Homework 5

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April 2, 2017

1 Problem 1

See crt.m function and problem1_diary.txt attached. Details are included in the comments of crt.m function.

2 Problem 2

- (a)
  See function probability.m. Details are included in the comments of probability.m function.

- (b)
  $\rho = 0.6078$ when $M = N = 10000000$. See problem2_diary.txt

- (c)
  Since p(two independently selected integers have common divisor of 2) + p(two independently selected integers don't have common divisor of 2) = 1,
  So p(two independently selected integers don't have common divisor of 2)
  = 1 - p(two independently selected integers have common divisor of 2)
  = 1 - p(first integer have divisor of 2)*p(second integer have divisor of 2)
  = 1 - $(\frac{1}{2})^2$
  This is because that first integer have divisor of 2 and that second integer have divisor of 2 are independent of each other. Similarly,
  p(two independently selected integers don't have common divisor of 3)
  = 1 - p(two independently selected integers have common divisor of 3)
  = 1 - p(first integer have divisor of 3)*p(second integer have divisor of 3)
  = 1 - $(\frac{1}{3})^2$

- (d)
  According to the result from (c), and fundamental theorem of arithmetic, all integers can be written as product of prime numbers, if two independently selected integers don't have any prime common divisor, they must don't have any common divisor > 1.

So ρ = p(two independently selected integers don't have common divisor greater than 1)
= p(two independently selected integers don't have common divisor of 2)
$*$ p(two independently selected integers don't have common divisor of 3)
$*$ p(two independently selected integers don't have common divisor of 5)
$*...$
$= \prod_{p\in\mathcal{P}}(1 - \frac{1}{p^2})$, where P denotes the set of all prime numbers.

- (e)
$\rho = 0.6079$ for all primes less than 10000. See function rho.m and problem2_diary.txt

3 Problem 3

- (a)
$\zeta(2) = 1.6449$ See function rz.m and problem3_diary.txt (approximated to 4 digits after decimal point)

- (b)
According to fundamental theorem of arithmetic, all integers can be written as product of prime numbers, so any $n = p_1^{a_1} * p_2^{a_2} * ... * p_n^{a_n}$, where $p_1, p_2, ..., p_n$ are all the prime numbers.
LHS:
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{\infty^s}$
$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{(p_1^{a_1} * p_2^{a_2} * ... * p_n^{a_n})^s} + ... + \frac{1}{\infty^s}$
$= \sum_{n=1}^{\infty} \frac{1}{(p_1^{a_1} * p_2^{a_2} * ... * p_n^{a_n})^s}$ for all the combinations of exponent's values a_1 through a_n.
RHS: $\prod_{p\in\mathcal{P}}(1 - \frac{1}{p^s})^{-1}$
$= (1 - \frac{1}{2^s})^{-1} + (1 - \frac{1}{3^s})^{-1} + ... + (1 - \frac{1}{p_i^s})^{-1} + ...$
$= ((\frac{1}{2^s})^0 + \frac{1}{2^s} + (\frac{1}{2^s})^2 + (\frac{1}{2^s})^3 + ...) * ((\frac{1}{3^s})^0 + \frac{1}{3^s} + (\frac{1}{3^s})^2 + (\frac{1}{3^s})^3 + ...) * ((\frac{1}{5^s})^0 + \frac{1}{5^s} + (\frac{1}{5^s})^2 + (\frac{1}{5^s})^3 + ...) * ... * ((\frac{1}{p_i^s})^0 + \frac{1}{p_i^s} + (\frac{1}{p_i^s})^2 + (\frac{1}{p_i^s})^3 + ...) * ...$
$= (\frac{1}{2^{0s}} + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + ...) * (\frac{1}{3^{0s}} + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + ...) * (\frac{1}{5^{0s}} + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + ...) * ... * (\frac{1}{p_i^{0s}} + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} + ...) * ...$
$= [(\frac{1}{2^s})^0 * (\frac{1}{3^s})^0 * (\frac{1}{5^s})^0 * ... * (\frac{1}{p_i^s})^0 * ...] + ... + [\frac{1}{2^s} * \frac{1}{3^s} * \frac{1}{5^s} * ... * \frac{1}{p_i^s} * ...] + ... + [(\frac{1}{2^s})^{a_1} * (\frac{1}{3^s})^{a_2} * (\frac{1}{5^s})^{a_3} * ... * (\frac{1}{p_i^s})^{a_i} * ...] + ...$
$= [((\frac{1}{2})^0 * (\frac{1}{3})^0 * (\frac{1}{5})^0 * ... * (\frac{1}{p_i})^0 * ...)^s] + ... + [(\frac{1}{2} * \frac{1}{3} * \frac{1}{5} * ... * \frac{1}{p_i} * ...)^s] + ... + [((\frac{1}{2})^{a_1} * (\frac{1}{3})^{a_2} * (\frac{1}{5})^{a_3} * ... * (\frac{1}{p_i})^{a_i} * ...)^s] + ...$
$= \frac{1}{(2^0 * 3^0 * 5^0 ... * p_i^0 * ...)^s} + ... + \frac{1}{(2^1 * 3^1 * 5^1 ... * p_i^1 * ...)^s} + ... + \frac{1}{(2^{a_1} * 3^{a_2} * 5^{a_3} ... * p_i^{a_i} * ...)^s} + ...$
$= \sum_{n=1}^{\infty} \frac{1}{(p_1^{a_1} * p_2^{a_2} * ... * p_n^{a_n})^s}$ for all the combinations of exponent's values a_1 through a_n.
So LHS = RHS, $\zeta(s) = \prod_{p\in\mathcal{P}}(1 - \frac{1}{p^s})^{-1}$

- (c)

Since $\rho = \prod_{p \in \mathcal{P}} (1 - \frac{1}{p^2})$, where P denotes the set of all prime numbers.
So $\rho = \frac{1}{\prod_{p \in \mathcal{P}} (1 - \frac{1}{p^s})^{-1}} = \frac{1}{\zeta(s)}$
So $\rho = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2}$