Solutions

pages	1 – 3	4 – 6	7 – 8	9 – 10	11 – 13	*total*
scores						

Exam #2, November 19, Calculus III, Fall, 2007, W. Stephen Wilson

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Name: ______________________ Date: ____________

TA Name and section: ______________________

NO CALCULATORS, NO PAPERS, SHOW WORK. (28 points total)

1. (2 points) Set up the triple integral for the volume of a cube (1 point), $0 \le x, y, z \le 1$ and evaluate it (1 point).

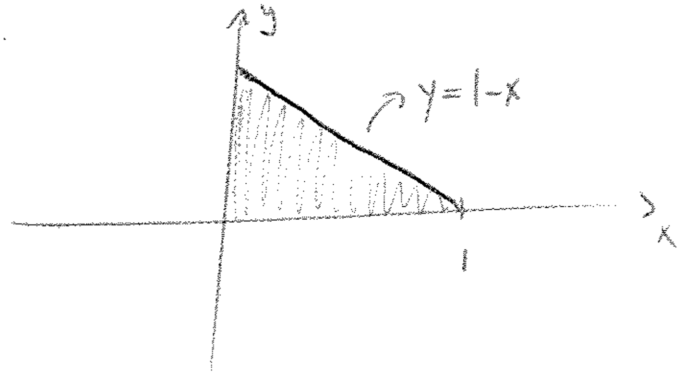
$\underline{1\text{ Pt}}$: $$\int_0^1\int_0^1\int_0^1 dx\,dy\,dz$$

$\underline{1\text{ Pt}}$: $$\int_0^1\int_0^1 x\Big|_0^1 dy\,dz = \int_0^1\int_0^1 dy\,dz = \int_0^1 y\Big|_0^1 dz$$

$$= \int_0^1 dz = 1$$

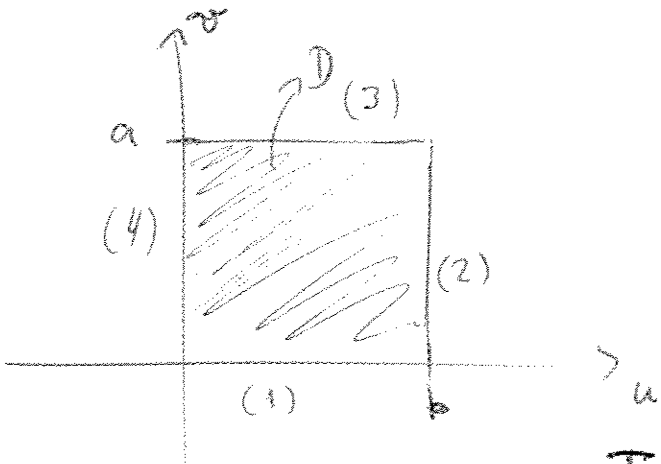
2. (2 points total) What region is the double integral, $\int_0^1 \int_0^{-x+1} f(x,y)\,dy\,dx$, taken over? Sketch and label (1 point). Change the order of integration (1 point).

1 pt:



1 pt: $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} f(x,y)\,dx\,dy$

3. (1 points total) Consider the map $T(u,v) = (u, (1-\frac{u}{b})v)$. Let D be the rectangular region $0 \le u \le b$, $0 \le v \le a$. What is the region $T(D)$? (Sketch and label.)



(1) $v=0$, $0 \le u \le b$

$T(u,0) = (u,0)$

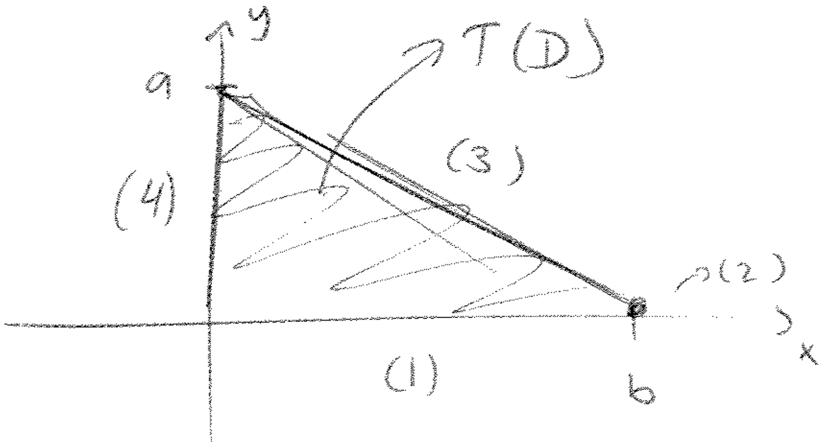
(2) $u=b$, $0 \le v \le a$

$T(b,v) = (b,(1-1)v) = (b,0)$

(3) $v=a$, $0 \le u \le b$

$T(u,a) = (u, a(1-\frac{u}{b}))$

(4) $u=0$, $0 \le v \le a$, $T(0,v) = (0,v)$



4. (2 points total) Set up a double integral on the region $T(D)$ from the previous problem to compute the area (1 point). Evaluate this integral to compute the area (1 point).

$$\text{Area} = \int_0^b \int_0^{a(1-\frac{x}{b})} 1\,dy\,dx = \int_0^b a\left(1-\frac{x}{b}\right)dx = a\left(b-\frac{1}{2}b\right)$$

$$= \frac{1}{2}ab$$

5. (3 points total) Use a change of variables to set up an integral on the region D to give the area of $T(D)$ from the previous problems. (1 point for the limits and 1 point for what is being integrated). Compute this integral over D to get the area of $T(D)$ (1 point).

$$\text{Area} = \iint_{T(D)} 1\,dx\,dy = \int_0^a \int_0^b 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \det \begin{pmatrix} 1, & 0 \\ -\frac{v}{b}, & 1-\frac{u}{b} \end{pmatrix} \right| = 1 - \frac{u}{b}$$

$$\int_0^a \int_0^b \left(1 - \frac{u}{b}\right) du\,dv = a \int_0^b \left(1 - \frac{u}{b}\right) du = a\left(b - \frac{1}{2}b\right)$$

$$= \frac{1}{2} a \cdot b \checkmark$$

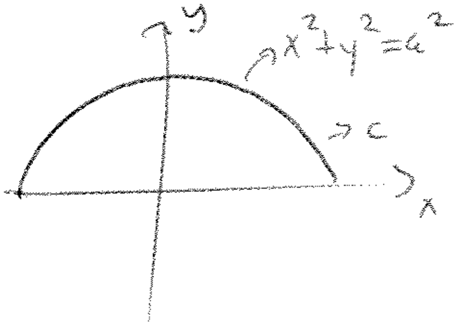
6. (1 points) Let $f(x,y,z) = e^{x+y^2+z^3}$. Take the curve given by the path $c(t) = (t^2, t^4, t^6)$ from $t = 0$ to $t = 1$. Consider the vector field ∇f and compute $\int_c \nabla f \cdot d\vec{s}$.

$$\int_c \nabla f \cdot d\vec{s} = f(\vec{c}(1)) - f(\vec{c}(0))$$

$$\vec{c}(1) = (1,1,1), \quad \vec{c}(0) = (0,0,0)$$

$$\int_c \nabla f \cdot d\vec{s} = e^3 - 1$$

7. (3 points) Define a function on the semi-circle $x^2 + y^2 = a^2$, $y \geq 0$, that takes a point on it to its y-coordinate. What is the average value of this function on the semi-circle? (2 points for setting up the integral (1 point for the limits and 1 point for what is integrated) and 1 point for getting the right answer using it.)



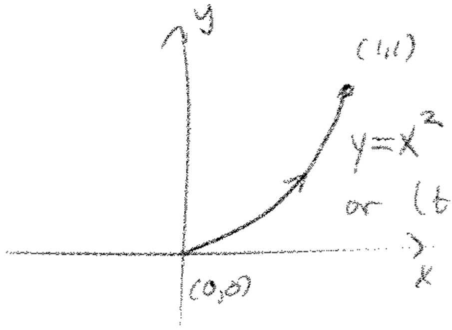
$$f(x,y) = y$$

$$f_{avg} = \frac{\int_c f\,ds}{\int_c ds}, \quad \int_c ds = \pi a$$

$$\int_c f\,ds = \int_0^{\pi} (a\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt = a^2 \int_0^{\pi} \sin t\,dt$$

$$= 2a^2 \Rightarrow f_{avg} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

8. (3 points) Consider the curve that goes from $(0,0)$ to $(1,1)$ along $y = x^2$. Consider the vector field $F(x,y) = (x^2y, xy^2)$ in the xy-plane. Compute the line integral of F on this curve. (1 point for getting the path right, 1 for setting up the integral and 1 for evaluating it correctly.)

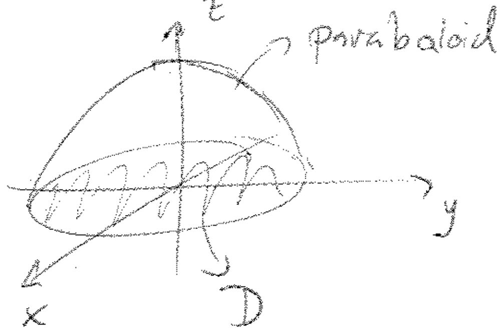


$$\int_c \vec{F}\cdot d\vec{s} = \int_0^1 (t^4, t^5)\cdot(1, 2t)\,dt$$

$$= \int_0^1 t^4 + 2t^6\,dt$$

$$= \frac{1}{5} + \frac{2}{7} = \frac{7+10}{35} = \frac{17}{35}$$

9. (3 points) Find the volume trapped between the graph of $f(x, y) = 1 - x^2 - y^2$ and the xy-plane. (1 point for the limits on the integral, 1 point for what is being integrated, and 1 point for getting the right answer.)



$$Vol = \iint_D \left(\int_0^{1-x^2-y^2} 1\, dz \right) dx\, dy$$

$$= \iint_D 1 - (x^2+y^2)\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^1 (1-r^2)\, r\, dr\, d\theta = 2\pi \int_0^1 r - r^3\, dr$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2}$$

10. (3 points) Find the surface area of the graph of $f(x, y) = 1 - x^2 - y^2$ where it is above the xy-plane. (1 point for the limits on the integral, 1 point for the thing you integrate and 1 point for getting the correct answer.)

$$\text{Surface Area} = \iint_S 1\,dS = \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

$$\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y \Rightarrow \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+4x^2+4y^2}$$

$$\iint_S dS = \iint_D \sqrt{1+4(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}\,dr\,d\theta$$

$$= 2\pi\int_0^1 r\sqrt{1+4r^2}\,dr. \quad \text{Let } u = 1+4r^2,\ du = 8r\,dr$$

$$= \frac{2\pi}{8}\int_1^5 \sqrt{u}\,du = \frac{\pi}{4}\left\{\frac{2}{3}u^{3/2}\right\}\Big|_1^5 = \frac{\pi}{6}\left(5^{3/2}-1\right)$$

answer to #12

11. (1 point) What is the average height of $f(x, y) = 1 - x^2 - y^2$ where it is above the xy-plane?

i.e. $z_{avg} = \dfrac{\iint_S z\,dS}{\iint_S dS}$, $\iint_S dS = \frac{\pi}{6}(5^{3/2}-1)$

$$\iint_S z\,dS = \iint_D (1-(x^2+y^2))\sqrt{1+4(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 r(1-r^2)\sqrt{1+4r^2}\,dr\,d\theta = 2\pi\int_0^1 r(1-r^2)\sqrt{1+4r^2}\,dr$$

$$= 2\pi\left\{\int_0^1 r\sqrt{1+4r^2}\,dr - \int_0^1 r^3\sqrt{1+4r^2}\,dr\right\}$$

$\int_0^1 r\sqrt{1+4r^2}\,dr = \frac{1}{12}(5^{3/2}-1)$, $\int_0^1 r^3\sqrt{1+4r^2}\,dr = \int_0^1 (r^2)\cdot r\sqrt{1+4r^2}\,dr$

$u = r^2$, $dv = r\sqrt{1+4r^2}\,dr \Rightarrow v = \frac{2}{3}\cdot\frac{1}{8}(1+4r^2)^{3/2} = \frac{1}{12}(1+4r^2)^{3/2}$

$$\int_0^1 r^3\sqrt{1+4r^2}\,dr = \frac{r^2}{12}(1+4r^2)^{3/2}\Big|_0^1 - \int_0^1 \frac{1}{6}r(1+4r^2)^{3/2}\,dr$$

$= \frac{1}{12}5^{3/2} - \frac{1}{6}\int_0^1 r(1+4r^2)^{3/2}\,dr$. Let $u = 1+4r^2$:

$= \frac{1}{12}5^{3/2} - \frac{1}{48}\int_1^5 u^{3/2}\,du = \frac{1}{12}5^{3/2} - \frac{1}{48}\cdot\frac{2}{5}(5^{5/2}-1)$

Now piece it together ...

 answer to #11

12. (2 points) Consider a function on the surface given by the graph of $f(x, y) = 1 - x^2 - y^2$ where it is above the xy-plane that assigns the z-coordinate to a point on the surface. Set up the integral for the average height of this function. Do not integrate.

$$\cancel{z_{avg} = \frac{\iint_D f(x,y)\,dx\,dy}{\iint_D dx}}$$

$$z_{avg} = \frac{\iiint z\,dx\,dy\,dz}{\text{Vol of hemisphere}}$$

$$= \frac{\iint_D \left(\int_0^{1-(x^2+y^2)} z\,dz\right) dx\,dy}{\frac{2}{3}\pi} = \frac{3}{4\pi} \iint_D \left(1-(x^2+y^2)\right)^2 dx\,dy$$

$$= \frac{3}{4\pi} \int_0^{2\pi}\int_0^1 (1-r^2)^2\, r\,dr\,d\theta = \frac{3}{2}\int_0^1 r(1-r^2)^2\,dr$$

Let $u = 1-r^2$. $\frac{3}{2}\int_0^1 \frac{1}{2} u^2\,du = \frac{1}{4}$

13. (2 points) Find a parameterization of the graph of $f(x, y) = 1 - x^2 - y^2$ where it is above the xy-plane that starts $\Phi(r, \theta) = (r\cos(\theta), -, -)$. Be sure and give the limits on r and θ.

$y = r\sin\theta, \quad z = 1 - r^2 \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1$