VI. Below K

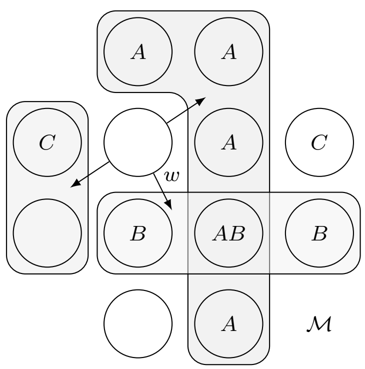
AS.150.498: Modal Logic and Its Applications
Johns Hopkins University, Spring 2017

So far we have considered a horde of *normal* modal logics extending the minimal logic **K** with additional axioms such as **T** and **5**. But there are also many *non-normal* modal logics that lie below **K**.

1 Neighborhood Semantics

In investigating these non-normal logics, it will be useful to replace the standard Kripke-model-based semantics with an alternative *neighborhood semantics* on which some principles of **K** can fail to hold. Instead of a binary accessibility relation between worlds in $\mathcal{W}$, our models will now include a relation between worlds and *neighborhoods* or *propositions*—sets of worlds in $2^{\mathcal{W}}$.

Definition 6.1. A **neighborhood model** $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ for $\mathcal{L}$ consists of a nonempty set $\mathcal{W}$ of possible worlds, a valuation $\mathcal{V} : At_{\mathcal{L}} \times \mathcal{W} \to \{T, F\}$ mapping each sentence letter $p \in At_{\mathcal{L}}$ and world $w \in \mathcal{W}$ to a truth value, and a binary relation $\mathcal{R} \subseteq \mathcal{W} \times 2^{\mathcal{W}}$ between worlds and (possibly empty) sets of worlds.



The relation $\mathcal{R}$ determines a **neighborhood function** $\mathcal{N} : \mathcal{W} \to 2^{2^{\mathcal{W}}}$ where $\mathcal{N}(w) = \{X : w\mathcal{R}X\}$ is the set of neighborhoods around w.

Definition 6.2. $[\varphi]_{\mathcal{M}} = \{w : [\![\varphi]\!]^w_{\mathcal{M}} = T\}$ is the **proposition expressed** by φ in $\mathcal{M}$.

Using the information supplied by a neighborhood model, we can evaluate every sentence $\varphi \in S_{\mathcal{L}}$ for **truth in a pointed neighborhood model**:

Definition 6.3. The following recursive definition of truth lifts $\mathcal{V}$ to the complete interpretation function $[\![\ ]\!]_{\mathcal{M}} : S_{\mathcal{L}} \times \mathcal{W} \to \{T, F\}$ for $\mathcal{L}$:

$$\begin{aligned}
[\![p]\!]^w_{\mathcal{M}} = T \quad &\text{iff} \quad \mathcal{V}(p, w) = T \\
[\![\bot]\!]^w_{\mathcal{M}} = T \quad &\text{iff} \quad 0 = 1 \\
[\![\neg\varphi]\!]^w_{\mathcal{M}} = T \quad &\text{iff} \quad [\![\varphi]\!]^w_{\mathcal{M}} = F \\
[\![(\varphi \wedge \psi)]\!]^w_{\mathcal{M}} = T \quad &\text{iff} \quad [\![\varphi]\!]^w_{\mathcal{M}} = [\![\psi]\!]^w_{\mathcal{M}} = T \\
[\![\Box\varphi]\!]^w_{\mathcal{M}} = T \quad &\text{iff} \quad [\varphi]_{\mathcal{M}} \in \mathcal{N}(w) \\
[\![\Diamond\varphi]\!]^w_{\mathcal{M}} = T \quad &\text{iff} \quad [\neg\varphi]_{\mathcal{M}} \notin \mathcal{N}(w)
\end{aligned}$$

In the above model, $[\![\Box A]\!]^w_{\mathcal{M}} = T$, $[\![\Box B]\!]^w_{\mathcal{M}} = T$, but $[\![\Box(A \wedge B)]\!]^w_{\mathcal{M}} = F$.

2 Non-Normal Logics

Definition 6.4. The **minimal non-normal modal logic E** has the following rules and axioms:[1]

(PL)	All (substitutions of) tautologies are axioms
(MP)	From φ and $\varphi \supset \psi$ infer ψ
(RE)	From $\varphi \equiv \psi$ infer $\Box\varphi \equiv \Box\psi$
(Duality)	Expressions involving $\Box$ and $\Diamond$ are interchangeable according to the duality $\Box \equiv \neg\Diamond\neg$

Other non-normal modal logics can be generated by extending **E** with **T**, **D**, **4**, **5**, **B**, and the following axiom schemata:

M $\Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)$
C $(\Box\varphi \wedge \Box\psi) \supset \Box(\varphi \wedge \psi)$
N $\Box\neg\bot$

3 Correspondence

There is also a nice correspondence between the above axiom schemata and constraints on the relation $\mathcal{R}$ in neighborhood models. For ease of exposition, let us now take a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ to consist both of a set of

[1]Note that $\equiv$ is the material biconditional.

worlds $\mathcal{W}$ and a neighborhood function $\mathcal{N}$. The correspondence lemmas will be stated using $\mathcal{N}$ (but these could, of course, be restated using $\mathcal{R}$).

Lemma 6.1. Given $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$, $\models_{\mathcal{F}} \Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)$ if and only if $\mathcal{N}$ is *closed under supersets*—that is, $(X \in \mathcal{N}(w) \wedge X \subseteq Y) \supset Y \in \mathcal{N}(w)$.

Proof: For the right-to-left direction, consider an arbitrary neighborhood model $\mathcal{M}$ based on $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where $\mathcal{N}$ is closed under supersets, pick an arbitrary world $w \in \mathcal{W}$, and suppose that $[\![\Box(\varphi \wedge \psi)]\!]^w_{\mathcal{M}} = T$. Since $[\varphi \wedge \psi]_{\mathcal{M}} \in \mathcal{N}(w)$ and $[\varphi \wedge \psi]_{\mathcal{M}} \subseteq [\varphi]_{\mathcal{M}}$, $[\varphi]_{\mathcal{M}} \in \mathcal{N}(w)$, so $[\![\Box\varphi]\!]^w_{\mathcal{M}} = T$. By similar reasoning, $[\![\Box\psi]\!]^w_{\mathcal{M}} = T$, so $[\![\Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)]\!]^w_{\mathcal{M}} = T$. Thus, $\models_{\mathcal{F}} \Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)$.

For the left-to-right direction, we prove the contrapositive. Consider a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where $\mathcal{N}$ is *not* closed under supersets—that is, there exists w and $X, Y \in 2^{\mathcal{W}}$ such that $X \in \mathcal{N}(w)$, $X \subseteq Y$, but $Y \notin \mathcal{N}(w)$. Define a model $\mathcal{M}$ based on $\mathcal{F}$ by setting $\mathcal{V}(A, v) = T$ iff $v \in X$, and $\mathcal{V}(B, v) = T$ iff $v \in Y$. Then $[\![\Box(A \wedge B)]\!]^w_{\mathcal{M}} = T$ but $[\![\Box B]\!]^w_{\mathcal{M}} = F$, so $[\![\Box(A \wedge B) \supset (\Box A \wedge \Box B)]\!]^w_{\mathcal{M}} = F$. Thus, $\not\models_{\mathcal{F}} \Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)$.

Lemma 6.2. Given $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$, $\models_{\mathcal{F}} (\Box\varphi \wedge \Box\psi) \supset \Box(\varphi \wedge \psi)$ if and only if $\mathcal{N}$ is *closed under intersections*—that is, $X, Y \in \mathcal{N}(w) \supset X \cap Y \in \mathcal{N}(w)$.

Lemma 6.3. Given $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$, $\models_{\mathcal{F}} \Box\neg\bot$ if and only if $\mathcal{N}$ *contains the unit*—that is, $\mathcal{W} \in \mathcal{N}(w)$.

Lemma 6.4. Given $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$, $\models_{\mathcal{F}} \Box\varphi \supset \varphi$ if and only if every world $w \in \mathcal{W}$ lies in each of its neighborhoods—that is, $\forall w \in \mathcal{W}(w \in \bigcap \mathcal{N}(w))$.

And so forth.

4 Soundness and Completeness

There are also the expected soundness and completeness results:

Theorem 6.1. $\vdash_{\mathbf{E}} \varphi$ iff $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$.

Theorem 6.2. $\vdash_{\mathbf{EM}} \varphi$ iff $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where $\mathcal{N}$ is closed under supersets.

Theorem 6.3. $\vdash_{\mathbf{EC}} \varphi$ iff $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where $\mathcal{N}$ is closed under intersections.

Theorem 6.4. $\vdash_{\mathbf{EN}} \varphi$ iff $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where $\mathcal{N}$ contains the unit.

Theorem 6.5. $\vdash_{\mathbf{ET}} \varphi$ iff $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{N} \rangle$ where every world $w \in \mathcal{W}$ lies in each of its neighborhoods.

And so forth.

5 The Landscape Below K

Adding both **M** and **C** to **E** is effectively adding the axiom (K). Adding **N** to **E** is effectively adding the rule (Nec). So **EMCN**=**K**. The landscape looks like this: