Cryptographic Engineering: ECC 3 - Scalar multiplication

Lecture 6

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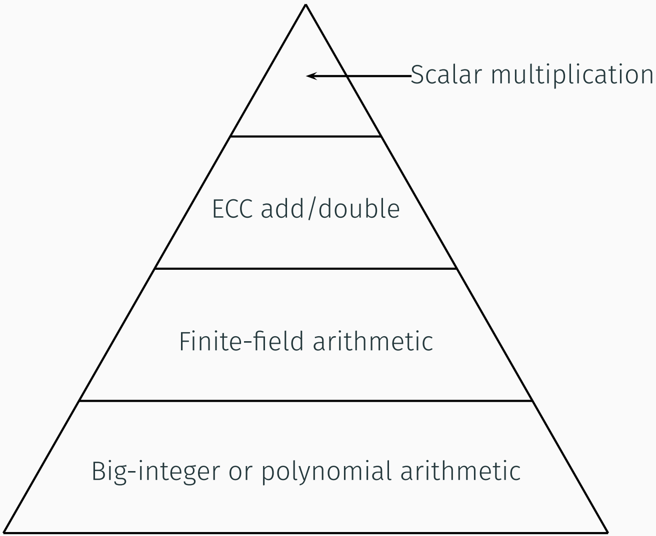
Version: v1.0.0

The ECC pyramid



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  - Compute $k \cdot P$ for $k \in \mathbb{Z}$ and $P \in G$

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- Setting for this lecture (peak of the pyramid):
  - Consider (finite, abelian) group G, written additively
  - Compute $k \cdot P$ for $k \in \mathbb{Z}$ and $P \in G$
  - This is the same as x^k for x in a multiplicative group G'
  - Same algorithms for scalar multiplication and exponentiation

The ECDLP

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Given two points P and Q on an elliptic curve, such that $Q \in \langle P \rangle$, find an integer k such that $kP = Q$.

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- Typical setting for cryptosystems:
  - P is a fixed system parameter,
  - k is the secret (private) key,
  - Q is the public key.
- Key generation needs to compute $Q = kP$, given k and P

EC Diffie-Hellman key exchange

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- Alice computes joint key as $K = k_A Q_B$
- Bob computes joint key as $K = k_B Q_A$

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$$\begin{aligned} R &= rP \\ S &= (r - H(R, M)k_A) \mod \ell \end{aligned}$$

- Verify: compute $\overline{R} = SP + H(R, M)Q_A$ and check that

$$H(\overline{R}, M) \stackrel{?}{=} H(R, M)$$

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- In the following: Distinguish these cases

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- Conclusion: we need algorithms that run in polynomial time (in the size of the scalar)

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- Cost: 6 doublings, 3 additions
- General algorithm: "Double and add"

  $R \leftarrow P$
  for $i \leftarrow n - 2$ downto 0 **do**
      $R \leftarrow 2R$
      if $(k)_2[i] = 1$ **then**
          $R \leftarrow R + P$
  return R

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- Handles single-scalar multiplication
- Running time clearly depends on the scalar: insecure for secret scalars!

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    - Compute $k_1 P_1$ ($n_1 - 1$ doublings, $m_1 - 1$ additions)
    - Compute $k_2 P_2$ ($n_2 - 1$ doublings, $m_2 - 1$ additions)
    - Add the results (1 addition)

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- We can do better ($\mathcal{O}$ denotes the neutral element):

$R \leftarrow \mathcal{O}$

for $i \leftarrow \max(n_1, n_2) - 1$ downto 0 **do**

  $R \leftarrow 2R$

  if $(k_1)_2[i] = 1$ **then**

    $R \leftarrow R + P_1$

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  return R
- $\max(n_1, n_2)$ doublings, $m_1 + m_2$ additions

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- Let's just precompute $T = P_1 + P_2$
- Modified algorithm (special case of Strauss' algorithm):

$R \leftarrow \mathcal{O}$
for $i \leftarrow \max(n_1, n_2) - 1$ downto 0 **do**
  $R \leftarrow 2R$
  if $(k_1)_2[i] = 1$ AND $(k_2)_2[i] = 1$ **then**
    $R \leftarrow R + T$
  else if $(k_1)_2[i] = 1$ **then**
    $R \leftarrow R + P_1$
  else if $(k_2)_2[i] = 1$ **then**
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- Problem: k is large. For a 256-bit k we would need a table of size

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- How about, for example, precompute $P, 2P, 4P, 8P, \ldots, 2^{n-1}P$
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- Modified scalar-multiplication algorithm:

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\begin{array}{l}
R \leftarrow \mathcal{O} \\
\textbf{for } i \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad \textbf{if } (k)_2[i] = 1 \textbf{ then} \\
\quad\quad R \leftarrow R + 2^i P \\
\textbf{return } R
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- Eliminated all doublings in fixed-basepoint scalar multiplication!

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- Still not constant time, more later...

Let's rewrite that a bit . . .

- We have a table $T = (\mathcal{O}, P)$
- Notation $T[0] = \mathcal{O}$, $T[1] = P$
- Scalar multiplication is

$$
\begin{array}{l}
R \leftarrow P \\
\textbf{for } i \leftarrow n-2 \text{ downto } 0 \textbf{ do} \\
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\quad R \leftarrow R + T[(k)_2[i]]
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- Disadvantage: 3 is just not nice (needs triplings)
- How about some nice numbers, like $4, 8, 16$?

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- This is the same as chopping the binary scalar into “windows” of fixed length w
- Compute scalar multiplication as

$R \leftarrow T[(k)_{2^w}[m-1]]$
for $i \leftarrow m-2$ downto 0 **do**
  for $j \leftarrow 1$ to w **do**
    $R \leftarrow 2R$
  $R \leftarrow R + T[(k)_{2^w}[i]]$

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- Number of additions in the loop is $\lceil n/w \rceil - 1$
- Larger w: More precomputation
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- For ≈ 256-bit scalars choose $w=4$ or $w=5$

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  - Remember that table lookups are generally not constant time!

Making it constant time

```
/* Sets r to the neutral element on the elliptic curve */
extern ec_point_setneutral(ec_point *r);

/* Adds p and q and stores the result in r */
extern ec_point_add(ec_point *r, const ec_point *p, const ec_point *q);

/* Doubles p and stores the result in r */
extern ec_point_double(ec_point *r, const ec_point *p);

/* For point P contains pre-computed multiples P, 2*P, 3*P,...,255*P */
extern ec_point precomputed[255];

ec_scalarmult_P(unsigned char scalar[32])
{
  int i,j;
  ec_point r;
  ec_setneutral(&r);

  for(i=31;i>=0;i--)
  {
    for(j=0;j<8;j++)
      ec_point_double(&r,&r);
    if(scalar[i] != 0)
      ec_point_add(&r,&r,precomputed[scalar[i]-1]);
  }
}
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    for(j=0;j<8;j++)
      ec_point_double(&r,&r);
    ec_point_add(&r,&r,precomputed[scalar[i]]);
  }
}
```

Making it constant time

```
/* Sets r to the neutral element on the elliptic curve */
extern ec_point_setneutral(ec_point *r);

/* Adds p and q and stores the result in r */
extern ec_point_add(ec_point *r, const ec_point *p, const ec_point *q);

/* Doubles p and stores the result in r */
extern ec_point_double(ec_point *r, const ec_point *p);

/* For point P contains pre-computed multiples 0, P, 2*P, 3*P,...,255*P */
extern ec_point precomputed[256];

ec_scalarmult_P(unsigned char scalar[32])
{
  int i,j;
  ec_point r,t;
  ec_setneutral(&r);

  for(i=31;i>=0;i--)
  {
    for(j=0;j<8;j++)
      ec_point_double(&r,&r);
    ec_point_lookup(&t,precomputed,scalar[i]);
    ec_point_add(&r,&r,&t);
  }
}
```

ec_point_lookup

```c
static void ec_point_lookup(ec_point *t, const ec_point *table, int pos)
{
  int i,j;
  unsigned char b;
  *t = table[0];
  for(i=0;i<256;i++)
  {
    b = (i == pos); // Not constant time!
    ec_point_cmov(t, &table[i], b); // Copy table[i] to t if b is 1
  }
}
```

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```
static void ec_point_lookup(ec_point *t, const ec_point *table, int pos)
{
  int i,j;
  unsigned char b;
  *t = table[0];
  for(i=0;i<256;i++)
  {
    b = int_eq(i, pos); // set b=1 if i==pos, else set b=0
    ec_point_cmov(t, &table[i], b); // Copy table[i] to t if b is 1
  }
}
```

int_eq and ec_point_cmov

```
unsigned char int_eq(int a, int b)
{
  unsigned long long t = a ^ b;
  t = (-t) >> 63;
  return 1-t;
}


void ec_point_cmov(ec_point *r, const ec_point *t, unsigned char b)
{
  unsigned char *u = (unsigned char *)r;
  unsigned char *v = (unsigned char *)t;
  int i;
  b = -b;
  for(i=0;i<sizeof(ec_point);i++)
    u[i] = (b & v[i]) ^ (~b & u[i]);
}
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`int_eq` and `ec_point_cmov`

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```

- Recent compilers may re-introduce a branch in **`ec_point_cmov`**
- One solution: move **`ec_point_cmov`** to separate file, compile without -flto

More offline precomputation

- Let's get back to fixed-basepoint multiplication
- So far we precomputed $P, 2P, 4P, 8P, \dots$

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- We can combine that with fixed-window scalar multiplication
- Precompute $T_i = (\mathcal{O}, P, 2P, 3P, \ldots, (2^w - 1)P) \cdot 2^i$ for $i = 0, w, 2w, 3w, \lceil n/w \rceil - 1$

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- Perform scalar multiplication as

  $R \leftarrow T_0[(k)_{2^w}[0]]$
  for $i \leftarrow 1$ to $\lceil n/w \rceil - 1$ **do**
  $\quad R \leftarrow R + T_{iw}[(k)_{2^w}[i]]$

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- No doublings, only $\lceil n/w \rceil - 1$ additions
- Can use huge w, but:
  - at some point the precomputed tables don't fit into cache anymore.
  - constant-time loads get slow for large w

Fixed-window limitations

- Consider the scalar $22 = (10110)_2$ and window size 2
  - Initialize R with P
  - Double, double, add P
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- Problem with fixed window: it's fixed.
- Idea: "slide" the window over the scalar

Sliding window scalar multiplication

- Choose window size w
- Rewrite scalar k as $k = (k_0, \ldots, k_m)$ with k_i in $\{0, 1, 3, 5, \ldots, 2^w - 1\}$ with at most one non-zero entry in each window of length w

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- Do this by scanning k from right to left, expand window from each 1-bit
- Precompute $P, 3P, 5P, \ldots, (2^w - 1)P$
- Perform scalar multiplication

  $R \leftarrow \mathcal{O}$
  for $i \leftarrow m$ to 0 **do**
      $R \leftarrow 2R$
      if $k_i \neq 0$ **then**
          $R \leftarrow R + k_iP$

Analysis of sliding window

- We still do $n-1$ doublings for an n-bit scalar
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- Expected number of additions in the main loop: $n/(w+1)$
- For the same w only half the precomputation compared to fixed-window scalar multiplication
- For the same w fewer additions in the main loop
- But: It's not running in constant time!
- Still nice (in double-scalar version) for signature verification

Differential addition

- Consider elliptic curves of the form $By^2 = x^3 + Ax^2 + x$.
- Montgomery in 1987 showed how to perform x-coordinate-based arithmetic:
  - Given the x-coordinate x_P of P, and
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  - compute the x-coordinate x_R of $R = P + Q$
- This is called *differential addition*
- Less efficient differential-addition formulas for other curve shapes
- Can be used for efficient computation of the x-coordinate of kP given only the x-coordinate of P
- For this, let's use projective representation $(X : Z)$ with $x = (X/Z)$

One Montgomery "ladder step"

const $a24 = (A+2)/4$ (A from the curve equation)
function $\textsc{LadderStep}(x_{Q-P}, X_P, Z_P, X_Q, Z_Q)$

$t_1 \leftarrow X_P + Z_P$
$t_6 \leftarrow t_1^2$
$t_2 \leftarrow X_P - Z_P$
$t_7 \leftarrow t_2^2$
$t_5 \leftarrow t_6 - t_7$
$t_3 \leftarrow X_Q + Z_Q$
$t_4 \leftarrow X_Q - Z_Q$
$t_8 \leftarrow t_4 \cdot t_1$
$t_9 \leftarrow t_3 \cdot t_2$
$X_{P+Q} \leftarrow (t_8 + t_9)^2$
$Z_{P+Q} \leftarrow x_{Q-P} \cdot (t_8 - t_9)^2$
$X_{2P} \leftarrow t_6 \cdot t_7$
$Z_{2P} \leftarrow t_5 \cdot (t_7 + a24 \cdot t_5)$
return $(X_{2P}, Z_{2P}, X_{P+Q}, Z_{P+Q})$

The Montgomery ladder

Require: A scalar $0 \leq k \in \mathbb{Z}$ and the x-coordinate x_P of some point P
Ensure: (X_{kP}, Z_{kP}) fulfilling $x_{kP} = X_{kP}/Z_{kP}$

$x_1 = x_P$; $X_2 = 1$; $Z_2 = 0$; $X_3 = x_P$; $Z_3 = 1$
for $i \leftarrow n-1$ downto 0 **do**
  if bit i of k is 1 **then**
    $(X_3, Z_3, X_2, Z_2) \leftarrow \text{LADDERSTEP}(x_1, X_3, Z_3, X_2, Z_2)$
  else
    $(X_2, Z_2, X_3, Z_3) \leftarrow \text{LADDERSTEP}(x_1, X_2, Z_2, X_3, Z_3)$
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$X_1 = x_P$; $X_2 = 1$; $Z_2 = 0$; $X_3 = x_P$; $Z_3 = 1$; $p = 0$
for $i \leftarrow n-1$ downto 0 **do**
  $b \leftarrow$ bit i of s
  $c \leftarrow b \oplus p$
  $p \leftarrow b$
  $(X_2, X_3) \leftarrow \textsc{cswap}(X_2, X_3, c)$
  $(Z_2, Z_3) \leftarrow \textsc{cswap}(Z_2, Z_3, c)$
  $(X_2, Z_2, X_3, Z_3) \leftarrow \textsc{ladderstep}(x_1, X_2, Z_2, X_3, Z_3)$
$(X_2, X_3) \leftarrow \textsc{cswap}(X_2, X_3, p)$
$(Z_2, Z_3) \leftarrow \textsc{cswap}(Z_2, Z_3, p)$
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Advantages of the Montgomery ladder

- Very regular structure, easy to protect against timing attacks
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- Very fast (at least if we don't compare to curves with efficient endomorphisms)
- Point compression/decompression is free
- Easy to implement
- No ugly special cases (see Bernstein's "Curve25519" paper)

Multi-scalar multiplication

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- We looked at $n = 2$ before, how about $n = 128$?

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- Assume $k_1 > k_2 > \cdots > k_n$.
- Use that $k_1 P_1 + k_2 P_2 = (k_1 - k_2)P_1 + k_2(P_1 + P_2)$
- Replace:
    - $(k_1 P_1)$ and $(k_2 P_2)$, with
    - $(k_1 - k_2)P_1$ and $k_2(P_1 + P_2)$
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- Can be very fast (but not constant-time)
- Requires fast access to the two largest scalars: put scalars into a heap
- Crucial for good performance: fast heap implementation

A fast heap

- Heap is a binary tree, each parent node is larger than the two child nodes
- Data structure is stored as a simple array, positions in the array determine positions in the tree
- Root is at position 0, left child node at position 1, right child node at position 2 etc.
- For node at position i, child nodes are at position $2 \cdot i + 1$ and $2 \cdot i + 2$, parent node is at position $\lfloor (i-1)/2 \rfloor$

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- Typical heap root replacement (pop operation): start at the root, swap down for a variable amount of times
- Floyd's heap: swap down to the bottom, swap up for a variable amount of times, advantages:
    - Each swap-down step needs only one comparison (instead of two)
    - Swap-down loop is more friendly to branch predictors

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- How about **variable** point, **fixed** scalar?
- Optimizing for the scalar means that the scalar has to be public
- Not the typical setting for ECC
- Some applications:
  - Inversion in finite fields (cmp. multiprecision lecture)
  - Elliptic-curve factorization method (not in this lecture)

Addition chains

Definition
Let k be a positive integer. A sequence $s_1, s_2, \ldots, s_m$ is called an addition chain of length m for k if

- $s_1 = 1$
- $s_m = k$
- for each s_i with $i > 1$ it holds that $s_i = s_j + s_\ell$ for some $j, \ell < i$

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- Signed-scalar representations are "addition-subtraction chains"
- For fixed scalar we can spend a lot of time to find a good addition chain at compile time
- This is what was used for inversion in $\mathbb{F}_{2^{255}-19}$
- Computing good addition chains? See https://github.com/mmcloughlin/addchain

Quiz: Scalar Multiplication

https://pingo.coactum.de/994716

Q1: ECDH

In ECDH key exchange, Alice computes the shared key as $K = k_A \cdot Q_B$. What type of scalar multiplication is this?

- A) Fixed-basepoint scalar multiplication with secret scalar
- B) Variable-basepoint scalar multiplication with secret scalar
- C) Fixed-basepoint scalar multiplication with public scalar
- D) Variable-basepoint scalar multiplication with public scalar

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Q2: Schnorr Signature

Schnorr signature verification computes $\overline{R} = SP + H(R, M)Q_A$, which requires two scalar multiplications. What is the nature of the two scalars S and $H(R, M)$?

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Q3: Double-and-add

What is the problem with the basic double-and-add algorithm when used with secret scalars?

- A) It produces incorrect results for certain scalar values
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Q4: Double-and-add always

What is the purpose of ”double-and-add always”?

- A) To make the algorithm faster by always performing additions
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Q5: Fixed-basepoint precomputation

In fixed-basepoint scalar multiplication with precomputed table $P, 2P, 4P, 8P, \ldots, 2^{n-1}P$, what is the cost of computing kP for an n-bit scalar k?

- A) $n - 1$ doublings and $\sim n/2$ additions
- B) $\sim n/2$ additions only (no doublings)
- C) $n - 1$ doublings only (no additions)
- D) No doublings and no additions

Q5: Fixed-basepoint precomputation

In fixed-basepoint scalar multiplication with precomputed table $P, 2P, 4P, 8P, \ldots, 2^{n-1}P$, what is the cost of computing kP for an n-bit scalar k?

- × A) $n - 1$ doublings and $\sim n/2$ additions
- ✓ B) $\sim n/2$ additions only (no doublings)
- × C) $n - 1$ doublings only (no additions)
- × D) No doublings and no additions

Q6: Fixed-window table size

For X25519 with 256-bit scalars and 32-byte points, if we use fixed-window scalar multiplication with window size $w = 8$, how large is the precomputed table?

- A) 8 KB (256 points)
- B) 16 KB (512 points)
- C) 256 bytes (8 points)
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Q7: Montgomery ladder

What is the key advantage of the Montgomery ladder for scalar multiplication?

- A) It eliminates all point additions
- B) It has a very regular structure that is easy to make constant-time
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Q8: Addition chains

Which of the following is a valid addition chain for 7?

- A) 1, 2, 3, 4, 5, 6, 7
- B) 1, 2, 4, 6, 7
- C) 1, 2, 3, 6, 7
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