| pages | 2 – 3 | 4 – 6 | 7 – 10 | 11 – 13 | 14 – 16 | 17 – 20 | total |
|---|---|---|---|---|---|---|---|
| scores | | | | | | | |

Final Exam, Dec 15, Calculus III, Fall, 2006, Eugene Ha and W. Stephen Wilson

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Name: SOLUTIONS Date: ____________

TA Name and section: ____________

**NO CALCULATORS, NO PAPERS, SHOW WORK**. (78 points total) To use cylindrical coordinates we have $\iiint f(x,y,z)dxdydz = \iiint f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$.

To use spherical coordinates we have

$\iiint f(x,y,z)dxdydz = \iiint f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi$.

To evaluate the average value of a function on a curve, surface or solid, you integrate the function over the object and divide by the length/area/volume of the object.

**1.** (2 points) Find the directional derivative of the function $g(x,y,z) = x^2y + yz + x - z$, in the direction $\mathbf{j} = (0,1,0)$ at an arbitrary point $(x,y,z)$.

Directional derivative of a function $f$ at a point $(x,y,z)$ in the direction of $\underline{v}$ (where $\underline{v}$ is a unit vector) $= \nabla f(x,y,z) \cdot \underline{v}$

For this problem, directional derivative $= \nabla g \cdot \underline{j} = \begin{pmatrix} 2xy+1 \\ x^2+z \\ y-1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$= x^2+z$

**2.** (4 points total) Find the direction that the directional derivative of the function $g(x,y,z) = x^2y + yz + x - z$, is maximal at the point $(1,1,1)$ (2 points) and evaluate that derivative (2 points).

The directional derivative is maximal in the direction of the gradient $= \nabla g(1,1,1) = \begin{pmatrix} 2xy+1 \\ x^2+z \\ y-1 \end{pmatrix}$ at $(1,1,1)$

$= 3\underline{i} + 2\underline{j} + 0\underline{k}$

Directional derivative in this direction $= \nabla g \cdot \underline{v} = \nabla g(1,1,1) \cdot \dfrac{\nabla g(1,1,1)}{\|\nabla g(1,1,1)\|}$

$= (3\underline{i} + 2\underline{j}) \cdot \dfrac{(3\underline{i} + 2\underline{j})}{\sqrt{3^2+2^2}}$

$= \dfrac{13}{\sqrt{13}} = \sqrt{13}$

(Note: we divided by the length to get a unit vector)

**3.** (6 points total) Let $z = f(x,y)$. Consider $z$ as a function of the polar coordinates $(r,\theta)$ (i.e. $(x,y) = (r\cos(\theta), r\sin(\theta))$). Compute $\partial z/\partial r$, $\partial z/\partial\theta$, and $\partial^2 z/\partial r\partial\theta$ in terms of the partials of $f$ with respect to $x$ and $y$ (2 points each).

$$z = f(x,y), \quad (x,y) = (r\cos\theta, r\sin\theta) \Rightarrow \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta,$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial\theta} = r\cos\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$= -\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta$$

$$\frac{\partial^2 z}{\partial r\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}(x,y)\cos\theta + \frac{\partial f}{\partial y}(x,y)\sin\theta\right)$$

$$= \frac{\partial f}{\partial x}(-\sin\theta) + \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial\theta}\right)\cos\theta$$

$$+ \frac{\partial f}{\partial y}\cos\theta + \left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial\theta}\right)\sin\theta$$

$$= -\frac{\partial f}{\partial x}\sin\theta + \left(\frac{\partial^2 f}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 f}{\partial y\partial x}(r\cos\theta)\right)\cos\theta$$

$$+ \frac{\partial f}{\partial y}\cos\theta + \left(\frac{\partial^2 f}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 f}{\partial y^2}(r\cos\theta)\right)\sin\theta$$

**4.** (4 points total) Use the method of Lagrange multipliers to find the minimum and maximum values of the function $f(x, y) = x/2 + y/5$ subject to the constraint $x^2 + y^2 - 1 = 0$. (2 points for setting up the correct equations and 2 points for solving and interpreting them.)

$f(x,y) = \frac{1}{2}x + \frac{1}{5}y$

Constraint: $x^2+y^2-1=0 \Rightarrow g(x,y) = x^2+y^2-1$

Using Lagrange multipliers we have: $\nabla f = \lambda \nabla g$

$$\Rightarrow \begin{pmatrix} 1/2 \\ 1/5 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$\Rightarrow$ We have the system of equations to solve for critical pts.:

$\frac{1}{2} = 2\lambda x$ --- (1)

$\frac{1}{5} = 2\lambda y$ --- (2)

$x^2+y^2-1=0$ --- (3)

(i) We consider the case $\lambda=0$: in eq. (1), $\frac{1}{2} = 2(0)x = 0 \Rightarrow$ no solution

(ii) In the case $\lambda \neq 0$, from (1) and (2) we have $x = \frac{1}{4\lambda}$, $y = \frac{1}{10\lambda}$

Substituting these in (3): $\left(\frac{1}{4\lambda}\right)^2 + \left(\frac{1}{10\lambda}\right)^2 = 1$

$$\Rightarrow \frac{1}{16\lambda^2} + \frac{1}{100\lambda^2} = 1 \Rightarrow \frac{1}{\lambda^2}\left(\frac{1}{16} + \frac{1}{100}\right) = 1 \Rightarrow \frac{1}{\lambda^2}\left(\frac{116}{1600}\right) = 1$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{116}{1600}} = \pm\frac{2\sqrt{29}}{40} = \pm\frac{\sqrt{29}}{20}$$

$\Rightarrow$ The critical pts. are (by substituting for $\lambda$ in (1) and (2)):

$\left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$ and $\left(-\frac{5}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\right)$

And $f\left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right) = \frac{1}{2}\frac{5}{\sqrt{29}} + \frac{1}{5}\frac{2}{\sqrt{29}}$

$f\left(-\frac{5}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\right) = -\frac{1}{2}\frac{5}{\sqrt{29}} - \frac{1}{5}\frac{2}{\sqrt{29}}$

$\Rightarrow$ Max. value is $\frac{5}{2\sqrt{29}} + \frac{2}{5\sqrt{29}}$ at $\left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$

and min. value is $-\frac{5}{2\sqrt{29}} - \frac{2}{\sqrt{29}}$ at $\left(-\frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\right)$

**5.** (4 points total) Consider the hemisphere, $x^2 + y^2 + z^2 = 1$, $z \geq 0$. What is the average height (z-height, i.e. distance from the $xy$-plane) of points on this hemisphere when considered as a function on the disk $x^2 + y^2 \leq 1$ in the plane? You can use the fact that the area of the circle of radius 1 is $\pi$ without having to compute it. (2 points for setting up the integral and 2 for evaluation)

Same problem as Fall '06 Midterm 2 #4.

See "Solutions to problems covered in review session" for Midterm 2 at the bottom of my Calc III webpage

**6.** (4 points total) Consider the hemisphere, $x^2 + y^2 + z^2 = 1$, $z \geq 0$. What is the average height (z-height, i.e. distance from the $xy$-plane) of points on this hemisphere when considered as a function on the hemisphere itself. You can use the fact that the area of the hemisphere is $2\pi$ without having to compute it. (2 points for setting up the integral and 2 for evaluation)

z-height = distance from xy-plane of a point $(x,y,z)$ on the hemisphere $x^2+y^2+z^2=1$

$$g(x,y,z) = z$$

Average height as a function on the hemisphere itself

$$= \frac{\iint_H g\, dS}{\text{Area of } H} = \frac{\iint_H g\, dS}{2\pi}$$

(where $H$ is the surface of the hemisphere)

To compute $\iint_H g\, dS$, we parameterize $H$ as:

$$\Phi(\phi,\theta) = (\underset{x}{\sin\phi\cos\theta}, \underset{y}{\sin\phi\sin\theta}, \underset{z}{\cos\phi})$$

(We used spherical coordinates here with $\rho = 1$)

And $\|T_\phi \times T_\theta\|$ = Jacobian for spherical coordinates $= \rho^2 \sin\phi$ (where $\rho = 1$ in our case)

$$= (1)^2 \sin\phi = \sin\phi$$

So
$$\iint_H g\, dS = \int_0^{2\pi}\int_0^{\pi/2} g(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\, \|T_\phi \times T_\theta\|\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (\cos\phi)(\sin\phi)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \tfrac{1}{2}\sin 2\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[-\tfrac{1}{4}\cos 2\phi\right]_0^{\pi/2} d\theta = \int_0^{2\pi} -\tfrac{1}{4}(-1) - \left(-\tfrac{1}{4}(1)\right) d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}\, d\theta = \pi$$

$\Rightarrow$ Average height $= \dfrac{\pi}{2\pi} = \dfrac{1}{2}$

**7.** (2 points) Consider the mapping, $T(u, v) = (x, y)$ defined by the equations $x = u + v$, $y = v - u^2$. Compute the derivative of this function.

Problems 7 – 11 here are exactly the same as Fall '06 Midterm 2 # 6 – 10

See solutions in "Solutions to problems covered in the review session" for Midterm 2 at the bottom of my Calc III webpage.

**8.** (2 points) Compute the Jacobian determinant $\partial(x, y)/\partial(u, v)$ of the function $T$ in the previous problem.

**9.** (2 points) A triangle region $W$ in the $uv$-plane has vertices $(0,0)$, $(0,1)$, $(1,0)$. Describe, by means of a sketch, its image, $S$, in the $xy$-plane, under the mapping $T$ from the previous page. (Hint: What is the image of the boundary of $W$?)

**10.** (4 points total) Express the area of $S$ (from the previous page) as an integral over $S$ (2 points) and compute the integral (2 points).

**11.** (4 points total) Express the area of $S$ (from the previous pages) as an integral over $W$ (2 points) and compute the integral (2 points).

**12.** (10 points total) Let $f(x,y) = (3-x)(3-y)(x+y-3)$. (a) Find all (4) of the critical points of $f$ (1 point each), and for each one, determine whether it's a (relative) minimum, a (relative) maximum, or a saddle point and give your reasons (1 point each). (It may be helpful to sketch the solution to the equation $f(x,y) = 0$ (and others).) (b) Does $f$ achieve an absolute minimum or absolute maximum value? Justify your answer (2 points).

(a) We need to solve $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ to find the critical points.

$$\frac{\partial f}{\partial x} = -(3-y)(x+y-3) + (3-x)(3-y) = 0 \Rightarrow (3-y)(3-x-x-y+3) = 0$$
$$\Rightarrow (3-y)(6-2x-y) = 0 \quad \text{..(1)}$$

$$\frac{\partial f}{\partial y} = -(3-x)(x+y-3) + (3-x)(3-y) = 0 \Rightarrow (3-x)(3-y-x-y+3) = 0$$
$$\Rightarrow (3-x)(6-2y-x) = 0 \quad \text{..(2)}$$

From (1) we have (i) $3-y=0 \Rightarrow y=3$ or (ii) $6-2x-y=0 \Rightarrow y = 6-2x$

Substituting case (i) in (2): $(3-x)(6-2(3)-x) = 0 \Rightarrow (3-x)(-x) = 0$
$$\Rightarrow x = 3 \text{ or } x = 0$$

We check that these are consistent with equation (1) and find that we have two of the solutions, $(3,3)$ and $(0,3)$.

Now, we substitute case (ii) in (2): $(3-x)(6-2(6-2x)-x) = 0$
$$\Rightarrow (3-x)(-6+3x) = 0$$
$$\Rightarrow x = 3 \text{ or } x = 2$$

Substituting these back into equation (ii) $y = 6-2x \Rightarrow y = 0$ or $2$ respectively

$\Rightarrow$ our last two solutions are: $(3,0)$ and $(2,2)$.

More space for this on the next page.

More space for the last problem

To classify the critical points we need:

(I) $\frac{\partial^2 f}{\partial x^2} = -(3-y) - (3-y) = -6+2y$

(II) The Discriminant, $D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$$= (-6+2y)(-6+2x) - \left(-(6-2x-y)-(3-y)\right)^2$$
$$= (-6+2y)(-6+2x) - (-9+2x+2y)^2$$

$\frac{\partial^2 f}{\partial x^2}(3,3) = 0$; $D(3,3) = 0 - (3)^2 < 0 \Rightarrow$ saddle pt. at $(3,3)$

$\frac{\partial^2 f}{\partial x^2}(0,3) = 0$; $D(0,3) = 0 - (-3)^2 < 0 \Rightarrow$ Saddle pt. at $(0,3)$

$\frac{\partial^2 f}{\partial x^2}(3,0) = -6 < 0$; $D(3,0) = (-6)(0) - (-3)^2 < 0 \Rightarrow$ saddle pt. at $(3,0)$

$\frac{\partial^2 f}{\partial x^2}(2,2) = -2 < 0$; $D(2,2) = (-2)(-2) - (-1)^2 = 3 > 0 \Rightarrow$ max. pt. at $(2,2)$

(b) $f$ is defined on all of $\mathbb{R}^2$ and there are no local minima. Therefore there is no absolute minimum.

We have a local max at $(2,2) = f(2,2) = (1)(1)(1) = 1$

But we can see that for example, when $x = y$,
$f(x,x) = (3-x)^2(2x-3)$ and for large enough $x$ this will be larger than 1

eg. $f(4,4) = (-1)^2(8-3) = 5 > 1$

$\Rightarrow$ There is no absolute max.

**13.** (4 points total) Consider the solid hemisphere, $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ of uniform density. By symmetry the center of mass is on the $z$-axis. What is its $z$-coordinate? (Another way to look at this is that this is the average height of all the points in half a solid ball viewed as a function defined on the solid hemisphere.) You can use the fact that the volume of the solid unit hemisphere is $\frac{2\pi}{3}$ without having to compute it. (2 points for setting up the integral and 2 for evaluation)

See solution to Fall '06 Midterm 2 #5 in "Solutions to problems covered in review session" for Midterm 2 at the bottom of my Calc III webpage

**14.** (10 points total) Let $C$ be the curve in the $xy$-plane which is the union of the curve $4x^2+y^2-4=0$, $y \geq 0$ and the line segment $|x| \leq 1$, $y=0$. Orient $C$ in the usual counter-clockwise direction. Let $\vec{F}$ be the vector field $(2+3yx)\vec{i}+16y\vec{j}$. (a) Compute the (oriented) line integral $\int_C \vec{F}\cdot d\vec{s}$ directly by using a parameterization of $C$ (One option is to think polar coordinates. If $x=r\cos(t)$, what would $y$ be?)(2 points for parameterizing the curve. 2 points for setting up the integral. 2 points for doing the integral). (b) This integral is, by Green's theorem, equal to some double integral. Set that double integral up (2 points). Compute the integral (2 points).

(a) See Solution to part (a) in "solutions to problems covered in review session" for Midterm 2 at the bottom of my Calc III webpage

(b) Green's Theorem says: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx$

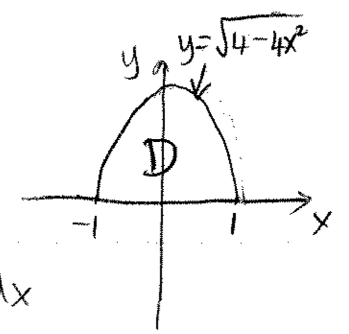
Our given line integral is $\int_C \vec{F}\cdot d\vec{s} = \int_C \vec{F}\cdot \underline{c}'(t)\,dt = \int_C \vec{F}\cdot\left(\frac{dx}{dt}\underline{i}+\frac{dy}{dt}\underline{j}\right)dt$

$= \int_C \left((2+3yx)\underline{i}+16y\underline{j}\right)\cdot\left(\frac{dx}{dt}\underline{i}+\frac{dy}{dt}\underline{j}\right)dt$

$= \int_C \left((2+3yx)\frac{dx}{dt}+16y\frac{dy}{dt}\right)dt = \int_C \overset{P}{(2+3yx)}\,dx + \overset{Q}{16y}\,dy$

$\Rightarrow P = 2+3yx,\ Q = 16y \Rightarrow \frac{\partial P}{\partial y} = 3x,\ \frac{\partial Q}{\partial x} = 0$

$\Rightarrow$ By Green's theorem $\int_C \vec{F}\cdot d\vec{s} = \iint_D (0-3x)\,dy\,dx$




$= \int_{-1}^{1}\int_0^{\sqrt{4-4x^2}} (-3x)\,dy\,dx = \int_{-1}^{1} -3x\sqrt{4-4x^2}\,dx$

Let $u = 4-4x^2 \Rightarrow du = -8x\,dx$
$\Rightarrow \frac{3}{8}du = -3x\,dx$

$= \int_0^0 \sqrt{u}\,\frac{3}{8}\,du$ (integration by substitution)

$= 0$

When $x=1$, $u=0$
when $x=-1$, $u=0$

More space for this on the next page.

More space for the last problem

**15.** (2 points) Let $C$ be the intersection of the surface $x^2+y^2+z^2=1$ with the surface $x+y+z=0$. Evaluate the line integrals $\int_C(x^2+y^2+z^2)\,ds$, $\int_C(x^2)\,ds$, $\int_C(y^2)\,ds$, and $\int_C(z^2)\,ds$. Explain how you got your answers.

We note that the plane $x+y+z=0$ passes through the origin, $(0,0,0)$ (because it is a solution of $x+y+z=0$)

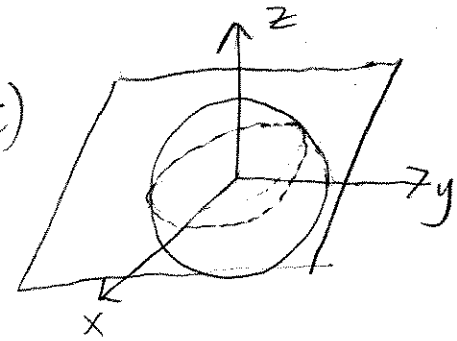
Since the surface $x^2+y^2+z^2=1$ is the unit sphere, the intersection with a plane through the origin must be a unit circle.

$\therefore \int_C (x^2+y^2+z^2)\,ds = \int_C 1\,ds$ (since $x^2+y^2+z^2=1$ at every point of $C$)

$=$ length of $C$

$=$ circumference of unit circle

$= 2\pi$




By the symmetry of the sphere and the plane about the origin, and hence the symmetry of $C$, $\int_C(x^2)\,ds = \int_C(y^2)\,ds = \int_C(z^2)\,ds$

$\Rightarrow \int_C (x^2+y^2+z^2)\,ds = 2\pi$

$\Rightarrow \int_C x^2\,ds + \int_C y^2\,ds + \int_C z^2\,ds = 2\pi$

$\Rightarrow 3\int_C x^2\,ds = 2\pi \Rightarrow \int_C x^2\,ds = \frac{2\pi}{3}$

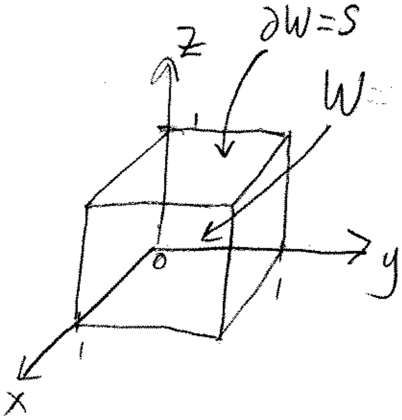
$\Rightarrow \int_C y^2\,ds = \int_C z^2\,ds = \frac{2\pi}{3}$

**16.** (8 points total) Let $S$ be the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and let $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$. Orient $S$ so that its unit normal points outward. The surface integral $\iint_S \vec{F} \cdot d\vec{S}$ can be computed directly or, by using Gauss' theorem, there is a triple integral way to compute it. Set up the integrals both ways (2 points each) and evaluate both of them (2 points each).

Computing directly: See solutions in "solutions to problems covered in review session" for midterm 2 at the bottom of my Calc III webpage.

Using Gauss' Theorem

(also called Divergence Theorem or Gauss' Divergence Thm)




Gauss' Thm says $\iiint_W (\operatorname{div} \underline{F})\,dV = \iint_{\partial W} \underline{F} \cdot d\underline{S}$

$\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k} \Rightarrow \operatorname{div}\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 2y + 2z$

So $\iint_S \underline{F} \cdot d\underline{S} = \iiint_W (\operatorname{div}\underline{F})\,dV = \int_0^1\int_0^1\int_0^1 (2x + 2y + 2z)\,dx\,dy\,dz$

$= \int_0^1\int_0^1 \left[x^2 + 2xy + 2xz\right]_0^1 dy\,dz = \int_0^1\int_0^1 (1 + 2y + 2z)\,dy\,dz$

$= \int_0^1 \left[y + y^2 + 2yz\right]_0^1 dz = \int_0^1 (2 + 2z)\,dz$

$= \left[2z + z^2\right]_0^1 = 3$

More space for this on the next page.

**17.** (6 points total) Let $Q$ be the square $[0,\pi]\times[0,\pi]$. Compute the integral $\iint_Q |\cos(x+y)|\,dx\,dy$. (2 points for describing where $\cos(x+y)$ is positive and negative, 2 points for setting up the integral, and 2 points for evaluating.)

On the square $[0,\pi]\times[0,\pi]$, $0 \le x+y \le 2\pi$

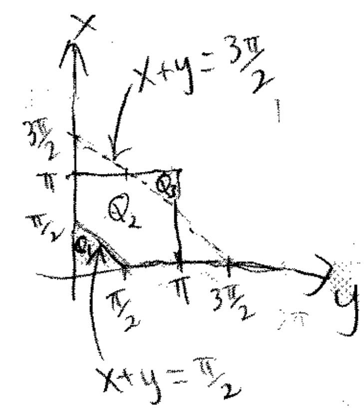
We know that for $0\le\theta\le 2\pi$, $\cos\theta = 0$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

and $\cos\theta \ge 0$ when $0\le\theta\le\frac{\pi}{2}$ or $\frac{3\pi}{2}\le\theta\le 2\pi$

and $\cos\theta < 0$ when $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$

Hence, $\cos(x+y) > 0$ in the regions $0\le x+y\le\frac{\pi}{2}$ and $\frac{3\pi}{2}\le x+y\le 2\pi$
and $\cos(x+y)<0$ in the region $\frac{\pi}{2}<x+y<\frac{3\pi}{2}$

$$\iint_Q |\cos(x+y)|\,dx\,dy = \iint_{Q_1} \cos(x+y)\,dx\,dy + \iint_{Q_2} -\cos(x+y)\,dx\,dy + \iint_{Q_2} \cos(x+y)\,dx\,dy$$




$$= \int_0^{\pi/2}\int_0^{\pi/2-x} \cos(x+y)\,dy\,dx \quad (\text{integral on } Q_1)$$

$$+ \int_0^{\pi/2}\int_{\pi/2-x}^{\pi} -\cos(x+y)\,dy\,dx + \int_{\pi/2}^{\pi}\int_0^{3\pi/2-x} -\cos(x+y)\,dy\,dx$$

(integral on $Q_2$ split into two parts)

$$+ \int_{\pi/2}^{\pi}\int_{3\pi/2-x}^{\pi} \cos(x+y)\,dy\,dx \quad (\text{integral on } Q_3)$$

$Q_1$ integral: $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2-x} \cos(x+y)\,dy\,dx = \int_0^{\pi/2}\Big[\sin(x+y)\Big]_0^{\pi/2-x} dx$

$$= \int_0^{\pi/2} \sin\left(x+\tfrac{\pi}{2}-x\right) - \sin(x+0)\,dx$$

$$= \int_0^{\pi/2} \sin\tfrac{\pi}{2} - \sin x\,dx = \int_0^{\pi/2} 1-\sin x\,dx$$

$$= \Big[x+\cos x\Big]_0^{\pi/2} = \tfrac{\pi}{2}+0-(0+1) = \tfrac{\pi}{2}-1$$

More space for this on the next page.

More space for the last problem

$Q_2$ integral (i): $\int_0^{\pi/2}\int_{\pi/2-x}^{\pi} -\cos(x+y)\,dy\,dx = \int_0^{\pi/2}\left[-\sin(x+y)\right]_{\pi/2-x}^{\pi} dx$

$= \int_0^{\pi/2} -\sin(x+\pi) + \sin(\tfrac{\pi}{2})\,dx = \int_0^{\pi/2} -\sin(x+\pi)+1\,dx$

$= \left[\cos(x+\pi) + x\right]_0^{\pi/2} = \cos(\tfrac{3\pi}{2}) + \tfrac{\pi}{2} - (\cos\pi + 0) = \tfrac{\pi}{2}+1$

$Q_2$ integral (ii): $\int_{\pi/2}^{\pi}\int_0^{3\pi/2-x} -\cos(x+y)\,dy\,dx = \int_{\pi/2}^{\pi}\left[\sin(x+y)\right]_0^{3\pi/2-x} dx$

$= \int_{\pi/2}^{\pi} -\sin(3\pi/2) + \sin x\,dx = \int_{\pi/2}^{\pi} 1+\sin x\,dx$

$= \left[x - \cos x\right]_{\pi/2}^{\pi} = \pi - \cos\pi - (\tfrac{\pi}{2} - \cos\tfrac{\pi}{2}) = \pi + 1 - \tfrac{\pi}{2}$

$= \tfrac{\pi}{2}+1$

$Q_3$ integral: $\int_{\pi/2}^{\pi}\int_{3\pi/2-x}^{\pi} \cos(x+y)\,dy\,dx = \int_{\pi/2}^{\pi}\left[\sin(x+y)\right]_{3\pi/2-x}^{\pi} dx$

$= \int_{\pi/2}^{\pi} \sin(x+\pi) - \sin(\tfrac{3\pi}{2})\,dx = \int_{\pi/2}^{\pi} \sin(x+\pi)+1\,dx$

$= \left[-\cos(x+\pi) + x\right]_{\pi/2}^{\pi} = -1+\pi-(0+\tfrac{\pi}{2}) = \tfrac{\pi}{2}-1$

Hence $\iint_Q |\cos(x+y)|\,dx\,dy = (\tfrac{\pi}{2}-1) + (\tfrac{\pi}{2}+1) + (\tfrac{\pi}{2}+1) + (\tfrac{\pi}{2}-1)$

$= 2\pi$