IV. K

AS.150.498: Modal Logic and Its Applications
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Validity in classical first-order logic is undecidable. There is no decision procedure for determining whether a given sentence in the language of first-order logic is valid. Still, there are proof systems that output all and only the validities of first-order logic. Likewise, there are systems in which a sentence $\varphi \in S_{\mathcal{L}}$ in the basic sentential modal language is provable if and only if φ is valid. Let us consider such a system.

1 The System K

Definition 4.1. The **minimal modal logic K** has the following rules and axioms:[1]

(PL)	All (substitutions of) tautologies are axioms
(MP)	From φ and $\varphi \supset \psi$ infer ψ
(Nec)	From φ infer $\Box\varphi$
(K)	For any φ, ψ, $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$ is an axiom
(Duality)	Expressions involving $\Box$ and $\Diamond$ are interchangeable according to the duality $\Box \equiv \neg\Diamond\neg$

Definition 4.2. A **proof in K** is a sequence of sentences $\langle \varphi_1, ..., \varphi_n \rangle$ in $S_{\mathcal{L}}$ such that for all $k \leq n$ one of the following conditions is met:

φ_k is an axiom

$\exists i, j \leq k(\varphi_i$ is $(\varphi_j \supset \varphi_k))$

$\exists i \leq k(\varphi_k$ is $\Box\varphi_i)$

$\exists i \leq k(\varphi_k$ is obtained from φ_i according to Duality)

If there is a proof in **K** ending with φ, then $\vdash_{\mathbf{K}} \varphi$.

Fact 4.1. $\vdash_{\mathbf{K}} \Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)$

1.	$(\varphi \wedge \psi) \supset \varphi$	PL
2.	$\Box((\varphi \wedge \psi) \supset \varphi)$	Nec 1
3.	$\Box((\varphi \wedge \psi) \supset \varphi) \supset (\Box(\varphi \wedge \psi) \supset \Box\varphi)$	K
4.	$\underline{\Box(\varphi \wedge \psi) \supset \Box\varphi}_A$	MP 3,2
5.	$(\varphi \wedge \psi) \supset \psi$	PL
6.	$\Box((\varphi \wedge \psi) \supset \psi)$	Nec 5
7.	$\Box((\varphi \wedge \psi) \supset \psi) \supset (\Box(\varphi \wedge \psi) \supset \Box\psi)$	K
8.	$\underline{\Box(\varphi \wedge \psi) \supset \Box\psi}_B$	MP 7,6
9.	$A \supset (B \supset (A \wedge B))$	PL
10.	$B \supset (A \wedge B))$	MP 9,4
11.	$(\underline{\Box(\varphi \wedge \psi)}_C \supset \underline{\Box\varphi}_D) \wedge (\underline{\Box(\varphi \wedge \psi)}_C \supset \underline{\Box\psi}_E)$	MP 10,8
12.	$((C \supset D) \wedge (C \supset E)) \supset (C \supset (D \wedge E))$	PL
13.	$\Box(\varphi \wedge \psi) \supset (\Box\varphi \wedge \Box\psi)$	MP 12,11

Fact 4.2. $\vdash_{\mathbf{K}} \Box(\varphi \supset \psi) \supset (\Diamond\varphi \supset \Diamond\psi)$

1.	$(\varphi \supset \psi) \supset (\neg\psi \supset \neg\varphi)$	PL
2.	$\Box((\varphi \supset \psi) \supset (\neg\psi \supset \neg\varphi))$	Nec 1
3.	$\Box((\varphi \supset \psi) \supset (\neg\psi \supset \neg\varphi)) \supset (\Box(\varphi \supset \psi) \supset \Box(\neg\psi \supset \neg\varphi))$	K
4.	$\underline{\Box(\varphi \supset \psi)}_A \supset \underline{\Box(\neg\psi \supset \neg\varphi)}_B$	MP 3,2
5.	$\Box(\neg\psi \supset \neg\varphi) \supset (\Box\neg\psi \supset \Box\neg\varphi)$	K
6.	$\Box(\neg\psi \supset \neg\varphi) \supset \underline{(\neg\Diamond\psi \supset \neg\Diamond\varphi)}_C$	Duality
7.	$(\neg\Diamond\psi \supset \neg\Diamond\varphi) \supset \underline{(\Diamond\varphi \supset \Diamond\psi)}_D$	PL
8.	$(A \supset B) \supset ((B \supset C) \supset ((C \supset D) \supset (A \supset D)))$	PL
9.	$(B \supset C) \supset ((C \supset D) \supset (A \supset D))$	MP 8,4
10.	$(C \supset D) \supset (A \supset D)$	MP 9,6
11.	$\Box(\varphi \supset \psi) \supset (\Diamond\varphi \supset \Diamond\psi)$	MP 10,7

2 Soundness

Theorem 4.1. If $\vdash_{\mathbf{K}} \varphi$ then $\models \varphi$.

Proof: It suffices to show that the axioms of **K** are validities and the rules of inference of **K** preserve validity.

(PL): If φ is (a substitution of) a tautology, then clearly $\models \varphi$ given the semantic clauses for $\bot$, $\neg$, and $\wedge$.

(MP): Suppose that $\models \varphi$ and $\models \varphi \supset \psi$. For any pointed model $\mathcal{M}, w$, $[\![\varphi]\!]^w_{\mathcal{M}} = T$ and $[\![\varphi \supset \psi]\!]^w_{\mathcal{M}} = T$, so $[\![\psi]\!]^w_{\mathcal{M}} = T$ given the semantic clauses for $\neg$ and $\wedge$. Thus $\models \psi$.

(Nec): Suppose that $\models \varphi$. For any pointed model $\mathcal{M}, w$, $[\![\Box\varphi]\!]^w_{\mathcal{M}} = T$ since $\forall v \in \{v : w\mathcal{R}v\}([\![\varphi]\!]^v_{\mathcal{M}} = T)$. Thus $\models \Box\varphi$.

(K): Suppose that $[\![\Box(\varphi \supset \psi)]\!]^w_{\mathcal{M}} = [\![\Box\varphi]\!]^w_{\mathcal{M}} = T$ for arbitrary $\mathcal{M}, w$. Then $\forall v \in \{v : w\mathcal{R}v\}([\![\varphi \supset \psi]\!]^v_{\mathcal{M}} = [\![\varphi]\!]^v_{\mathcal{M}} = T)$, so $\forall v \in \{v : w\mathcal{R}v\}([\![\psi]\!]^v_{\mathcal{M}} = T)$.

[1] Recall that $\varphi \supset \psi$ abbreviates $\neg(\varphi \wedge \neg\psi)$.

That is, $[\![\Box\psi]\!]^w_{\mathcal{M}} = T$, so $[\![\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)]\!]^w_{\mathcal{M}} = T$. Thus $\models \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$.

(Duality): Suppose that $\models \varphi$ and φ' is obtained from φ by appealing to the duality of $\Box$ and $\Diamond$. Since $[\![\varphi]\!]^w_{\mathcal{M}} = [\![\varphi']\!]^w_{\mathcal{M}}$ for any pointed model $\mathcal{M}, w$, $\models \varphi'$.

3 Completeness

Theorem 4.2. If $\models \varphi$ then $\vdash_{\mathbf{K}} \varphi$.

Proof: For the purposes of the completeness proof, let us introduce the notion of **K**-*consistency*.

Definition 4.3. A set of sentences $\Gamma \subset S_{\mathcal{L}}$ is **K-consistent** just in case there are no sentences $\varphi_1, ..., \varphi_n \in \Gamma$ such that $\vdash_{\mathbf{K}} (\varphi_1 \wedge ... \wedge \varphi_n) \supset \bot$.

Note: If $\nvdash_{\mathbf{K}} \varphi$ then $\{\neg\varphi\}$ is **K**-consistent. Why? If $\{\neg\varphi\}$ is **K**-inconsistent, then $\vdash_{\mathbf{K}} \neg\varphi \supset \bot$ and since $(\neg\varphi \supset \bot) \supset \varphi$ is a (substitution of) a tautology, $\vdash_{\mathbf{K}} \varphi$.

To prove completeness, it suffices to establish the contrapositive: if $\nvdash_{\mathbf{K}} \varphi$ then $\not\models \varphi$.

Given the previous note, it thus suffices to show that if $\{\neg\varphi\}$ is **K**-consistent then $\not\models \varphi$; that is, $\{\neg\varphi\}$ is **K**-consistent only if there is some pointed model $\mathcal{M}, w$ such that $[\![\neg\varphi]\!]^w_{\mathcal{M}} = T$.

In fact, the proof will establish something much stronger—*viz.*, that there is a single model $\mathcal{M}$ where *every* **K**-consistent set of sentences in $S_{\mathcal{L}}$ is satisfied at some world in this model.

Definition 4.4. $\Gamma \subset S_{\mathcal{L}}$ is a **maximal K-consistent set** if and only if Γ is **K**-consistent and there is no $\Delta \subset S_{\mathcal{L}}$ such that $\Gamma \subset \Delta$ and Δ is **K**-consistent.

Using these maximal sets, we can construct the model we are after:

Definition 4.5. The **canonical model for K** is the model $\mathcal{M}^{\mathbf{K}}$ where:

$\mathcal{W}^{\mathbf{K}}$ is the set of all maximal **K**-consistent sets
$\mathcal{R}^{\mathbf{K}} = \{\langle\Gamma, \Delta\rangle : \text{for all } \varphi \in S_{\mathcal{L}}, \text{ if } \Box\varphi \in \Gamma \text{ then } \varphi \in \Delta\}$
$\mathcal{V}^{\mathbf{K}}(p, \Gamma) = T$ iff $p \in \Gamma$

Every **K**-consistent set is satisfied at some world $\Gamma \in \mathcal{W}^{\mathbf{K}}$. This follows immediately from the following two lemmas.

Lemma 4.1. (Lindenbaum Lemma) If $\Gamma \subset S_{\mathcal{L}}$ is **K**-consistent, then there is a maximal **K**-consistent set Δ such that $\Gamma \subseteq \Delta$.

Proof: Enumerate all of the sentences in $S_{\mathcal{L}}$: $\varphi_1, \varphi_2, \varphi_3, ...$ The set Δ can then be constructed in stages:

$$\Delta_0 = \Gamma$$
$$\Delta_n = \begin{cases} \Delta_{n-1} \cup \{\varphi_n\} & \text{if this is } \mathbf{K}\text{-consistent} \\ \Delta_{n-1} \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$
$$\Delta = \bigcup_n \Delta_n$$

Lemma 4.2. (Truth Lemma) $[\![\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\varphi \in \Gamma$.

Proof: By induction on the complexity of sentences in $S_{\mathcal{L}}$.

Atomic case: By the definition of $\mathcal{V}^{\mathbf{K}}$.

$\bot$ case: Since Γ is **K**-consistent, $\bot \notin \Gamma$. By the semantic clause for $\bot$, $[\![\bot]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = F$. Thus $[\![\bot]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\bot \in \Gamma$.

$\neg$ case: $[\![\neg\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $[\![\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = F$ iff $\varphi \notin \Gamma$ by the semantic clause for $\neg$ and the induction step. Since Γ is a maximal **K**-consistent set, $\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$. Thus $[\![\neg\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\neg\varphi \in \Gamma$.

$\wedge$ case: $[\![\varphi \wedge \psi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $[\![\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = [\![\psi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\varphi, \psi \in \Gamma$ by the semantic clause for $\wedge$ and the induction step. Since Γ is a maximal **K**-consistent set, $\varphi, \psi \in \Gamma$ iff $\varphi \wedge \psi \in \Gamma$. Thus $[\![\varphi \wedge \psi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\varphi \wedge \psi \in \Gamma$.

$\Box$ case (right-to-left): $\Box\varphi \in \Gamma$ only if $\forall\Delta \in \{\Delta : \Gamma\mathcal{R}^{\mathbf{K}}\Delta\}(\varphi \in \Delta)$ only if $\forall\Delta \in \{\Delta : \Gamma\mathcal{R}^{\mathbf{K}}\Delta\}([\![\varphi]\!]^{\Delta}_{\mathcal{M}^{\mathbf{K}}} = T)$ by the definition of $\mathcal{R}^{\mathbf{K}}$ and the induction step. Thus, $\Box\varphi \in \Gamma$ only if $[\![\Box\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ by the semantic clause for $\Box$.

$\Box$ case (left-to-right): Assume $\Box\varphi \notin \Gamma$. Then $\Delta = \{\neg\varphi\} \cup \{\psi : \Box\psi \in \Gamma\}$ is **K**-consistent. If not, there are sentences $\psi_1, ..., \psi_n \in \Delta$ such that:

$\vdash_{\mathbf{K}} (\psi_1 \wedge ... \wedge \psi_n) \supset \bot$
$\vdash_{\mathbf{K}} (\psi_1 \wedge ... \wedge \psi_n \wedge \neg\varphi) \supset \bot$
$\vdash_{\mathbf{K}} (\psi_1 \wedge ... \wedge \psi_n) \supset \varphi$
$\vdash_{\mathbf{K}} \Box(\psi_1 \wedge ... \wedge \psi_n) \supset \Box\varphi$
$\vdash_{\mathbf{K}} (\Box\psi_1 \wedge ... \wedge \Box\psi_n) \supset \Box\varphi$ (from converse of Fact 4.1)

But then since $\Box\psi_1, ..., \Box\psi_n \in \Gamma$ and Γ is a maximal **K**-consistent set, $\Box\varphi \in \Gamma$, contradicting the opening assumption. (What if one of the chosen ψ_i is $\neg\varphi$? Then we're already at the second step in the above argument.)

Now, since Δ is **K**-consistent, there is a maximal **K**-consistent set Δ' such that $\Delta \subseteq \Delta'$ by the Lindenbaum Lemma. $[\![\varphi]\!]^{\Delta'}_{\mathcal{M}^{\mathbf{K}}} = F$ by the induction step and $\Gamma \mathcal{R}^{\mathbf{K}} \Delta'$. Thus $[\![\Box\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = F$ by the semantic clause for $\Box$.

$\Diamond$ case: $[\![\Diamond\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $[\![\neg\Box\neg\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\neg\Box\neg\varphi \in \Gamma$ by the duality of $\Diamond$ and $\Box$, and previous cases. Since Γ is **K**-consistent, $\neg\Box\neg\varphi \in \Gamma$ iff $\Diamond\varphi \in \Gamma$. Thus $[\![\Diamond\varphi]\!]^{\Gamma}_{\mathcal{M}^{\mathbf{K}}} = T$ iff $\Diamond\varphi \in \Gamma$.

That ends the completeness proof for **K**.

4 Strong Completeness

There is an even stronger result in the vicinity.

Let us write $\Gamma \vdash_{\mathbf{K}} \psi$ when there are sentences $\varphi_1, ..., \varphi_n \in \Gamma$ such that $\vdash_{\mathbf{K}} (\varphi_1 \wedge ... \wedge \varphi_n) \supset \psi$. Definition 1.5 can also be extended to handle the case where Γ is infinite: $\Gamma \models \psi$ just in case there is no pointed model $\mathcal{M}, w$ such that $[\![\varphi]\!]^{w}_{\mathcal{M}} = T$ for each $\varphi \in \Gamma$ but $[\![\psi]\!]^{w}_{\mathcal{M}} = F$.

Theorem 4.3. $\Gamma \vdash_{\mathbf{K}} \psi$ if and only if $\Gamma \models \psi$.

Proof: The left-to-right soundness result was effectively proven in §2. For the right-to-left completeness result, assume $\Gamma \nvdash_{\mathbf{K}} \psi$. Then $\Gamma \cup \{\neg\psi\}$ is **K**-consistent, so $\Gamma \cup \{\neg\psi\} \subseteq \Delta$ where Δ is a maximal **K**-consistent set. But then $[\![\varphi]\!]^{\Delta}_{\mathcal{M}^{\mathbf{K}}} = T$ for each $\varphi \in \Gamma$ and $[\![\psi]\!]^{\Delta}_{\mathcal{M}^{\mathbf{K}}} = F$. Thus $\Gamma \nvdash_{\mathbf{K}} \psi$ only if $\Gamma \not\models \psi$.