I. Basic Sentential Modal Language

AS.150.498: Modal Logic and Its Applications
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A **modal operator** qualifies the truth of a statement. These come in a variety of different flavors:

epistemic	According to what McNulty knows, it is possible that... McNulty knows that...
metaphysical	It could have been that... It is necessary that...
temporal	Going forward, it will sometime be that... Going forward, it will always be that...
deontic	It is permissible that... It is obligatory that...

And so forth. One of our primary goals in this course is to develop formal techniques for systematically determining whether an argument involving modal operators is logically valid.

1 Syntax

To do this, we will be working with formal languages. Here is the simplest one:[1]

Definition 1.1. The **basic sentential modal language** $\mathcal{L}$ has the following syntax (in *Backus-Naur* notation):

$p ::= A, B, C, ...$
$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi$

$At_{\mathcal{L}} = \{A, B, ...\}$ is the set of atoms in $\mathcal{L}$, and $S_{\mathcal{L}}$ is the set of well-formed sentences in $\mathcal{L}$.

Note that this language is a bit redundant since '$\bot$' and '$\Diamond$' can be regarded as abbreviations for '$(A \wedge \neg A)$' and '$\neg\Box\neg$' respectively. The remaining logical connectives '$\vee$,' '$\supset$,' and '$\equiv$' can also be defined in the usual fashion.

[1] Going forward, I'll be loose about use and mention, omitting Quinean quasi-quotes and the like.

At various points in the course, we will work with a *polymodal* language with multiple modalities.

Definition 1.2. The **modal depth** of $\varphi \in S_{\mathcal{L}}$ is defined recursively as follows:

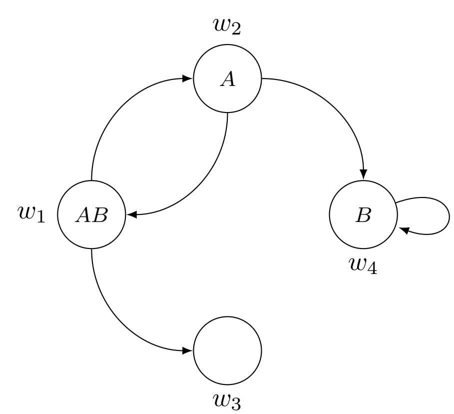
$md(A) = 0$
$md(\bot) = 0$
$md(\neg\varphi) = md(\varphi)$
$md((\varphi \wedge \psi)) = \max(md(\varphi), md(\psi))$
$md(\Box\varphi) = md(\varphi) + 1$
$md(\Diamond\varphi) = md(\varphi) + 1$

For example, $md(\Diamond\Diamond A) = 2$ and $md(\Box(A \wedge \Box(A \wedge \Box A))) = 3$.

2 Semantics

A *model* for a formal language is, roughly, something that provides enough information to determine the extension of each sentence in this language. For a sentential language without modals, a model is (basically) just a reference row of a truth table—more precisely, an assignment of truth values to all sentence letters in this language. For the modal language $\mathcal{L}$, models are a bit more complicated.

Definition 1.3. A **Kripke model** $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ for $\mathcal{L}$ consists of a nonempty set $\mathcal{W}$ of possible world states, a binary accessibility relation $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ between worlds, and a valuation $\mathcal{V} : At_{\mathcal{L}} \times \mathcal{W} \to \{T, F\}$ mapping each sentence letter $p \in At_{\mathcal{L}}$ and world $w \in \mathcal{W}$ to a truth value.



This figure represents the model $\mathcal{M}$ where:

$\mathcal{W} = \{w_1, w_2, w_3, w_4\}$
$\mathcal{R} = \{\langle w_1, w_2\rangle, \langle w_1, w_3\rangle, \langle w_2, w_1\rangle, \langle w_2, w_4\rangle, \langle w_4, w_4\rangle\}$
$\mathcal{V}(A, w_1) = \mathcal{V}(A, w_2) = T$, $\mathcal{V}(A, w_3) = \mathcal{V}(A, w_4) = F$
$\mathcal{V}(B, w_1) = \mathcal{V}(B, w_4) = T$, $\mathcal{V}(B, w_2) = \mathcal{V}(B, w_3) = F$

Using this information, we can evaluate every sentence $\varphi \in S_{\mathcal{L}}$ for **truth in a pointed model** $\mathcal{M}, w$.

Definition 1.4. The following recursive definition of truth lifts $\mathcal{V}$ to the complete interpretation function $[\![\ ]\!]_{\mathcal{M}} : S_{\mathcal{L}} \times \mathcal{W} \to \{T, F\}$ for $\mathcal{L}$ mapping each sentence and world to a truth value:

$$\begin{array}{rll} [\![p]\!]^w_{\mathcal{M}} = T & \text{iff} & \mathcal{V}(p, w) = T \\ [\![\bot]\!]^w_{\mathcal{M}} = T & \text{iff} & 0 = 1 \\ [\![\neg\varphi]\!]^w_{\mathcal{M}} = T & \text{iff} & [\![\varphi]\!]^w_{\mathcal{M}} = F \\ [\![(\varphi \wedge \psi)]\!]^w_{\mathcal{M}} = T & \text{iff} & [\![\varphi]\!]^w_{\mathcal{M}} = [\![\psi]\!]^w_{\mathcal{M}} = T \\ [\![\Box\varphi]\!]^w_{\mathcal{M}} = T & \text{iff} & \forall v \in \{v : w\mathcal{R}v\}([\![\varphi]\!]^v_{\mathcal{M}} = T) \\ [\![\Diamond\varphi]\!]^w_{\mathcal{M}} = T & \text{iff} & \exists v \in \{v : w\mathcal{R}v\}([\![\varphi]\!]^v_{\mathcal{M}} = T) \end{array}$$

Note that $\Box$ is a kind of restricted universal quantifier and $\Diamond$ is a kind of restricted existential quantifier.

Given the above model, for example, $[\![B]\!]^{w_2}_{\mathcal{M}} = F$, $[\![\Diamond(A \wedge B)]\!]^{w_2}_{\mathcal{M}} = T$, and $[\![\Box B]\!]^{w_2}_{\mathcal{M}} = T$.

We can also define a family of formal logical notions in terms of truth in a pointed model:

Definition 1.5. The argument from premises $\varphi_1, ..., \varphi_n$ to conclusion ψ is **logically valid**, $\{\varphi_1, ..., \varphi_n\} \models \psi$, just in case there is no pointed model $\mathcal{M}, w$ such that $[\![\varphi_1]\!]^w_{\mathcal{M}} = ... = [\![\varphi_n]\!]^w_{\mathcal{M}} = T$ and $[\![\psi]\!]^w_{\mathcal{M}} = F$.

Definition 1.6. The sentence φ is a **logical validity**, $\models \varphi$, just in case there is no pointed model $\mathcal{M}, w$ such that $[\![\varphi]\!]^w_{\mathcal{M}} = F$.

Definition 1.7. The sentences $\varphi_1, ..., \varphi_n$ are **logically consistent** just in case there is a pointed model $\mathcal{M}, w$ such that $[\![\varphi_1]\!]^w_{\mathcal{M}} = ... = [\![\varphi_n]\!]^w_{\mathcal{M}} = T$.

And so forth. Now, let us put this formalism to use. Consider the following argument:

(P1) It must be the case that Avon is in the tower.

(P2) It must be the case that Stringer is in the tower.

(C) It must be the case that both Avon and Stringer are in the tower.

Is this argument logically valid? Well, we now have some helpful resources to answer this question. The first step is to translate the English argument into the formal language $\mathcal{L}$:

(P1) $\Box A$

(P2) $\Box S$

(C) $\Box(A \wedge S)$

The second step is to determine whether $\{\Box A, \Box S\} \models \Box(A \wedge S)$.

Suppose that $[\![\Box A]\!]^w_{\mathcal{M}} = T$ and $[\![\Box S]\!]^w_{\mathcal{M}} = T$.

By the clause for $\Box$ in Def 1.4, $\forall v \in \{v : w\mathcal{R}v\}([\![A]\!]^v_{\mathcal{M}} = [\![S]\!]^v_{\mathcal{M}} = T)$.

By the clause for $\wedge$ in Def 1.4, $\forall v \in \{v : w\mathcal{R}v\}([\![A \wedge S]\!]^v_{\mathcal{M}} = T)$.

By the clause for $\Box$ in Def 1.4, $[\![\Box(A \wedge S)]\!]^w_{\mathcal{M}} = T$.

Thus, $\{\Box A, \Box S\} \models \Box(A \wedge S)$ and assuming that $\models$ explicates our target informal notion of validity, the original English argument is logically valid.

Next, consider this argument:

(P1) It might be the case that Avon is in the tower.

(P2) It might be the case that Stringer is in the tower.

(C) It might be the case that both Avon and Stringer are in the tower.

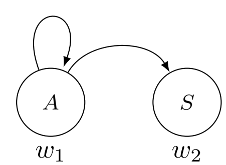
Is this argument logically valid? Its translation into $\mathcal{L}$ is this:

(P1) $\Diamond A$

(P2) $\Diamond S$

(C) $\Diamond(A \wedge S)$

But $\{\Diamond A, \Diamond S\} \not\models \Diamond(A \wedge S)$. Here is a counter-model:



$[\![\Diamond A]\!]^{w_1}_{\mathcal{M}} = [\![\Diamond S]\!]^{w_1}_{\mathcal{M}} = T$ but $[\![\Diamond(A \wedge S)]\!]^{w_1}_{\mathcal{M}} = F$. If $\models$ explicates our target informal notion of validity, then the second English argument is logically invalid.