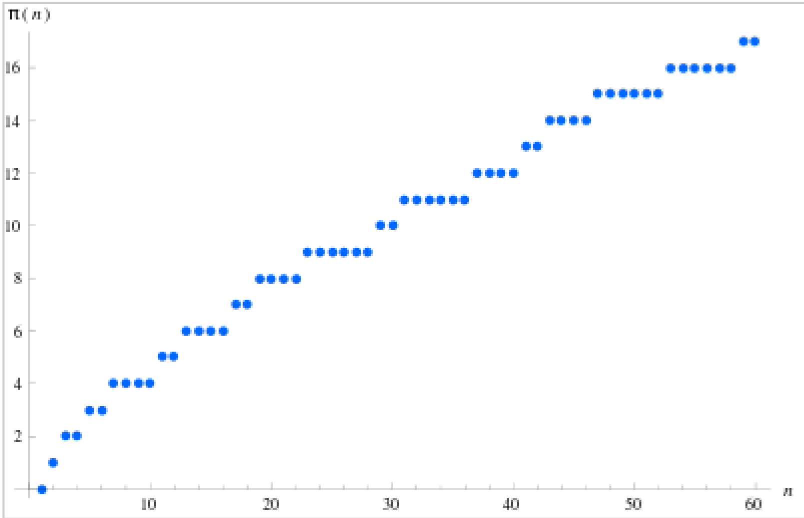
Prime-counting function

From Wikipedia, the free encyclopedia

In mathematics, the prime-counting function is the function counting the number of prime numbers less than or equal to some real number x.[1][2] It is denoted by π(x) (unrelated to the number π).

Contents



The values of π(n) for the first 60 integers

History

Of great interest in number theory is the growth rate of the prime-counting function.[3][4] It was conjectured in the end of the 18th century by Gauss and by Legendre to be approximately

$$x/\ln(x)$$

in the sense that

$$\lim_{x\to\infty}\frac{\pi(x)}{x/\ln(x)}=1.$$

This statement is the prime number theorem. An equivalent statement is

$$\lim_{x\to\infty}\pi(x)/\operatorname{li}(x)=1$$

where li is the logarithmic integral function. The prime number theorem was first proved in 1896 by Jacques Hadamard and by Charles de la Vallée Poussin independently, using properties of the Riemann zeta function introduced by Riemann in 1859.

More precise estimates of $\pi(x)$ are now known; for example

$$\pi(x)=\operatorname{li}(x)+O\big(xe^{-\sqrt{\ln x}/15}\big)$$

where the O is big O notation. For most values of x we are interested in (i.e., when x is not unreasonably large) $\operatorname{li}(x)$ is greater than $\pi(x)$, but infinitely often the opposite is true. For a discussion of this, see Skewes' number.

Proofs of the prime number theorem not using the zeta function or complex analysis were found around 1948 by Atle Selberg and by Paul Erdős (for the most part independently).[5]

Table of π(x), x / ln x, and li(x)

The table shows how the three functions π(x), x / ln x and li(x) compare at powers of 10. See also,[3][6][7] and[8]

x	π(x)	π(x) − x / ln x	li(x) − π(x)	x / π(x)
10	4	−0.3	2.2	2.500
10^2	25	3.3	5.1	4.000
10^3	168	23	10	5.952
10^4	1,229	143	17	8.137
10^5	9,592	906	38	10.425
10^6	78,498	6,116	130	12.740
10^7	664,579	44,158	339	15.047
10^8	5,761,455	332,774	754	17.357
10^9	50,847,534	2,592,592	1,701	19.667
10^{10}	455,052,511	20,758,029	3,104	21.975
10^{11}	4,118,054,813	169,923,159	11,588	24.283
10^{12}	37,607,912,018	1,416,705,193	38,263	26.590
10^{13}	346,065,536,839	11,992,858,452	108,971	28.896
10^{14}	3,204,941,750,802	102,838,308,636	314,890	31.202
10^{15}	29,844,570,422,669	891,604,962,452	1,052,619	33.507
10^{16}	279,238,341,033,925	7,804,289,844,393	3,214,632	35.812
10^{17}	2,623,557,157,654,233	68,883,734,693,281	7,956,589	38.116
10^{18}	24,739,954,287,740,860	612,483,070,893,536	21,949,555	40.420
10^{19}	234,057,667,276,344,607	5,481,624,169,369,960	99,877,775	42.725
10^{20}	2,220,819,602,560,918,840	49,347,193,044,659,701	222,744,644	45.028
10^{21}	21,127,269,486,018,731,928	446,579,871,578,168,707	597,394,254	47.332
10^{22}	201,467,286,689,315,906,290	4,060,704,006,019,620,994	1,932,355,208	49.636
10^{23}	1,925,320,391,606,803,968,923	37,083,513,766,578,631,309	7,250,186,216	51.939
10^{24}	18,435,599,767,349,200,867,866	339,996,354,713,708,049,069	17,146,907,278	54.243
10^{25}	176,846,309,399,143,769,411,680	3,128,516,637,843,038,351,228	55,160,980,939	56.546
10^{26}	1,699,246,750,872,437,141,327,603	28,883,358,936,853,188,823,261	155,891,678,121	58.850

In the On-Line Encyclopedia of Integer Sequences, the π(x) column is sequence A006880, π(x) − x / ln x is sequence A057835, and li(x) − π(x) is sequence A057752.

The value for π(10^{24}) was originally computed by J. Buethe, J. Franke, A. Jost, and T. Kleinjung assuming the Riemann hypothesis.[9] It was later verified unconditionally in a computation by D. J. Platt.[10] The value for π(10^{25}) is due to J. Buethe, J. Franke, A. Jost, and T. Kleinjung.[11] The value for π(10^{26}) was computed by D. B. Staple.[12] All other entries in this table were also verified as part of that work.

Algorithms for evaluating π(x)

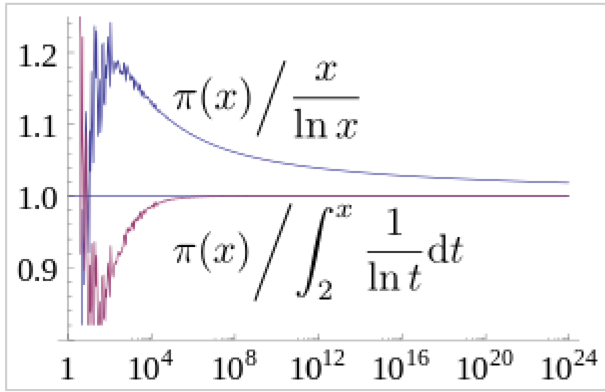
A simple way to find $\pi(x)$, if x is not too large, is to use the sieve of Eratosthenes to produce the primes less than or equal to x and then to count them.

A more elaborate way of finding $\pi(x)$ is due to Legendre: given x, if $p_1, p_2, \ldots, p_n$ are distinct prime numbers, then the number of integers less than or equal to x which are divisible by no p_i is

$$\lfloor x \rfloor - \sum_i \left\lfloor \frac{x}{p_i} \right\rfloor + \sum_{i<j} \left\lfloor \frac{x}{p_i p_j} \right\rfloor - \sum_{i<j<k} \left\lfloor \frac{x}{p_i p_j p_k} \right\rfloor + \cdots$$

(where $\lfloor \cdots \rfloor$ denotes the floor function). This number is therefore equal to

$$\pi(x) - \pi\left(\sqrt{x}\right) + 1$$



Graph showing ratio of the prime-counting function π(x) to two of its approximations, x/ln x and Li(x). As x increases (note x axis is logarithmic), both ratios tend towards 1. The ratio for x/ln x converges from above very slowly, while the ratio for Li(x) converges more quickly from below.

when the numbers $p_1, p_2, \ldots, p_n$ are the prime numbers less than or equal to the square root of x.

In a series of articles published between 1870 and 1885, Ernst Meissel described (and used) a practical combinatorial way of evaluating $\pi(x)$. Let p_1, $p_2, \ldots, p_n$ be the first n primes and denote by $\Phi(m,n)$ the number of natural numbers not greater than m which are divisible by no p_i. Then

$$\Phi(m,n) = \Phi(m,n-1) - \Phi\left(\frac{m}{p_n}, n-1\right)$$

Given a natural number m, if $n = \pi\left(\sqrt[3]{m}\right)$ and if $\mu = \pi\left(\sqrt{m}\right) - n$, then

$$\pi(m) = \Phi(m,n) + n(\mu+1) + \frac{\mu^2-\mu}{2} - 1 - \sum_{k=1}^{\mu} \pi\left(\frac{m}{p_{n+k}}\right)$$

Using this approach, Meissel computed $\pi(x)$, for x equal to 5×10^5, 10^6, 10^7, and 10^8.

In 1959, Derrick Henry Lehmer extended and simplified Meissel's method. Define, for real m and for natural numbers n and k, $P_k(m,n)$ as the number of numbers not greater than m with exactly k prime factors, all greater than p_n. Furthermore, set $P_0(m,n) = 1$. Then

$$\Phi(m,n) = \sum_{k=0}^{+\infty} P_k(m,n)$$

where the sum actually has only finitely many nonzero terms. Let y denote an integer such that $\sqrt[3]{m} \le y \le \sqrt{m}$, and set $n = \pi(y)$. Then $P_1(m,n) = \pi(m) - n$ and $P_k(m,n) = 0$ when $k \geqslant 3$. Therefore,

$$\pi(m) = \Phi(m,n) + n - 1 - P_2(m,n)$$

The computation of $P_2(m,n)$ can be obtained this way:

$$P_2(m,n) = \sum_{y<p\le\sqrt{m}} \left(\pi\left(\frac{m}{p}\right) - \pi(p) + 1\right),$$

where the sum is over prime numbers.

On the other hand, the computation of $\Phi(m,n)$ can be done using the following rules:

1. $\Phi(m,0) = \lfloor m \rfloor$
2. $\Phi(m,b) = \Phi(m,b-1) - \Phi\left(\frac{m}{p_b}, b-1\right)$

Using his method and an IBM 701, Lehmer was able to compute $\pi\left(10^{10}\right)$.

Further improvements to this method were made by Lagarias, Miller, Odlyzko, Deléglise and Rivat.[13]

Other prime-counting functions

Other prime-counting functions are also used because they are more convenient to work with. One is Riemann's prime-counting function, usually denoted as $\Pi_0(x)$ or $J_0(x)$. This has jumps of $1/n$ for prime powers p^n, with it taking a value halfway between the two sides at discontinuities. That added detail is because then it may be defined by an inverse Mellin transform. Formally, we may define $\Pi_0(x)$ by

$$\Pi_0(x) = \frac{1}{2}\left(\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \le x} \frac{1}{n}\right)$$

where p is a prime.

We may also write

$$\Pi_0(x) = \sum_2^x \frac{\Lambda(n)}{\ln n} - \frac{1}{2}\frac{\Lambda(x)}{\ln x} = \sum_{n=1}^\infty \frac{1}{n}\pi_0(x^{1/n})$$

where $\Lambda(n)$ is the von Mangoldt function and

$$\pi_0(x) = \lim_{\varepsilon \to 0} \frac{\pi(x-\varepsilon) + \pi(x+\varepsilon)}{2}.$$

The Möbius inversion formula then gives

$$\pi_0(x) = \sum_{n=1}^\infty \frac{\mu(n)}{n}\Pi_0(x^{1/n})$$

Knowing the relationship between log of the Riemann zeta function and the von Mangoldt function Λ, and using the Perron formula we have

$$\ln \zeta(s) = s \int_0^\infty \Pi_0(x) x^{-s-1}\, dx$$

The Chebyshev function weights primes or prime powers p^n by $\ln(p)$:

$$\theta(x) = \sum_{p \le x} \ln p$$

$$\psi(x) = \sum_{p^n \le x} \ln p = \sum_{n=1}^\infty \theta(x^{1/n}) = \sum_{n \le x} \Lambda(n).$$

Formulas for prime-counting functions

Formulas for prime-counting functions come in two kinds: arithmetic formulas and analytic formulas. Analytic formulas for prime-counting were the first used to prove the prime number theorem. They stem from the work of Riemann and von Mangoldt, and are generally known as explicit formulas.[14]

We have the following expression for ψ:

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \ln 2\pi - \frac{1}{2}\ln(1 - x^{-2})$$

where

$$\psi_0(x) = \lim_{\varepsilon \to 0} \frac{\psi(x-\varepsilon) + \psi(x+\varepsilon)}{2}.$$

Here ρ are the zeros of the Riemann zeta function in the critical strip, where the real part of ρ is between zero and one. The formula is valid for values of x greater than one, which is the region of interest. The sum over the roots is conditionally convergent, and should be taken in order of increasing absolute value of the imaginary part. Note that the same sum over the trivial roots gives the last subtrahend in the formula.

For $\Pi_0(x)$ we have a more complicated formula

$$\Pi_0(x) = \operatorname{li}(x) - \sum_\rho \operatorname{li}(x^\rho) - \ln 2 + \int_x^\infty \frac{dt}{t(t^2-1)\ln t}.$$

Again, the formula is valid for x > 1, while ρ are the nontrivial zeros of the zeta function ordered according to their absolute value, and, again, the latter integral, taken with minus sign, is just the same sum, but over the trivial zeros. The first term li(x) is the usual logarithmic integral function; the expression li(x^ρ) in the second term should be considered as Ei(ρ ln x), where Ei is the analytic continuation of the exponential integral function from positive reals to the complex plane with branch cut along the negative reals.

Thus, Möbius inversion formula gives us[15]

$$\pi_0(x) = \operatorname{R}(x) - \sum_\rho \operatorname{R}(x^\rho) - \frac{1}{\ln x} + \frac{1}{\pi}\arctan\frac{\pi}{\ln x}$$

valid for x > 1, where

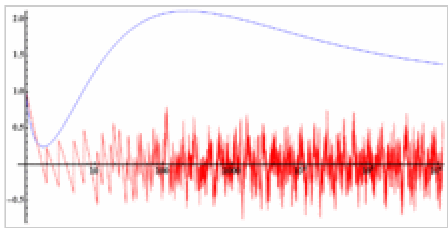
$$\operatorname{R}(x) = \sum_{n=1}^\infty \frac{\mu(n)}{n}\operatorname{li}(x^{1/n}) = 1 + \sum_{k=1}^\infty \frac{(\ln x)^k}{k!k\zeta(k+1)}$$

is so-called Riemann's R-function.[16] The latter series for it is known as Gram series [17] and converges for all positive x.

The sum over non-trivial zeta zeros in the formula for $\pi_0(x)$ describes the fluctuations of $\pi_0(x)$, while the remaining terms give the "smooth" part of prime-counting function,[18] so one can use

$$\operatorname{R}(x) - \frac{1}{\ln x} + \frac{1}{\pi}\arctan\frac{\pi}{\ln x}$$

as the best estimator (http://primefan.ru:8014/WWW/stuff/primes/best_estimator.gif) of $\pi(x)$ for x > 1.


Δ-function (red line) on log scale

The amplitude of the "noisy" part is heuristically about $\sqrt{x}/\ln x$, so the fluctuations of the distribution of primes may be clearly represented with the Δ-function:

$$\Delta(x) = \left(\pi_0(x) - \mathrm{R}(x) + \frac{1}{\ln x} - \frac{1}{\pi}\arctan\frac{\pi}{\ln x}\right)\frac{\ln x}{\sqrt{x}}.$$

An extensive table of the values of Δ(x) is available.[7]

Inequalities

Here are some useful inequalities for π(x).

$$\frac{x}{\ln x} < \pi(x) < 1.25506\frac{x}{\ln x}$$ for x ⩾ 17.[19]

The left inequality holds for x ⩾ 17 and the right inequality holds for x > 1.

An explanation of the constant 1.25506 is given at (sequence A209883 in the OEIS).

Pierre Dusart proved in 2010:

$$\frac{x}{\ln x - 1} < \pi(x)$$ for $x \geq 5393$, and

$$\pi(x) < \frac{x}{\ln x - 1.1}$$ for $x \geq 60184$.[20]

Here are some inequalities for the nth prime, p_n.[21]

$$n(\ln(n \ln n) - 1) < p_n < n\ln(n \ln n)$$ for n ⩾ 6.

The left inequality holds for n ⩾ 1 and the right inequality holds for n ⩾ 6.

An approximation for the nth prime number is

$$p_n = n(\ln(n\ln n) - 1) + \frac{n(\ln\ln n - 2)}{\ln n} + O\left(\frac{n(\ln\ln n)^2}{(\ln n)^2}\right).$$

In his well-known notebooks, Ramanujan[22] proves that the inequality

$$\pi(x)^2 < \frac{ex}{\log x}\pi\left(\frac{x}{e}\right)$$

holds for all sufficiently large values of x.

The Riemann hypothesis

The Riemann hypothesis is equivalent to a much tighter bound on the error in the estimate for $\pi(x)$, and hence to a more regular distribution of prime numbers,

$$\pi(x) = \mathrm{li}(x) + O(\sqrt{x}\log x).$$

Specifically,[23]

$$|\pi(x) - \mathrm{li}(x)| < \frac{1}{8\pi}\sqrt{x}\,\log x, \qquad \text{for all } x \geq 2657.$$

See also

- Bertrand's postulate
- Oppermann's conjecture
- Foias constant

References

1. Bach, Eric; Shallit, Jeffrey (1996). Algorithmic Number Theory. MIT Press. volume 1 page 234 section 8.8. ISBN 0-262-02405-5.
2. Weisstein, Eric W. "Prime Counting Function". MathWorld.
3. "How many primes are there?". Chris K. Caldwell. Retrieved 2008-12-02.
4. Dickson, Leonard Eugene (2005). History of the Theory of Numbers, Vol. I: Divisibility and Primality. Dover Publications. ISBN 0-486-44232-2.
5. Ireland, Kenneth; Rosen, Michael (1998). A Classical Introduction to Modern Number Theory (Second ed.). Springer. ISBN 0-387-97329-X.
6. "Tables of values of pi(x) and of pi2(x)". Tomás Oliveira e Silva. Retrieved 2008-09-14.
7. "Values of π(x) and Δ(x) for various x's". Andrey V. Kulsha. Retrieved 2008-09-14.
8. "A table of values of pi(x)". Xavier Gourdon, Pascal Sebah, Patrick Demichel. Retrieved 2008-09-14.
9. "Conditional Calculation of pi(10^{24})". Chris K. Caldwell. Retrieved 2010-08-03.
10. Platt, David J. (2012). "Computing π(x) Analytically)". arXiv:1203.5712 [math.NT].
11. "How Many Primes Are There?". J. Buethe. Retrieved 2015-09-01.
12. "The combinatorial algorithm for computing pi(x)". Dalhousie University. Retrieved 2015-09-01.
13. "Computing π(x): The Meissel, Lehmer, Lagarias, Miller, Odlyzko method" (PDF). Marc Deléglise and Jöel Rivat, Mathematics of Computation, vol. 65, number 33, January 1996, pages 235–245. Retrieved 2008-09-14.
14. Titchmarsh, E.C. (1960). The Theory of Functions, 2nd ed. Oxford University Press.
15. Riesel, Hans; Göhl, Gunnar (1970). "Some calculations related to Riemann's prime number formula". Mathematics of Computation. American Mathematical Society. 24 (112): 969–983. doi:10.2307/2004630. ISSN 0025-5718. JSTOR 2004630. MR 0277489.
16. Weisstein, Eric W. "Riemann Prime Counting Function". MathWorld.
17. Weisstein, Eric W. "Gram Series". MathWorld.
18. "The encoding of the prime distribution by the zeta zeros". Matthew Watkins. Retrieved 2008-09-14.
19. Rosser, J. Barkley; Schoenfeld, Lowell (1962). "Approximate formulas for some functions of prime numbers". Illinois J. Math. 6: 64–94. ISSN 0019-2082. Zbl 0122.05001.
20. Dusart, Pierre. "Estimates of Some Functions Over Primes without R.H." (PDF). Retrieved 22 April 2014.
21. Inequalities for the n-th prime number at function.wolfram, retrieved March 22, 2013
22. Berndt, Bruce C. (2012-12-06). Ramanujan's Notebooks. Springer Science & Business Media. ISBN 9781461209652.
23. Schoenfeld, Lowell (1976). "Sharper bounds for the Chebyshev functions θ(x) and ψ(x). II". Mathematics of Computation. American Mathematical Society. 30 (134): 337–360. doi:10.2307/2005976. ISSN 0025-5718. JSTOR 2005976. MR 0457374.

External links

- Chris Caldwell, The Nth Prime Page (http://primes.utm.edu/nthprime/) at The Prime Pages.
- Tomás Oliveira e Silva, Tables of prime-counting functions (http://sweet.ua.pt/tos/primes.html).