On the Complexity for Epistemic Modal logics with Multiple Agents

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1 Introduction

Reasoning about knowledge has been an important topic for various fields of computer science, and computational complexity has always been an intriguing topic for people in order to find how quickly computers can solve or verify a given problem. The problem regarding the complexity on epistemic modals is an interesting topic to discover in modal logic, in which the problem is especially related to the area of computer science.

After studying the chapters for deciding validity and epistemic modal logic in class, I'm interested in discovering more about the complexity of K, T, S4, S5, KD45 modal logics, as well as them with more than one agent. Specifically, I'm interested in which complexity classes these logics belong to.

After consulting the paper on completeness and complexity for modal logics of knowledge and belief as well as the paper on the complexity of fragments of modal logics, in this paper, I'm focusing on modeling knowledge using possible-worlds semantics that are imposed with different conditions on accessibility relations. For example, we require that knowledge relations to be transitive, symmetric, euclidean or reflexive on the models. And under these frames, I discuss how hard is it to decide if a given formula is valid, as well as how hard is it to decide if such given formula is satisfiable. Then we extend the discussion to include cases where there are multiple agents involved in the models, and talk about how they will affect the hardness of decidability under the same class of logic.

In general, the paper will go over the basic semantics employed here, with multiple agents involved, which is slightly different from what I've learned in class. And the relationship between complexity classes that is important to know before going deep in decidability of validity and of satisfiability for formulas. Then in the third section, we go into details about why the validity problem for $K_n, T_n, S4_n, S5_n, KD45_n$ are decidable and how it can be checked. In the following section, we provide lower and upper bounds for the complexity of deciding satisfiability for $K_n, T_n, S4_n, S5_n, KD45_n$, as well as for such modal logics with more agents included. And we show how with more agents involved, $S5_n, KD45_n$ are much harder to reason their knowledge.

We end up with a conclusion, summarizing what we discussed and further applications of complexity in modal logics.

2 Basic Semantics and Axiom Systems

2.1 Syntax and Semantics

The formulas, or sentences are primitive propositions closed under negation, conjunction, and the modal operators $K_1, ...K_n$. $K_i(\phi)$ is denoted as "agent i knows ϕ". These are defined in the same way as what we did in class. Moreover:

The size of a formula ϕ, $|\phi|$, is the length over the formula.

The depth of a formula $dep(\phi)$ is 0 is ϕ is a primitive proposition, where $dep(\neg\phi) = dep(\phi), dep(\phi \wedge \psi) = max(dep(\phi), dep(\psi)), dep(K_i(\phi)) = dep(\phi) + 1$. Note that it is always the case that $dep(\phi) < |\phi|$.

ψ is a subformula for ϕ if it is a formula that is a substring of ϕ. Let $Sub(\phi)$ be the set of all subformulas of ϕ, $|Sub(\phi)| \leq |\phi|$.

The above definitions are similar to what was discussed in class. However, we need to redefine the Kripke structure for n agents slightly differently:

A Kripke structure for n agents is a tuple $M = (S, \pi, K_1, ...K_n)$, where S is the set of possible worlds, π is the truth assignment to the primitive propositions for each state $s \in S$, and K_i is the binary relation on the states of S for n agents. Note that the size of the structure M means the number of states in S.

For example, for M,if the only primitive proposition is p, $n = 2$, and there are 3 states, then a possible $M = (S, \pi, K_1, K_2)$, where $S = \{s, t, u\}, \pi(s)(p) = T, \pi(s)(t) = T, \pi(s)(u) = F, K_1 = \{(s, t), (u, u)\}, K_2 = \{(s, u), (u, u)\}$

We then formally define the binary relation $\vDash$ between ϕ and (M, s), where $(M, s) \vDash \phi$ read as (M, s) satisfies ϕ:

$(M, s) \vDash \phi$ iff $\pi(s)(\phi) = T$
$(M, s) \vDash \phi \wedge \psi$ iff $\pi(s)(\phi) = T$ and $\pi(s)(\psi) = T$
$(M, s) \vDash \neg\phi$ iff $\neg(M, s) \vDash \phi$
$(M, s) \vDash K_i(\phi)$ iff $(M, s) \vDash \phi$ for all t satisfying $(s, t) \in K_i$

All the above definitions share the same logic as what was discussed in class, except that they use different symbols. The following theorem captures some of the formal properties of $\vDash$ that are important to notice:

If ϕ is an instance of propositional tautology, then $M \vDash \phi$
If $M \vDash \phi$ and $M \vDash \phi \Rightarrow \psi$, then $M \vDash \psi$
$M \vDash (K_i(\phi) \wedge K_i(\phi \Rightarrow \psi)) \Rightarrow K_i(\psi)$
If $M \vDash \phi$ then $M \vDash K_i(\phi)$

These are same as what was proven in the last weeks of classes.

2.2 Basic Complexity Classes

It is also important to know basic terms regarding complexity in modal logic as well as in computer science:

Decidability: the term decidable refers to the decision problem, the question of the existence of an effective method for determining membership in a set of formulas, or, more precisely, an algorithm that can and will return a boolean true or false value that is correct (instead of looping indefinitely, crashing, returning "don't know" or returning a wrong answer). (From Wikipedia) Note that decidability does not require efficiency, because as long as the time for determining membership is finite, the problem is considered as decidable.

Non-deterministic: a non-deterministic algorithm is an algorithm that, even for the same input, can exhibit different behaviors on different runs, as opposed to a deterministic algorithm.

P: P is a complexity class that represents the set of all decision problems that can be solved in polynomial time. That is, given an instance of the problem, the answer yes or no can be decided in polynomial time.

NP: NP is a complexity class that represents the set of all decision problems for which the instances where the answer is "yes" have proofs that can be verified in polynomial time. This means that if someone gives us an instance of the problem and a certificate (sometimes called a witness) to the answer being yes, we can check that it is correct in polynomial time.

NP-hard: A problem H is NP-hard when every problem L in NP can be reduced in polynomial time to H. These are the problems that are at least as hard as the NP-complete problems.

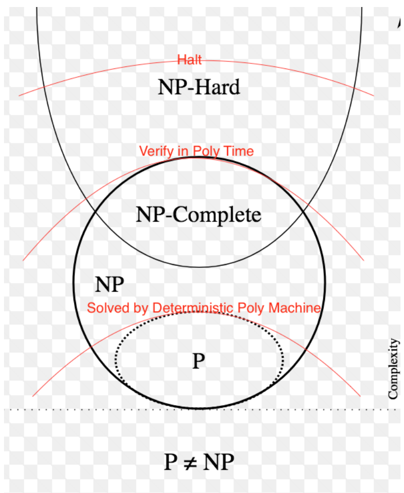
NP-complete: NP-Complete is a complexity class which represents the set of all problems X in NP for which it is possible to reduce any other NP problem Y to X in polynomial time. A decision problem is NP-complete when it is both in NP and NP-hard. (From Stackoverflow)

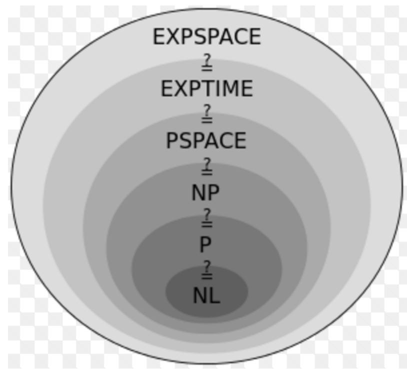
PSPACE: PSPACE is the set of all decision problems that can be solved by a Turing machine using a polynomial amount of space. (From Wikipedia)

PSPACE-hard: These are the problems that are solved using as least a polynomial amount of space.

EXPTIME: The complexity class EXPTIME (sometimes called EXP or DEXPTIME) is the set of all decision problems that have exponential runtime, i.e., that are solvable by a deterministic Turing machine in O(2p(n)) time, where p(n) is a polynomial function of n. (From Wikipedia)

Note that $P \subseteq NP \subseteq PSPACE \subseteq EXPTIME$ and $P \neq EXPTIME$, which is important for later discussion in section 4, below are the diagrams representing the relationships among complexity classes regarding space or time:





2.3 Axiom Systems

We define an axiom system K_n, which characterizes Kripke structures for knowledge. K_n consists of two axioms:

A1. All instances of tautologies of the propositional calculus
A2. $(K_i(\phi) \wedge K_i(\phi \Rightarrow \psi)) \Rightarrow K_i(\psi), i = 1, ..., n$

K_n also consists of two rules:

R1. From $\vdash \phi$ and $\vdash \phi \Rightarrow \psi$ infer $\vdash \psi$
A2. From $\vdash \phi$ infer $\vdash K_i(\phi)$

Besides Axioms that K_n consists of, there are also other ways to characterize knowledge:

A3. $K_i(\phi) \Rightarrow \phi$, which states that only true facts can be known. This is similar to what we discussed in class as $\Box\phi \implies \phi$ for T axiom in normal modal logic.

A4. $K_i(\phi) \Rightarrow K_i(K_i(\phi))$, which is the positive introspection axiom, which states that an agent knows what fact he knows. This is similar to what we discussed in class as $\Box\phi \implies \Box\Box\phi$ for 4 axiom in normal modal logic.

A5. $\neg K_i(\phi) \Rightarrow K_i(\neg K_i(\phi))$, which is known as the negative introspection axiom, saying that an agent knows what facts he does not know. This is rejected by philosophers as we discussed in class.

A6. $\neg K_i(false)$, saying an agent does not know inconsistent facts. This is similar to N axiom discussed in class.

Specifically, $K + A3$ has been called T, $T + A4$ has been called $S4$, $S4 + A5$ has been called $S5$, and $K + A4 + A5 + A6$ has been called $KD45$.

2.4 Correspondence between Model Structures and Axioms

We say that a structure M is a model of K_n if every K_n-provable formula is valid in M. Ans similarly, M is a model of $T_n, S4_n, S5_n, KD45_n$, if if every $T_n, S4_n, S5_n, KD45_n$-provable formula is valid in M, respectively.

Since K_n is a sound and complete axiomatization with respect to M(we have shown similar proof of soundness and completeness of system K in class), we can conclude that:

T_n is a sound and complete axiomatization with respect to M^r.
$S4_n$ is a sound and complete axiomatization with respect to M^{rt}.
$S5_n$ is a sound and complete axiomatization with respect to M^{rst}.
$KD45_n$ is a sound and complete axiomatization with respect to M^{elt}.

We can observe that these four correspondence are corresponding to what we discussed in class. For example, for T_n, every reflexive structure satisfies all the axioms of T_n and thus is a model of T_n. T_n, specifically, is $K + A3$, and $A3$ is analogous to the T axiom in normal modal logic, which also perfectly corresponds to reflexivity that was proven in class. As another instance, $S4_n$ has the rule $A4$ added, which refers to 4 axiom in normal modal logic.

The only difference between such correspondences here and correspondence in normal modal logic is that we involve multiple agents here. However, we just modify M that was discussed in class to be $M^r = (S, \pi, K_1^r, ...K_n^r)$. As long as each of K_i as epistemic relations or epistemic relations are reflexive themselves, such proof can be considered as the same process.

Before going to complexity discussion, here are two important propositions to keep in mind:

If a formula ϕ is $S5$ consistent, then ϕ is satisfiable in a structure $M = (S, \pi, K_1)$, where K_1 is universal.

If a formula ϕ is $KD45$ consistent, then ϕ is satisfiable in a structure $M = (\{s_0\} \cup S, \pi, K_1)$, where S is nonempty and $K_1 = \{(s,t) : s \in \{s_0\} \cup S, t \in S\}$
These are provable and important for differentiating there complexity from $K, T, S4$.

3 Deciding the Validity of Formulas

Now it comes to the meat of the paper that the validity problem for M, M^r, M^{rt}, M^{rst}, M^{elt} are provable.

3.1 Validity and Satisfiability

We consider how validity and satisfiability are defined differently from what we discussed in class. Here, validity and satisfiability are defined as follows:

Validity: ϕ is valid in M, if $(M, s) \vDash \phi$ for every state $s \in S$. Written as $M \vDash \phi$.
We say ϕ is valid with respect to class M, if ϕ is valid in all structures in M.
Satisfiability: ϕ is satisfiable in M, if $(M, s) \vDash \phi$ for some state $s \in S$.
We say ϕ is satiafiable with respect to class M, if ϕ is valid in some structures in M.

It is important to note, same as what is discussed in class, that ϕ is valid in M iff $\neg\phi$ is not satisfiable in M. So validity and satisfiability are regarded as complementary problems, and this point will be revisited in section 4.

3.2 Decidability of M_n and the Provability for K_n

In order to know if the validity problem for M_n are the probability problem for K_n are decidable, we have to come with two observations:

First, given a structure M and a formula ϕ, there is an algorithm for checking if ϕ is satisfied in M that runs in time $c * (|M| * |\phi|)$:
This is because we have to check at each state whether ϕ is true. To check on a single state whether ϕ is true, we consider all the subformulats of ϕ, and the number of subformulas $Sub(\phi) \leq |\phi|$. So running time for checking single state is $O(|\phi|)$. And there are $|M|$ states in total, so the time will be in $O(|M| * |\phi|)$. The only part that is different from what we discussed in class is that ϕ involves relations of knowledge K. However, this is not a big problem since each $K_i(\psi)$ can be carried out in time at most of $O(|M|)$, by tracing back to t, if $(s, t) \in K_i$. Overall, the time is still polynomial.

Second, If ϕ is K_n consistent then ϕ is satisfiable in a structure in M_n with at most $2^{|\phi|}$ states where every primitive proposition which is not a subformula of ϕ is false at every state:
Proof: Let $Sub(\phi)^+$ consist of all the subformulas and their negations. Let $Con(\phi)$ be a set of maximal K_n consistent subsets of $Sub(\phi)^+$. A member that belongs to $Con(\phi)$ is either ψ or $\neg\psi$ for all formula ψ which is a subformula of ϕ. We can assign 0 or 1 to true or false of such ψ in a member of $Con(\phi)$. So we can easily deduce that $|Con(\phi)|$ is at most $2^{|\phi|}$, because number of ψ is $|Sub(\phi)| \leq |\phi|$.

So we just construct $2^{|\phi|}$: this many M_n and in each structure we construct, we use the checking algorithm mentioned in the first part to determine whether ϕ is satisfiable. This procedure is not different from what is discussed in class except for involving more agents during the checking algorithm. But satisfiability can still be decided in finite time, and thus validity can be decided for

K_n.

3.3 Extending Decidability to $T_n, S4_n, S5_n, KD45_n$

Not only in the case of K_n, we can decide whether a formula ϕ is provable, similarly, we can decide whether a formula is provable, thus determining validity for $T_n, S4_n, S5_n, KD45_n$:

This is because during the model checking process, if we restrict our attention to particular structures, we can slightly modify the algorithm to check whether ϕ holds at a particular state s in M. It basically undergoes the same process without affecting the complexity.

Thus, validity problem for $M, M^r, M^{rt}, M^{rst}, M^{elt}$, with respect to K_n, T_n, $S4_n, S5_n, KD45_n$ are all decidable.

4 Deciding the Satisfiability of Formulas with Respect to Complexity Classes

We are interested in comparing difficulties of determining satisfiability of a given formula ϕ over different classes of models in this section. Specifically, we consider $K_n, T_n, S4_n, S5_n, KD45_n$ with one or more agents. If not specified, it is referred to single agent case.

4.1 Complexity Classes and Cook's Theorem

Usually the difficulty of determining if a candidate belongs to a set is measured under time and space required. In this section we are going to show that $S5, KD45$ belong to $NP{-}complete$ complexity class whereas $K, T, S4, S5_2, KD45_2$ belong to $PSPACE - complete$ complexity class.

It is important to know Cook's Theorem beforehand:

The problem of determining whether a formula of propositional logic is satisfiable is NP-hard.

Another fact to notice is that given a complexity class C, the class $co - C$ includes all the sets whose complement is a member in C. So, since validity and satisfiability are complementary problems, and since satisfiability is NP-complete, validity if co-NP-complete as a result.

4.2 NP-completeness for $S5, KD45$

Here, we prove the $NP - complete$ tight bound of $S5$'s satisfiability given formula ϕ as a representation.

4.2.1 Lower Bound for $S5$

By Cook's Theorem, deciding $S5$ satisfiability is $NP-hard$, as hard as NP, because propositional calculus is part of $S5$.

4.2.2 Upper Bound for $S5$

As what we discussed in Section 2, If a formula ϕ is $S5$ consistent, then ϕ is satisfiable in a structure $M = (S, \pi, K_1)$, where K_1 is universal, It is provable that an $S5$ formula ϕ is satisfiable iff it is satisfiable in a structure in M^{rst} with at most $|\phi|$ states.

Now we prove that such decision problem is actually in NP:

We do this by "guessing" a structure M non deterministically with at most $|\phi|$ states, suppose the formula given is ϕ. Say $M = (S, \pi, K)$, and S contains at most $|\phi|$ states, since K doesn't leave room for us to "guess", we only have π needing to "guess". We "guess" the truth value of $\pi(s)(q)$. This can be done at $O(|\phi|^2)$ non deterministically, because we have $|\phi|$ choices for both s, p. Then we can use the checking algorithm discussed in section 3 to know whether ϕ is satisfied in such guessed structure.

This is exactly what is defined as NP problem discussed in section 2.

4.2.3 Tight Bound for $S5$

So, since deciding $S5$ satisfiability is both $NP - hard$ and NP, we can conclude that such problem is in $NP - complete$ class.

4.2.4 Extending to $KD45$

Similarly, by proving based on another key proposition from section 2, we can prove essentially the same result for $KD45$: a $KD45$ formula ϕ is satisfiable iff it is satisfiable in a structure in M^{elt} with at most $|\phi|$ states.

4.3 PSPACE-completeness for $K, T, S4, S5_2, KD45_2$

We first use K_n system as an example to show that it belongs to $PSPACE-$ *complete* class.

4.3.1 Lower Bound for K

As shown from section 3, if a formula ϕ is K_n satisfiable then the satisfiability takes place in a structure of size $\leq 2^{|\phi|}$. However, as a comparison, $S5$ satisfiablility takes place in a structure of size of at most $|\phi|$. This indicates the reason why K is not able to give an NP decision procedure for its satisfiability is that checking whether ϕ is satisfied at some state s cannot be done deterministically in polynomial time.

Why is this the case? Briefly, according to what Ladner says, we consider quantified boolean formulae(QBF) in the form of $Q_1p_1Q_2p_2...Q_mp_mA'$, where

$Q_i \in \{\forall, \exists\}$, and A' is a propositional formula whose only primitive propositions are among $p_1 ... p_m$. We determine the truth of such QBF by replacing each subformula $\forall$ by $\wedge$, $\exists$ by $\vee$. In the case of $S4$, suppose we are given a QBF, we construct ϕ like a binary tree accordingly by $d_0 \wedge \neg d_1 \wedge K(depth \wedge determined \wedge branching_A \wedge (d_m \Rightarrow A'))$, where d_i is true iff we are at a depth $\geq i$. "Determined" means the truth value of p_i is determined by depth i in the tree, such that if p_i is true at depth $j \geq i$ of node s, then p_i is true for all successors of s. "Branching" means it is possible two have two successors at next depth such that p_{i+1} is true and false, respectively.

So we can see ϕ is satisfiable in a structure if A is true. And since the size of ϕ is in polynomial of the size A, $S4$ satisfiability is taking at least as much space as $PSPACE$, so it is a $PSPACE - hard$ problem.

As for the lower bounds construction for $S5_2$, we can modify the structure M^{rt} from $S4$ to get M_2^{rst}, and thus providing the same lower bound, details are omitted. One of the steps is that we replace every K edge in M by two edges, one for K_1 and the other for K_2. And then define K_1 and K_2 to both be r, s, t closures $\{(s_{(s,t)}, t : (s,t) \in K)\}$. The same underlying idea is for $KD45_2$. This invariantly shows that decidining satisfiability of $KD45_2$ is also a $PSPACE - hard$ problem.

4.3.2 Upper Bound for K

We argue for the upper bound for K as PSPACE-complete, that is, can be solved using an amount of memory that is polynomial in the input length, by employing the tableau method.

The key lemma here is that the formula ϕ is K_n (resp. $T_n, S4_n, S5_n, KD45_n$) satisfiable iff there is a propositional K_n (resp. $T_n, S4_n, S5_n, KD45_n$) tableau for ϕ.

A K_n tableau is a tuple $T = (S, L, K_1, ..., K_n)$ where S, K_i are same as modals, and L is the labeling function regarding each state $s \in S$.

K_n tableau includes formulas that $L(s)$ is a propositional tableau without contradiction of propositions.

$K_i\psi \in L(s), (s,t) \in K_i$, then $\psi \in L(t)$.

$\neg K_i\psi \in L(s)$, then $\exists (s,t) \in K_i and \neg\psi \in L(t)$.

Note that T is a K_n tableau for ϕ if T is a K_n tableau and $\phi \in L(s)$ for some $s \in S$.

A T_n tableau takes reflexivity in addition: if $K_i\psi \in L(s)$, then $\psi \in L(t)$.

Respectively, A $S4_n, S5_n$ tableau takes transitivity, seriality in addition, and a $KD45_n$ tableau takes euclidean property plus transitivity in addition to a K_n tableau.

In the following discussion, we show how to construct a K_n tableau for ϕ and see if the construction is successful in order to tell if ϕ is K_n satisfiable. Then we prove that there is a verification process that takes space polynomial to input size $|\phi|$ to tell whether construction is successful, which indicates K_n satisfiability problem is in complexity class of $PSPACE$. Given this result, we arrive at a tight bound for such problem as $PSPACE - complete$ problem.

4.3.3 Tight Bound for K

Here's the brief summary of construction of such tableau given:

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Step1: Construct a tree with single node root
Step2: Repeat until non of (a)-(d) applies:
    (a). Forming a propositional tableau: if s is a leaf,
    L(s) is not inconsistent,
    L(s) is not formed completely, p is the least witness
    (as defined in the original
    paper):
        i. If p is !!p',create successor s' of s
        such that L(s')=L(s)U{p'}
        ii.If p is p'Vp'',create successor s' of s
        such that L(s')=L(s)U{p',p''}
        iii.If p is !(p'&p''),create successor s',s'' of s
        such that L(s')=L(s)U{!p'},
        L(s'')=L(s)U{!p''}
    (b). Forming a fully expanded tableau
    (c). Creating successor nodes
    (d). Marking nodes "satisfiable":
Step3: If the root is satisfiable, return satisfiable,
    otherwise return unsatisfiable
    (some details are omitted)
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(An example of K_n tableau construction for $\phi_0 = (p \wedge \neg(p \wedge q)) \wedge (K_1 \neg p \wedge \neg K_1 K_2 q)$ is shown on the top of next page.)

It is proven that such construction will eventually terminate, and let's prove the correctness of the algorithm:

" $\Rightarrow$ ": If the construction for ϕ returns "satisfiable", then there exists a K_n tableau for ϕ, by the key lemma, ϕ is k_n provable(satisfiable).

" $\Leftarrow$ ": Prove by contrapositive. If the root is not marked "satisfiable" by the algorithm, then $\neg\phi$ is provable. Specifically, if the root is not marked "satisfiable", thus the conjunction of all the formulas ψ in $L(root)$ is inconsistent, and then ϕ is not provable in K_n.(this can be proven by induction on the height of $root$ using bottom leaves as base cases: If $h = 0$, then $L(s)$ is either inconsistent(not marked "satisfiable") or it has no formulas of form $\neg K_i \phi$ (marked "satisfiable"). If $h > 0$, s is marked "satisfiable" iff one of its successors is marked "satisfiable"). So if ϕ is k_n provable, the root is marked "satisfiable" by the algorithm.

Thus the construction for ϕ returns "satisfiable" iff ϕ is k_n satisfiable.

The basic idea for this tableau construction algorithm is to permute, and see if there is any possible way to make ϕ's "nature" consistent.

There's an algorithm deciding satisfiability of K_n formula occupying polinomial space:

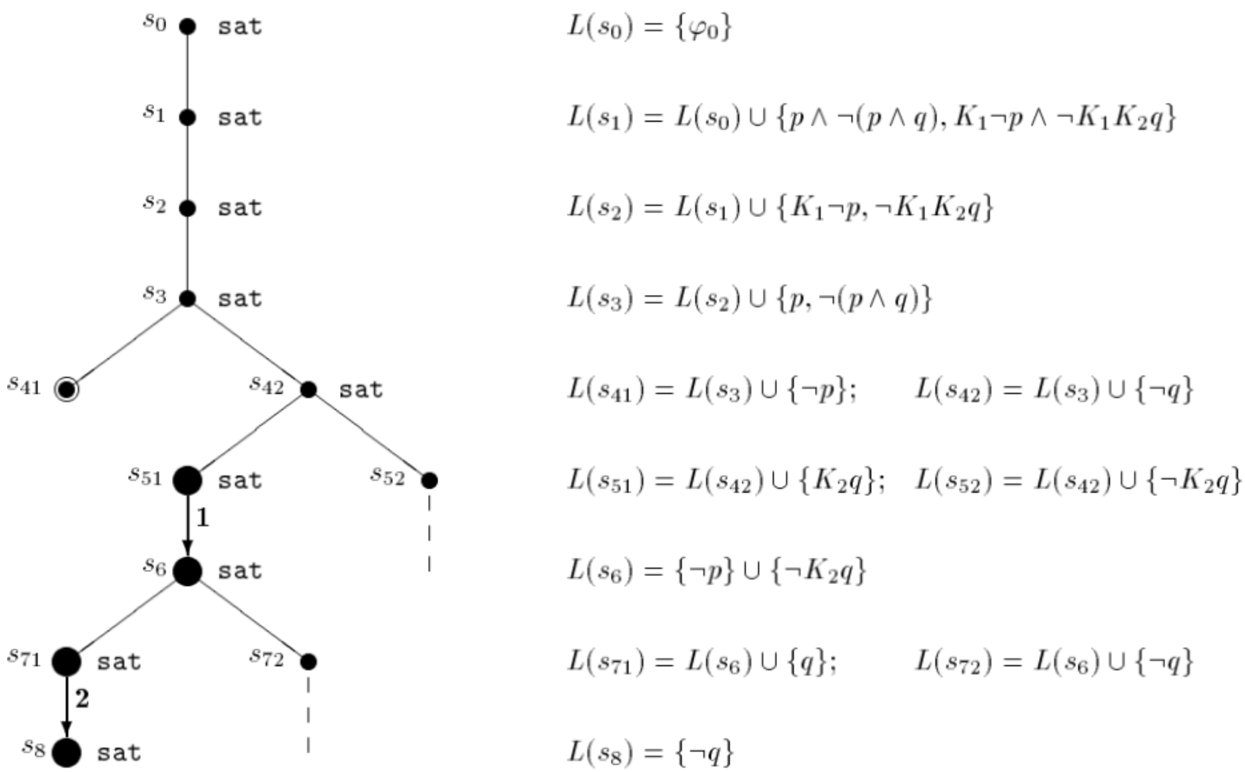

Figure 3: K_n tableau construction for $\varphi_0 = (p \wedge \neg(p \wedge q)) \wedge (K_1\neg p \wedge \neg K_1 K_2 q)$.

How: A state s only needs to use $3hm + m$ bits in total to store satisfiability states, where h is the height of the tree after construction terminates, and m is $|\phi|$. specifically, We use m bits to store ϕ, and $3hm$ bits to store information below it. We can use $2m$ bits to record information of $Sub^+(\phi)$ at a successor s',(setting 1 as T and 0 as F at the corresponding bit, since there are at most $2m$ subformulas in $Sub^+(\phi)$), and s' also keeps m bits to store what nodes have yet explored. Since we only care about the final information stored at root s, we can reuse space over and over, ignoring the information stored at subnode's subnodes.

So we can verify if ϕ is satisfiable using space $O(m^3)$, which is polynomial.

This show s K_n satisfiability problem is in $PSPACE$. Given the lower bound, we can conclude that K_n satisfiability problem is in $PSPACE{-}complete$.

4.3.4 Extending to $T, S4, S5_2, KD45_2$

We can easily modify the procedure from K_n's algorithm to make it work for T_n by modifying some step 2(d) from the construction of tableau.

Similarly, we can modify the algorithm accordingly to construct $S4, S5_2, KD45_2$ tableau. Details are omitted here.

It is also proven that there exists an algorithm for deciding satisfiability of $S4_n$ formulas that runs in polynomial space:

We show by induction on the height h of a node s. We suppose X is a list of labels appeared in ancestors of s. In this case, if we start the tableau construction with $L(s)$ as such node's label, we can determine how the node will be marked taking at most $(2h+3) * |\phi| + c + |X|$ bits of space. According to the fact that a node has at most m^4 ancestors, and requirement of storage for each label is $2m$ same as before, we can compute the labeling in polynomial space of $O(m^5)$ bits.

Similarly we can prove the existence of algorithm for deciding satisfiability of $T_n, S5_2, KD45_2$ formulas that runs in polynomial space(for example, an algorithm running in $O(m^2)$ bits for $S5_2$).

Finally, we can conclude that $T, S4, S5_2, KD45_2$ satisfiability problems are also in $PSPACE - complete$ complexity class.

The key reason to make deciding satisfiability for $S5, KD45$ modal logics in a much "quicker" class than the others' is the fact that we only need to verify them on a model with at most $|\phi|$ states, guaranteeing polynomial time verification whereas other 5 only guaranteeing polynomial space verification.

5 Conclusion

In this paper, we went over basic semantics of epistemic logic incorporated with multiple agents, and discovered the similarities shared with correspondence theories discussed in class. We identified axiom systems regarding ways to characterize knowledge.

We then identified and distinguished definitions of decidability, determinism, basic complexity classes. Then we examined decidability of validity of epistemic modal logics with multiple agents involved and how the involvements of multiple agents affect the decidability of validity for a given formula. The conclusion is such involvement may add complexity but it will not affect successfully deciding if a given formula is valid.

We then went into steps in deciding the satisfiability of formulas and attributing $K, S5, KD45, T, S4, S5_2, KD45_2$ into different complexity classes and prove that they meet the requirements of respective complexity classes' tight bounds. We also found that the involvement of multiple agents actually adds complexity in deciding the satisfiability, and that deciding if a given formula is satisfied is more complex if epistemic relation is under $K, T, S4, S5_2, KD45_2$ systems.

Besides what we discussed so far, topics such as knowledge common to a group of agents as well as distributed knowledge are also interesting to be modeled and their complexities are meaningful to analyze, which could provide practical use in lots of areas, such as computer science. Moreover, broader range of modal logics such as KB, KDB or temporal modal logics are good models to be analyzed of their complexity. The analyze can also involve analyzing probability under randomized algorithms.

6 References

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