Pappus chain

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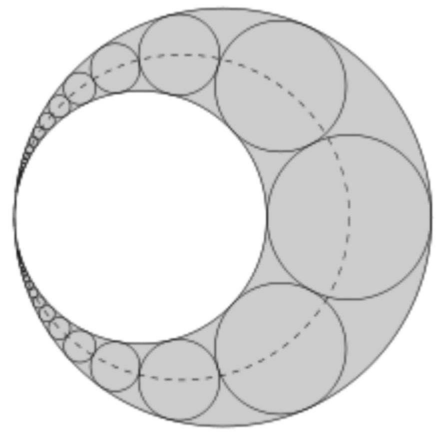
In geometry, the **Pappus chain** is a ring of circles between two tangent circles investigated by Pappus of Alexandria in the 3rd century AD.



A Pappus chain

Contents

Construction

The arbelos is defined by two circles, C_{U} and C_{V}, which are tangent at the point **A** and where C_{U} is enclosed by C_{V}. Let the radii of these two circles be denoted as r_{U} and r_{V}, respectively, and let their respective centers be the points **U** and **V**. The Pappus chain consists of the circles in the shaded grey region, which are externally tangent to C_{U} (the inner circle) and internally tangent to C_{V} (the outer circle). Let the radius, diameter and center point of the n^{th} circle of the Pappus chain be denoted as r_n, d_n and $\mathbf{P}_n$, respectively.

Properties

Centers of the circles

Ellipse

All the centers of the circles in the Pappus chain are located on a common ellipse, for the following reason. The sum of the distances from the n^{th} circle of the Pappus chain to the two centers **U** and **V** of the arbelos circles equals a constant

$$\overline{\mathbf{P}_n\mathbf{U}} + \overline{\mathbf{P}_n\mathbf{V}} = (r_U + r_n) + (r_V - r_n) = r_U + r_V$$

Thus, the foci of this ellipse are **U** and **V**, the centers of the two circles that define the arbelos; these points correspond to the midpoints of the line segments **AB** and **AC**, respectively.

Coordinates

If $r = AC/AB$, then the center of the nth circle in the chain is:

$$(x_n, y_n) = \left(\frac{r(1+r)}{2[n^2(1-r)^2+r]}\,,\ \frac{nr(1-r)}{n^2(1-r)^2+r}\right)$$

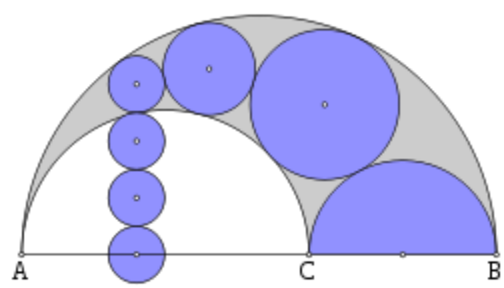
Radii of the circles

If $r = AC/AB$, then the radius of the nth circle in the chain is:

$$r_n = \frac{(1-r)r}{2[n^2(1-r)^2+r]}$$

Circle inversion

The height h_n of the center of the n^{th} circle above the base diameter **ACB** equals n times d_n.[1] This may be shown by inverting in a circle centered on the tangent point **A**. The circle of inversion is chosen to intersect the n^{th} circle perpendicularly, so that the n^{th} circle is transformed into itself. The two arbelos circles, C_U and C_V, are transformed into parallel lines tangent to and sandwiching the n^{th} circle; hence, the other circles of the Pappus chain are transformed into similarly sandwiched circles of the same diameter. The initial circle C_0 and the final circle C_n each contribute ½d_n to the height h_n, whereas the circles C_1–C_{n-1} each contribute d_n. Adding these contributions together yields the equation $h_n = n\ d_n$.



Under a particular inversion centered on **A**, the four initial circles of the Pappus chain are transformed into a stack of four equally sized circles, sandwiched between two parallel lines. This accounts for the height formula $h_n = n\ d_n$ and the fact that the original points of tangency lie on a common circle.

The same inversion can be used to show that the points where the circles of the Pappus chain are tangent to one another lie on a common circle. As noted above, the inversion centered at point **A** transforms the arbelos circles C_U and C_V into two parallel lines, and the circles of the Pappus chain into a stack of equally sized circles sandwiched between the two parallel lines. Hence, the points of tangency between the transformed circles lie on a line midway between the two parallel lines. Undoing the inversion in the circle, this line of tangent points is transformed back into a circle.

Steiner chain

In these properties of having centers on an ellipse and tangencies on a circle, the Pappus chain is analogous to the Steiner chain, in which finitely many circles are tangent to two circles.

References

1. Ogilvy, pp. 54–55.

Bibliography

- Ogilvy, C. S. (1990). *Excursions in Geometry*. Dover. pp. 54–55. ISBN 0-486-26530-7.
- Bankoff, L. (1981). "How did Pappus do it?". In Klarner, D. A. *The Mathematical Gardner*. Boston: Prindle, Weber, & Schmidt. pp. 112–118.
- Johnson, R. A. (1960). *Advanced Euclidean Geometry: An elementary treatise on the geometry of the triangle and the circle* (reprint of 1929 edition by Houghton Miflin ed.). New York: Dover Publications. pp. 116–117. ISBN 978-0-486-46237-0.
- Wells, D. (1991). *The Penguin Dictionary of Curious and Interesting Geometry*. New York: Penguin Books. pp. 5–6. ISBN 0-14-011813-6.

External links

- Floer van Lamoen and Eric W. Weisstein. "Pappus Chain" (http://mathworld.wolfram.com/PappusChain.html). *MathWorld*.

- Tan, Stephen. "Arbelos" (http://www.math.ubc.ca/~cass/courses/m308/projects/tan/html/home.html).