V. The Modal Zoo

AS.150.498: Modal Logic and Its Applications
Johns Hopkins University, Spring 2017

There are a host of **normal modal logics** extending the minimal modal logic **K** with additional axiom schemata:

T	$\Box\varphi \supset \varphi$
D	$\Box\varphi \supset \Diamond\varphi$
4	$\Box\varphi \supset \Box\Box\varphi$
5	$\Diamond\varphi \supset \Box\Diamond\varphi$
B	$\varphi \supset \Box\Diamond\varphi$

K$\varphi_1 ... \varphi_n$ is the weakest logic obtained by extending **K** with the axiom schemata $\varphi_1, ..., \varphi_n$. For instance, **KD45** extends **K** with **D**, **4**, and **5**. **KT4** and **KT5** are abbreviated **S4** and **S5** respectively.

1 Correspondence

One of the most beautiful parts of the theory of modal logic is the tight correspondence between the above axioms and structural constraints on the accessibility relation in Kripke models.

Definition 5.1. A **frame** $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ consists of a nonempty set $\mathcal{W}$ of possible world states and a binary accessibility relation $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ between worlds. Model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ **is based on frame** $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$.

Definition 5.2. The sentence $\varphi \in S_{\mathcal{L}}$ is **valid on frame** $\mathcal{F}$, $\models_{\mathcal{F}} \varphi$, just in case there is no pointed model $\mathcal{M}, w$ with $\mathcal{M}$ based on $\mathcal{F}$ such that $[\![\varphi]\!]^w_{\mathcal{M}} = F$.

Lemma 5.1. Given frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, $\models_{\mathcal{F}} \Box\varphi \supset \varphi$ if and only if $\mathcal{R}$ is *reflexive*—that is, $\forall w (w\mathcal{R}w)$.

Proof: For the right-to-left direction, consider an arbitrary model $\mathcal{M}$ based on $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is reflexive, pick an arbitrary world $w \in \mathcal{W}$, and suppose $[\![\Box\varphi]\!]^w_{\mathcal{M}} = T$. Since $w\mathcal{R}w$, $[\![\varphi]\!]^w_{\mathcal{M}} = T$, so $[\![\Box\varphi \supset \varphi]\!]^w_{\mathcal{M}} = T$. Thus, $\models_{\mathcal{F}} \Box\varphi \supset \varphi$.

For the left-to-right direction, we prove the contrapositive. Consider a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is *not* reflexive—that is, $\neg w\mathcal{R}w$ for some $w \in \mathcal{W}$. Define a model $\mathcal{M}$ based on $\mathcal{F}$ by setting $\mathcal{V}(A, w) = F$ but $\mathcal{V}(A, x) = T$ for all $x \neq w$ (the valuation on other sentence letters is unimportant). Then $[\![\Box A]\!]^w_{\mathcal{M}} = T$ but $[\![A]\!]^w_{\mathcal{M}} = F$, so $[\![\Box A \supset A]\!]^w_{\mathcal{M}} = F$. Thus, $\not\models_{\mathcal{F}} \Box\varphi \supset \varphi$.

Lemma 5.2. Given frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, $\models_{\mathcal{F}} \Box\varphi \supset \Diamond\varphi$ if and only if $\mathcal{R}$ is *serial*—that is, $\forall w \exists v (w\mathcal{R}v)$.

Proof: For the right-to-left direction, consider an arbitrary model $\mathcal{M}$ based on $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is serial, pick an arbitrary world $w \in \mathcal{W}$, and suppose $[\![\Box\varphi]\!]^w_{\mathcal{M}} = T$. Given the seriality of $\mathcal{R}$, $w\mathcal{R}v$ for some $v \in \mathcal{W}$. Since $[\![\varphi]\!]^v_{\mathcal{M}} = T$, $[\![\Diamond\varphi]\!]^w_{\mathcal{M}} = T$, so $[\![\Box\varphi \supset \Diamond\varphi]\!]^w_{\mathcal{M}} = T$. Thus, $\models_{\mathcal{F}} \Box\varphi \supset \Diamond\varphi$.

For the left-to-right direction, we prove the contrapositive. Consider a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is *not* serial—that is, some $w \in \mathcal{W}$ is a dead-end state. Since all $\Box$-sentences are true at a dead-end state and all $\Diamond$-sentences are false, $[\![\Box A \supset \Diamond A]\!]^w_{\mathcal{M}} = F$ for any $\mathcal{M}$ based on $\mathcal{F}$ and $w \in \mathcal{W}$. Thus, $\not\models_{\mathcal{F}} \Box\varphi \supset \Diamond\varphi$.

Lemma 5.3. Given frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, $\models_{\mathcal{F}} \Box\varphi \supset \Box\Box\varphi$ if and only if $\mathcal{R}$ is *transitive*—that is, $\forall w, v, u((w\mathcal{R}v \wedge v\mathcal{R}u) \supset w\mathcal{R}u)$.

Proof (left-to-right only): Consider a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is *not* transitive—that is, $w\mathcal{R}v$ and $v\mathcal{R}u$ but $\neg w\mathcal{R}u$ for some $w, v, u \in \mathcal{W}$. Define a model $\mathcal{M}$ based on $\mathcal{F}$ by setting $\mathcal{V}(A, u) = F$ but $\mathcal{V}(A, x) = T$ for all $x \neq u$. Then $[\![\Box A]\!]^w_{\mathcal{M}} = T$ but $[\![\Box\Box A]\!]^w_{\mathcal{M}} = F$ since $[\![\Box A]\!]^v_{\mathcal{M}} = F$, so $[\![\Box A \supset \Box\Box A]\!]^w_{\mathcal{M}} = F$. Thus, $\not\models_{\mathcal{F}} \Box\varphi \supset \Box\Box\varphi$.

Lemma 5.4. Given frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, $\models_{\mathcal{F}} \Diamond\varphi \supset \Box\Diamond\varphi$ if and only if $\mathcal{R}$ is *Euclidean*—that is, $\forall w, v, u((w\mathcal{R}v \wedge w\mathcal{R}u) \supset v\mathcal{R}u)$.

Proof (left-to-right only): Consider a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is *not* Euclidean—that is, $w\mathcal{R}v$ and $w\mathcal{R}u$ but $\neg v\mathcal{R}u$ for some $w, v, u \in \mathcal{W}$. Define a model $\mathcal{M}$ based on $\mathcal{F}$ by setting $\mathcal{V}(A, u) = T$ but $\mathcal{V}(A, x) = F$ for all $x \neq u$. Then $[\![\Diamond A]\!]^w_{\mathcal{M}} = T$ but $[\![\Box\Diamond A]\!]^w_{\mathcal{M}} = F$ since $[\![\Diamond A]\!]^v_{\mathcal{M}} = F$, so $[\![\Diamond A \supset \Box\Diamond A]\!]^w_{\mathcal{M}} = F$. Thus, $\not\models_{\mathcal{F}} \Diamond\varphi \supset \Box\Diamond\varphi$.

Lemma 5.5. Given frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, $\models_{\mathcal{F}} \varphi \supset \Box\Diamond\varphi$ if and only if $\mathcal{R}$ is *symmetric*—that is, $\forall w, v (w\mathcal{R}v \supset v\mathcal{R}w)$.

Proof (left-to-right only): Consider a frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ where $\mathcal{R}$ is *not* symmetric—that is, $w\mathcal{R}v$ but $\neg v\mathcal{R}w$ for some $w, v \in \mathcal{W}$. Define a model $\mathcal{M}$ based on $\mathcal{F}$ by setting $\mathcal{V}(A, w) = T$ but $\mathcal{V}(A, x) = F$ for all $x \neq w$. Then $[\![A]\!]^w_{\mathcal{M}} = T$ but $[\![\Box\Diamond A]\!]^w_{\mathcal{M}} = F$ since $[\![\Diamond A]\!]^v_{\mathcal{M}} = F$, so $[\![A \supset \Box\Diamond A]\!]^w_{\mathcal{M}} = F$. Thus, $\not\models_{\mathcal{F}} \varphi \supset \Box\Diamond\varphi$.

Here are a few more correspondence results without proofs:

Lemma 5.6. $\models_{\mathcal{F}} \Diamond\varphi \supset \Box\varphi$ if and only if $\mathcal{R}$ is a *partial function*—that is, $\forall w, v, u((w\mathcal{R}v \wedge w\mathcal{R}u) \supset v = u)$.

Lemma 5.7. $\models_{\mathcal{F}} \Box(\Box\varphi \supset \varphi)$ if and only if $\mathcal{R}$ is *shift reflexive*—that is, $\forall w, v(w\mathcal{R}v \supset v\mathcal{R}v)$.

Lemma 5.8. $\models_{\mathcal{F}} \Diamond\Box\varphi \supset \Box\Diamond\varphi$ if and only if $\mathcal{R}$ is *convergent*—that is, $\forall w, v, u((w\mathcal{R}v \wedge w\mathcal{R}u) \supset \exists x(v\mathcal{R}x \wedge u\mathcal{R}x))$.

Lemma 5.9. $\models_{\mathcal{F}} (\Diamond\varphi\wedge\Diamond\psi) \supset (\Diamond(\varphi\wedge\psi)\vee\Diamond(\varphi\wedge\Diamond\psi)\vee\Diamond(\psi\wedge\Diamond\varphi))$ if and only if $\mathcal{R}$ is *right linear*—that is, $\forall w, v, u((w\mathcal{R}v\wedge w\mathcal{R}u) \supset (v = u\vee v\mathcal{R}u\vee u\mathcal{R}v))$.

However, not all axioms correspond to first-order definable properties of $\mathcal{R}$:

McKinsey axiom: $\Box\Diamond\varphi \supset \Diamond\Box\varphi$

Löb axiom: $\Box(\Box\varphi \supset \varphi) \supset \Box\varphi$

Conversely, not all first-order definable properties of $\mathcal{R}$ have a corresponding modal axiom:

Irreflexivity: $\forall w(\neg w\mathcal{R}w)$

Antisymmetry: $\forall w, v((w\mathcal{R}v \wedge v\mathcal{R}w) \supset w = v)$

2 Soundness and Completeness

The various proof systems extending **K** are also sound and complete with respect to validity on different kinds of frames.

Theorem 5.1. $\vdash_{\mathbf{KT}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is reflexive.

Proof: The left-to-right soundness result follows from Lemma 5.1. Since $\Box\varphi \supset \varphi$ is valid over the class of reflexive frames, all of the axioms in **KT** are valid over reflexive frames and the rules of **KT** preserve validity over reflexive frames.

For the right-to-left completeness result, we can tweak the completeness proof for **K**. The canonical model $\mathcal{M}^{\mathbf{KT}}$ for **KT** is constructed as before except we now use maximal **KT**-consistent sets. Importantly, $\mathcal{R}^{\mathbf{KT}}$ is reflexive. For any $\Gamma \in \mathcal{W}^{\mathbf{KT}}$, $\Box\varphi \supset \varphi \in \Gamma$, so $\Box\varphi \in \Gamma$ iff $\varphi \in \Gamma$. Thus by construction, $\Gamma\mathcal{R}^{\mathbf{KT}}\Gamma$.

Theorem 5.2. $\vdash_{\mathbf{KD}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is serial.

Theorem 5.3. $\vdash_{\mathbf{K4}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is transitive.

Theorem 5.4. $\vdash_{\mathbf{K5}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is Euclidean.

Theorem 5.5. $\vdash_{\mathbf{KB}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is symmetric.

Theorem 5.6. $\vdash_{\mathbf{S4}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is reflexive and transitive.

Theorem 5.7. $\vdash_{\mathbf{S5}} \varphi$ if and only if $\models_{\mathcal{F}} \varphi$ for each frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is reflexive and Euclidean.

3 The Landscape

How are these logics related? The following partial ordering is based on proof-theoretic strength:

Definition 5.3. L_1 is a **sublogic** of L_2, $L_1 \leq L_2$, just in case $\vdash_{L_1} \varphi$ implies $\vdash_{L_2} \varphi$ for each $\varphi \in S_{\mathcal{L}}$. L_1 is a **proper sublogic** of L_2, $L_1 < L_2$, just in case $L_1 \leq L_2$ but $L_2 \not\leq L_1$.

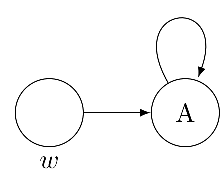
Here is an example:

Fact 5.1. KD < KT.

Proof: To show that $\mathbf{KD} \leq \mathbf{KT}$, it suffices to show that $\vdash_{\mathbf{KT}}$ **D**:

1.	$\Box\varphi \supset \varphi$	**T** Axiom
2.	$\Box\neg\varphi \supset \neg\varphi$	**T** Axiom
3.	$\varphi \supset \neg\Box\neg\varphi$	Contraposition 2
4.	$\varphi \supset \Diamond\varphi$	Duality 3
5.	$\Box\varphi \supset \Diamond\varphi$	Transitivity 1,4

To show that $\mathbf{KT} \not\leq \mathbf{KD}$, it suffices to show that $\not\models_{\mathcal{F}}$ **T** for some frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ where $\mathcal{R}$ is serial. Note that $[\![\Box A \supset A]\!]^w_{\mathcal{M}} = F$ in the serial model $\mathcal{M}$ below:



$\mathcal{M}$

The landscape looks like this: