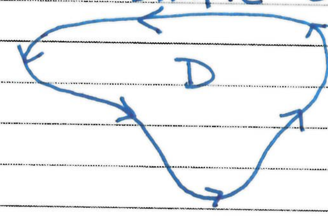
Green's theorem:

- $D \subseteq \mathbb{R}^2$ simple region, Suppose ∂D (Boundary of D) is a piecewise C^1, oriented counter clockwise simple closed curve.



$$\partial D = C^+ = c(t), \quad a \leq t \leq b$$

- If $F(x,y) = (P(x,y), Q(x,y))$ is a C^1 vect. field on D,

$$\rightsquigarrow \int_{C^+} F \cdot ds = \int_{C^+} P\,dx + Q\,dy \quad \text{(line integral)}$$

$$= \int_a^b F(c(t)) \cdot c'(t)\, dt$$

Recall: The line integral depends only on orientation & parametrization.

"Green's theorem relates this line integral to an integral over the interior D."

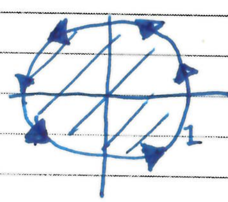
Green's theorem:

$$\int_{C^+} F \cdot ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Let's verify Green's theorem on an example.

Example: $F(x,y) = (x, xy)$

$$D = \{ (x,y) \mid x^2 + y^2 \leq 1 \}$$



$\partial D =$ unit circle $= C^+$

$c(t) = (\cos(t), \sin(t)) \quad 0 \leq t \leq 2\pi$

line integral

$$\int_{C^+} F \cdot ds = \int_0^{2\pi} (\cos(t), \cos(t)\sin(t)) \cdot (-\sin(t), \cos(t)) \; dt$$

$$= \int_0^{2\pi} -\sin(t)\cos(t) + \sin(t)\cos^2(t) \; dt$$

Note:

- $\frac{d}{dt}\left(\frac{1}{2}\cos^2(t)\right) = -\sin(t)\cos(t).$
- $\frac{d}{dt}\left(-\frac{1}{3}\cos^3(t)\right) = \sin(t)\cos^2(t)$

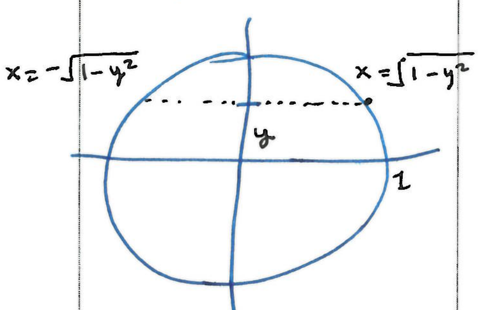
=D line int

$$\int_{C^+} F \cdot ds = \left[\frac{1}{2}\cos^2(t) - \frac{1}{3}\cos^3(t)\right]_0^{2\pi}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) = 0.$$

Compare:

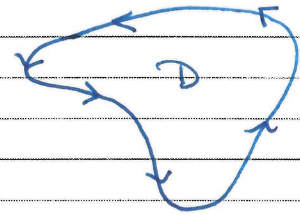
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dxdy = \int_{-1}^{1}\left[\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y - 0 \; dx\right] dy$$



$$= \int_{-1}^{1}\Bigg(\underbrace{y\,x\,\Big|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}}}_{2y\sqrt{1-y^2}}\Bigg) dx$$

$$= \left[\frac{-2}{3}(1-y^2)^{3/2}\right]_{-1}^{1} = \frac{-2}{3}\cdot 0^{3/2} - \frac{-2}{3}\,0^{3/2} = 0.$$

"Interesting consequence of green's theorem: Can calculate Area as integral over boundary."



as before

$C^+ = \partial D$ oriented counter clockwise.

Prop: Area of $D = \frac{1}{2} \int_{C^+} x\,dy - y\,dx$.

why? $F(x,y) = (-y, x)$

$$\frac{1}{2}\int_{C^+} x\,dy - y\,dx = \frac{1}{2}\int_{C^+} F \cdot ds$$

$$= \frac{1}{2}\iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dx\,dy$$

$$= \frac{1}{2} \cdot 2 \iint_D 1\,dx\,dy = \text{Area}(D).$$

Vector form of Green's theorem:

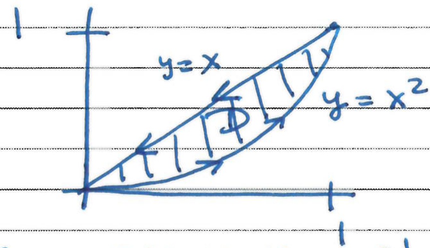
Recall that $\text{curl}(F) = \nabla \times F$

If $F(x,y) = (P(x,y), Q(x,y))$, then

$$\text{curl}(F) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$\Rightarrow \int_{C^+} F \cdot ds = \iint_D \text{curl}(F) \cdot \hat{k}\,dx\,dy$$

Example:



$\partial D = c_1 + c_2$ piecewise C^1.

$c_1(t) = (1-t, 1-t) \qquad 0 \leq t \leq 1$

$c_2(t) = (t, t^2) \qquad 0 \leq t \leq 1.$

$\text{Area}(D) = \frac{1}{2}\int_{\partial D} F \cdot ds \quad , \quad F(x,y) = (-y, x).$

$= \frac{1}{2}\left(\int_{c_1} F \cdot ds + \int_{c_2} F \cdot ds \right)$

$\cdot \int_{c_1} F \cdot ds = \int_0^1 \underbrace{(-1+t, 1-t) \cdot (-1,-1)}_{1-t-1+t=0} \, dt$

$= 0$

$\cdot \int_{c_2} F \cdot ds = \int_0^1 \underbrace{(-t^2, t) \cdot (1, 2t)}_{-t^2+2t^2 = t^2} \, dt$

$= \frac{1}{3} t^3 \Big|_0^1 = 1/3$

$\Rightarrow \text{Area} = \frac{1}{2} \cdot \frac{1}{3} = 1/6.$

Analog of Green's thm for line integrals in $\mathbb{R}^3$.

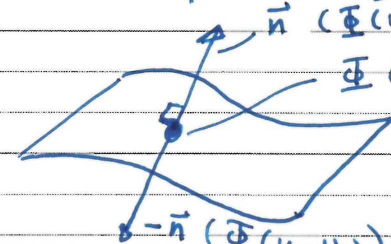
Last time:

- $S \subseteq \mathbb{R}^3$ surface, $\Phi : D \to \mathbb{R}^3$, $\Phi(D) = S$.
  "satisfying all properties from last time"

  surface integral of $F : S \to \mathbb{R}^3$ C^1:

  $$\iint_{\Phi} F \cdot dS = \iint_{D} F \cdot (T_u \times T_v)\, du\, dv.$$

- This depends (up to ± 1) on the orientation of the parametrization.



$\|\vec{n}_1\| = \|-\vec{n}_1\| = 1$
unit normals.

an orientation on S is a choice of "positive" side and "negative side."

⚠ Not all surfaces can be oriented in a meaningful way.

- $\Phi : D \to \mathbb{R}^3$ is orientation preserving

  if $\dfrac{T_{u_0} \times T_{v_0}}{\|T_{u_0} \times T_{v_0}\|} = \vec{n}(\Phi(u_0,v_0))$

  orientation reversing if $= -\vec{n}(\Phi(u_0,v_0))$.

Thm: Let $\Phi_1, \Phi_2 : D \to \mathbb{R}^3$ two regular parametrizations of S.

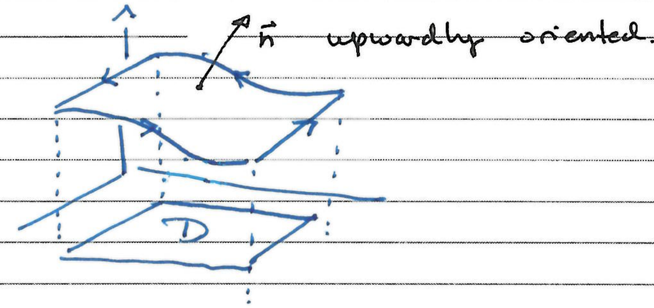
$$\iint_{\Phi_1} F \cdot dS = \pm \iint_{\Phi_2} F \cdot dS$$

w/ $+1$ if both orientation preserving/reversing.
-1 if one O.P., the other O.R.

Stokes' theorem

$g : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad C^2$,

$\rightsquigarrow$ S = surface defined by g.



Give ∂S the <u>positive</u> orientation.

∂S = oriented path in $\mathbb{R}^3$

F vector field defined on S

$\Rightarrow$ $\int_{\partial S} F \cdot ds$ (line integral) has meaning.

Stokes' theorem:

$$\int_{\partial S} F \cdot ds = \iint_{\Phi} \mathrm{curl}(F) \cdot dS$$

Φ = orientation-preserving parametrisation of S. , ends.

Example $F(x,y,z) = (ye^z, xe^z, xye^z)$

Let S = graph of any function as before.

claim $\int_{\partial S} F \cdot ds = 0$.

$F = \nabla f$, $f(x,y,z) = xye^z$.

$\Rightarrow$ $\text{Curl}(F) = \vec{0}$

Stokes' theorem $\Rightarrow$

$$\int_{\partial S} F \cdot ds = \iint_S \text{Curl}(F) \cdot dS$$

$$= 0.$$