Johns Hopkins University, Department of Mathematics
Calc III - Fall 2012
Final Exam

Instructions: This exam has 7 pages and is out of 100 points. No calculators, books or notes are allowed. Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper. Be sure to clearly label each problem and attach them to the exam. You have **3 HOURS**.

Academic Honesty Certification

I certify that I have taken this exam without the aid of unauthorized people or objects.

Signature: Nicholas Marshburn Date: 12/12/12

Name: Nicholas Marshburn Section: 0

Problem	Score
1	
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Bonus	
Total	

1. (20 pts, 2 pts each) True/False. Write "True" or "False" below each of the following statements:

(a) Green's Theorem states that for a simple region D in $\mathbb{R}^2$ with counterclockwise-oriented boundary C, $\int_C Pdx + Qdy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$.

True

(b) Distinct level curves of a function $f(x, y)$ do not intersect.

True

(c) A particle moving in $\mathbb{R}^2$ with constant speed has zero acceleration.

False

(d) If the curl of an everywhere-defined vector field F on $\mathbb{R}^3$ is zero, then F is a gradient field.

True

(e) If $\vec{v} \times \vec{w} = 0$ and $\vec{w} \neq 0$, then $\vec{v}$ is a scalar multiple of $\vec{w}$.

True

(f) $dS = T_u \times T_v \, dudv$.

False

(g) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(r, \theta) = (r\cos\theta, r\sin\theta)$ is one-to-one and onto.

False

(h) A path integral over a curve C is independent of the orientation of C.

True

(i) If a surface S is regular with respect to a parametrization Φ, then Φ is orientation-preserving.

False

(j) If for a function $f(x, y)$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist at (0,0), then a tangent plane to the graph of f exists at (0,0).

False

2. (15 pts, 5 pts each) Give an example of the following objects:

(a) A nonzero vector field F in $\mathbb{R}^2$ such that the line integral of F around any simple closed curve is zero.

$f(x,y) = x$

$\boxed{F = \nabla f = (1,0)}$

(b) A plane in $\mathbb{R}^3$ containing the vector $\vec{i} - \vec{j} + 2\vec{k}$. $= (1,-1,2)$

$(1,1,0) \cdot (1,-1,2) = 0$

$\boxed{x+y=0}$

(c) A function $f(x,y)$ that is continuous at (0,0) but not differentiable at (0,0).

$\boxed{f(x,y) = |x|}$

$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx}|x| \Big|_{x=0}$ DNE

3. (5 pts) Let $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2 + 1}$. For $c = 0$, $c = .5$, and $c = 1$, determine which of the level sets of f with value c are nonempty. Sketch the nonempty level sets.

$$\frac{x^2-y^2}{x^2+y^2+1} = c$$

$$x^2 - y^2 = c(x^2+y^2+1)$$

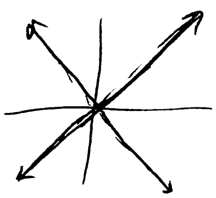
$$(1-c)x^2 + (-1-c)y^2 = c$$

$\underline{c=0}$

$$x^2 - y^2 = 0$$

$$x^2 = y^2$$

$$x = \pm y$$

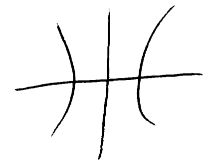


$\underline{c=.5}$

$$.5x^2 - 1.5y^2 = .5$$

$$x^2 - 3y^2 = 1$$

hyperbola

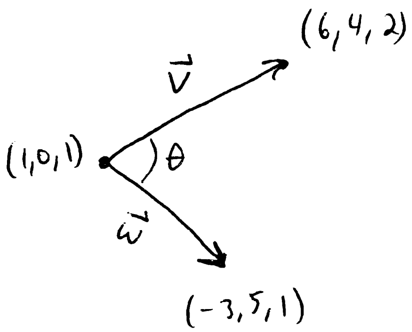


$\underline{c=1}$

$$-2y^2 = 1$$

empty

4. (5 pts) For the triangle in $\mathbb{R}^3$ with vertices (1,0,1), (6,4,2), and (-3,5,1), find the measure of the angle at (1,0,1).



$$\vec{v} = (6,4,2) - (1,0,1) = (5,4,1)$$

$$\vec{w} = (-3,5,1) - (1,0,1) = (-4,5,0)$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = 0$$

$$\therefore \boxed{\theta = \frac{\pi}{2}}$$

5. (5 pts) Let $f(x, y) = ye^{xy}$, where $x(u, v) = u^2v + v^3$ and $y(u, v) = e^v - e^{uv}\cos v$. Compute $\frac{\partial f}{\partial v}$ at the point $(u, v) = (1, 0)$.

$\frac{\partial f}{\partial x} = y^2 e^{xy}$ $\quad \frac{\partial f}{\partial y} = e^{xy} + xye^{xy}$ $\quad x(1,0) = 0 \quad y(1,0) = 0$

$\frac{\partial x}{\partial v} = u^2 + 3v^2$ $\quad \frac{\partial y}{\partial v} = e^v - ue^{uv}\cos v + e^{uv}\sin v$

$$\frac{\partial f}{\partial v}(1,0) = \frac{\partial f}{\partial x}(0,0)\frac{\partial x}{\partial v}(1,0) + \frac{\partial f}{\partial y}(0,0)\frac{\partial y}{\partial v}(1,0)$$

$$= 0 \cdot 1 + 1 \cdot 0 = 0$$

6. (5 pts) Let $f(x, y) = x^2 + y^2 + kxy$, where k is a real number. For which values of k is (0,0) a local minimum?

$\frac{\partial f}{\partial x} = 2x + ky$ $\quad \frac{\partial f}{\partial y} = 2y + kx$

$\frac{\partial^2 f}{\partial x^2} = 2$ $\quad \frac{\partial^2 f}{\partial y^2} = 2$ $\quad \frac{\partial^2 f}{\partial x \partial y} = k$

$D = 2 \cdot 2 - k \cdot k = 4 - k^2$

$D > 0$ when $-2 < k < 2$ $\quad \frac{\partial^2 f}{\partial x^2} > 0$ $\quad \therefore$ local min

$D = 0$ when $k = \pm 2$ test fails

$k = 2$

$f(x,y) = x^2 + y^2 + 2xy$
$= (x+y)^2$

$\therefore (0,0) =$ local min

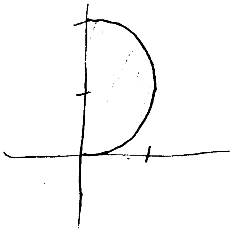
$k = -2$

$f(x,y) = x^2 + y^2 - 2xy$
$= (x-y)^2$

$\therefore (0,0) =$ local min

$\boxed{-2 \leq k \leq 2}$

7. (5 pts) Change the order of integration: $\int_0^2 \int_0^{\sqrt{1-(y-1)^2}} f(x,y)\,dx\,dy$.



$$x = \sqrt{1-(y-1)^2}$$
$$x^2 = 1-(y-1)^2$$
$$(y-1)^2 = 1-x^2$$
$$y-1 = \pm\sqrt{1-x^2}$$
$$y = 1 \pm \sqrt{1-x^2}$$

$$\boxed{\int_0^1 \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} f(x,y)\,dy\,dx}$$

8. (5 pts) A particle in $\mathbb{R}^2$ moves along the parabola $y = x^2$, starting at (0,0) and stopping at (2,4). Calculate the work done on the particle by the vector field $F(x,y) = (-y,x)$.

$$c(t) = (t,t^2), \quad 0 \le t \le 2$$
$$c'(t) = (1,2t)$$
$$F(c(t)) = (-t^2,t)$$

$$\int_c F \cdot d\vec{s} = \int_0^2 F(c(t)) \cdot c'(t)\,dt = \int_0^2 (-t^2,t)\cdot(1,2t)\,dt$$

$$= \int_0^2 t^2\,dt = \left.\frac{t^3}{3}\right|_0^2 = \frac{8}{3}$$

9. (5 pts) Let W be the region in $\mathbb{R}^3$ bounded by the cylinder $x^2 + y^2 = 1$, the plane $z = 0$, and the cone $z = \sqrt{x^2 + y^2}$. Set up the following integral in cylindrical coordinates (but do not evaluate it):

$$\iiint_W z^3 e^{x^2+y^2} \sqrt{x^2 + y^2}\, dx dy dz$$

$$\left\{ \begin{matrix} 0 \le z \le \sqrt{x^2+y^2} \\ x^2+y^2 \le 1 \end{matrix} \right\} \Longrightarrow \left\{ \begin{matrix} 0 \le z \le r \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix} \right\}$$

rectangular

cylindrical

$$\int_0^{2\pi}\int_0^1\int_0^r z^3 e^{r^2} r \cdot r\, dz\, dr\, d\theta = \boxed{\int_0^{2\pi}\int_0^1\int_0^r z^3 r^2 e^{r^2}\, dz\, dr\, d\theta}$$

10. (5 pts) Parametrize the part of the plane $2x + 4y - z = 1$ that is inside the cylinder $x^2 + y^2 = 1$.

$$z = 2x + 4y - 1$$

$$\Phi(x,y) = (x, y, 2x+4y-1), \quad x^2+y^2 \le 1$$

or

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 2r\cos\theta + 4r\sin\theta - 1), \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

11. (5 pts) ~~Calculate~~ Set up $\iint_S xy\,dS$, where S is the surface in $\mathbb{R}^3$ parametrized by $\Phi(u,v) = (u^2, v, uv)$, with $0 \le u \le 1$ and $0 \le v \le 1$.

$$T_u = (2u, 0, v) \qquad T_v = (0, 1, u)$$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ 2u & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -2u^2, 2u)$$

$$\iint_S xy\,dS = \int_0^1\int_0^1 u^2 v\,\|T_u \times T_v\|\,du\,dv = \boxed{\int_0^1\int_0^1 u^2 v\sqrt{4u^4 + 4u^2 + v^2}\,du\,dv}$$

12. (10 pts) Use Stokes' Theorem to evaluate $\int_C F \cdot d\vec{s}$, where $F(x, y, z) = (z^2, x^2, y^2)$ and C is the rectangle formed by traveling in straight lines between the points (0,0,0), (1,0,0), (1,1,1), (0,1,1), and back to the origin, in that order.

$\int_C F \cdot d\vec{s} = \iint_S (\nabla \times F) \cdot d\vec{S}$, where S is the interior of the rectangle, which is part of the plane $z = y$. The orientation on S is given by the upward-pointing normals.

Parametrize S: $\Phi(x, y) = (x, y, y)$, $0 \le x, y \le 1$

$$T_x = (1, 0, 0)$$

$$T_y = (0, 1, 1)$$

$$T_x \times T_y = (0, -1, 1)$$

$\therefore$ $T_x \times T_y$ gives the upward-pointing normals and $\therefore$ Φ is orientation-preserving.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & x^2 & y^2 \end{vmatrix} = (2y, 2z, 2x)$$

$$(\nabla \times F) \circ \Phi = (2y, 2y, 2x)$$

$$\int_C F \cdot d\vec{s} = \iint_S (\nabla \times F) \cdot d\vec{S} = \int_0^1 \int_0^1 (2y, 2y, 2x) \cdot (0, -1, 1)\, dx\, dy$$

$$= \int_0^1 \int_0^1 (2x - 2y)\, dx\, dy = \boxed{0}$$

13. (10 pts) Calculate $\iint_S F \cdot d\vec{S}$, where $F(x,y,z) = (x^3, y^3 + 3yz^2, -5)$ and S is the sphere of radius 4 centered at the origin.

Use Gauss' Th.: $\iint_S F \cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV$, where W is the interior of the sphere

$$\operatorname{div} F = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_W \operatorname{div} F \, dV = \iiint_W (3x^2 + 3y^2 + 3z^2) \, dx\, dy\, dz$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^4 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^4 3\rho^4 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{3 \cdot 4^5}{5} \sin\phi \, d\theta\, d\phi = \int_0^{\pi} \frac{6\pi \cdot 4^5}{5} \sin\phi \, d\phi$$

$$= -\frac{6\pi \cdot 4^5}{5} \cos\phi \Big|_0^{\pi} = \frac{12\pi \cdot 4^5}{5} = \frac{12288\pi}{5}$$

14. (2 pts, 1 pt each) Bonus!

(a) Give an example of a surface S and two parametrizations of S, Φ_1 and Φ_2, such that S is regular with respect to Φ_1 but is not regular with respect to Φ_2.

S = unit disk in xy plane

$\Phi_1(x,y) = (x, y, 0)$, $x^2+y^2 \leq 1$ regular

$\Phi_2(r,\theta) = (r\cos\theta, r\sin\theta, 0)$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ not regular where $r=0$

(b) Draw a picture involving (but not limited to!) a lamp, a ladder, and a lion.