X. Temporal Logic

AS.150.498: Modal Logic and Its Applications
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Let us now turn to our last application: *temporal logic*.

1 Syntax

We will be working mostly with the following polymodal language:

Definition 10.1. The **Priorean language** $\mathcal{L}_t$ has this syntax:

$p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathrm{G}\varphi \mid \mathrm{H}\varphi$

Read Gφ as 'Henceforth φ' and Hφ as 'Hitherto φ.' The duals of G and H are defined as follows: $\mathrm{F} \equiv \neg\mathrm{G}\neg$ and $\mathrm{P} \equiv \neg\mathrm{H}\neg$.

Definition 10.2. The **mirror image** of φ is the sentence obtained from φ by switching G and H operators.

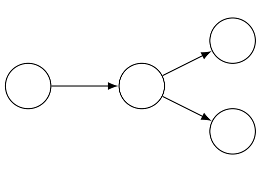
For example, $(\mathrm{F}\varphi \vee \mathrm{H}\varphi) \wedge (\mathrm{G}\varphi \vee \mathrm{P}\varphi)$ is the image of $(\mathrm{P}\varphi \vee \mathrm{G}\varphi) \wedge (\mathrm{H}\varphi \vee \mathrm{F}\varphi)$.

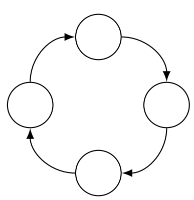
2 Semantics

Models for $\mathcal{L}_t$ are Kripke models based on a restricted class of frames:

Definition 10.3. A **flow of time** is a frame $\mathcal{F} = \langle \mathcal{T}, < \rangle$ where the **precedence relation** $<$ between points of time in $\mathcal{T}$ is transitive and irreflexive—that is, $\forall t, t', t''((t < t' < t'') \supset t < t'')$ and $\forall t(t \not< t)$.

A flow of time cannot be *circular*. While the left frame is a flow of time, the right frame is not—why? (Note: following convention, the arrows required for transitivity are omitted.)





The recursive specification of truth in a pointed flow of time model is just the standard semantics supplemented with a backwards-looking clause for H:

Definition 10.4. The following recursive clauses lift $\mathcal{V}$ to the complete interpretation function $[\![\ ]\!]_{\mathcal{M}} : S_{\mathcal{L}_t} \times \mathcal{T} \to \{T, F\}$ for $\mathcal{L}_t$:

$$\begin{array}{rcl}
[\![p]\!]^t_{\mathcal{M}} = T & \text{iff} & \mathcal{V}(p,t) = T \\
[\![\bot]\!]^t_{\mathcal{M}} = T & \text{iff} & 0 = 1 \\
[\![\neg\varphi]\!]^t_{\mathcal{M}} = T & \text{iff} & [\![\varphi]\!]^t_{\mathcal{M}} = F \\
[\![(\varphi \wedge \psi)]\!]^t_{\mathcal{M}} = T & \text{iff} & [\![\varphi]\!]^t_{\mathcal{M}} = [\![\psi]\!]^t_{\mathcal{M}} = T \\
[\![\mathrm{G}\varphi]\!]^t_{\mathcal{M}} = T & \text{iff} & \forall t' \in \{t' : t < t'\}([\![\varphi]\!]^{t'}_{\mathcal{M}} = T) \\
[\![\mathrm{H}\varphi]\!]^t_{\mathcal{M}} = T & \text{iff} & \forall t' \in \{t' : t' < t\}([\![\varphi]\!]^{t'}_{\mathcal{M}} = T)
\end{array}$$

Since $y > x$ iff $x < y$, the last clause for H can be restated thus:

$$[\![\mathrm{H}\varphi]\!]^t_{\mathcal{M}} = T \quad \text{iff} \quad \forall t' \in \{t' : t > t'\}([\![\varphi]\!]^{t'}_{\mathcal{M}} = T)$$

3 Correspondence

Many structural properties of time can be pinned down with sentences in the Priorean language. Here are some examples:

Lemma 10.1. $\models_{\mathcal{F}} \mathrm{F}\varphi \supset \mathrm{G}(\mathrm{P}\varphi \vee \varphi \vee \mathrm{F}\varphi)$ if and only if time has a *non-branching future*—$\forall t, t', t''((t < t' \wedge t < t'') \supset (t' = t'' \vee t' < t'' \vee t'' < t'))$.

Lemma 10.2. $\models_{\mathcal{F}} \mathrm{P}\varphi \supset \mathrm{H}(\mathrm{F}\varphi \vee \varphi \vee \mathrm{P}\varphi)$ if and only if time has a *non-branching past*—$\forall t, t', t''((t' < t \wedge t'' < t) \supset (t' = t'' \vee t' < t'' \vee t'' < t'))$.

Lemma 10.3. $\models_{\mathcal{F}} \mathrm{F}\neg\bot$ if and only if time has *no end*—$\forall t \exists t'(t < t')$.

Lemma 10.4. $\models_{\mathcal{F}} \mathrm{H}\bot \vee \mathrm{PH}\bot$ if and only if time has *beginning points*—$\forall t, t'(t < t' \supset \exists t''(t'' < t' \wedge \neg\exists t'''(t''' < t'')))$.[1]

Lemma 10.5. $\models_{\mathcal{F}} \mathrm{F}\varphi \supset \mathrm{FF}\varphi$ if and only if time is *dense*—$\forall t, t'(t < t' \supset \exists t''(t < t'' < t'))$.

As we will see in a moment, many properties also have correspondents within the restricted class of *linear* flows of time that have a non-branching future and past (and no parallel time lines)—$\forall t, t'(t < t' \vee t' < t \vee t = t')$.

[1] The condition that time has a *single* beginning point—$\exists t \forall t'(t < t' \vee t = t')$—has no frame correspondent.

4 Proof Systems

There are many proof systems for reasoning about time. Here are some of them:

Definition 10.5. The **minimal temporal logic $\mathbf{K}_t$** has the following rules and axioms:

(PL)	All (substitutions of) tautologies are axioms
(MP)	From φ and $\varphi \supset \psi$ infer ψ
(TG)	From φ infer $\mathrm{G}\varphi$ From φ infer $\mathrm{H}\varphi$
(DB)	For any φ, ψ, $\mathrm{G}(\varphi \supset \psi) \supset (\mathrm{G}\varphi \supset \mathrm{G}\psi)$ is an axiom For any φ, ψ, $\mathrm{H}(\varphi \supset \psi) \supset (\mathrm{H}\varphi \supset \mathrm{H}\psi)$ is an axiom
(4)	For any φ, $\mathrm{G}\varphi \supset \mathrm{GG}\varphi$ is an axiom
(CV)	For any φ, $(\varphi \supset \mathrm{GP}\varphi) \wedge (\varphi \supset \mathrm{HF}\varphi)$ is an axiom

Theorem 10.1. $\mathbf{K}_t$ is sound and complete with respect to the class of all flows of time.

Definition 10.6. The logic **Lin** is obtained by adding the correspondents for non-branching future and past to $\mathbf{K}_t$:

(NB) $\mathrm{F}\varphi \supset \mathrm{G}(\mathrm{P}\varphi \vee \varphi \vee \mathrm{F}\varphi)$
$\mathrm{P}\varphi \supset \mathrm{H}(\mathrm{F}\varphi \vee \varphi \vee \mathrm{P}\varphi))$

Theorem 10.2. **Lin** is sound and complete with respect to the class of all linear flows of time.

Definition 10.7. The logic **Lin.**$\mathbb{N}$ is obtained from **Lin** by adding these correspondents (within the class of linear frames):

Beginning of Time	$\forall t, t'(t < t' \supset \exists t''(t'' < t' \wedge \neg\exists t'''(t''' < t''))$ $\mathrm{H}\bot \vee \mathrm{PH}\bot$
No End of Time	$\forall t \exists t'(t < t')$ $\mathrm{F}\neg\bot$
Finite Intervals	$\forall t, t'(\exists^{\text{finite}} t''(t < t'' < t'))$ $(\mathrm{G}(\mathrm{G}\varphi \supset \varphi) \supset (\mathrm{FG}\varphi \supset \mathrm{G}\varphi))$ $(\mathrm{H}(\mathrm{H}\varphi \supset \varphi) \supset (\mathrm{PH}\varphi \supset \mathrm{H}\varphi))$

Theorem 10.3. **Lin.**$\mathbb{N}$ is sound and complete with respect to $\langle \mathbb{N}, < \rangle$.

Definition 10.8. The logic **Lin.**$\mathbb{Z}$ is obtained from **Lin** by adding these correspondents (within the class of linear frames):

No Beginning of Time	$\forall t \exists t'(t' < t)$ $\mathrm{P}\neg\bot$
No End of Time	$\forall t \exists t'(t < t')$ $\mathrm{F}\neg\bot$
Finite Intervals	$\forall t, t'(\exists^{\text{finite}} t''(t < t'' < t'))$ $(\mathrm{G}(\mathrm{G}\varphi \supset \varphi) \supset (\mathrm{FG}\varphi \supset \mathrm{G}\varphi))$ $(\mathrm{H}(\mathrm{H}\varphi \supset \varphi) \supset (\mathrm{PH}\varphi \supset \mathrm{H}\varphi))$

Theorem 10.4. **Lin.**$\mathbb{Z}$ is sound and complete with respect to $\langle \mathbb{Z}, < \rangle$.

Definition 10.9. The logic **Lin.**$\mathbb{Q}$ is obtained from **Lin** by adding these correspondents (within the class of linear frames):

No Beginning of Time	$\forall t \exists t'(t' < t)$ $\mathrm{P}\neg\bot$
No End of Time	$\forall t \exists t'(t < t')$ $\mathrm{F}\neg\bot$
Density	$\forall t, t'(t < t' \supset \exists t''(t < t'' < t')$ $\mathrm{F}\varphi \supset \mathrm{FF}\varphi$

Theorem 10.5. **Lin.**$\mathbb{Q}$ is sound and complete with respect to $\langle \mathbb{Q}, < \rangle$.

Definition 10.10. The logic **Lin.**$\mathbb{R}$ is obtained from **Lin** by adding these correspondents (within the class of linear frames):

No Beginning of Time	$\forall t \exists t'(t' < t)$ $\mathrm{P}\neg\bot$
No End of Time	$\forall t \exists t'(t < t')$ $\mathrm{F}\neg\bot$
Density	$\forall t, t'(t < t' \supset \exists t''(t < t'' < t')$ $\mathrm{F}\varphi \supset \mathrm{FF}\varphi$
Dedekind Continuity	$\forall X(\forall t, t'((t \in X \wedge t' \notin X) \supset t < t') \supset$ $\exists t''(\forall t, t'((t \neq t'' \neq t' \wedge t \in X \wedge t' \notin X) \supset$ $(t < t'' < t'))))$ $(\mathrm{FH}\varphi \wedge \mathrm{F}\neg\varphi \wedge \mathrm{G}(\neg\varphi \supset \mathrm{G}\neg\varphi)) \supset$ $\mathrm{F}((\varphi \wedge \mathrm{G}\neg\varphi) \vee (\neg\varphi \wedge \mathrm{H}\varphi))$

Theorem 10.6. **Lin.**$\mathbb{R}$ is sound and complete with respect to $\langle \mathbb{R}, < \rangle$.

5 More Operators

We can extend the Priorean language $\mathcal{L}_t$ with other temporal operators besides G and H. The **progressive** operator Π is intended to capture something being in progress:

$$\begin{array}{lll} [\![\Pi\varphi]\!]^t_{\mathcal{M}} = T & \text{iff} & \exists t', t''(t' < t < t'' \wedge \\ & & \forall t''' \in \{t''' : t' < t''' < t''\}([\![\varphi]\!]^{t'''}_{\mathcal{M}} = T)) \end{array}$$

The **nexttime** or **tomorrow** operator $\mathcal{X}$ allows one to talk about the next state of a process ($\mathcal{X}$ applies only to discrete flows of time):

$$[\![\mathcal{X}\varphi]\!]^t_{\mathcal{M}} = T \quad \text{iff} \quad [\![\varphi]\!]^{t+1}_{\mathcal{M}} = T$$

The dyadic operators $\mathcal{S}$ and $\mathcal{U}$ are intended to formalize 'since' and 'until' respectively:

$$\begin{array}{lll} [\![\mathcal{S}\varphi\psi]\!]^t_{\mathcal{M}} = T & \text{iff} & \exists t' < t([\![\varphi]\!]^{t'}_{\mathcal{M}} = T) \wedge \\ & & \forall t'' \in \{t'' : t' < t'' < t\}([\![\psi]\!]^{t''}_{\mathcal{M}} = T) \end{array}$$

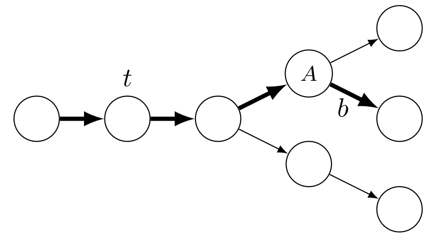
$$\begin{array}{lll} [\![\mathcal{U}\varphi\psi]\!]^t_{\mathcal{M}} = T & \text{iff} & \exists t' > t([\![\varphi]\!]^{t'}_{\mathcal{M}} = T) \wedge \\ & & \forall t'' \in \{t'' : t < t'' < t'\}([\![\psi]\!]^{t''}_{\mathcal{M}} = T) \end{array}$$

It can be shown (using bisimulation) that adding any of these operators to $\mathcal{L}_t$ increases expressive power. But note that Π and $\mathcal{X}$ can be defined in terms of $\mathcal{S}$ and $\mathcal{U}$: $\Pi\varphi \equiv \mathcal{S}\varphi\varphi \wedge \varphi \wedge \mathcal{U}\varphi\varphi$ and $\mathcal{X}\varphi \equiv \mathcal{U}\varphi\bot$. In fact, so can F and P: $\mathrm{F}\varphi \equiv \mathcal{U}\varphi\neg\bot$ and $\mathrm{P}\varphi \equiv \mathcal{S}\varphi\neg\bot$.

6 Branching Time

Another motivation for extending $\mathcal{L}_t$ arises with branching time. If time branches in the future, then 'It will be the case that' is ambiguous. On a strong reading, it will be the case that φ just in case φ happens *come what may*—that is, φ occurs in every possible future. On a weaker reading, it will be the case that φ just in case φ occurs in the *actual* future. These two readings correspond to the Peircean and Ockhamist schools respectively.

Let us now consider flows of time that are **trees**—there must be a path along $< \cup >$ between any two times $t, t' \in \mathcal{T}$ but the flow does not branch in the past. A **branch** b is a maximal linearly ordered subset of $\mathcal{T}$. If $t \in b$, then t **lies on** b and b **passes through** t.



On the Peircean model, the Priorean language $\mathcal{L}_t$ is supplemented with an operator $\mathrm{F}_\Box$ with the following truth clause:

$$[\![\mathrm{F}_\Box\varphi]\!]^t_{\mathcal{M}} = T \quad \text{iff} \quad \forall b(t \in b \supset \exists t' \in \{t' \in b : t < t'\}([\![\varphi]\!]^{t'}_{\mathcal{M}} = T))$$

That is, $\mathrm{F}_\Box\varphi$ is true at t in $\mathcal{M}$ just in case there is some future time t' lying on every branch b passing through t where φ is true at t'.

On the Ockhamist model, sentences are evaluated for truth relative *both* to a time $t \in \mathcal{T}$ *and* to a branch $b \subset \mathcal{T}$ where $t \in b$ (the actual history):

$$\begin{array}{rll} [\![p]\!]^{t,b}_{\mathcal{M}} = T & \text{iff} & \mathcal{V}(p, t) = T \\ [\![\bot]\!]^{t,b}_{\mathcal{M}} = T & \text{iff} & 0 = 1 \\ [\![\neg\varphi]\!]^{t,b}_{\mathcal{M}} = T & \text{iff} & [\![\varphi]\!]^{t,b}_{\mathcal{M}} = F \\ [\![(\varphi \wedge \psi)]\!]^{t,b}_{\mathcal{M}} = T & \text{iff} & [\![\varphi]\!]^{t,b}_{\mathcal{M}} = [\![\psi]\!]^{t,b}_{\mathcal{M}} = T \\ [\![\mathrm{G}\varphi]\!]^{t,b}_{\mathcal{M}} = T & \text{iff} & \forall t' \in \{t' \in b : t < t'\}([\![\varphi]\!]^{t',b}_{\mathcal{M}} = T) \\ [\![\mathrm{H}\varphi]\!]^{t,b}_{\mathcal{M}} = T & \text{iff} & \forall t' \in \{t' \in b : t' < t\}([\![\varphi]\!]^{t',b}_{\mathcal{M}} = T) \end{array}$$

F and P are still defined as follows: $\mathrm{F} \equiv \neg\mathrm{G}\neg$ and $\mathrm{P} \equiv \neg\mathrm{H}\neg$.

In the above model, for example, $[\![\mathrm{F}_\Box A]\!]^t_{\mathcal{M}} = F$ in the Peircean language but $[\![\mathrm{F}A]\!]^{t,b}_{\mathcal{M}} = T$ in the Ockhamist language.

In addition to the temporal operators G and H, the Ockhamist language also includes another necessity modal $\Box$ that quantifies over branches:

$$[\![\Box\varphi]\!]^{t,b}_{\mathcal{M}} = T \quad \text{iff} \quad \forall c(t \in c \supset [\![\varphi]\!]^{t,c}_{\mathcal{M}} = T)$$

The following equivalence holds: $[\![\mathrm{F}_\Box\varphi]\!]^t_{\mathcal{M}} = [\![\Box\mathrm{F}\varphi]\!]^{t,b}_{\mathcal{M}}$. So the Peircean language can be regarded as a fragment of the Ockhamist language.

7 Intervals

So far, we have been working in the point-based paradigm. But it is worth briefly noting that there are also interval-based semantics that evaluate sentences for truth relative to an interval $[s, t]$. Within such a semantics, we can introduce temporal modals like the following **during** operator $\mathcal{D}$:

$$[\![\mathcal{D}\varphi]\!]_{\mathcal{M}}^{[s,t]} = T \quad \text{iff} \quad \exists s', t'(s \leq s' \leq t' \leq t \wedge [\![\varphi]\!]_{\mathcal{M}}^{[s',t']} = T)$$

8 The Master Argument

Let us close with an application of temporal logic to ancient philosophy. Consider the fatalistic Master Argument of Diodorus Cronus as described by Epictetus:

(P1) Everything that is past and true is necessary.

(P2) The impossible does not follow from the possible.

(C) Nothing is possible which neither is nor will be true.

Fitting and Mendelsohn [1998] formally interpret this Master Argument as follows (for the moment, we won't worry too much about the semantics of $\Box$ and $\Diamond$):

(P1) corresponds to this schema: $\mathrm{P}\varphi \supset \Box\mathrm{P}\varphi$

(P2) corresponds to this rule: From $\varphi \supset \psi$ infer $\Diamond\varphi \supset \Diamond\psi$

(C) corresponds to this schema: $(\neg\varphi \wedge \neg\mathrm{F}\varphi) \supset \neg\Diamond\varphi$

It is also assumed that $(*)$ time is discrete and has a non-branching future. Then the argument runs like this:

1.	$(\neg\varphi \wedge \neg\mathrm{F}\varphi) \supset \mathrm{P}\neg\mathrm{F}\varphi$	$*$
2.	$\mathrm{P}\neg\mathrm{F}\varphi \supset \Box\mathrm{P}\neg\mathrm{F}\varphi$	P1
3.	$(\neg\varphi \wedge \neg\mathrm{F}\varphi) \supset \Box\mathrm{P}\neg\mathrm{F}\varphi$	From 1,2
4.	$\varphi \supset \neg\mathrm{P}\neg\mathrm{F}\varphi$	CV
5.	$\Diamond\varphi \supset \Diamond\neg\mathrm{P}\neg\mathrm{F}\varphi$	P2
6.	$\Box\mathrm{P}\neg\mathrm{F}\varphi \supset \neg\Diamond\varphi$	From 5
7.	$(\neg\varphi \wedge \neg\mathrm{F}\varphi) \supset \neg\Diamond\varphi$	From 3,6

Given this result, we might now define $\Diamond$ and $\Box$ in the Priorean language as follows (indeed, Diodorus seems to have thought about possibility and necessity in this way):

$\Diamond\varphi \equiv (\varphi \vee \mathrm{F}\varphi)$
$\Box\varphi \equiv (\varphi \wedge \mathrm{G}\varphi)$