler to arrive at one of the two possible fates ?

Gambler's ruin

The gambler's situation can be modeled by a Markov chain on the state space $\{0, 1, ..., N\}$:

- X_0 : initial money in purse
- X_n : the gambler's fortune at time n
- $\mathbb{P}[X_{n+1} = X_n + 1 \mid X_n] = 1/2$,
- $\mathbb{P}[X_{n+1} = X_n - 1 \mid X_n] = 1/2$.
- The states 0 and N are absorbing.
- τ : the time that the gambler stops.

Answer to the questions

Theorem

Assume that $X_0 = k$ for some $0 \leq k \leq N$. Then

$$\mathbb{P}[X_\tau = N] = \frac{k}{N}, \quad \mathbb{E}[\tau] = k(N-k).$$

Coupon collecting

A company issues N different types of coupons. A collector desires a complete set.
Question :
How many coupons must he obtain so that his collection contains all N types.
Assumption : each coupon is equally likely to be each of the N types.

Coupon collecting

The collector's situation can be modeled by a Markov chain on the state space $\{0, 1, ..., N\}$:

- $X_0 = 0$
- X_n : the number of different types among the collector's first n coupons.
- $\mathbb{P}[X_{n+1} = k + 1 \mid X_n = k] = (N - k)/N$,
- $\mathbb{P}[X_{n+1} = k \mid X_n = k] = k/N$.
- τ : the first time that the collector obtains all N types.

Coupon collecting

Answer to the question.

Theorem

$$\mathbb{E}[\tau] = N \sum_{k=1}^{N} \frac{1}{k} \approx N \log N.$$

A more precise answer.

Theorem

For any $c > 0$, we have that

$$\mathbb{P}[\tau > N \log N + cN] \leq e^{-c}.$$

Notations

Ω : state space
μ : measure on Ω
P, Q : transition matrices $|\Omega| \times |\Omega|$
f : function on Ω

Notations

- μP : measure on Ω
- PQ : transition matrix
- Pf : function on Ω

Associative

- $(\mu P)Q = \mu(PQ)$
- $(PQ)f = P(Qf)$

Notations

Consider a Markov chain with state space Ω and transition matrix P. Recall that

$$\mathbb{P}[X_{n+1} = y \mid X_n = x] = P(x, y).$$

- μ_0 : the distribution of X_0
- μ_n : the distribution of X_n

Then we have that

- $\mu_{n+1} = \mu_n P$.
- $\mu_n = \mu_0 P^n$.
- $\mathbb{E}[f(X_n)] = \mu_0 P^n f$.

Stationary distribution

Consider a Markov chain with state space Ω and transition matrix P. Recall that

$$\mathbb{P}[X_{n+1} = y \mid X_n = x] = P(x, y).$$

- μ_0 : the distribution of X_0
- μ_n : the distribution of X_n

Definition

We call a probability measure π is stationary if

$$\pi = \pi P.$$

If π is stationary and the initial measure μ_0 equals π, then

$$\mu_n = \pi, \quad \forall n.$$

Random walks on graphs

Definition

A graph $G = (V, E)$ consists of a vertex set V and an edge set E :

- V : set of vertices
- E : set of pairs of vertices
- When $(x, y) \in E$, we write $x \sim y$: x and y are joined by an edge. We say y is a neighbor of x.
- For $x \in V$, $deg(x)$: the number of neighbors of x.

Definition

Given a graph $G = (V, E)$, we define simple random walk on G to be the Markov chain with state space V and transition matrix :

$$P(x, y) = \begin{cases} 1/deg(x) & \text{if } y \sim x \\ 0 & \text{else} \end{cases}.$$

Random walks on graphs

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Theorem

Define

$$\pi(x) = \frac{deg(x)}{2|E|}, \quad \forall x \in V.$$

Then π is a stationary distribution for the simple random walk on the graph.