Progress Report

Revisiting `ASkewSGD`: New Theoretical Guarantees for Quantization-Aware Deep Neural Network Optimization

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Contents

Abstract

The question of enforcing quantization on the weights and activations in Deep Neural Networks (DNNs) has come to the forefront in recent years due to its relevance to restricted memory and/or computational resources. While low-precision fixed integer values could substantially reduce the memory footprint and latency, inaccurate quantization on model parameters is susceptible to significant accuracy drops. In this paper, we will continue to pave the way for quantization algorithms during neural network training and provide stronger guarantees for the Annealed Skewed Stochastic Gradient Descent algorithm (`ASkewSGD`) proposed by Leconte et al. [1]. In particular, we attempt to retain the algorithm's convergence guarantees without hinging on a highly differentiable loss function. Numerical experiments show that `ASkewSGD` are able to produce state-of-the-art results in classical benchmarks, justifying its effectiveness as a robust optimization algorithm.

Contributions

- We improve the understanding of `ASkewSGD` and provide new convergence insights for `ASkewSGD` proposed by Leconte et al. [1] under weaker assumptions. We identify a loophole in their proof on the loss function's Lipschitzness, and further weaken the differentiability requirement of the loss function. Several theoretical guarantees are provided to address this issue.
- (Multi-bit assessment to be completed.) We evaluate the performance of `ASkewSGD` along with other SOTA quantization-aware training methods (`BinaryConnect`, `ProxQuant`) by numerical experiments on `MNIST`, `CIFAR-10`, and `ImageNet`. We made the related codes available at `https://github.com/SWongHF/Experiment-2025-Summer`.

1 Background

We are interested in solving the optimization problem related to learning a quantized neural network (QNN),

$$\min_{\mathbf{w}\in\mathcal{Q}} \ell(\mathbf{w}), \text{ where } \ell(\mathbf{w}) = \mathbb{E}_{(\mathbf{x},y)\sim p_{\text{data}}}[\ell(f(\mathbf{x},\mathbf{w}),y)],$$

where $\ell : \mathbb{R}^d \to \mathbb{R}$ denotes the training loss, $\mathcal{Q} \subset \mathbb{R}^d$ is the set of quantization levels, d is the number of parameters in the neural network, p_{data} is the training distribution [1].

We do not necessarily aim to solve the above (hard, combinational) optimization problem globally and optimally. Instead, we want to find discrete weights $\mathbf{w} \in \mathcal{Q}$ that remain "satisfactory" when compared to the non-quantized continuous weights [2].

Goal. Given a continuously differentiable and reasonably smooth function ℓ, develop an algorithm with certain guarantees that converges to a close-by quantized weight of a locally, or globally, optimal continuous weight with similar performance.

2 Existing Works

2.1 BinaryConnect

Courbariaux et al. [3] considered an aggresive quantization scheme using binary networks, where $\mathcal{Q} = \{\pm 1\}^d$. The `BinaryConnect` algorithm updates the weights by the following scheme:

$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} - \gamma_k \widehat{\nabla}\ell(\mathsf{P}(\mathbf{w}^{(k)})), \text{where } [\mathsf{P}(\mathbf{w})]_i = \begin{cases} +1, & w_i \geq 0, \\ -1, & \text{otherwise.} \end{cases}$$

While the above method is deterministic, yet another scheme quantizes the weights stochastically, which is given by:

$$[\mathsf{P}(\mathbf{w})]_i = \begin{cases} +1, & \text{with probability } \sigma(w_i), \\ -1, & \text{with probability } 1 - \sigma(w_i), \end{cases} \qquad \text{where } \sigma(x) = \text{clip}\left(\frac{x+1}{2}, 0, 1\right).$$

For the general quantized neural network, STE can easily be adapted by adding a quantization step that maps a real number input $w \in [0, 1]$ to a k-bit number output w_q [4]:

$$w_q = \text{round}((2^k - 1)w) - (2^{k-1} - 1).$$

The behaviour of BC is analyzed by [2], stating that the updates of BC are formally the same as the dual averaging (DA) algorithm (as a non-convex counterpart). The work [2] also generalizes BC into `ProxConnect` with rigorous convergence guarantees.

2.2 ProxQuant

Bai et al. [5] proposed the proximal gradient method for quantization-aware training, which is a variant of the proximal operator

$$\mathbf{w}^{(k+1)} \leftarrow \mathsf{P}\left(\mathbf{w}^{(k)} - \eta_k \widehat{\nabla}\ell(\mathbf{w}^{(k)})\right),$$

where the proximal operator is defined as

$$\mathsf{P}(\mathbf{w}) = \underset{\bar{\mathbf{w}} \in \mathbb{R}^n}{\arg\min} \left(\frac{1}{2}\|\mathbf{w} - \bar{\mathbf{w}}\|_2^2 + \lambda R(\bar{\mathbf{w}})\right), \text{where } R(\bar{\mathbf{w}}) = \inf_{\hat{\mathbf{w}} \in \mathcal{Q}} \|\hat{\mathbf{w}} - \bar{\mathbf{w}}\|_2.$$

This method is theoretically sound, but the proximal operator is expensive to compute when $\mathcal{Q}$ is large.

3 The ASkewSGD Algorithm

We turn to consider the smoothed sequence of interval-constrained optimization problems ($\mathcal{P}_\varepsilon$),

$$\min_{\mathbf{w}\in C_\varepsilon} \ell(\mathbf{w}) := \frac{1}{N}\sum_{j=1}^{N} \ell_j(\mathbf{w}), C_\varepsilon = \{\mathbf{w}\in\mathbb{R}^d : g_\varepsilon(\mathbf{w}) \geq 0\},$$

where N is the size of the training set, ℓ_j is the loss associated with the j-th observation, and ε is an annealing parameter that converges to zero (a fixed value, however, for analysis).

Previously, Muehlebach and Jordan [6] reformulated the position constraints into forward "velocity" constraints by considering a linear and convex approximation of the original feasible set with the objective function satisfying the Mangasarian-Fromovitz condition.

Definition 1 (**Mangasarian-Fromovitz Constraint Qualification**). $\forall \mathbf{x}\in\mathbb{R}^d$, $\exists \mathbf{w}\in\mathbb{R}^n$ *s.t.* $\nabla g_i(\mathbf{x})^\top \mathbf{w} > 0$ *for all* $i \in I(\mathbf{x})$, *where* $I(\mathbf{x}) = \{i\in\mathbb{Z} \mid g_i(\mathbf{x}) \leq 0\}$.

By MFCQ, the set $V_\alpha(\mathbf{w})$, $V_\alpha(\mathbf{w}) = \{\mathbf{v}\in\mathbb{R}^d : \nabla g_i(\mathbf{w})^\top \mathbf{v} \geq -\alpha g_i(\mathbf{w})\}$, is considered as an extension of the tangent cone outside of the feasible set, and is always a convex polyhedron.

Algorithm 1: Muehlebach-Jordan Algorithm

for $k = 1, 2, \ldots$ **do**
- $\mathbf{v}^{(k)} = \arg\min_{\mathbf{v}\in V_\alpha(\mathbf{w}^{(k)})} (1/2)\|\mathbf{v} + \nabla\ell(\mathbf{w}^{(k)})\|^2$.
- $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \gamma_k \mathbf{v}^{(k)}$.

We note that α controls the trade-off betweeen two objectives: for large α, the emphasis is on the convergence to the feasible set, while for small α, the focus is on reducing the objective function [6].

Muehlebach and Jordan's algorithm [6] is then extended by Leconte et al. [1] to the quantization-aware training algorithm ASkewSGD.

Definition 2 (**Remoteness Measurement of the Quantization Set**). *Let* $\varepsilon \in (0, 1]$, $w_i \in \mathcal{Q}_i, i \in [d]$, *where* $\mathcal{Q}_i = \{q_i^{(1)}, \ldots, q_i^{(K_i)}\}$ *are sets of quantization values defined coordinate-wise. We define the piecewise function*

$$\psi_i(w_i;\varepsilon) := \begin{cases} \varepsilon - (q_i^{(1)} - w_i)^2 & w_i < q_i^{(1)}, \\ \varepsilon - (w_i - q_i^{(j-1)})^2 (w_i - q_i^{(j)})^2 & q_i^{(j-1)} \leq w_i < q_i^{(j)}, j = 2, \ldots, K, \\ \varepsilon - (w_i - q_i^{(K_i)})^2 & w_i \geq q_i^{(K_i)}, \end{cases}$$

for all $w_i \in \mathbb{R}$ *and* $i \in [n]$.

We impose constraints on each parameter of the neural network, by setting $g_i(\mathbf{w}) = \psi_i(w_i;\varepsilon)$. This then forms the feasible set C_ε. Furthermore, $V_{\varepsilon,\alpha} = \{\mathbf{v}\in\mathbb{R}^d : v_i\psi_i'(w_i;\varepsilon) \geq -\alpha\psi_i(w_i;\varepsilon) \text{ for } i \in I_\varepsilon(\mathbf{w})\}$, where $I_\varepsilon(\mathbf{w}) = \{i\in[d] : \psi_i(w_i;\varepsilon) \leq 0\}$, the normal cone of C_ε is given by $N_{C_\varepsilon} = \{-\sum_{i\in I_\varepsilon(\mathbf{w})} \lambda_i \nabla g_i(\mathbf{w}), \lambda_i \in \mathbb{R}_+\}$.

Algorithm 2: ASkewSGD Algorithm for QNN Training

for $k = 1, 2, \ldots$ **do**
  Obtain a stochastic gradient $\widehat{\nabla}\ell(\mathbf{w}^{(k)}) = 1/N_b \sum_{i=1}^{N_b} \nabla \ell_{j_i}(\mathbf{w}^{(k)})$.
  $\widehat{\mathbf{v}}^{(k)} = \arg\min_{\mathbf{v} \in V_{\varepsilon,\alpha}(\mathbf{w}^k)} (1/2)\|\mathbf{v} + \widehat{\nabla}\ell(\mathbf{w}^{(k)})\|^2$.
  $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \gamma_k \widehat{\mathbf{v}}^{(k)}$.

The optimization problem

$$\arg\min_{\mathbf{v} \in V_{\varepsilon,\alpha}(\mathbf{w})} (1/2)\|\mathbf{v} + \mathbf{u}\|^2$$

has an explicit solution (we set $[s_{\varepsilon,\alpha}(\mathbf{g}, \mathbf{w})]_i = M_c$ if $w_i = (q_i^{(j)} + q_i^{(j+1)})/2$ by convention):

$$[\mathbf{s}_{\varepsilon,\alpha}(\mathbf{g}, \mathbf{w})]_i := \begin{cases} -g_i & \text{if } \psi_i(w_i;\varepsilon) > 0 \text{ or } -g_i \cdot \psi_i'(w_i;\varepsilon) \geq -\alpha\psi_i(w_i;\varepsilon) \geq 0, \\ \text{clip}(-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon), M_c) & \text{otherwise.} \end{cases}$$

Furthermore, the set of stationary points are given by the Karush-Kuhn-Tucker condition:

$$\mathcal{Z}_\varepsilon := \{\mathbf{w} \in C_\varepsilon : \mathbf{0} \in -\nabla\ell(\mathbf{w}) - N_{C_\varepsilon}(\mathbf{w})\}.$$

That is, $\mathbf{w} \in \mathcal{Z}_\varepsilon$ if and only if $[\nabla\ell(\mathbf{w})]_i = 0$ when $\psi_i(w_i;\varepsilon) > 0$ and $\text{sign}([\nabla\ell(\mathbf{w})]_i) = \text{sign}(\psi_i'(w_i;\varepsilon))$ when $\psi_i(w_i;\varepsilon) = 0$.

4 Existing Convergence Analysis

To set up the analysis, we will introduce several assumptions on the loss function and step sizes.

Assumption 1. *The function ℓ is coercive, i.e.*

$$\lim_{\|\mathbf{w}\| \to \infty} \ell(\mathbf{w}) \to +\infty.$$

Assumption 2. *For $j \in \{1, \ldots, N\}$, the function ℓ_j is continuously differentiable and M_ℓ-smooth, i.e.*

$$\|\nabla\ell_j(\mathbf{x}) - \nabla\ell_j(\mathbf{y})\|^2 \leq M_{\ell_j}\|\mathbf{x} - \mathbf{y}\|^2 \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Assumption 3. *For $j \in \{1, \ldots, N\}$, the function ℓ_j is continuously differentiable and L_ℓ-Lipschitz continuous, i.e.*

$$|\ell_j(\mathbf{x}) - \ell_j(\mathbf{y})|^2 \leq L_{\ell_j}\|\mathbf{x} - \mathbf{y}\|^2 \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Assumption 4. *For $j \in \{1, \ldots, N\}$, the function ℓ_j is d-times continuously differentiable.*

Assumption 5. *The stepsizes $(\gamma_k)_{k \geq 0}$ are positive, non-summable and square-summable, i.e.*

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \sum_{k=0}^{\infty} \gamma_k^2 < \infty.$$

Lemma 1 (**Convergence to the Feasible Set**). *Under A3 and A5, it holds that* $\limsup_{k\to\infty} d(\mathbf{w}^{(k)}, C_\varepsilon) = 0$ *almost surely.*

Proof. See Leconte et al. [1], an asymptotic argument.

Theorem 1 (**Asymptotic Convergence Guarantees with Lipschitz Continuity**). *Assume A3, A4, A5 and* $0 < \varepsilon \leq \inf_{1\leq i\leq d} \inf_{1\leq j<K_i} |q_i^{(j)} - q_i^{(j+1)}|^4/16$ *holds, where* $\{q_i^{(j)}\}$ *are the quantization levels,* $\ell(\mathbf{w}^{(k)})$ *converges and* $\lim_{k\to\infty} d(\mathbf{w}^{(k)}, \mathcal{Z}_\varepsilon) = 0$ *almost surely.*

Proof. See Leconte et al. [1]. Sard's Theorem makes A4 necessary for Leconte's proof.

Remark. A3 is a special case of A2 and it rules out many functions (such as the quadratic function) as a consequence. A4 is non-standard. It represents strong differentiability of the loss function's differentiability, which is not always the case for NNs. Our work is to remove A3 (replaced by A2), A4. We also attempt to investigate the general behavior of ASkewSGD without A5.

5 Our Work

We provide four convergence guarantees different from Leconte et al.'s [1] for a modified variant of ASkewSGD (on the remoteness measurement by ψ) using deterministic and stochastic oracles. Note that Leconte et al.'s proof still holds in our modified version.

Definition 3 (**Quadratic Remoteness Measurement of the Quantization Set**). *Let* $\varepsilon \in (0, 1]$, $w_i \in \mathcal{Q}_i, i \in [d]$, *where* $\mathcal{Q}_i = \{q_i^{(1)}, \ldots, q_i^{(K_i)}\}$ *are sets of quantization values defined coordinate-wise. We define the piecewise function*

$$\psi_i(w_i;\varepsilon) := \begin{cases} \varepsilon - (q_i^{(1)} - w_i) & w_i < q_i^{(1)}, \\ \varepsilon - (q_i^{(j-1)} - w_i)(w_i - q_i^{(j)}) & q_i^{(j-1)} \leq w_i < q_i^{(j)}, j = 2, \ldots, K_i, \\ \varepsilon - (w_i - q_i^{(K_i)}) & w_i \geq q_i^{(K_i)}, \end{cases}$$

for all $w_i \in \mathbb{R}$.

Remark. This function has a lower order compared to Leconte et al.'s (see Definition 2), as we shall see in Lemma 2.

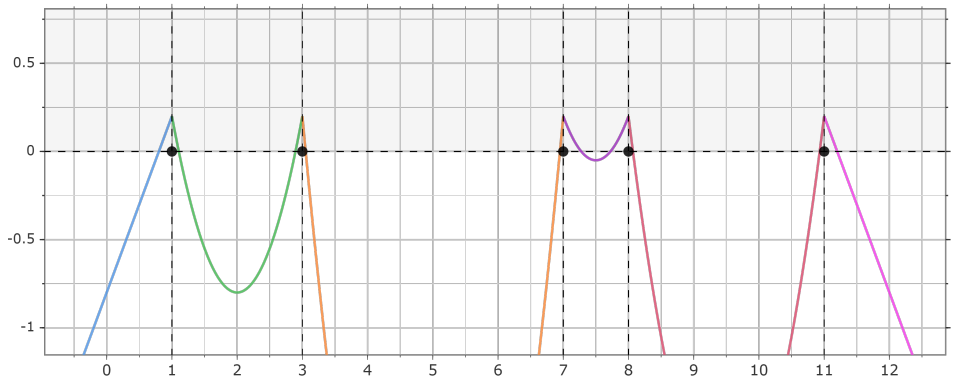


Figure 1: $\psi_i(w_i; \varepsilon)$ where $\varepsilon = 0.2$ and $\mathcal{Q}_i = \{1, 3, 7, 8, 11\}$.

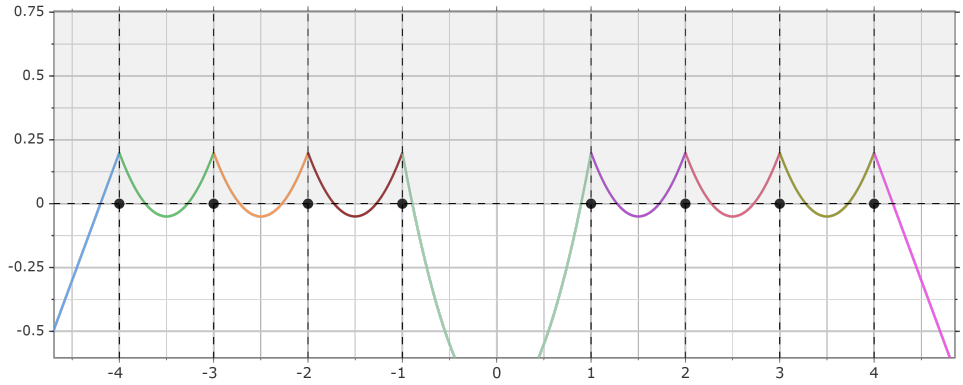


Figure 2: The INT3 uniform quantization scheme, where $\varepsilon = 0.2$ and $\mathcal{Q}_i = \{-4, -3, -2, -1, 1, 2, 3, 4\}$.

Lemma 2 (A Distance Bound on the Skewing Force). *Fix an arbitrary* $0 < \varepsilon \le \inf_{1\le i\le d} \inf_{1\le j<K_i} |q_i^{(j)} - q_i^{(j+1)}|^2/4$, *where* $\{q_i^{(j)}\}$ *are the quantization levels. For* $j \in \{2, \dots, K_i - 1\}$, *denote* $[q_-, q_+]$ *as the set* $C_{\varepsilon,i} \cap [(q_i^{(j)} + q_i^{(j-1)})/2, (q_i^{(j)} + q_i^{(j+1)})/2)$, *where* $C_{\varepsilon,i}$ *is the projection of* C_ε *on the* i*-th coordinate. Let* $0 < \delta_1 < (q_i^{(j)} + q_i^{(j+1)} - 2q_+)/3$ *and* $0 < \delta_2 < (2q_- - q_i^{(j)} - q_i^{(j-1)})/3$ *be some small perturbations on a quantization level. Then,*

(a)
$$\left|\frac{-\alpha\psi_i(q_+ + \delta_1; \varepsilon)}{\psi_i'(q_+ + \delta_1; \varepsilon)}\right| \le 2\alpha\delta_1;$$

(b)
$$\left|\frac{-\alpha\psi_i(q_- - \delta_2; \varepsilon)}{\psi_i'(q_- - \delta_2; \varepsilon)}\right| \le 2\alpha\delta_2.$$

For $j \in \{1, K_i\}$, *denote* $q_- = q_i^{(1)} - \varepsilon, q_+ = q_i^{(K_i)} + \varepsilon$, *and let* $\delta_1 > 0, \delta_2 > 0$, *we have*

$$\left|\frac{-\alpha\psi_i(q_+ + \delta_1; \varepsilon)}{\psi_i'(q_+ + \delta_1; \varepsilon)}\right| = \alpha\delta_1, \left|\frac{-\alpha\psi_i(q_- - \delta_2; \varepsilon)}{\psi_i'(q_- - \delta_2; \varepsilon)}\right| = \alpha\delta_2.$$

Proof. By symmetry, we only need to prove the statement as in (a). Note that

$$\psi_i(q_+; \varepsilon) = \varepsilon - (q_+ - q_i^{(j-1)})(q_i^{(j)} - q_+) = 0 \text{ and } \psi_i'(w; \varepsilon) = 2w - q_i^{(j+1)} - q_i^{(j)}.$$

Using (1), we expand the numerator of the left-hand side of (a) and obtain

$$\begin{aligned}\psi_i(q_+ + \delta_1; \varepsilon) &= \varepsilon + (q_+ + \delta_1 - q_i^{(j)})(q_+ + \delta_1 - q_i^{(j+1)}) \\ &= \varepsilon + (q_+ - q_i^{(j)})(q_+ - q_i^{(j+1)}) + \delta_1(2q_+ - q_i^{(j+1)} - q_i^{(j)}) + \delta_1^2 \\ &= \delta_1(2q_+ - q_i^{(j+1)} - q_i^{(j)}) + \delta_1^2 < 0.\end{aligned}$$

Since $\delta_1 > 0$, we have $\psi_i'(q_+ + \delta_1; \varepsilon) = 2q_+ + \delta_1 - q_i^{(j+1)} - q_i^{(j)} < 0$ by (2). Then,

$$\begin{aligned}\left|\frac{-\alpha\psi_i(q_+ + \delta_1; \varepsilon)}{\psi_i'(q_+ + \delta_1; \varepsilon)}\right| &= \frac{\alpha\delta_1(2q_+ - q_i^{(j)} - q_i^{(j+1)} + \delta_1)}{(2q_+ - q_i^{(j)} - q_i^{(j+1)}) + 2\delta_1} \\ &\le \frac{\alpha\delta_1(2q_+ - q_i^{(j)} - q_i^{(j+1)})}{(2q_+ - q_i^{(j)} - q_i^{(j+1)}) + 2\delta_1} \\ &= \alpha\delta_1 - \frac{\alpha\delta_1^2}{(2q_+ - q_i^{(j)} - q_i^{(j+1)}) + 2\delta_1}.\end{aligned}$$

Since $\delta_1 \le (q_i^{(j)} + q_i^{(j+1)} - 2q_+)/3$, we have

$$\left|\frac{-\alpha\psi_i(q_+ + \delta_1; \varepsilon)}{\psi_i'(q_+ + \delta_1; \varepsilon)}\right| \le 2\alpha\delta_1.$$

□

Remark. If we have $\sup_{k\ge k_0} \mathrm{d}(\mathbf{w}^{(k)}, C_\varepsilon) \le \delta$, then $|-\alpha\psi_i(w_i^{(k)}; \varepsilon)/\psi_i'(w_i^{(k)}; \varepsilon)| \le 2\alpha\delta$ for all $k \ge k_0$.

5.1 Deterministic Full Gradient Convergence Proof

Recall the update

$$[\mathbf{s}_{\varepsilon,\alpha}(\nabla\ell(\mathbf{w}),\mathbf{w})]_i := \begin{cases} -[\nabla\ell(\mathbf{w})]_i & \text{if } \psi_i(w_i;\varepsilon) > 0 \text{ or} \\ & -[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \geq -\alpha\psi_i(w_i;\varepsilon) \geq 0, \\ \mathrm{clip}(-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon), M_c) & \text{otherwise.} \end{cases}$$

We simplify the "otherwise" direction with the notation $\mathbf{u}$ and refer this to the "skewing force."

Definition 4 (**Categorization of the Update Direction**). *For the update on the i-th coordinate, we define*

$$i \in S_\varepsilon^+(\mathbf{w}) \iff \psi_i(w_i;\varepsilon) \leq 0 \text{ and } -\alpha\psi_i(w_i;\varepsilon) > -[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \geq 0,$$
$$i \in S_\varepsilon^-(\mathbf{w}) \iff \psi_i(w_i;\varepsilon) \leq 0 \text{ and } -\alpha\psi_i(w_i;\varepsilon) \geq 0 > -[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon),$$
$$i \notin S_\varepsilon(\mathbf{w}) := S_\varepsilon^+(\mathbf{w}) \cup S_\varepsilon^-(\mathbf{w}) \iff \psi_i(w_i;\varepsilon) > 0 \text{ or } -[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \geq -\alpha\psi_i(w_i;\varepsilon) \geq 0.$$

5.1.1 Finite Time Convergence with Lipschitz Continuity and Constant Stepsize

In the sequel, we introduce three lemmata that provides non-asymptotic guarantees for ASkewSGD under Assumption A3 and a small enough step size γ (which will be regulated by the lemmata).

Lemma 3 (**Confinement to the Quantization Interval, General Case**). *Assume that A3 holds. Fix an $i \in [d]$ and $2 \leq j \leq K_i - 1$. Let $[q_-, q_+] = C_{\varepsilon,i} \cap [(q_i^{(j-1)} + q_i^{(j)})/2, (q_i^{(j)} + q_i^{(j+1)})/2)$, where $C_{\varepsilon,i}$ is the projection of C_ε on the i-th coordinate, and $0 < \gamma \cdot \max\{L_\ell, M_c\} < \min\{q_+ - q_-, q_- - (q_i^{(j-1)} + q_i^{(j)})/2, (q_i^{(j-1)} + q_i^{(j)})/2 - q_+\}$. If there is a moment k_0 such that $w_i^{(k_0)} \in ((q_i^{(j)} + q_i^{(j-1)})/2, (q_i^{(j)} + q_i^{(j+1)})/2)$, then $w_i^{(k)}$ is confined to the interval $((q_i^{(j)} + q_i^{(j-1)})/2, (q_i^{(j)} + q_i^{(j+1)})/2)$ for every $k \geq k_0$.*

Proof. Notice for any $k > 0$, $\|\nabla\ell(\mathbf{w}^{(k)})\| \leq L_\ell$ and $|v_i^{(k)}| \leq \max\{L_\ell, M_c\}$. If $(q_i^{(j)} + q_i^{(j-1)})/2 \leq w_i^{(k_0)} < q_-$, then $w_i^{(k_0)}$ is pushed to the right. We have $w_i^{(k_0)} \leq w_i^{(k_0+1)} := w_i^{(k_0)} + \gamma\max\{L_\ell, M_c\} < q_+$. If $q_+ < w_i^{(k)} < (q_i^{(j)} + q_i^{(j+1)})/2$, then $w_i^{(k_0)}$ is pushed to the left, and $w_i^{(k_0)} \geq w_i^{(k_0+1)} := w_i^{(k_0)} - \gamma\max\{M_\ell, M_c\} \leq q_-$. Finally, if $w_i^{(k_0)} \in [q_-, q_+]$, then $(q_i^{(j-1)} + q_i^{(j)})/2 < w_i^{(k_0)} - \gamma L_\ell \leq w_i^{(k_0+1)} \leq w_i^{(k_0)} + \gamma L_\ell < (q_i^{(j)} + q_i^{(j+1)})/2$. The same argument is inductively true for all $k \geq k_0$, thus concluding the proof. □

Lemma 4 (**Confinement to the Quantization Interval, Edge Case**). *Assume that A3 holds. Fix an $i \in [d]$. Let $[q_-, q_+] = C_{\varepsilon,i} \cap (-\infty, (q_i^{(1)} + q_i^{(2)})/2)$, where $C_{\varepsilon,i}$ is the projection of C_ε on the i-th coordinate, and $0 < \gamma \cdot \max\{L_\ell, M_c\} < \min\{q_+ - q_-, (q_i^{(1)} + q_i^{(2)})/2 - q_+\}$. If there is a moment k_0 such that $w_i^{(k_0)} \in (-\infty, (q_i^{(1)} + q_i^{(2)})/2)$, then $w_i^{(k)}$ is confined to the interval $(-\infty, (q_i^{(1)} + q_i^{(2)})/2)$ for every $k \geq k_0$. The same conclusion holds for $[q_-, q_+] = C_{\varepsilon,i} \cap ((q_i^{(K_i-1)} + q_i^{(K_i)})/2, +\infty)$ and $0 < \gamma \cdot \max\{L_\ell, M_c\} < q_+ - q_-$ with $w_i^{(k)}$ being confined to the interval $((q_i^{(K_i-1)} + q_i^{(K_i)})/2, +\infty)$ for every $k \geq k_0$.*

Proof. Similar to Lemma 3a. If $w_i^{(k_0)} < q_-$, then $w_i^{(k_0)}$ is pushed to the right. We have $w_i^{(k_0)} \leq$

$w_i^{(k_0+1)} := w_i^{(k_0)} + \gamma \max\{L_\ell, M_c\} < q_+$. If $w_i^{(k_0)} \in [q_-, q_+]$, then $w_i^{(k_0+1)} \leq w_i^{(k_0)} + \gamma L_\ell < (q_i^{(1)} + q_i^{(2)})/2$. □

Lemma 5 (**Convergence to Neighborhood of The Feasible Region**). *Assume that A3 holds. Let γ be such that each coordinate i of $\mathbf{w}^{(k_0)}$ is confined to an interval I_i as in Lemma 3 and 4. Let $\eta = \min_i \inf_{w \notin C_\varepsilon}\{\text{clip}(|\alpha\psi_i(a_i;\varepsilon)|/|\psi_i'(a_i;\varepsilon)|, M_c)\}$. Then, $\mathrm{d}(\mathbf{w}^{(k)}, C_\varepsilon) \leq \gamma L_\ell \sqrt{d}$ for all $k \geq k_0 + \lceil(\mathrm{d}(\mathbf{w}^{(k_0)}, C_\varepsilon) - \gamma L_\ell)/(\eta\gamma)\rceil$.*

Proof. Consider the case where $\mathbf{w}_i^{(k)} \in C_{\varepsilon,i}$. After an update, $\mathrm{d}(\mathbf{w}^{(k+1)}, C_\varepsilon) \leq \sqrt{\sum_i (\gamma_k L_\ell)^2} = \gamma_k L_\ell \sqrt{d}$. Otherwise, the skewing force activates at $k+1$. For each infeasible coordinate i, the skewing correction is at least $\eta\gamma$ by definition (since the skewing force diminishes as $w_i^{(k)}$ gets closer to the feasible region, and the gradient pushes stronger than the skewing force). Hence, $\mathrm{d}_i(w_i^{(k+1)}, C_{\varepsilon,i}) \leq \mathrm{d}_i(w_i^{(k)}, C_{\varepsilon,i}) - \eta\gamma$. Therefore, in at most an extra $\lceil(\mathrm{d}(\mathbf{w}^{(k_0)}, C_\varepsilon) - \gamma L_\ell)/(\eta\gamma)\rceil$ steps, we can guarantee that the distance to the feasible set is at an order of $\mathcal{O}(\gamma)$. □

Theorem 2 (**Finite Time Convergence with Lipschitz Continuity and Small Constant Step-size**). *Let $(\gamma_k)_{k\geq 0}$ be a γ-constant sequence, where $\gamma < 1/L_\ell$ is small enough and k_1 be such that $\mathrm{d}(\mathbf{w}^{(k)}, C_\varepsilon) \leq \gamma L_\ell \sqrt{d} =: \delta$ and $|\mathbf{u}_i^{(k)}| \leq 2\alpha\delta$ for all $k \geq k_1$ and $i \in [d]$, as in Lemma 2 and Lemma 5. Then, under assumption A3, we have*

$$\min_{k=k_1}^{k_1+T} \|\mathbf{v}^{(k)}\|^2 \leq \frac{2(\ell(\mathbf{w}^{(k_1)}) - \ell(\mathbf{w}^{(k_1+T+1)}))}{\gamma(T+1)} + 4\alpha d\gamma L_\ell^2 + 4\alpha^2 d\gamma^3 L_\ell^3 + 8\alpha^2 d\gamma L_\ell^2.$$

with a convergence rate of $\mathcal{O}(1/T)$ and

$$\min_{k=k_1}^{k_1+T} \|\mathbf{v}^{(k)}\|^2 \to \mathcal{O}(\gamma),$$

as T tends to infinity.

Proof. First, apply the descent lemma on the L_ℓ-smooth function ℓ. Then, we discuss the two cases of the update coordinate-wisely. We obtain the following inequality:

$$\begin{aligned}
\ell(\mathbf{w}^{(k+1)}) &\leq \ell(\mathbf{w}^{(k)}) + \gamma(\mathbf{v}^{(k)})^\top \nabla\ell(\mathbf{w}^{(k)}) + \frac{\gamma^2 L_\ell}{2}\|\mathbf{v}^{(k)}\|_2^2 \\
&= \ell(\mathbf{w}^{(k)}) + \gamma \sum_{i \in S_\epsilon(\mathbf{w}^{(k)})} u_i^{(k)} [\nabla\ell(\mathbf{w}^{(k)})]_i - \gamma \sum_{i \notin S_\epsilon(\mathbf{w}^{(k)})} [\nabla\ell(\mathbf{w}^{(k)})]_i^2 \\
&\quad + \frac{\gamma^2 L_\ell}{2} \sum_{i \in S_\epsilon(\mathbf{w}^{(k)})} (u_i^{(k)})^2 + \frac{\gamma^2 L_\ell}{2} \sum_{i \notin S_\epsilon(\mathbf{w}^{(k)})} [\nabla\ell(\mathbf{w}^k)]_i^2.
\end{aligned}$$

Since $\gamma < 1/L_\ell$, we have $1 - \gamma L_\ell/2 > 1/2$, then

$$\ell(\mathbf{w}^{(k+1)}) \leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla\ell(\mathbf{w}^{(k)})]_i^2 + \gamma \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} u_i^{(k)} [\nabla\ell(\mathbf{w}^{(k)})]_i + \frac{\gamma^2 L_\ell}{2} \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} (u_i^{(k)})^2.$$

Since the gradient of ℓ is bounded by L_ℓ and $|u_i^{(k)}| \leq 2\alpha\gamma L_\ell\sqrt{d}$ as a consequence, we have

$$\begin{aligned}\ell(\mathbf{w}^{(k+1)}) &\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2}\sum_{i\notin S_\varepsilon(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2 + \gamma\sum_{i\in S_\epsilon(\mathbf{w}^{(k)})}(2\alpha\gamma L_\ell)L_\ell + \frac{\gamma^2 L_\ell}{2}\sum_{i\in S_\varepsilon(\mathbf{w}^{(k)})}(2\alpha\gamma L_\ell)^2 \\ &\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2}\sum_{i\notin S_\varepsilon(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2 + \gamma(2\alpha\gamma L_\ell)L_\ell d + \frac{\gamma^2 L_\ell}{2}(2\alpha\gamma L_\ell)^2 d \\ &\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2}\sum_{i\notin S_\varepsilon(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2 + 2\alpha d\gamma^2 L_\ell^2 + 2\alpha^2 d\gamma^4 L_\ell^3.\end{aligned}$$

Rearrange the inequality and summing over $T+1$ epochs from k_1 to k_1+T, we obtain

$$\frac{\gamma}{2}\sum_{k=k_1}^{k_1+T}\sum_{i\notin T_\varepsilon(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2 \leq \ell(\mathbf{w}^{(k_1)}) - \ell(\mathbf{w}^{(k_1+T+1)}) + 2(T+1)\alpha d\gamma^2 L_\ell^2 + 2(T+1)\alpha^2 d\gamma^4 L_\ell^3.$$

Completing the entire update direction with the skewing force, we can bound $\|\mathbf{v}^{(k)}\|$ as we have

$$\begin{aligned}\frac{\gamma}{2}\sum_{k=k_1}^{k_1+T}\left(\sum_{i\notin S_\varepsilon(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2 + \sum_{i\in S_\varepsilon(\mathbf{w}^{(k)})}(u_i^{(k)})^2\right) &\leq \ell(\mathbf{w}^{(k_1)}) - \ell(\mathbf{w}^{(k_1+T+1)}) + 2(T+1)\alpha d\gamma^2 L_\ell^2 \\ &\quad + 2(T+1)\alpha^2 d\gamma^4 L_\ell^3 + 4(T+1)\alpha^2 d\gamma^2 L_\ell^2 \\ \min_{k=k_1}^{k_1+T}\left(\sum_{i\notin S_\varepsilon(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2 + \sum_{i\in S_\varepsilon(\mathbf{w}^{(k)})}(u_i^{(k)})^2\right) &\leq \frac{2(\ell(\mathbf{w}^{(k_1)}) - \ell(\mathbf{w}^{(k_1+T+1)}))}{\gamma(T+1)} + 4\alpha d\gamma L_\ell^2 \\ &\quad + 4\alpha^2 d\gamma^3 L_\ell^3 + 8\alpha^2 d\gamma L_\ell^2.\end{aligned}$$

Finally,

$$\min_{k=k_1}^{k_1+T}\|\mathbf{v}^{(k)}\|^2 \leq \frac{2(\ell(\mathbf{w}^{(k_1)}) - \ell(\mathbf{w}^{(k_1+T+1)}))}{\gamma(T+1)} + 4\alpha d\gamma L_\ell^2 + 4\alpha^2 d\gamma^3 L_\ell^3 + 8\alpha^2 d\gamma L_\ell^2.$$

As T tends to infinity, we have

$$\min_{k=k_1}^{k_1+T}\|\mathbf{v}^{(k)}\|^2 \to \mathcal{O}(\gamma),$$

with convergence rate of $\mathcal{O}(1/T)$. □

Theorem 3 (**Finite Time Convergence with Lipschitz Continuity and General Constant Stepsize**). *Let $\gamma < 1/L$. Under assumption A3, we have*

$$\min_{k=0}^{T}\|\mathbf{v}^{(k)}\|^2 \leq \frac{2(\ell(\mathbf{w}^{(0)}) - \ell(\mathbf{w}^{(T+1)}))}{\gamma(T+1)} + 4\alpha d\gamma L_\ell^2 + 2M_c L_\ell d + \gamma L_\ell M_c^2 d + M_c^2 d.$$

with a convergence rate of $\mathcal{O}(1/T)$ and

$$\min_{k=0}^{T}\|\mathbf{v}^{(k)}\|^2 \to \mathcal{O}(\gamma + M_c d(1+L_\ell)),$$

as T tends to infinity.

Proof. Starting from the descent inequality obtained previously, we have

$$\ell(\mathbf{w}^{(k+1)}) \leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \gamma \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} u_i^{(k)} [\nabla \ell(\mathbf{w}^{(k)})]_i + \frac{\gamma^2 L_\ell}{2} \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} (u_i^{(k)})^2.$$

Since the gradient of ℓ is bounded by L_ℓ and $|u_i^{(k)}|$ is bounded by M_c, we have

$$\begin{aligned}
\ell(\mathbf{w}^{(k+1)}) &\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \gamma \sum_{i \in S_\epsilon(\mathbf{w}^{(k)})} M_c L_\ell + \frac{\gamma^2 L_\ell}{2} \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} M_c^2 \\
&\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \gamma M_c L_\ell d + \frac{\gamma^2 L_\ell M_c^2 d}{2}.
\end{aligned}$$

Rearrange the inequality and summing over $T+1$ epochs from 0 to T, we obtain

$$\frac{\gamma}{2} \sum_{k=0}^{T} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 \leq \ell(\mathbf{w}^{(0)}) - \ell(\mathbf{w}^{(T+1)}) + (T+1)\gamma M_c L_\ell d + (T+1)\frac{\gamma^2 L_\ell M_c^2 d}{2}.$$

Completing the entire update direction with the skewing force, we can bound $\|\mathbf{v}^{(k)}\|$ as we have

$$\begin{aligned}
\frac{\gamma}{2} \sum_{k=0}^{T} \left(\sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} (u_i^{(k)})^2 \right) &\leq \ell(\mathbf{w}^{(0)}) - \ell(\mathbf{w}^{(T+1)}) + 2(T+1)\alpha d \gamma^2 L_\ell^2 \\
&\quad + (T+1)\gamma M_c L_\ell d + (T+1)\frac{\gamma^2 L_\ell M_c^2 d}{2} + \frac{\gamma}{2}(T+1) M_c^2 d, \\
\min_{k=0}^{T} \left(\sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} (u_i^{(k)})^2 \right) &\leq \frac{2(\ell(\mathbf{w}^{(0)}) - \ell(\mathbf{w}^{(T+1)}))}{\gamma(T+1)} + 4\alpha d \gamma L_\ell^2 \\
&\quad + 2M_c L_\ell d + \gamma L_\ell M_c^2 d + M_c^2 d.
\end{aligned}$$

Finally,

$$\min_{k=0}^{T} \|\mathbf{v}^{(k)}\|^2 \leq \frac{2(\ell(\mathbf{w}^{(0)}) - \ell(\mathbf{w}^{(T+1)}))}{\gamma(T+1)} + 4\alpha d \gamma L_\ell^2 + 2M_c L_\ell d + \gamma L_\ell M_c^2 d + M_c^2 d.$$

As T tends to infinity, we have

$$\min_{k=0}^{T} \|\mathbf{v}^{(k)}\|^2 \to \mathcal{O}(\gamma + M_c d(1 + L_\ell)),$$

with convergence rate of $\mathcal{O}(1/T)$. □

Why is an extra upper bound on γ necessary? Consider the case where $\mathcal{Q} = \{\pm 1\}$ and ℓ is decreasing away from $(-1, 1)$. Let $[q_-, q_+] = \{w : \psi(w; \varepsilon) \geq 0\}$. Let $w < q_-$. There exists $\gamma = 2w/(\alpha(\varepsilon + w + 1)))$ such that w enters a cycle.

To see this, first note that $\psi(w; \varepsilon) = \varepsilon - (-1 - w) < 0$. We compute that

$$u = \frac{-\alpha\psi(w)}{\psi'(w)} = -\alpha(\varepsilon + w + 1).$$

Then, $w' = w + \gamma u = w - \gamma \cdot \alpha(\varepsilon + w + 1)) = w - 2w = -w$. By symmetry, we have $w'' = -w' = w$.

Why do we have an extra M_c term? We cannot be sure whether $u_i^{(k)}$ converges to $C_{\varepsilon,i}$. Large gradient could disrupt the distance to feasible set for general γ.

Significance. We have derived a finite-time non-asymptotic convergence bound for constant stepsize full-gradient ASkewSGD on the update $\mathbf{v}^{(k)}$ (which is an indicator for stationarity), with removal of the strong differentiability assumption A4.

Implications. For $\gamma < 1/L_\ell$, we can bound the update due to bounded gradient and skewing forces. However, if we further restrict on γ, then the distance will become an upper bound for the skewing update and force the parameter to converge to the (neighborhood of the) quantization values.

5.1.2 Asymptotic Convergence with Lipschitz Smoothness and Robbins-Monro Stepsizes

Theorem 4 (**Asymptotic Convergence with Lipschitz Smoothness and Robbins-Monro Stepsizes**). *Let $\gamma_k < 1/L$ for all $k \geq k_0$. Under assumptions A2 and A5, we have*

$$\frac{1}{\sum_{k=k_0}^{T} \gamma_k} \sum_{k=k_0}^{T} \frac{\gamma_k}{2} \|\mathbf{v}^{(k)}\|^2 \longrightarrow \mathcal{O}(M_c^2 d), \quad \text{as } T \text{ tends to infinity.}$$

Proof. Note that when $i \in S_\varepsilon^+(\mathbf{w})$, $|u_i^{(k)}| \geq |[\nabla \ell(\mathbf{w}^{(k)})]_i|$.

Starting from the descent inequality obtained previously, we have

$$\begin{aligned}
\ell(\mathbf{w}^{(k+1)}) &\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma_k}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \gamma_k \sum_{i \in S_\varepsilon^+(\mathbf{w}^{(k)})} u_i^{(k)} [\nabla \ell(\mathbf{w}^{(k)})]_i \\
&\quad + \gamma_k \sum_{i \in S_\varepsilon^-(\mathbf{w}^{(k)})} u_i^{(k)} [\nabla \ell(\mathbf{w}^{(k)})]_i + \frac{\gamma_k^2 M_\ell}{2} \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} (u_i^{(k)})^2 \\
&\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma_k}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \gamma_k \sum_{i \in S_\epsilon^+(\mathbf{w}^{(k)})} |u_i^{(k)}|^2 \\
&\quad - \gamma_k \sum_{i \in S_\epsilon^-(\mathbf{w}^{(k)})} |u_i^{(k)}| |[\nabla \ell(\mathbf{w}^{(k)})]_i| + \frac{\gamma_k^2 M_\ell}{2} \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} M_c^2 \\
&\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma_k}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \gamma_k \sum_{i \in S_\epsilon^+(\mathbf{w}^{(k)})} |u_i^{(k)}|^2 + \frac{\gamma_k^2 M_\ell M_c^2 d}{2}.
\end{aligned}$$

Completing the entire update direction with the skewing force, we bound $\|\mathbf{v}^{(k)}\|^2$ by

$$\begin{aligned}
\frac{\gamma_k}{2} \|\mathbf{v}^{(k)}\|^2 &= \frac{\gamma_k}{2} \sum_{i \notin S_\varepsilon(\mathbf{w}^{(k)})} [\nabla \ell(\mathbf{w}^{(k)})]_i^2 + \frac{\gamma_k}{2} \sum_{i \in S_\varepsilon(\mathbf{w}^{(k)})} |u_i^{(k)}|^2 \\
&\leq \ell(\mathbf{w}^{(k)}) - \ell(\mathbf{w}^{(k+1)}) + \frac{3\gamma_k}{2} \sum_{i \in S_\epsilon(\mathbf{w}^{(k)})} |u_i^{(k)}|^2 + \frac{\gamma_k^2 M_\ell M_c^2 d}{2} \\
&\leq \ell(\mathbf{w}^{(k)}) - \ell(\mathbf{w}^{(k+1)}) + \frac{3\gamma_k d M_c^2}{2} + \frac{\gamma_k^2 M_\ell M_c^2 d}{2}.
\end{aligned}$$

Summing from epoch k_0 to T, we have

$$\begin{aligned}
\sum_{k=k_0}^{T} \frac{\gamma_k}{2} \|\mathbf{v}^{(k)}\|^2 &\leq \ell(\mathbf{w}^{(k_0)}) - \ell(\mathbf{w}^{(K+1)}) + \frac{3dM_c^2}{2} \sum_{k=k_0}^{T} \gamma_k + \frac{M_\ell M_c^2 d}{2} \sum_{k=k_0}^{T} \gamma_k^2 \\
\frac{1}{\sum_{k=k_0}^{T} \gamma_k} \sum_{k=k_0}^{T} \frac{\gamma_k}{2} \|\mathbf{v}^{(k)}\|^2 &\leq \frac{\ell(\mathbf{w}^{(k_0)}) - \ell(\mathbf{w}^{(T+1)})}{\sum_{k=k_0}^{T} \gamma_k} + \frac{3dM_c^2}{2} + \frac{M_\ell M_c^2 d}{2} \frac{\sum_{k=k_0}^{T} \gamma_k^2}{\sum_{k=k_0}^{T} \gamma_k}.
\end{aligned}$$

As T tends to infinity, we have

$$\frac{1}{\sum_{k=k_0}^{T} \gamma_k} \sum_{k=k_0}^{T} \frac{\gamma_k}{2} \|\mathbf{v}^{(k)}\|^2 \longrightarrow \mathcal{O}(M_c^2 d).$$

□

5.2 Convergence Analysis under Stochasticity

Let $\pi(\cdot)$ be a probability density function defined on the probability space $\mathtt{Z}$ and ξ be a random parameter, such that

$$\ell(\mathbf{x}) = \int_{\mathtt{Z}} \ell(\mathbf{x};\xi)\pi(\xi)\mathrm{d}\xi.$$

Assumption 6. *The stochastic gradient is unbiased, i.e.*

$$\mathbb{E}_{\xi\sim\pi}[\widehat{\nabla}\ell(\mathbf{x};\xi)] = \nabla\ell(\mathbf{x}), \forall \mathbf{x}\in\mathbb{R}^n.$$

Assumption 7. *The stochastic gradient has a bounded variance, i.e.*

$$\mathbb{E}_{\xi\sim\pi}\left[\left\|\widehat{\nabla}\ell(\mathbf{x};\xi) - \nabla\ell(\mathbf{x})\right\|_2^2\right] \le \sigma^2, \forall \mathbf{x}\in\mathbb{R}^n.$$

Recall the (stochastic version of) update

$$[\hat{\mathbf{s}}_{\varepsilon,\alpha}(\hat{\mathbf{g}},\mathbf{w})]_i := \begin{cases} -\hat{g}_i & \text{if } \psi_i(w_i;\varepsilon) > 0 \text{ or} \\ & -\hat{g}_i\cdot\psi_i'(w_i;\varepsilon) \ge -\alpha\psi_i(w_i;\varepsilon) \ge 0, \\ \mathrm{clip}(-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon), M_c) & \text{otherwise.} \end{cases}$$

We simplify the "otherwise" direction with the notation $\mathbf{u}$ and refer this to the "skewing force."

Remark. For the stochastic case, we do not have full information for the true update direction and thus we need to discuss in total of four possible matches between the perturbed direction and the true update direction given by the true gradient. Furthermore, the magnitude of the skewing force does not depend on the (possibly perturbed) gradient.

Definition 5 (**Categorization of the Update Direction**). *Consider the update in each iteration k, $\hat{\mathbf{v}}^{(k)}$, we categorize the update direction $\hat{\mathbf{v}}^{(k)}$ into four types (with the indices collected by $\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})$, $\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})$, $\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})$, $\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})$ and $\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)}) = [d]$). Formally, for the update on i-th coordinate*

$$\begin{aligned}
&i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)}) \iff \hat{v}_i^{(k)} = -[\widehat{\nabla}\ell(\mathbf{w}^{(k)})]_i \text{ and } v_i^{(k)} = -[\nabla\ell(\mathbf{w}^{(k)})]_i.\\
&i\in\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)}) \iff \hat{v}_i^{(k)} = -[\widehat{\nabla}\ell(\mathbf{w}^{(k)})]_i \text{ and } \mathbf{v}_i^{(k)} = u_i^{(k)}.\\
&i\in\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)}) \iff \hat{v}_i^{(k)} = \hat{u}_i^{(k)} = v_i^{(k)} = u_i^k.\\
&i\in\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)}) \iff \hat{v}_i^{(k)} = \hat{u}_i^{(k)} \text{ and } v_i^{(k)} = -[\nabla\ell(\mathbf{w}^{(k)})]_i.
\end{aligned}$$

Remark. Our goal is to bound the true gradient. We know that the type 1 update is standard. The type 2 and 3 updates are captured by the fact that u can be infinitely small as δ progresses to zero. The type 4 update is tricky - as it represents a trade-off between the quantization value and decrease of function value. We alleviate this situation by hinging on the fact that the stationary point is reached after projecting the $\mathbf{w}$ to the quantization set on these coordinates (while the distance to this quantization value is merely δ). We allow (only) gradients on these coordinates to be non-zero because our problem is a constrained optimization problem.

5.2.1 Asymptotic Convergence with Lipschitz Continuity and Robbins-Monro Stepsizes

Idea. We first construct a bound for $[\nabla \ell(\mathbf{w}_\tau)]_i$ summed over $i \in \hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)}) \cup \hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})$ (note that $\mathbf{w}_\tau$ is a randomized variable). We then use Lemma 2 again for the gradient at coordinates in $\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})$. Coordinates in $\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})$ is justified by the smoothness of the gradient and the algorithm's special dynamics.

Theorem 5 (Asymptotic Convergence with Lipschitz Continuity and Robbins-Monro Stepsizes). *Assuming that A3, A5, A6, A7, $0 < \varepsilon \leq \inf_{1\leq i\leq d} \inf_{1\leq j<K_i} |q_i^{(j)} - q_i^{(j+1)}|^2/4$ holds, where $\{q_i^{(j)}\}$ are the quantization levels, then (i) there is a k_0 with $\forall k \geq k_0, \gamma_{k_0} \leq 1/L$, $d(\mathbf{w}^{(k)}, C_\varepsilon) \leq \delta, \|\boldsymbol{u}^{(k)}\| \leq C\delta$; (ii) $\lim_{K\to\infty} d(w^{(\tau_K)}, \mathcal{Z}_\varepsilon) \to 0$ almost surely, where τ_K is a randomized index at iteration K taking value from k_0 to K, generated with the probability distribution proportional to the stepsizes $(\gamma_k)_{k_0\leq k\leq K}$.*

Proof. The first statement is justified by assumptions A3, A5 and Lemma 1, 2.

We focus on the second statement. Notice that we can break down the gradient norm coordinate-wisely. The first part of the second statement follows from the standard SGD analysis. First, apply the descent lemma on the L_ℓ-smooth function ℓ with care. We have

$$
\begin{aligned}
\mathbb{E}_{\mathbf{w}^{(k)}}[\ell(\mathbf{w}^{(k+1)})] &\leq \ell(\mathbf{w}^{(k)}) + \gamma_k \mathbb{E}_{\mathbf{w}^{(k)}}[(\hat{\mathbf{v}}^{(k)})^\top]\nabla\ell(\mathbf{w}^{(k)}) + \frac{\gamma_k^2 L_\ell}{2}\mathbb{E}_{\mathbf{w}^{(k)}}[\|\hat{\mathbf{v}}^{(k)}\|_2^2|\mathbf{w}^{(k)}] \\
&= \ell(\mathbf{w}^{(k)}) - \gamma_k \mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})} [\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] + \frac{\gamma_k^2 L_\ell}{2}\mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})} [\widehat{\nabla\ell}(\mathbf{w}^k)]_i^2\right] \\
&\quad + \gamma_k \mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})} u_i^{(k)}[\nabla\ell(\mathbf{w}^{(k)})]_i\right] + \frac{\gamma_k^2 L_\ell}{2}\mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})} (u_i^{(k)})^2\right] \\
&\leq \ell(\mathbf{w}^{(k)}) - \gamma_k \mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})} [\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] \\
&\quad + \frac{\gamma_k^2 L_\ell}{2}\mathbb{E}_{\mathbf{w}^{(k)}}\left[\left(\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})} [\nabla\ell(\mathbf{w}^{(k)})]_i^2\right) + \sigma^2\right] \\
&\quad + \gamma_k \mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})} 2\alpha\delta L_\ell\right] + \frac{\gamma_k^2 L_\ell}{2}\mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})} 4\alpha^2\delta^2\right]
\end{aligned}
$$

$$\leq \ell(\mathbf{w}^{(k)}) - \frac{\gamma_k}{2}\mathbb{E}_{\mathbf{w}^{(k)}}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] + \frac{L_\ell\gamma_k^2\sigma^2}{2} + 2dL_\ell\gamma_k\alpha\delta + 2dL_\ell\gamma_k^2\alpha^2\delta^2,$$

where we have assumed $1-\gamma_{k_0}L_\ell/2 > 1/2$ in the last step.

Now, we can take full expectation with constant $D \geq 2dL_\ell\alpha + 2d\alpha^2\delta \geq 2dL_\ell(\alpha+\gamma_k\alpha^2\delta)$ and derive

$$\begin{aligned}\frac{\gamma_k}{2}\mathbb{E}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] &\leq \mathbb{E}[\ell(\mathbf{w}^{(k)})] - \mathbb{E}[\ell(\mathbf{w}^{(k+1)})] + \frac{L_\ell\gamma_k^2\sigma^2}{2} + 2dL_\ell\gamma_k\alpha\delta + 2dL_\ell\gamma_k^2\alpha^2\delta^2 \\ &\leq \mathbb{E}[\ell(\mathbf{w}^{(k)})] - \mathbb{E}[\ell(\mathbf{w}^{(k+1)})] + \frac{L_\ell\gamma_k^2\sigma^2}{2} + D\gamma_k\delta.\end{aligned}$$

Summing from epoch k_0 to K, we have

$$\sum_{k=k_0}^{K}\frac{\gamma_k}{2}\mathbb{E}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] \leq \left(\mathbb{E}[\ell(\mathbf{w}^{(k_0)})] - \mathbb{E}[\ell(\mathbf{w}^{(K+1)})]\right) + \frac{L_\ell\sigma^2}{2}\sum_{k=k_0}^{K}\gamma_k^2 + D\delta\sum_{k=k_0}^{K}\gamma_k.$$

Let $H_K = \sum_{k=k_0}^{K}\gamma_k$ and hence $\tau_K = k$ with probability γ_k/H_K, we can take the expectation of the randomized squared-gradient:

$$\mathbb{E}\left[[\nabla\ell(\mathbf{w}^{(\tau_K)})]_i^2\right] = \sum_{k=k_0}^{K}\mathbb{P}(\tau_K=k)\mathbb{E}\left[[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] = \sum_{k=k_0}^{K}\frac{\gamma_k\mathbb{E}\left[[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right]}{H_K} = \frac{2}{H_K}\sum_{k=k_0}^{K}\frac{\gamma_k}{2}\mathbb{E}\left[[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right].$$

$$\begin{aligned}\mathbb{E}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(\tau_K)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(\tau_K)})}[\nabla\ell(\mathbf{w}^{(\tau_K)})]_i^2\right] &= \frac{2}{H_K}\sum_{k=k_0}^{K}\frac{\gamma_k}{2}\mathbb{E}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(k)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(k)})}[\nabla\ell(\mathbf{w}^{(k)})]_i^2\right] \\ &\leq \frac{2}{H_K}\left(\mathbb{E}[\ell(\mathbf{w}^{(k_0)})] - \mathbb{E}[\ell(\mathbf{w}^{(K+1)})]\right) + \frac{L_\ell\sigma^2}{H_K}\sum_{k=k_0}^{K}\gamma_k^2 + \frac{2D\delta}{H_K}\sum_{k=k_0}^{K}\gamma_k \\ &\overset{K\to\infty}{\longrightarrow} 2D\delta,\end{aligned}$$

since assumption 5 ensures that $\sum_{k=k_0}^{\infty}\gamma_k = \infty, \sum_{k=k_0}^{\infty}\gamma_k^2 < \infty$.

In the second part of the second statement, we consider the case where $i \in \hat{S}_{3,\varepsilon}(\mathbf{w}^{(k)})$. Note that the true update direction is $v_i^{(k)} = u_i^{(k)} \leq C\delta$ given that $k \geq k_0$.

In the third part of the second statement, we consider the case where $i \in \hat{S}_{4,\varepsilon}(\mathbf{w}^{(k)})$. For this case to happen, we must first have $\psi_i(w_i^{(k)};\varepsilon) \leq 0$. Let $c_- < c_+$ be such that $\psi_i(c_-;\varepsilon) = \psi_i(c_+;\varepsilon) = 0$. By Lemma 1, for all k large enough, there are three possible cases: (a) $w_i^{(k)}$ stays at the left of $[c_-, c_+]$, $w_i^{(k)}$ stays at the right of $[c_-, c_+]$, or (b) $w_i^{(k)}$ visits $[c_-, c_+]$ infinitely often. If $w_i^{\infty} \in (c_-, c_+)$, then there exists a subsequence $\{w_i^{(k_j)}\}_{j\geq 1} \subseteq [c_-, c_+]$ such that $[\nabla\ell(\mathbf{w}^{(k_j)})]_i \to 0$. As a result, either $w_i^{\infty} \to c_-$ or $w_i^{\infty} \to c_+$, as $\mathrm{d}(\mathbf{w}^{(k)}, C_\varepsilon) \to 0$.

We verify the stationarity condition for w_i^{∞}. It suffices to consider the case for $w_i^{\infty} \to c_-$ without loss of generality. We replace $w_i^{(k)}$ by c_- and denote it as $\bar{\mathbf{w}}^{(k)}$. Violation of the stationary condition

only happens when $-[\nabla\ell(\bar{\mathbf{w}}^{(k)})]_i \cdot \psi_i'(w_i;\varepsilon) > 0$. However, the algorithm will take an update with $-[\nabla\ell(\bar{\mathbf{w}})]_i \neq 0$, which violates w_i's convergence to c_-.

Denote $\bar{\bar{\mathbf{w}}}$ as a result of projecting $\mathbf{w}$ to C_ε for coordinates $i \in S_{4,\varepsilon}(\mathbf{w})$. It is clear that the smoothness condition gives $\|\nabla\ell(\mathbf{w}) - \nabla\ell(\bar{\bar{\mathbf{w}}})\| \leq L_\ell\|\bar{\bar{\mathbf{w}}} - \mathbf{w}\| \leq L_\ell\delta$. This implies $\|\nabla\ell(\mathbf{w})\| \leq \|\nabla\ell(\bar{\bar{\mathbf{w}}})\| + L_\ell\delta$ and hence $\|\nabla\ell(\mathbf{w})\|^2 \leq \|\nabla\ell(\bar{\bar{\mathbf{w}}})\|^2 + 2L_\ell\delta\|\nabla\ell(\bar{\bar{\mathbf{w}}})\| + L_\ell^2\delta^2$.

$$
\begin{aligned}
\mathbb{E}\left[\|\nabla\ell(\mathbf{w}^{(\tau_K)})]\|^2\right] &= \mathbb{E}\left[\sum_{i\in[n]}[\nabla\ell(\mathbf{w}^{(\tau_K)})]_i^2\right] \\
&= \mathbb{E}\left[\sum_{i\in\hat{S}_{1,\varepsilon}(\mathbf{w}^{(\tau_K)})\cup\hat{S}_{2,\varepsilon}(\mathbf{w}^{(\tau_K)})}[\nabla\ell(\mathbf{w}^{(\tau_K)})]_i^2\right] \\
&\quad + \mathbb{E}\left[\sum_{i\in\hat{S}_{3,\varepsilon}(\mathbf{w}^{(\tau_K)})}[\nabla\ell(\mathbf{w}^{(\tau_K)})]_i^2\right] + \mathbb{E}\left[\sum_{i\in\hat{S}_{4,\varepsilon}(\mathbf{w}^{(\tau_K)})}[\nabla\ell(\mathbf{w}^{(\tau_K)})]_i^2\right] \\
&\leq 2D\delta + C\delta + L_\ell^2\delta^2 + 2L_\ell\delta\mathbb{E}\left[\sum_{i\in\hat{S}_{4,\varepsilon}(\mathbf{w}^{(\tau_K)})}\left|\left[\nabla\ell(\bar{\bar{\mathbf{w}}}^{(\tau_K)})\right]_i\right|\right] \\
&\quad + \mathbb{E}\left[\sum_{i\in\hat{S}_{4,\varepsilon}(\mathbf{w}^{(\tau_K)})}\left[\nabla\ell(\bar{\bar{\mathbf{w}}}^{(\tau_K)})\right]_i^2\right] \quad \text{(as } K \text{ approaches infinity)} \\
&\xrightarrow{K\to\infty\Rightarrow\delta\to 0} \mathbb{E}\left[\sum_{i\in\hat{S}_{4,\varepsilon}(\mathbf{w}^{(\tau_K)})}\left[\nabla\ell(\bar{\bar{\mathbf{w}}}^{(\tau_K)})\right]_i^2\right].
\end{aligned}
$$

From the above we know that we can be arbitrarily close to a stationary point in $\mathcal{Z}_\varepsilon$. □

5.2.2 Asymptotic Convergence with Lipschitz Smoothness and Robbins-Monro Stepsizes

Theorem 6 (**Asymptotic Convergence with Lipschitz Smoothness and Robbins-Monro Stepsizes**). *Assuming that A1, A2, A5, A6, A7,* $0 < \varepsilon \leq \inf_{1\leq i\leq d}\inf_{1\leq j<K_i}|q_i^{(j)} - q_i^{(j+1)}|^2/4$ *holds, where* $\{q_i^{(j)}\}$ *are the quantization levels, then* $\lim_{k\to\infty} d(w^{(k)}, \mathcal{Z}_\varepsilon) \to 0$ *almost surely.*

Significance. This has largely improved the results of ASkewSGD's convergence with incorporation of the mainstream stochastic analysis method. We can even generalize the constraint function ψ. This has left us with a class of candidate functions for selection (given that ψ is Lipschitz-smooth).

Proof. The ASkewSGD update is

$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} + \gamma_k\hat{\mathbf{v}}^{(k)},$$

where $\hat{\mathbf{v}}^{(k)} = \mathbf{s}_{\varepsilon,\alpha}(\widehat{\nabla\ell}(\mathbf{w}^{(k)}), \mathbf{w}^{(k)})$, with the mean direction is $\mathbf{h}(\mathbf{w}) = \mathbb{E}[\mathbf{s}_{\varepsilon,\alpha}(\widehat{\nabla\ell}(\mathbf{w}), \mathbf{w})|\mathbf{w}]$.

Consider the Stochastic Approximation (SA) Scheme, where we express the update as:

$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} + \gamma_k(\mathbf{h}(\mathbf{w}^{(k)}) + \boldsymbol{\xi}^{(k)}),$$

where $\boldsymbol{\xi}^{(k)} = \hat{\mathbf{v}}^{(k)} - \mathbf{h}(\mathbf{w}^{(k)})$ is a martingale difference noise.

Define the Lyapunov function

$$L(\mathbf{w}) = \ell(\mathbf{w}) + \frac{\alpha}{2}\sum_{i=1}^{d}[\min(0, \psi_i(\mathbf{w}))]^2.$$

Since $\ell \in C^1$, and the penalty term $[\min(0, \psi_i(w_i;\varepsilon))]^2$ is C^1 due to the piecewise C^1 structure of ψ_i and the derivative continuity at boundaries, we know that L is C^1. Note that ℓ is also radially bounded (if ℓ is also coercive), with the gradient

$$\nabla L(\mathbf{w}) = \nabla\ell(\mathbf{w}) + \mathbf{r}(\mathbf{w}), \text{where } r_i(\mathbf{w}) = \begin{cases} 0, & \psi_i(w_i;\varepsilon) > 0, \\ \alpha\psi_i(w_i;\varepsilon)\psi_i'(w_i;\varepsilon), & \psi_i(w_i;\varepsilon) \le 0. \end{cases}$$

Moreover, $\mathbf{h}(\mathbf{w}) = \mathbf{0}$ if and only if $\mathbf{w} \in \mathcal{Z}_\varepsilon$.

We first prove the "if" direction. When $\mathbf{w} \in \mathcal{Z}_\varepsilon$ (which is a subset of C_ε), $\psi_i(w_i;\varepsilon) \ge 0$. If $\psi_i(w_i;\varepsilon) > 0$, then $[\nabla\ell(\mathbf{w})]_i = 0$. Otherwise, $\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon) = 0$ and we just need to consider the case $-[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \ge 0$, this is impossible as $\text{sign}([\nabla\ell(\mathbf{w})]_i) = \text{sign}(\psi_i'(w_i;\varepsilon))$.

We then investigate the "only-if" direction. If $\psi_i(w_i;\varepsilon) > 0$, we can make use of the fact that $[\nabla\ell(\mathbf{w})]_i = [h(\mathbf{w})]_i = 0$. If $\psi_i(w_i;\varepsilon) < 0$ and $-[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \le \alpha\psi_i(w_i;\varepsilon)$, then $h(\mathbf{w}) \ne 0$, which is a contradiction. This implies that $0 = -[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \ge \alpha\psi_i(w_i;\varepsilon) > 0$, which is also impossible. If $\psi_i(w_i;\varepsilon) = 0$ and $-[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \le \alpha\psi_i(w_i;\varepsilon) = 0$, then indeed we have $\text{sign}([\nabla\ell(\mathbf{w})]_i) = \text{sign}(\psi_i'(w_i;\varepsilon))$. Now, we are left with the conditions that takes $[h(\mathbf{w})]_i = -[\nabla\ell(\mathbf{w})]_i = 0$ and $\psi_i(w_i;\varepsilon) \ge 0$, which is always satisfactory.

We need to prove the following claim.

Claim. Almost surely, $\langle\nabla L(\mathbf{w}), \mathbf{h}(\mathbf{w})\rangle \le -c\|\mathbf{h}(\mathbf{w})\|^2$ for some $c > 0$ when $\mathbf{h}(\mathbf{w}) \ne 0$.

Case 1. If w_i is strictly feasible, i.e. $\psi_i(\mathbf{w}) > 0$, then $[\nabla L(\mathbf{w})]_i = [\nabla\ell(\mathbf{w})]_i$ and $[h(\mathbf{w})]_i = -[\nabla\ell(\mathbf{w})]_i$. Hence,

$$[\nabla L(\mathbf{w})]_i[\mathbf{h}(\mathbf{w})]_i = -[\nabla\ell(\mathbf{w})]_i^2 = -[\mathbf{h}(\mathbf{w})]_i^2.$$

Case 2. If w_i is not strictly feasible, i.e. $\psi_i(\mathbf{w}) \le 0$, then $[\nabla L(\mathbf{w})]_i = [\nabla\ell(\mathbf{w})]_i + \alpha\psi_i(w_i;\varepsilon)\psi_i'(w_i;\varepsilon)$.

(a) The gradient condition holds, i.e. $-[\nabla\ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) \ge -\alpha\psi_i(w_i;\varepsilon) \ge 0$. Then, $[\mathbf{h}(\mathbf{w})]_i = -[\nabla\ell(\mathbf{w})]_i$,

$$\begin{aligned}[\nabla L(\mathbf{w})]_i \cdot [\mathbf{h}(\mathbf{w})]_i &= -[\nabla\ell(\mathbf{w})]_i^2 - \alpha\psi_i(w_i;\varepsilon)\psi_i'(w_i;\varepsilon) \cdot [\nabla\ell(\mathbf{w})]_i \\ &\le -[\nabla\ell(\mathbf{w})]_i^2 - \alpha^2\psi_i^2(w_i;\varepsilon) \le -[\mathbf{h}(\mathbf{w})]_i^2,\end{aligned}$$

where

$$-\alpha\psi_i(w_i;\varepsilon)\psi_i'(w_i;\varepsilon) \cdot [\nabla\ell(\mathbf{w})]_i \le -\alpha\psi_i(w_i;\varepsilon) \cdot (\alpha\psi_i(w_i;\varepsilon)) = -\alpha^2\psi_i(w_i;\varepsilon)^2.$$

(b) The gradient condition fails, i.e. $-[\nabla \ell(\mathbf{w})]_i \cdot \psi_i'(w_i;\varepsilon) < -\alpha\psi_i(w_i;\varepsilon)$. Then, $[\mathbf{h}(\mathbf{w})]_i = \text{clip}(-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon), M_c)$.
If $|[\mathbf{h}(\mathbf{w})]_i| \le M_c$, then

$$\begin{aligned}
[\nabla L(\mathbf{w})]_i \cdot [\mathbf{h}(\mathbf{w})]_i &= ([\nabla \ell(\mathbf{w})]_i + \alpha\psi_i(w_i;\varepsilon)\psi_i'(w_i;\varepsilon)) \cdot (-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon)) \\
&= -\alpha \frac{\psi_i(w_i;\varepsilon) \cdot [\nabla \ell(\mathbf{w})]_i}{\psi_i'(w_i;\varepsilon)} - \alpha^2 \psi_i(w_i;\varepsilon) \\
&< -\alpha^2 \psi_i^2(w_i;\varepsilon) \left(\frac{1}{(\psi_i'(w_i;\varepsilon))^2} + 1 \right) \\
&\le -\left(\frac{\alpha\psi_i(w_i;\varepsilon)}{\psi_i'(w_i;\varepsilon)} \right)^2 = -[\mathbf{h}(\mathbf{w})]_i^2.
\end{aligned}$$

Otherwise, assume that clipping is activated and $[\mathbf{h}(\mathbf{w})]_i = \varsigma M_c$, where $\varsigma = \text{sign}(-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon))$. Since the case where $\psi_i'(w_i;\varepsilon) = 0$ is measure-zero, then almost surely,

$$\begin{aligned}
[\nabla L(\mathbf{w})]_i \cdot [\mathbf{h}(\mathbf{w})]_i &= [\nabla L(\mathbf{w})]_i \cdot (-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon)) \cdot \frac{\varsigma M_c}{-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon)} \\
&\le -(\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon))^2 \cdot \frac{\varsigma M_c}{-\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon)} \\
&= (\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon)) \cdot \varsigma M_c \\
&\le -\varsigma^2 M_c^2 \quad (\text{since } |\alpha\psi_i(w_i;\varepsilon)/\psi_i'(w_i;\varepsilon)| \ge M_c) \\
&= -M_c^2.
\end{aligned}$$

Thus, $L(\mathbf{w}(t))$ is almost surely non-increasing along trajectories. Moreover, L is radially unbounded, then $L(\mathbf{w}(t))$ converges to some $L^* \ge \inf L$ as $t \to \infty$.

Define the equilibrium set:

$$E = \left\{ \mathbf{w} : \frac{\mathrm{d}}{\mathrm{d}t} L(\mathbf{w}) = 0 \right\} \overset{\text{a.s.}}{=} \{\mathbf{w} : h(\mathbf{w}) = 0\} = \mathcal{Z}_\varepsilon$$

By LaSalle's Invariance Principle, the largest invariant set in E is $\mathcal{Z}_\varepsilon$ (since $h(\mathbf{w}) = 0$ implies $\mathbf{w}(t)$ is constant). Hence,

$$\lim_{t\to\infty} d(\mathbf{w}(t), Z_\varepsilon) = 0 \quad \text{a.s..}$$

For any $\epsilon > 0$, choose $\xi > 0$ such that

$$\mathbf{w}(0) \in B_\xi(\mathcal{Z}_\varepsilon) \implies L(\mathbf{w}(0)) < \min_{\mathbf{w}\in\partial B_\epsilon(\mathcal{Z}_\varepsilon)} L(\mathbf{w}).$$

Since L decreases along trajectories, $\mathbf{w}(t) \in B_\epsilon(\mathcal{Z}_\varepsilon)$ for all $t \ge 0$, and the set Z_ε is almost surely globally asymptotically stable.

Via Kushner-Clark theorem, we verify that $\{\mathbf{w}^{(k)}\}$ bounded a.s., $\mathbb{E}[\boldsymbol{\xi}^{(k)} \mid \mathcal{F}_k] = 0$, $\mathbb{E}[\|\boldsymbol{\xi}^{(k)}\|^2 \mid \mathcal{F}_k] \le \sigma^2$, $\sum \gamma_k = \infty$, $\sum \gamma_k^2 < \infty$, and Z_ε is almost surely globally asymptotically stable for ODE, and

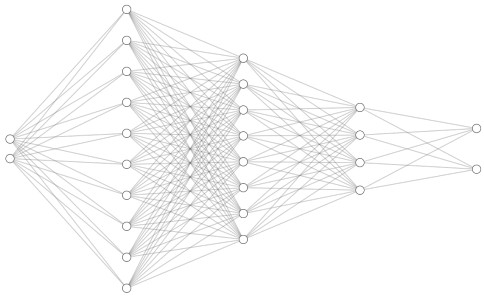
conclude that

$$\lim_{k\to\infty} d(\mathbf{w}^{(k)}, \mathcal{Z}_\varepsilon) = 0 \quad \text{a.s..}$$

□

6 Experimental Results

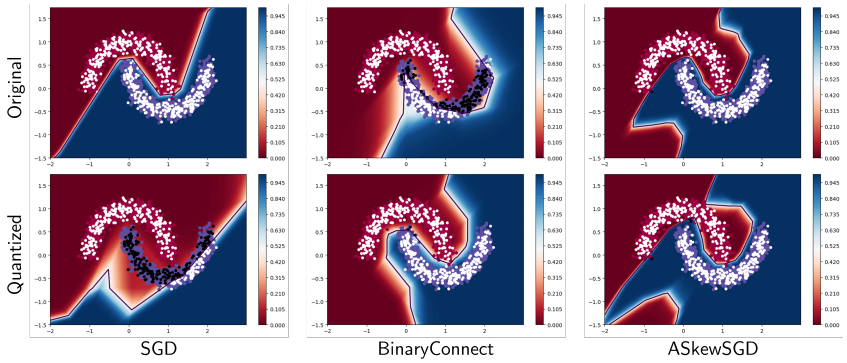
Two Moons Classification. We train a five-layer MLP (shown as below) with a 2-dimensional input, and ReLU as activation function on the non-convex two moon classification task for $T = 60$ epochs. Our dataset consists of $n = 2000$ training samples (in batches of 200 per iteration, points are colored in blue and red) and $m = 500$ test samples (color in black and white), which is generated by `scikit-learn` library's `make_moons` data generator. We have also added a random Gaussian noise with the variance of $\xi = 0.1$ to offset the point from its original position. We apply the logistic loss function in this experiment. The learning rate lr and the hyperparameter for `ASkewSGD` α are set to 0.2 and 0.2, respectively.



We compare SGD, BinaryConnect, and ASkewSGD to obtain the following results for the two moons classification problem. The first row shows the performance of different optimizers on the original net before quantization, and the second row depicts the quantized model with the quantization set $\mathcal{Q} = \{-1, +1\}$.

Table 1: Logistic loss after 60 epochs.

Method	Loss	Quantized Loss
Full Precision [W32/A32]	0.00002	0.59982
Deterministic BinaryConnect [W1/A32]	0.57799	0.03260
`ASkewSGD` [W1/A32]	**0.00367**	**0.00513**



The original net has its contour completely perturbed by the weight quantization process. BinaryConnect is aware of the quantization scheme, thus generalizing well on the test set, but the original model is completely off from the groundtruth. ASkewSGD, on the other hand, has only a slight skew away from the sensible region, and the quantized model is still able to generalize well on the test set, which demonstrates the strength of this optimization method.

Computer Vision Task. We train `ResNet-18` without bias and tunable parameters on the batch normalization layer on the CIFAR-10 dataset for $T = 150$ epochs. The dataset consists of $n = 50000$ training samples and $m = 10000$ test samples. We apply the cross-entropy loss function in this experiment. We set the learning rate lr and the hyperparameter for ASkewSGD α to 0.06 and 0.2, respectively. The quantization set $\mathcal{Q}$ is set to $\{-1, +1\}$.

Table 2: Performance of ResNet-18 on CIFAR10 after 60 epochs.

Method	Final Accuracy	Test Loss
Full Precision [W32/A32]	88.91%	0.1
ASkewSGD [W1/A32]	86.91% (quantized)	0.8669 (quantized)

It is worthwhile to note that the accuracy of the quantized model skyrocketed to 50% only after running for 40 epochs, meaning that quantized models are sensitive to the accumulation of positional errors and the uncertain landscape of the loss function.

7 Future Works

We have not yet finished the experiments on stochastic BinaryConnect and ProxQuant. For full comparison, we will also need to conduct experiments on 2-bit and 4-bit quantization. We also need to thoroughly verify all the proofs. For the stochastic case, we may leverage Theorem 2 for non-asymptotic convergence.

8 Conclusion

Previous optimization algorithms show empirical success for quantization-aware training of deep neural networks. While there are advancements in different quantization schemes, we should keep on investigating the theoretical guarantees provided by these rules in order to keep the error at a justifiable level. As a clear goal, we have managed to show fruitful convergence guarantees of the modified variant of ASkewSGD under weaker assumptions that is dedicated to covering a wider family of neural network architectures.

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Index:	**1**
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).
Source Content:	If C is nonconvex, nonemptiness of \(V_\alpha (x)\) for all x in a neighborhood of C is guaranteed if the Mangasarian-Fromovitz constraint qualification holds for all \(x\in C\).
Source:	https://link.springer.com/article/10.1007/s10107-025-02224-1
From:	Internet

Index:	**2**
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).

Source Content:	Let f be 1-smooth, let g satisfy the Mangasarian-Fromovitz constraint qualification, and let either f be convex or \(2\delta - \beta > 0\).
Source:	https://link.springer.com/article/10.1007/s10107-025-02224-1
From:	Internet

Index:	**3**
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).
Source Content:	Let \(\bar{C}\) be a compact set and let g satisfy the Mangasarian-Fromovitz constraint qualification on \(\bar{C}\).
Source:	https://link.springer.com/article/10.1007/s10107-025-02224-1
From:	Internet

Index:	**4**
Suspected Sentence:	It suffices to consider the case for w∞ i → c− without loss of generality.
Source Content:	We therefore consider the case \(\lambda \ne 0\) and without loss of generality assume that \(|\lambda |=1\).
Source:	https://link.springer.com/article/10.1007/s10107-025-02224-1
From:	Internet

Index:	**5**
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).
Source Content:	Mangasarian-Fromovitz constraint qualification
Source:	https://en.wikipedia.org/wiki/Karush%E2%80%93Kuhn%E2%80%93Tucker_conditions
From:	Internet

Index:	**6**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**7**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**8**
Suspected Sentence:	, N}, the function ℓj is d-times continuously differentiable.
Source Content:	, N}, the function j is d-times continuously differentiable and has M j - Lipschitz continuous gradients.

Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**9**
Suspected Sentence:	Under A3 and A5, it holds that lim supk→∞ d(w(k) , Cε) = 0 almost surely.
Source Content:	Under assumptions of Theorem 1 it holds that lim supk→∞ d(wk, C) = 0 almost surely.
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**10**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**11**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**12**
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Indeed, fix such a j and denote [c−, c+] the set Ci ∩ [(ci j + ci j−1)/2, (ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**13**
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Denote, as previously, [c−, c+] the set Ci ∩ [(ci j + ci j−1)/2, (ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**14**

Suspected Sentence:	The learning rate lr and the hyperparameter for ASkewSGD α are set to 0.2 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**15**
Suspected Sentence:	We set the learning rate lr and the hyperparameter for ASkewSGD α to 0.06 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://arxiv.org/pdf/2211.03741
From:	Internet

Index:	**16**
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).
Source Content:	The Mangasarian-Fromovitz constraint qualification (MFCQ) condition.
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	**17**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	**18**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	**19**
Suspected Sentence:	, N}, the function ℓj is d-times continuously differentiable.
Source Content:	,|D|}, the function fj is d-times continuously differentiable and has Lfj - Lipschitz continuous gradients.
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf

From:	Internet

Index:	20
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	21
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	22
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Indeed, fix such a j and denote [c−,c+] the set Ci ∩ [(ci j + ci j−1)/2,(ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	23
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Denote, as previously, [c−,c+] the set Ci ∩[(ci j +ci j−1)/2,(ci j +ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	24
Suspected Sentence:	The learning rate lr and the hyperparameter for ASkewSGD α are set to 0.2 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	25
Suspected Sentence:	We set the learning rate lr and the hyperparameter for ASkewSGD α to 0.06 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://theses.hal.science/tel-04649643v1/file/142638_LECONTE_2024_archivage.pdf
From:	Internet

Index:	26
Suspected Sentence:	In the second part of the second statement, we consider the case where i ∈ ˆS3,ε(w(k)).
Source Content:	As for the second part of the statement, we consider the case X = H0.
Source:	https://arxiv.org/pdf/2203.12970
From:	Internet

Index:	27
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	28
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	29
Suspected Sentence:	, N}, the function ℓj is d-times continuously differentiable.
Source Content:	, N}, the function j is d-times continuously differentiable and has M j - Lipschitz continuous gradients.
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	30
Suspected Sentence:	Under A3 and A5, it holds that lim supk→∞ d(w(k) , Cε) = 0 almost surely.
Source Content:	Under assumptions of Theorem 1 it holds that lim supk→∞ d(wk, C) = 0 almost surely.
Source:	https://hal.science/hal-04063706/document

From:	Internet

Index:	31
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	32
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	33
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Indeed, fix such a j and denote [c−, c+] the set Ci ∩ [(ci j + ci j−1)/2, (ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	34
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Denote, as previously, [c−, c+] the set Ci ∩ [(ci j + ci j−1)/2, (ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	35
Suspected Sentence:	The learning rate lr and the hyperparameter for ASkewSGD α are set to 0.2 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	36
Suspected Sentence:	We set the learning rate lr and the hyperparameter for ASkewSGD α to 0.06 and 0.2, respectively.

Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://hal.science/hal-04063706/document
From:	Internet

Index:	**37**
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).
Source Content:	The Mangasarian-Fromovitz constraint qualification (MFCQ) condition.
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	**38**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	**39**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	**40**
Suspected Sentence:	, N}, the function ℓj is d-times continuously differentiable.
Source Content:	,|D|}, the function fj is d-times continuously differentiable and has Lfj - Lipschitz continuous gradients.
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	**41**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	**42**
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1,...
Source:	https://theses.hal.science/tel-04649643/document

From:	Internet

Index:	43
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Indeed, fix such a j and denote [c−,c+] the set Ci ∩ [(ci j + ci j−1)/2,(ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	44
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Denote, as previously, [c−,c+] the set Ci ∩[(ci j +ci j−1)/2,(ci j +ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	45
Suspected Sentence:	The learning rate lr and the hyperparameter for ASkewSGD α are set to 0.2 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	46
Suspected Sentence:	We set the learning rate lr and the hyperparameter for ASkewSGD α to 0.06 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://theses.hal.science/tel-04649643/document
From:	Internet

Index:	47
Suspected Sentence:	We do not necessarily aim to solve the above (hard, combinational) optimization problem globally and optimally.
Source Content:	In other words, we do not necessarily aim to solve (the hard, combinatorial) problem (1) globally and optimally.
Source:	https://proceedings.neurips.cc/paper/2021/file/6e0cf80a83327822a972bcde3c1d9740-Paper.pdf
From:	Internet

Index:	48
Suspected Sentence:	Instead, we want to find discrete weights w ∈ Q that remain “satisfactory” when compared to the non-quantized continuous weights [2].
Source Content:	Instead, we want to find discrete weights w ∈ Q that remain satisfactory when compared to the non-quantized continuous weights.
Source:	https://proceedings.neurips.cc/paper/2021/file/6e0cf80a83327822a972bcde3c1d9740-Paper.pdf
From:	Internet

Index:	49
Suspected Sentence:	Definition 1 (Mangasarian-Fromovitz Constraint Qualification).
Source Content:	The Mangasarian-Fromovitz constraint qualification (MFCQ) condition.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	50
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	51
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	52
Suspected Sentence:	, N}, the function ℓj is continuously differentiable and Lℓ-Lipschitz continuous, i.e.
Source Content:	, N}, the function ℓj is d-times continuously differentiable and has Mℓj -Lipschitz continuous gradients.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	53
Suspected Sentence:	, N}, the function ℓj is d-times continuously differentiable.
Source Content:	, N}, the function ℓj is d-times continuously differentiable and has Mℓj -Lipschitz continuous gradients.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf

From:	Internet

Index:	54
Suspected Sentence:	Under A3 and A5, it holds that lim supk→∞ d(w(k) , Cε) = 0 almost surely.
Source Content:	Under assumptions of Theorem 1 it holds that lim supk→∞ d(wk, C) = 0 almost surely.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	55
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	The set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	56
Suspected Sentence:	, q (Ki) i } are sets of quantization values defined coordinate-wise.
Source Content:	, INT4): the set of quantization values Q is defined coordinate wise: {ci 1, .
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	57
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Indeed, fix such a j and denote [c−, c+] the set Ci ∩ [(ci j + ci j−1)/2, (ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	58
Suspected Sentence:	, Ki − 1}, denote [q−, q+] as the set Cε,i ∩ [(q (j) i + q (j−1) i)/2, (q (j) i + q (j+1) i)/2), where Cε,i is the projection of Cε on the i-th coordinate.
Source Content:	Denote, as previously, [c−, c+] the set Ci ∩ [(ci j + ci j−1)/2, (ci j + ci j+1)/2), where Ci is the projection of C onto the i-th coordinate.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	59
Suspected Sentence:	The learning rate lr and the hyperparameter for ASkewSGD α are set to 0.2 and 0.2, respectively.

Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet

Index:	**60**
Suspected Sentence:	We set the learning rate lr and the hyperparameter for ASkewSGD α to 0.06 and 0.2, respectively.
Source Content:	We perform cross-validation of the hyperparameters, such as the learning rate, the tradeoff between constraints α, the rate of increase of the annealing hyperparameter, and their respective schedules.
Source:	https://proceedings.mlr.press/v206/leconte23a/leconte23a.pdf
From:	Internet