Coding Review (I2P 2019)

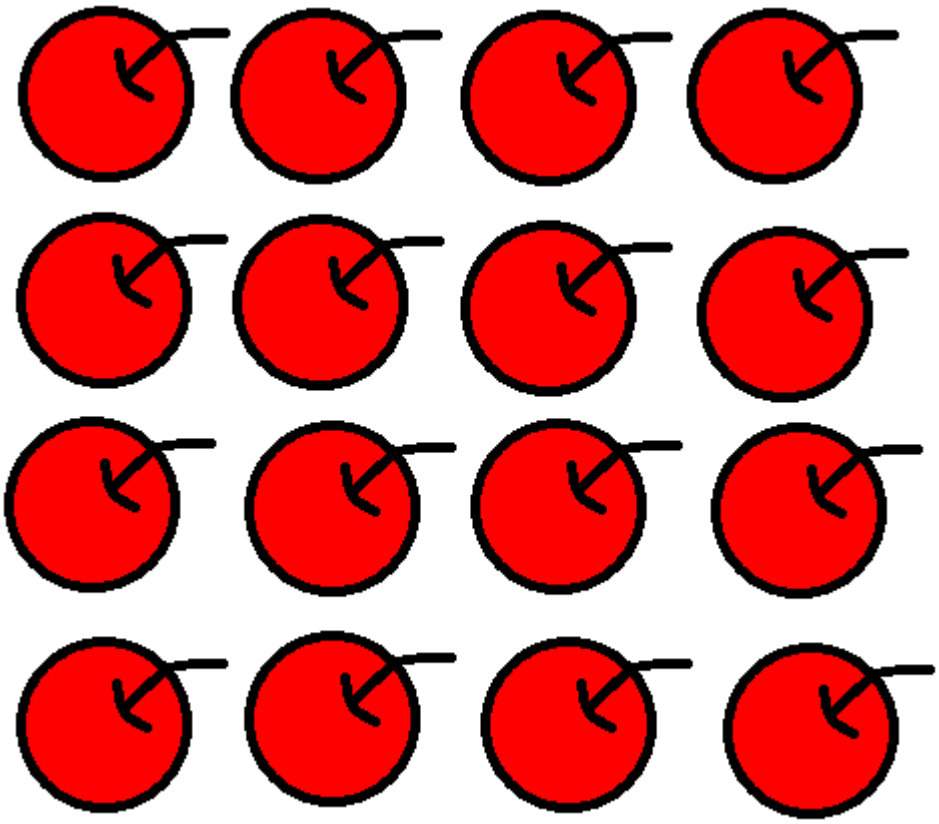
Binary Representations v1.0

Review of binary representations.

Number Basis (Radix)

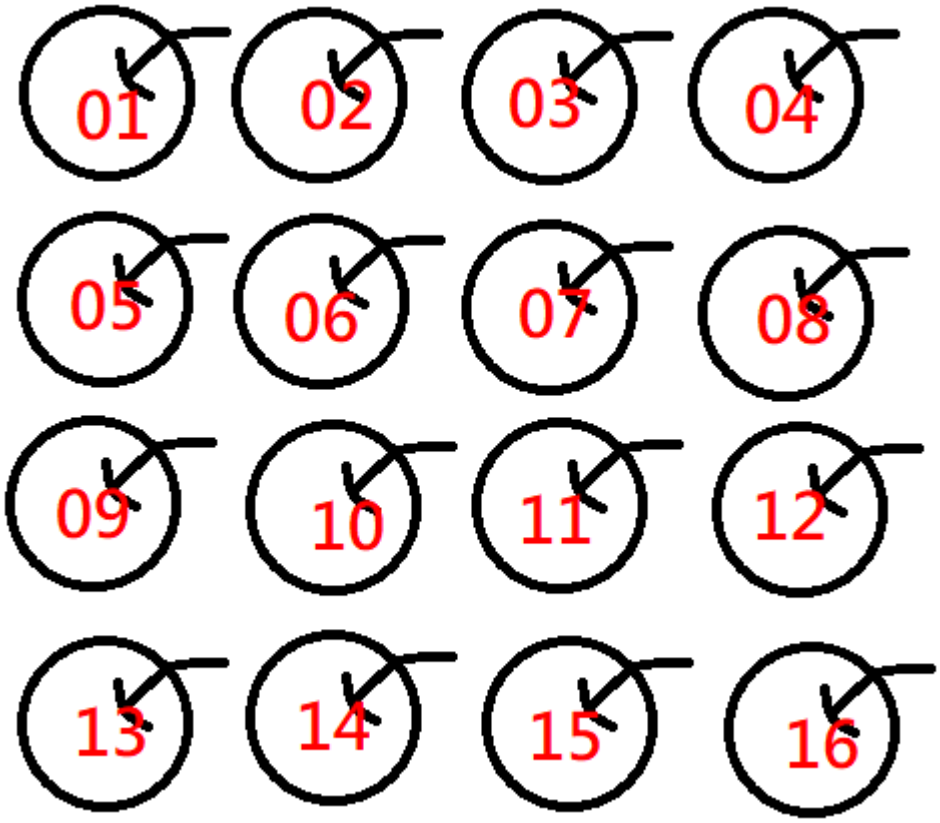
Decimal, Binary, Octal, Hexadecimal are the common number basis seen in Computer Science courses.

- Decimal: All digits are 0 to 9, carry occurs at ten, so each digits have ten different values. (0, 1, ..., 9)
- Binary: All digits are 0 to 1, carry occurs at two, so each digits have two different values. (0, 1)
- Octal: All digits are 0 to 7, carry occurs at eight, so each digits have eight different values. (0, 1, ..., 7)
- Hexadecimal: All digits are 0 to 15 (F), carry occurs at sixteen, so each digits have sixteen different values. (0, 1, ..., 9, A, B, C, D, E, F)



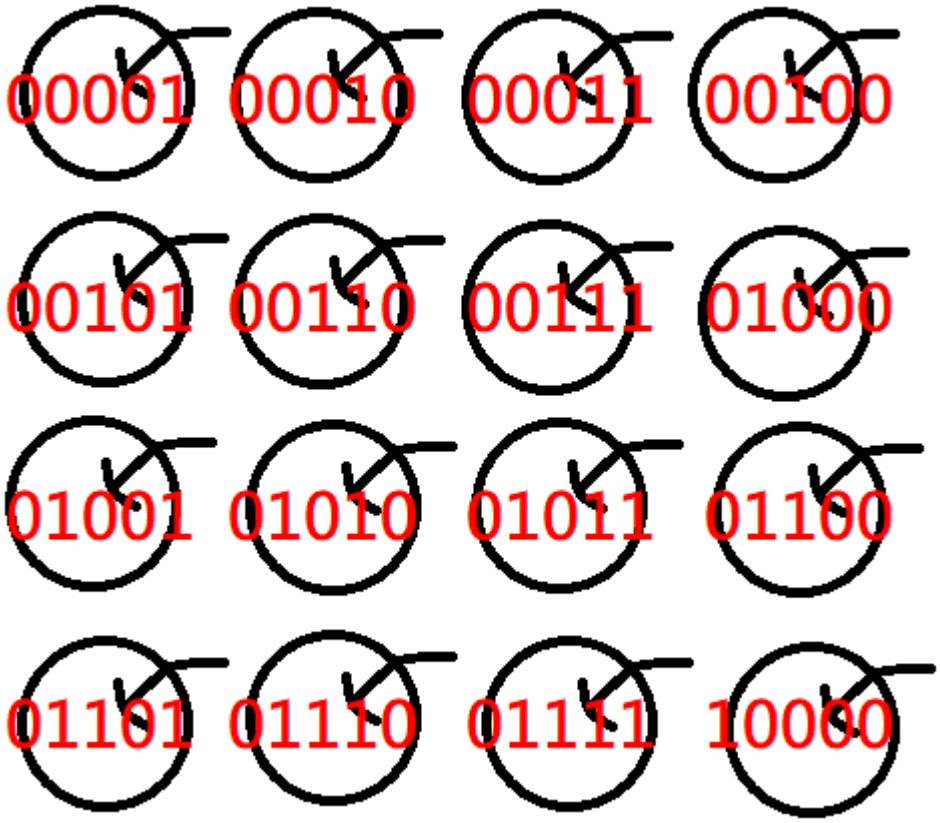
1. So, if we have these apples, counting in Decimal should be:

   Answer:



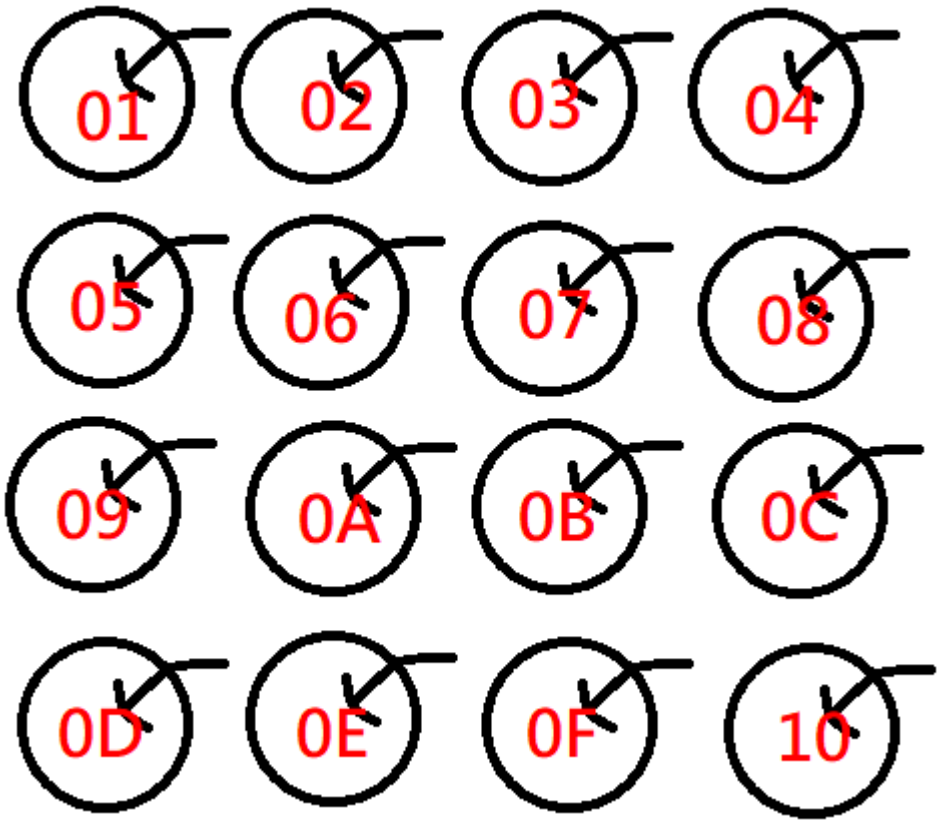
2. Counting in Binary should be:

   Answer:



3. Counting in Hexadecimal should be:

Answer:



Integer Conversion between Number Bases

Any Number Basis to Decimal

Since we're used to Decimal, so converting any other number basis to Decimal is an easy task for us. We can simply multiply each digits by the power of the number basis.

Take sixty one ($61_{10}, 111101_2, 3\mathrm{D}_{16}$) as example:

1. Decimal to decimal (61_{10})

   Answer:

10^6	10^5	10^4	10^3	10^2	10^1	10^0
0	0	0	0	0	6	1

   In decimal form: $61_{10} = 0 \cdot 10^2 + 6 \cdot 10^1 + 1 \cdot 10^0 = 61_{10}$

2. Binary to Decimal (111101_2)

   Answer:

2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	1	1	1	1	0	1

In decimal form: $111101_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 61_{10}$

3. Hexadecimal to Decimal ($3\mathrm{D}_{16}$)

   Answer:

16^6	16^5	16^4	16^3	16^2	16^1	16^0
0	0	0	0	0	3	D

   In decimal form: $3\mathrm{D}_{16} = 0 \cdot 16^2 + 3 \cdot 16^1 + 13 \cdot 16^0 = 61_{10}$

Some more practices:

1. Binary to Decimal (1010111_2)

   Answer:

   87_{10}

2. Hexadecimal to Decimal ($6\mathrm{B}_{16}$)

   Answer:

   107_{19}

Decimal to Any Number Basis

Converting to any other number basis is also quite easy. We can simply keep dividing by the number basis and write down the remainder from right to left after each division. This is actually an inverse operation of converting to Decimal.

Take sixty one ($61_{10}, 111101_2, 3\mathrm{D}_{16}$) as example:

1. Decimal to Hexadecimal (61_{10})

   Answer:

   Calculate in Decimal.

   $61/16 = 3....13$, write down 13_{10}, which is D_{16}.

16^6	16^5	16^4	16^3	16^2	16^1	16^0
						D

   $3/16 = 0....3$, write down 3_{16}.

16^6	16^5	16^4	16^3	16^2	16^1	16^0
					3	D

   The process is over once the quotient becomes zero.

2. Decimal to Binary (61_{10})

   Answer:

   Calculate in Decimal.

   $61/2 = 30....1$, write down 1_2.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
						1

   $30/2 = 15....0$, write down 0_2.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
					0	1

   $15/2 = 7....1$, write down 1_2.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
				1	0	1

   $7/2 = 3....1$, write down 1_2.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
			1	1	0	1

   $3/2 = 1....1$, write down 1_2.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
		1	1	1	0	1

   $1/2 = 0....1$, write down 1_2.

2^6	2^5	2^4	2^3	2^2	2^1	2^0
	1	1	1	1	0	1

   The process ends since the quotient becomes zero.

Some more practices:

1. Decimal to Binary (173_{10})

   Answer:

   10101101_2

2. Decimal to Hexadecimal (173_{10})

Answer:

AD_{16}

Any Basis to Any Basis

You can convert use Decimal as an intermediate basis when converting between any basis. The direct conversion process is similar to the process above. You can try to come up with the direct conversion solution by yourself.

1. Binary to Hexadecimal (11111011_2)

   Answer:

   FB_{16}

2. Hexadecimal to Binary (AC_{16})

   Answer:

   10101100_2

Binary and Hexadecimal are the number bases commonly used in programming. You can find the special relationship ($2^4 = 16$) between them, that is each four binary digits can be converted independently into one hexadecimal digit.

Fraction Conversion between Number Bases

1. Binary to Decimal (10.1101_2)

   Answer:

   2.8125_{10}

   Follow the same process:

   $10.1101_2 = 6.8125_{10}$

2^1	2^0	.	2^{-1}	2^{-2}	2^{-3}	2^{-4}
1	0	.	1	1	0	1

   In decimal form: $10.1101_2 = 2 + 0.5 + 0.25 + 0.0625 = 2.8125_{10}$

2. Decimal to Binary (3.6875_{10})

   Answer:

   11.1011_{10}

   This requires some thinking. You can perform the same operation on the digits before the radix point. And then, we clear those converted digits, and start multiplying by the basis, while keep taking away the digits before the radix point.

$11.1011_2 = 3.6875_{10}$

2^1	2^0	.	2^{-1}	2^{-2}	2^{-3}	2^{-4}
1	1	.	1	0	1	1

In decimal form:

$0.6875 \cdot 2 = 1 + 0.375$

$0.375 \cdot 2 = 0 + 0.75$

$0.75 \cdot 2 = 1 + 0.5$

$0.5 \cdot 2 = 1 + 0.0$

3. Decimal to Binary (0.87_{10})

   Answer:

   $0.11011110....$

   Some numbers cannot be represented precisely using other number bases. This also happens in some Decimal fractions.

Online calculators can be found here:

- Binary to Decimal converter
- Decimal to Binary converter
- Exploring Binary

Binary Representations

Representation of Integers

Now we know how to represent positive integers in binary format, but how about negative integers?

We can simply add a bit to represent whether the number is negative. However, it's not straight forward to do additions and subtractions in the hardware level.

So we use 2's complement representation to make addition easier. The concept is simple, we want $N + (-N) = 0$, so if we have the binary representation of N, we can calculate $(-N)$ by inverting the digits and adding one.

1. What is the 2's complement binary representation of $(-6)_{10}$ if we have a total of four bits?

   Answer:

   1010_2

   So if we add 6_{10} with $(-6)_{10}$,

```
 0110
+1010
------
 0000
```

The carry bit is discarded.

4-bit 2's Complement Table:

Bit Pattern	2's Complement
0111	7
0110	6
...	...
0010	2
0001	1
0000	0
1111	-1
1110	-2
...	...
1001	-7
1000	-8

It's easy to see that the representable range for N-bit signed integer is $[-2^{N-1}, 2^{N-1}-1]$. The fact that the positive range is smaller than the negative range can be easily seen by the left-most bit, since 0 is only represented once here.

Representation of Fractions

For representing fractions, we can simply use a signed-bit since the addition cannot be simplified as seen in 2's complement.

Floating point representation is used instead of Fixed point representation to increase the representable range.

We can further use Excess Notation to speed up comparison between numbers. The concept is to make the exponent term directly comparable through bit-patterns, that is, make the bit pattern of the smallest representable exponent as $0...0$, and keep adding up. This exponent is put at front of the mantissa so that we can compare two floating-points like integers. See the table below:

Excess-8 Notation Table:

Bit Pattern	Excess-8 Notation

Bit Pattern	Excess-8 Notation
1111	7
1110	6
...	...
1010	2
1001	1
1000	0
0111	-1
0110	-2
...	...
0001	-7
0000	-8

1. What is the floating-point representation of -2.625_{10}, with 4-bit fraction and 3-bit exponent using IEEE Standard.

   Answer:

   11010101_2

   $-2.625_{10} = -10.101_2 = -1.0101_2 \cdot (2_{10})^{1_2}$

Sign	Exponent	Fraction (Mantissa)
1	101	0101

   The `Sign`, `Exponent`, `Mantissa` order here makes the number comparable as if they were 2's complement integers.

2. What is the truncation error of 0.1_{10} (direct truncation), with 4-bit fraction and 3-bit exponent using IEEE Standard.

   Answer:

   0.00234375

   $0.1_{10} = 0.000110011..._2 = 1.10011..._2 \cdot (2_{10})^{-4_2}$

Sign	Exponent	Fraction (Mantissa)
0	000	1001

   Converting back: $1.1001_2 \cdot (2_{10})^{-4_2} = 0.09765625 \quad |0.1 - 0.09765625| = 0.00234375$

The direct truncation is used for simplicity, different rounding techniques are used in the real world.

For further information, there's also a method for 2's complement multiplication, which is much harder to think intuitively. And there's also a 1's complement representation, which is quite straight forward. For floating-point representation there are Infinities, NaNs and Denormalized Numbers.

To wrap up, the common used representations are:

- Signed Magnitude
- One's Complement
- Two's Complement
- Excess-N Notation

Overflow & Underflow & Truncation

- In unsigned representation

  When two unsigned numbers are added, overflow occurs if there is a carry out of the leftmost bit.

- In signed 2's complement representation

  Overflow occurs when both operands are positive and the result is negative. Or both operands are negative but the result is positive.

- In float representation

  Underflow occurs when the result of a floating point representation is smaller than the smallest value representable.

  When a number cannot be precisely represented, the number is truncated.

Try it by Yourself

Here's a C code that shows the binary representation of any variable.

```c
#include <stdio.h>
#include <limits.h>

void printBits_LittleEndian(size_t const size, void const* const ptr) {
    unsigned char* b = (unsigned char*) ptr;
    unsigned char byte;
    int i, j;

    for (i = size - 1; i >= 0; i--) {
        for (j = 7; j >= 0; j--) {
            byte = (b[i] >> j) & 1;
            printf("%u", byte);
        }
    }
    puts("");
```

```
}

void printBits_BigEndian(size_t const size, void const* const ptr) {
    unsigned char* b = (unsigned char*) ptr;
    unsigned char byte;
    int i, j;

    for (i = size - 1; i >= 0; i--) {
        for (j = 0; j < 8; j++) {
            byte = (b[i] >> j) & 1;
            printf("%u", byte);
        }
    }
    puts("");
}

int main(void) {
    int i = 23;
    unsigned int ui = UINT_MAX;
    float f = 23.45f;
    puts("For Little Endian:");
    printBits_LittleEndian(sizeof i, &i);
    printBits_LittleEndian(sizeof ui, &ui);
    printBits_LittleEndian(sizeof f, &f);
    // puts("For Big Endian:");
    // printBits_BigEndian(sizeof i, &i);
    // printBits_BigEndian(sizeof ui, &ui);
    // printBits_BigEndian(sizeof f, &f);
    return 0;
}
```

The output should be:

```
For Little Endian:
00000000000000000000000000010111
11111111111111111111111111111111
01000001101110111001100110011010
```

If your output is different than above, try un-commenting the Big Endian code below and see the results. This part is just for fun and you can see the actual binary representation for any type of variables. You'll learn more about Endians in Computer Architecture course.

If you look closely enough, you'll find that IEEE-754 actually uses a Excess-127 (127 bias) notation instead of Excess-128. If you are interested in the rationale of this, see:

- Why does the IEEE 754 standard use a 127 bias?
- What is a “bias value” of floating-point numbers?

By thinking the process inside out and think of the intuition behind the representations, you never need to memorize these representations or calculations.

For the next assignment, we'll review the basic concept of arrays and pointers.

Epilogue

> Q: Why do programmers confuse Halloween and Christmas?
>
> A: Because Oct 31 equals Dec 25.

If there's any typo, please discuss on iLMS or email `j3soon@gapp.nthu.edu.tw`, I appreciate your help.