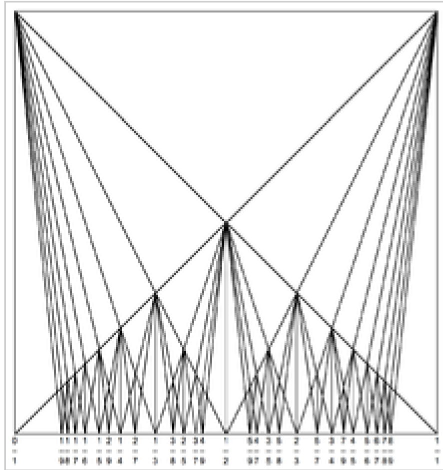
Farey sequence

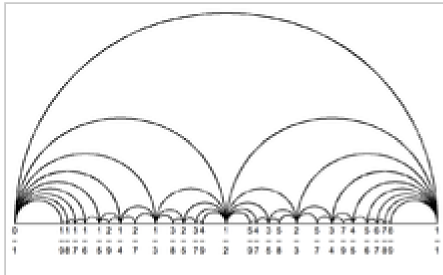
From Wikipedia, the free encyclopedia

In mathematics, the Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which when in lowest terms have denominators less than or equal to n, arranged in order of increasing size.



Farey diagram to F_9.

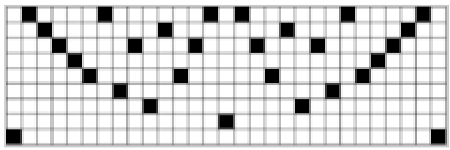
Each Farey sequence starts with the value 0, denoted by the fraction $\frac{0}{1}$, and ends with the value 1, denoted by the fraction $\frac{1}{1}$ (although some authors omit these terms).



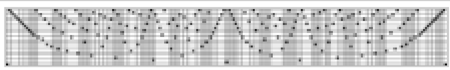
Farey diagram to F_9.

A Farey sequence is sometimes called a Farey series, which is not strictly correct, because the terms are not summed.

Contents



Symmetrical pattern made by the denominators of the Farey sequence, F_9.



Symmetrical pattern made by the denominators of the Farey sequence, F_{25}.

Examples

The Farey sequences of orders 1 to 8 are :

$$F_1 = \{ \tfrac{0}{1}, \tfrac{1}{1} \}$$
$$F_2 = \{ \tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1} \}$$
$$F_3 = \{ \tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1} \}$$
$$F_4 = \{ \tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1} \}$$
$$F_5 = \{ \tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1} \}$$
$$F_6 = \{ \tfrac{0}{1}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{1}{1} \}$$
$$F_7 = \{ \tfrac{0}{1}, \tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{6}{7}, \tfrac{1}{1} \}$$
$$F_8 = \{ \tfrac{0}{1}, \tfrac{1}{8}, \tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{6}{7}, \tfrac{7}{8}, \tfrac{1}{1} \}$$

Centered

$$F_1 = \{ \tfrac{0}{1}, \tfrac{1}{1} \}$$
$$F_2 = \{ \tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1} \}$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

$$F_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\}$$

$$F_7 = \left\{ \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1} \right\}$$

$$F_8 = \left\{ \frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1} \right\}$$

Sorted

```
F1 = {0/1,                                                                                                    1/1}
F2 = {0/1,                                               1/2,                                                 1/1}
F3 = {0/1,                               1/3,            1/2,            2/3,                                 1/1}
F4 = {0/1,                     1/4,      1/3,            1/2,            2/3,      3/4,                       1/1}
F5 = {0/1,                1/5, 1/4,      1/3,       2/5, 1/2,       3/5, 2/3,      3/4, 4/5,                  1/1}
F6 = {0/1,           1/6, 1/5, 1/4,      1/3,       2/5, 1/2,       3/5, 2/3,      3/4, 4/5, 5/6,             1/1}
F7 = {0/1,      1/7, 1/6, 1/5, 1/4, 2/7, 1/3,       2/5, 3/7, 1/2, 4/7, 3/5,      2/3, 5/7, 3/4, 4/5, 5/6, 6/7,      1/1}
F8 = {0/1, 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8, 1/1}
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History

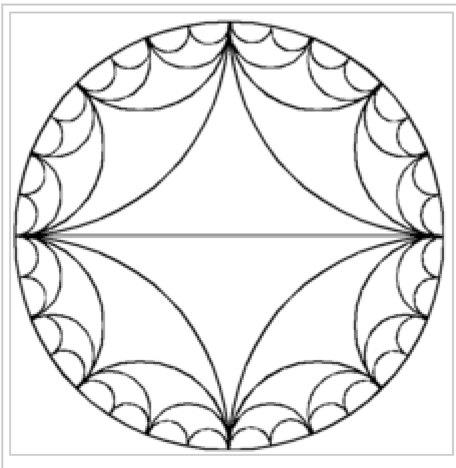
> The history of 'Farey series' is very curious — Hardy & Wright (1979) Chapter III[1]

> ... once again the man whose name was given to a mathematical relation was not the original discoverer so far as the records go. — Beiler (1964) Chapter XVI[2]

Farey sequences are named after the British geologist John Farey, Sr., whose letter about these sequences was published in the Philosophical Magazine in 1816. Farey conjectured, without offering proof, that each new term in a Farey sequence expansion is the mediant of its neighbours. Farey's letter was read by Cauchy, who provided a proof in his Exercices de mathématique, and attributed this result to Farey. In fact, another mathematician, Charles Haros, had published similar results in 1802 which were not known either to Farey or to Cauchy.[2] Thus it was a historical accident that linked Farey's name with these sequences. This is an example of Stigler's law of eponymy.

Properties

Sequence length and index of a fraction



The Farey sequence of order n contains all of the members of the Farey sequences of lower orders. In particular F_n contains all of the members of F_{n-1} and also contains an additional fraction for each number that is less than n and coprime to n. Thus F_6 consists of F_5 together with the fractions $\frac{1}{6}$ and $\frac{5}{6}$.

The middle term of a Farey sequence F_n is always $\frac{1}{2}$, for $n > 1$. From this, we can relate the lengths of F_n and F_{n-1} using Euler's totient function $\varphi(n)$:

$$|F_n| = |F_{n-1}| + \varphi(n).$$

Using the fact that $|F_1| = 2$, we can derive an expression for the length of F_n :

$$|F_n| = 1 + \sum_{m=1}^{n} \varphi(m).$$

We also have :

$$|F_n| = \frac{1}{2}\left(3 + \sum_{d=1}^{n} \mu(d) \left\lfloor \tfrac{n}{d} \right\rfloor^2\right),$$

and by a Möbius inversion formula :

$$|F_n| = \frac{1}{2}(n+3)n - \sum_{d=2}^{n} |F_{\lfloor n/d \rfloor}|,$$

where μ(d) is the number-theoretic Möbius function, and $\lfloor \frac{n}{d} \rfloor$ is the floor function.

The asymptotic behaviour of $|F_n|$ is :

$$|F_n| \sim \frac{3n^2}{\pi^2}.$$

The index $I_n(a_{k,n}) = k$ of a fraction $a_{k,n}$ in the Farey sequence $F_n = \{a_{k,n} : k = 0, 1, \ldots, m_n\}$ is simply the position that $a_{k,n}$ occupies in the sequence. This is of special relevance as it is used in an alternative formulation of the Riemann hypothesis, see below. Various useful properties follow:

$$I_n(0/1) = 0,$$
$$I_n(1/n) = 1,$$
$$I_n(1/2) = (|F_n| - 1)/2,$$
$$I_n(1/1) = |F_n| - 1,$$
$$I_n(h/k) = |F_n| - 1 - I_n((k-h)/k).$$

Farey neighbours

Fractions which are neighbouring terms in any Farey sequence are known as a Farey pair and have the following properties.

If $\frac{a}{b}$ and $\frac{c}{d}$ are neighbours in a Farey sequence, with $\frac{a}{b} < \frac{c}{d}$, then their difference $\frac{c}{d} - \frac{a}{b}$ is equal to $\frac{1}{bd}$. Since

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd},$$

this is equivalent to saying that

$$bc - ad = 1.$$

Thus $\frac{1}{3}$ and $\frac{2}{5}$ are neighbours in F_5, and their difference is $\frac{1}{15}$.

The converse is also true. If

$$bc - ad = 1$$

for positive integers a, b, c and d with a < b and c < d then $\frac{a}{b}$ and $\frac{c}{d}$ will be neighbours in the Farey sequence of order max(b, d).

If $\frac{p}{q}$ has neighbours $\frac{a}{b}$ and $\frac{c}{d}$ in some Farey sequence, with

$$\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$$

then $\frac{p}{q}$ is the mediant of $\frac{a}{b}$ and $\frac{c}{d}$ — in other words,

$$\frac{p}{q} = \frac{a+c}{b+d}.$$

This follows easily from the previous property, since if bp−aq = qc−pd = 1, then bp+pd = qc+aq, p(b+d)=q(a+c), $\frac{p}{q} = \frac{a+c}{b+d}$

It follows that if $\frac{a}{b}$ and $\frac{c}{d}$ are neighbours in a Farey sequence then the first term that appears between them as the order of the Farey sequence is increased is

$$\frac{a+c}{b+d},$$

which first appears in the Farey sequence of order b + d.

Thus the first term to appear between $\frac{1}{3}$ and $\frac{2}{5}$ is $\frac{3}{8}$, which appears in F_8.

The Stern–Brocot tree is a data structure showing how the sequence is built up from 0 (= $\frac{0}{1}$) and 1 (= $\frac{1}{1}$), by taking successive mediants.

Fractions that appear as neighbours in a Farey sequence have closely related continued fraction expansions. Every fraction has two continued fraction expansions — in one the final term is 1; in the other the final term is greater than 1. If $\frac{p}{q}$, which first appears in Farey sequence F_q, has continued fraction expansions

$$[0;\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ a_n,\ 1]$$
$$[0;\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ a_n + 1]$$

then the nearest neighbour of $\frac{p}{q}$ in F_q (which will be its neighbour with the larger denominator) has a continued fraction expansion

$$[0;\ a_1,\ a_2,\ \ldots,\ a_n]$$

and its other neighbour has a continued fraction expansion

$$[0;\ a_1,\ a_2,\ \ldots,\ a_{n-1}]$$

For example, $\frac{3}{8}$ has the two continued fraction expansions [0; 2, 1, 1, 1] and [0; 2, 1, 2], and its neighbours in F_8 are $\frac{2}{5}$, which can be expanded as [0; 2, 1, 1]; and $\frac{1}{3}$, which can be expanded as [0; 2, 1].

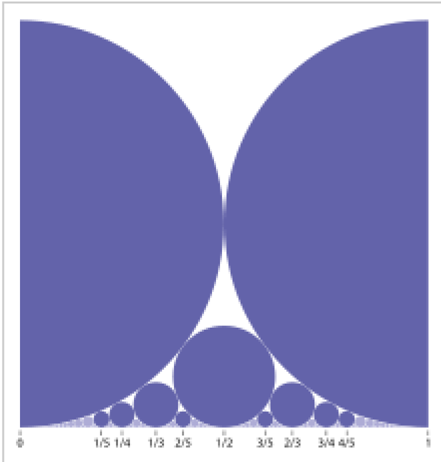
Applications

Farey sequences are very useful to find rational approximations of irrational numbers [3] (http://nrich.maths.org/6596).

In physics systems featuring resonance phenomena Farey sequences provide a very elegant and efficient method to compute resonance locations in 1D [3] and 2D [4]

Ford circles

There is a connection between Farey sequence and Ford circles.



Ford circles

For every fraction p/q (in its lowest terms) there is a Ford circle C[p/q], which is the circle with radius $1/(2q^2)$ and centre at $(p/q, 1/(2q^2))$. Two Ford circles for different fractions are either disjoint or they are tangent to one another—two Ford circles never intersect. If 0 < p/q < 1 then the Ford circles that are tangent to C[p/q] are precisely the Ford circles for fractions that are neighbours of p/q in some Farey sequence.

Thus C[2/5] is tangent to C[1/2], C[1/3], C[3/7], C[3/8] etc.

Riemann hypothesis

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_n are $\{a_{k,n} : k = 0, 1, \ldots, m_n\}$. Define $d_{k,n} = a_{k,n} - k/m_n$, in other words $d_{k,n}$ is the difference between the kth term of the nth Farey sequence, and the kth member of a set of the same number of points, distributed evenly on the unit interval. In 1924 Jérôme Franel[5] proved that the statement

$$\sum_{k=1}^{m_n} d_{k,n}^2 = O(n^r) \quad \forall r > -1$$

is equivalent to the Riemann hypothesis, and then Edmund Landau[6] remarked (just after Franel's paper) that the statement

$$\sum_{k=1}^{m_n} |d_{k,n}| = O(n^r) \quad \forall r > 1/2$$

is also equivalent to the Riemann hypothesis.

Next term

A surprisingly simple algorithm exists to generate the terms of F_n in either traditional order (ascending) or non-traditional order (descending). The algorithm computes each successive entry in terms of the previous two entries using the mediant property given above. If $\frac{a}{b}$ and $\frac{c}{d}$ are the two given entries, and $\frac{p}{q}$ is the unknown next entry, then $\frac{c}{d} = \frac{a + p}{b + q}$. Since $\frac{c}{d}$ is in lowest terms, there must be an integer k such that $kc = a + p$ and $kd = b + q$, giving $p = kc - a$ and $q = kd - b$. If we consider p and q to be functions of k, then

$$\frac{p(k)}{q(k)} - \frac{c}{d} = \frac{cb - da}{d(kd - b)}$$

so the larger k gets, the closer $\frac{p}{q}$ gets to $\frac{c}{d}$.

To give the next term in the sequence k must be as large as possible, subject to $kd - b \leqslant n$ (as we are only considering numbers with denominators not greater than n), so k is the greatest integer $\leqslant \frac{n+b}{d}$. Putting this value of k back into the equations for p and q gives

$$p = \left\lfloor \frac{n+b}{d} \right\rfloor c - a$$
$$q = \left\lfloor \frac{n+b}{d} \right\rfloor d - b$$

This is implemented in Python as:

```
def farey( n, asc=True ):
    """Python function to print the nth Farey sequence, either ascending or descending."""
    if asc:
        a, b, c, d = 0, 1,  1 , n     # (*)
    else:
        a, b, c, d = 1, 1, n-1, n     # (*)
    print "%d/%d" % (a,b)
    while (asc and c <= n) or (not asc and a > 0):
        k = int((n + b)/d)
        a, b, c, d = c, d, k*c - a, k*d - b
        print "%d/%d" % (a,b)
```

Brute-force searches for solutions to Diophantine equations in rationals can often take advantage of the Farey series (to search only reduced forms). The lines marked (*) can also be modified to include any two adjacent terms so as to generate terms only larger (or smaller) than a given term.[7]

See also

- Stern-Brocot tree
- Euler's totient function

References

1. Hardy, G.H. & Wright, E.M. (1979) An Introduction to the Theory of Numbers (Fifth Edition). Oxford University Press. ISBN 0-19-853171-0
2. Beiler, Albert H. (1964) Recreations in the Theory of Numbers (Second Edition). Dover. ISBN 0-486-21096-0. Cited in Farey Series, A Story (http://www.cut-the-knot.org/blue/FareyHistory.shtml) at Cut-the-Knot
3. A. Zhenhua Li, W.G. Harter, "Quantum Revivals of Morse Oscillators and Farey-Ford Geometry", arXiv:1308.4470v1
4. http://prst-ab.aps.org/abstract/PRSTAB/v17/i1/e014001
5. "Les suites de Farey et le problème des nombres premiers", Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1924, 198-201, web link:[1] (http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN00250653X)
6. "Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel", Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1924, 202-206, web link: [2] (http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN002506548)
7. Norman Routledge, "Computing Farey Series," The Mathematical Gazette, Vol. 92 (No. 523), 55–62 (March 2008).

Further reading

- Allen Hatcher, Topology of Numbers (http://www.math.cornell.edu/~hatcher/TN/TNpage.html)
- Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, Concrete Mathematics: A Foundation for Computer Science, 2nd Edition (Addison-Wesley, Boston, 1989); in particular, Sec. 4.5 (pp. 115–123), Bonus Problem 4.61 (pp. 150, 523–524), Sec. 4.9 (pp. 133–139), Sec. 9.3, Problem 9.3.6 (pp. 462–463). ISBN 0-201-55802-5.

- Linas Vepstas. The Minkowski Question Mark, GL(2,Z), and the Modular Group. http://linas.org/math/chap-minkowski.pdf reviews the isomorphisms of the Stern-Brocot Tree.
- Linas Vepstas. Symmetries of Period-Doubling Maps. http://linas.org/math/chap-takagi.pdf reviews connections between Farey Fractions and Fractals.
- Scott B. Guthery, A Motif of Mathematics: History and Application of the Mediant and the Farey Sequence, (Docent Press, Boston, 2010). ISBN 1-4538-1057-9.
- Cristian Cobeli and Alexandru Zaharescu, The Haros-Farey Sequence at Two Hundred Years. A Survey, Acta Univ. Apulensis Math. Inform. no. 5 (2003) 1 - 38, pp. 1 - 20 (http://www.emis.de/journals/AUA/acta5/survey3.ps_pages1-20.pdf) pp. 21 - 38 (http://www.emis.de/journals/AUA/acta5/survey3.ps_pages21-38.pdf)
- A.O. Matveev, A Note on Boolean Lattices and Farey Sequences III, arXiv:1507.07236 (http://arxiv.org/pdf/1507.07236.pdf)
- A.O. Matveev, Neighboring Fractions in Farey Subsequences, arXiv:0801.1981 (http://arxiv.org/pdf/0801.1981.pdf)

External links

- Alexander Bogomolny. Farey series (http://www.cut-the-knot.org/blue/Farey.shtml) and Stern-Brocot Tree (http://www.cut-the-knot.org/blue/Stern.shtml) at Cut-the-Knot
- Ettore Pennestri'. A Brocot table of base 120 (https://www.researchgate.net/publication/297000899_Kinematic_synthesis_of_gear_trains_A_Brocot_table_of_base_120)
- Hazewinkel, Michiel, ed. (2001), "Farey series", Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4
- Weisstein, Eric W. "Stern-Brocot Tree". MathWorld.

- Farey Sequence (http://oeis.org/A005728) from The On-Line Encyclopedia of Integer Sequences (http://oeis.org/).
- Bonahon, Francis. "Funny Fractions and Ford Circles" (YouTube video). Brady Haran. Retrieved 9 June 2015.

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