III. Deciding Validity

AS.150.498: Modal Logic and Its Applications
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Recall the definition of validity for sentences in $S_{\mathcal{L}}$:

Definition 3.1. The sentence φ is a **logical validity** just in case there is no pointed model $\mathcal{M}, w$ such that $[\![\varphi]\!]^w_{\mathcal{M}} = F$.

We can also define the following related notion:

Definition 3.2. The sentence φ is **satisfiable** just in case there is a pointed model $\mathcal{M}, w$ such that $[\![\varphi]\!]^w_{\mathcal{M}} = T$.

Note that φ is valid if and only if $\neg\varphi$ is not satisfiable.

It turns out that like validity in sentential logic and monadic predicate logic but unlike validity in classical first-order logic, the validity of a sentence $\varphi \in S_{\mathcal{L}}$ is **decidable**—that is, there is an algorithmic procedure that for each $\varphi \in S_{\mathcal{L}}$ decides after a finite number of operations whether φ is valid.[1]

There are a number of ways to see this. Let us consider two of them here.

1 Selection

To prove decidability, it suffices to show that basic modal logic has the **effective finite model property**. The finite model property is this:

Theorem 3.1. The sentence $\varphi \in S_{\mathcal{L}}$ is satisfiable just in case there is a *finite* pointed model $\mathcal{M}, w$ such that $[\![\varphi]\!]^w_{\mathcal{M}} = T$.

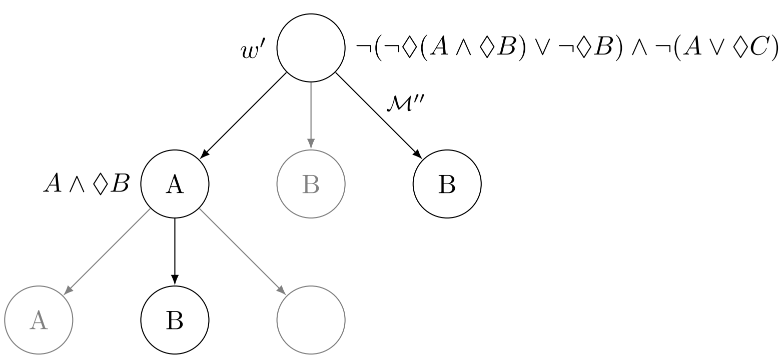
The effective finite model property is stronger:

Theorem 3.2. The sentence $\varphi \in S_{\mathcal{L}}$ is satisfiable just in case there is a pointed model $\mathcal{M}, w$ such that $[\![\varphi]\!]^w_{\mathcal{M}} = T$ and $|\mathcal{W}^{\mathcal{M}}| \leq f(|\varphi|)$ where f is a computable function and $|\varphi|$ is the length of φ.

To decide the validity of $\varphi \in S_{\mathcal{L}}$, we can first compute the effective bound $f(|\neg\varphi|)$ on the size of a verifying model for $\neg\varphi$. Since there are only finitely many models of a fixed size (up to isomorphism) when the valuation function is restricted to the finitely many sentence letters occurring in $\neg\varphi$, we can then check in a finite number of steps whether $\neg\varphi$ is true in any pointed model of size $\leq f(|\neg\varphi|)$. If so, φ is invalid. If not, φ is valid.

The first proof of the effective finite model property involves **selection**.

Suppose that φ is satisfied in the pointed model $\mathcal{M}, w$—that is, $[\![\varphi]\!]^w_{\mathcal{M}} = T$. Let $\mathcal{M}', w'$ be the tree unraveling of $\mathcal{M}$ around w. Since $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$, $[\![\varphi]\!]^{w'}_{\mathcal{M}'} = T$. Now we prune this tree. After converting $\Box$ operators in φ to $\Diamond$ operators using the duality $\Box \equiv \neg\Diamond\neg$, notice that the resulting logically equivalent sentence can be regarded as a truth-functional combination of sentence letters and sentences of the form $\Diamond\psi$. For each such subsentence $\Diamond\psi$ where $[\![\Diamond\psi]\!]^{w'}_{\mathcal{M}'} = T$, select one of the worlds v' such that $w'\mathcal{R}^{\mathcal{M}'}v'$ and $[\![\psi]\!]^{v'}_{\mathcal{M}'} = T$. Remove every unselected world u' such that $w'\mathcal{R}^{\mathcal{M}'}u'$ to obtain the thinner tree $\mathcal{M}''$. It is not hard to see that $[\![\varphi]\!]^{w'}_{\mathcal{M}''} = T$. Repeat this one level down for each of the ψs. The end result of this process is a finite pointed model that verifies φ and has size $\leq md(\varphi) \times |\varphi|^{md(\varphi)}$ where $md(\varphi)$ and $|\varphi|$ are computable.



2 Filtration

The second proof of the effective finite model property involves **filtration**.

Definition 3.3. The set $sub(\varphi)$ of subsentences of $\varphi \in S_{\mathcal{L}}$ is the smallest set such that:

$\varphi \in sub(\varphi)$
if $\neg\psi \in sub(\varphi)$ then $\psi \in sub(\varphi)$
if $(\psi \wedge \xi) \in sub(\varphi)$ then $\psi, \xi \in sub(\varphi)$
if $\Box\psi \in sub(\varphi)$ then $\psi \in sub(\varphi)$
if $\Diamond\psi \in sub(\varphi)$ then $\psi \in sub(\varphi)$

[1] I focus on the validity of sentences here but much the same goes for the validity of arguments since $\{\varphi_1, ..., \varphi_n\} \models \psi$ iff $\models \neg\varphi_1 \vee ... \vee \neg\varphi_n \vee \psi$.

Given model $\mathcal{M}$, we can define the following equivalence relation on $\mathcal{W}$: $w \sim_\varphi v$ iff for all $\psi \in sub(\varphi)$, $[\![\psi]\!]^w_{\mathcal{M}} = [\![\psi]\!]^v_{\mathcal{M}}$.

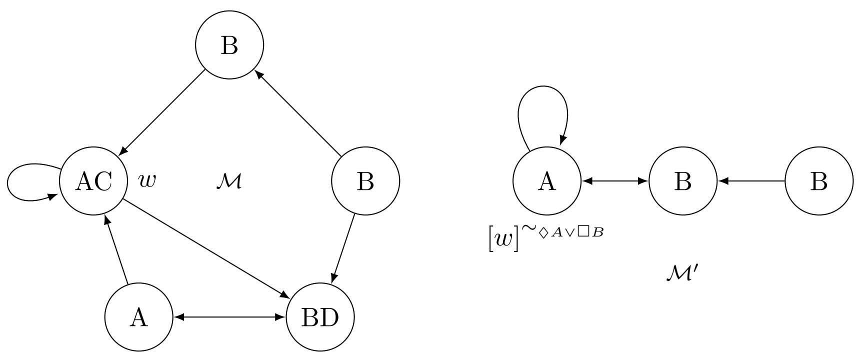
Let $|w|^{\sim_\varphi} = \{v \in \mathcal{W} : w \sim_\varphi v\}$.

This facilitates our next transformation of $\mathcal{M}$:

Definition 3.4. Given $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, the **filtration of $\mathcal{M}$ through φ** is the model $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \mathcal{V}' \rangle$ where:

$\mathcal{W}' = \{[w]^{\sim_\varphi} : w \in \mathcal{W}\}$
$\mathcal{R}' = \{\langle [w]^{\sim_\varphi}, [v]^{\sim_\varphi} \rangle$: there is $x \in [w]^{\sim_\varphi}$ and $y \in [v]^{\sim_\varphi}$ such that $x\mathcal{R}y\}$
$\mathcal{V}'(p, [w]^{\sim_\varphi}) = \begin{cases} \mathcal{V}(p, w) \text{ if } p \in sub(\varphi) \\ F \text{ otherwise} \end{cases}$

For example, the filtration of $\mathcal{M}$ through $\Diamond A \vee \Box B$ is this:



Theorem 3.3. If $\mathcal{M}'$ is the filtration of $\mathcal{M}$ through φ, then for each $w \in \mathcal{W}$ and $\psi \in sub(\varphi)$, $[\![\psi]\!]^w_{\mathcal{M}} = [\![\psi]\!]^{[w]^{\sim_\varphi}}_{\mathcal{M}'}$.

The proof is a straightforward induction on the complexity of sentences in $S_{\mathcal{L}}$.

Note that $|\mathcal{W}'| \leq 2^{|sub(\varphi)|} \leq 2^{|\varphi|}$. So we have shown again that basic modal logic has the effective finite model property.