1. 1) False

$x = \cos^2 t \quad y = \sin^2 t$

$\Rightarrow x + y = 1 \quad 0 \leq x \leq 1 \quad 0 \leq y \leq 1$

$\Rightarrow$ Line segment.

2). True

Direction of the line : $(3, 2, -1)$

normal of the plane : $(3, 2, -1)$

3). False

Ex. consider $\vec{\gamma}(t) = (t, t^2)$

$\Rightarrow \vec{a}(t) = \vec{\gamma}''(t) = (0, 2)$ constant vector field

But in Cartisian coordinate this curve is $y = x^2$

4) False

Ex. $\vec{a} = (1, 1) \quad \vec{b} = (-1, 1) \quad \vec{c} = (2, -2)$

[Caution: the following statement is true:

if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ <u>for all $\vec{a}$</u> then $\vec{b} = \vec{c}$]

5) False

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$

if $|\vec{a}| = |\vec{b}| = 1 \Rightarrow |\vec{a} \times \vec{b}| = |\sin\theta|$ ~~may not be~~

$\Rightarrow \vec{a} \times \vec{b}$ may not be a unit vector.

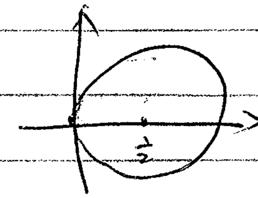
6) True

Since curl is divergence free.

2. There are two ways to set up the polar coordinates.

Way 1:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} 0 \le r \le \cos\theta \\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2} \end{aligned}$$



$$\Rightarrow \text{Integral} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} r^3 \, dr \, d\theta = \frac{3}{32}\pi$$

Way 2: ~~Let~~ Notice $y = \pm\sqrt{x - x^2} \iff (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

Let $\begin{cases} x = \frac{1}{2} + r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} 0 \le r \le \tfrac{1}{2} \\ 0 \le \theta \le 2\pi \end{aligned}$

Jacobian is still r

$$\Rightarrow \text{integral} = \int_0^{2\pi} \int_0^{\frac{1}{2}} \left[\left(\frac{1}{2} + r\cos\theta\right)^2 + r^2\sin^2\theta \right] r \, dr \, d\theta$$

$$= \frac{3}{32}\pi$$

[Rm: in Way one, you want to use: $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$]

3. Since the circle is a simple closed smooth curve (positively oriented) and both of components of $\vec{F}$ are C' - functions.

Green's thm $\Rightarrow$ $\int_{\vec{C}} \vec{F} \cdot d\vec{s} = \iint_D \frac{\partial}{\partial x}(x^3 + \sin\sin(y)) - \frac{\partial}{\partial y}(-y^3 + \sin(\sin x))\, dx dy$

where D is the disk: $x^2 + y^2 \le 1$

$\Rightarrow \iint_D 3x^2 + 3y^2 \, dx dy = 3\int_0^{2\pi}\int_0^1 r^3 \, dr d\theta = \frac{3}{4} \cdot 2\pi = \frac{3}{2}\pi$

$\Rightarrow \int_{\vec{C}} \vec{F} \cdot d\vec{s} = \frac{3}{2}\pi$

4. Area $= \iint_D 1 \, dx dy$

Let $\begin{cases} u = xy & 1 \le u \le 3 \\ v = \frac{y}{x} & 1 \le v \le 3 \end{cases}$

~~Jacobian~~ $= \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = 2\frac{y}{x} = 2v$

The Jacobian should be 1/2v,

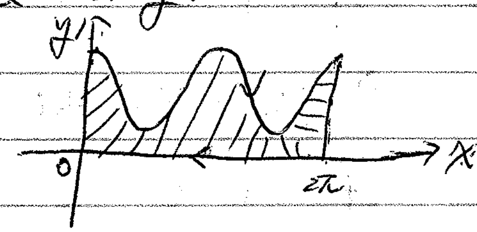
$\Rightarrow$ Area $= \iint_D 2v \, dv du = \int_1^3 \int_1^3 2v \, dv du = 16$

Final answer should be ln 3.

The integrand should be 1/2v

The integrand should be 1/2v

5. Let $P = xe^{x} - y^2$ $\quad Q = \sin y$.

The region looks like 

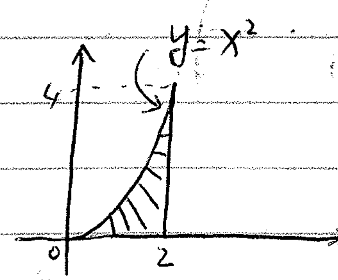
By Green's thm.

$$\int_{C} \vec{F} \cdot d\vec{s} = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^{2+\cos x} 2y \, dy \, dx$$

$$= \int_0^{2\pi} (2+\cos x)^2 \, dx = 9\pi$$

$\left[\cos^2 x = \frac{1+\cos 2x}{2}\right]$

6. $$\int_0^4 \int_{\sqrt{y}}^2 \cos(1+x^3)\, dx\, dy$$ 

$$= \int_0^2 \int_0^{x^2} \cos(1+x^3)\, dy\, dx$$

$$= \int_0^2 x^2 \cos(1+x^3)\, dx = \frac{1}{3}\sin(1+x^3)\Big|_0^2 = \frac{1}{3}(\sin 9 - \sin 1)$$

7. $z = x^2 + y^2 + cxy$.

$\Rightarrow \frac{\partial z}{\partial x} = 2x + cy \qquad \frac{\partial z}{\partial y} = 2y + cx$

$\Rightarrow \frac{\partial z}{\partial x}(0,0) = 0$ & $\frac{\partial z}{\partial y}(0,0) = 0$

$\Rightarrow (0,0)$ is the critical point of z.

$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 2 \quad \frac{\partial^2 z}{\partial x \partial y} = c.$

$D(x,y) = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 4 - c^2.$

in particular, $D(0,0) = 4 - c^2$

Case I: if $4 - c^2 > 0$ i.e. $-2 < c < 2$ then $D(0,0) > 0$

$\Rightarrow (0,0)$ is the local min of z.

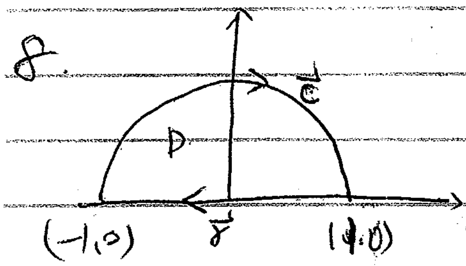
Case II: if $4 - c^2 = 0$ i.e. $c = 2$ or -2 then $D(0,0) = 0$

$\Rightarrow$ No conclusion.

Case III. if $4 - c^2 < 0$ i.e. $c > 2$ or $c < -2$ then $D^{(0,0)} < 0$

$\Rightarrow (0,0)$ is the saddle point of z.

8.



[Add a curve & use Green's thm]

Let $\vec{\gamma}$ be the ~~cur~~ Line segment from $(0,1)$ to $(-1,0)$ on x-axis

$\Rightarrow$ $\vec{C}+\vec{\gamma}$ is a simple closed curve.

$$\Rightarrow \int_{\vec{C}+\vec{\gamma}} P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= -\iint_D \left(1 + xy e^{xy} + e^{xy} - e^{xy} - xy e^{xy}\right) dx\,dy$$

$$= -\iint_D 1\, dx\,dy = -\text{Area}(D) = -\frac{1}{2}\pi$$

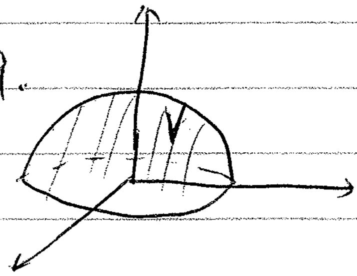
$$\Rightarrow \int_{\vec{C}} P\,dx + Q\,dy = -\frac{1}{2}\pi - \int_{\vec{\gamma}} P\,dx + Q\,dy.$$

$$= -\frac{1}{2}\pi - \int_1^{-1} (x + \sin x)\,dx$$

$$= -\frac{1}{2}\pi + \int_{-1}^{1} (x + \sin x)\,dx$$

$$= -\frac{1}{2}\pi.$$

[Notice $x + \sin x$ is an odd function $\Rightarrow \int_{-1}^{1} (x+\sin x)\,dx = 0$]

9. 

[Add "bottom" surface and use Gauss thm]

Let the bottom surface be
$S_0 = \{ x^2+y^2 \le 1,\ z=0 \}$ with "outer" normal.

$$\Rightarrow \underset{S_0+S}{\oiint} \vec{F}\cdot d\vec{S} = \underset{V}{\iiint} (\operatorname{div}\vec{F})\,dx\,dy\,dz$$

$$= \underset{V}{\iiint} \left(1 + ye^{yz} + xe^{xz} - xe^{xz} - ye^{yz}\right) dx\,dy\,dz$$

$$= \underset{V}{\iiint} 1\,dx\,dy\,dz = \mathrm{Vol}(V) = \frac{2}{3}\pi$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \frac{2}{3}\pi - \iint_{S_0} \vec{F}\cdot d\vec{S}$$

$$= \frac{2}{3}\pi + \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$= \frac{5}{3}\pi$$

[On S_0, you can use parametrization $\langle r\cos\theta,\ r\sin\theta,\ 0 \rangle$]

Rm: [If I don't have time to show you examples of Gauss thm, you don't have to worry about the application of it]

10 a) No intersection point with ~~z~~ xy-plane ($z=0$)
($0 = 0 - 4$ no solution)

xz-plane $\Leftrightarrow y=0 \Rightarrow z^3 = -4$ $z = -\sqrt[3]{4}$
$\Rightarrow$ a line in xz-plane parallel to x-axis

yz-plane $\Leftrightarrow$ ~~z~~ $x=0 \Rightarrow z^3 = -4$ $z = -\sqrt[3]{4}$
$\Rightarrow$ a line in yz-plane parallel to y-axis.

10b) Let $F(x,y,z) = z^3 - xyz + 4$

$\nabla F = \langle -yz, -xz, 3z^2 - xy \rangle$

$\Rightarrow \nabla F(2,3,2) = \langle -6, -4, 6 \rangle$ ~ normal vector

$\Rightarrow$ tangent plane equ. is

$-6(x-2) - 4(y-3) + 6(z-2) = 0$

10c) By b), the tangent plane equ is $z = x + \frac{2}{3}y - \cancel{6}\ 2$

Notice that the surface $z^3 = xyz - 4$ determines a function $z = f(x,y)$, although it's not easy to write down explicitly f.

The question is asking finding the linear approximation of z at $x = 1.95$ $y = 3.05$

use $z = x + \frac{2}{3}y - 2$.

$\Rightarrow z \approx 1.95 + \frac{2}{3} \cdot 3.05 - 2 = \frac{5.95}{3}$

11 Area of graph $= \iint_D \sqrt{1+f_x^2+f_y^2}\, dx\,dy$.

$\frac{\partial f}{\partial x} = x^{\frac{1}{2}}$. $\frac{\partial f}{\partial y} = y^{\frac{1}{2}}$

$\Rightarrow$ Area $= \int_0^1\int_0^1 \sqrt{1+x+y}\; dx\,dy$

$= \frac{4}{15}\left(3^{\frac{5}{2}} - 2^{\frac{7}{2}} + 1\right)$

12. Use spherical coordinate.

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} \qquad \begin{aligned} &1 \le r \le \sqrt{2} \\ &0 \le \theta \le 2\pi \\ &0 \le \varphi \le \frac{\pi}{2} \end{aligned}$$

$\cancel{\iiint_D \exp(r^2)^{\frac{3}{2}}\, dr\, d\varphi\, d\theta} =$

$\iiint_D \exp\left[(x^2+y^2+z^2)^{\frac{3}{2}}\right] dx\,dy\,dz$.

$= \iiint_D \left(\exp(r^3)\right)\cdot r^2\sin\varphi\; d\varphi\, d\theta\, dr$

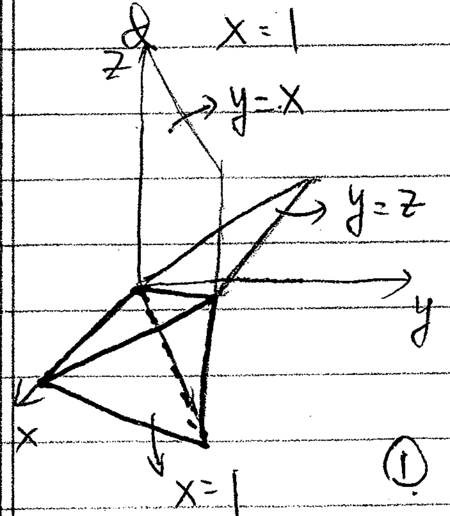
$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{\sqrt{2}} r^2\exp(r^3)\,\sin\varphi\; dr\, d\varphi\, d\theta$

$= \frac{2\pi}{3}\left[\exp(2^{\frac{3}{2}}) - \exp(1)\right]$

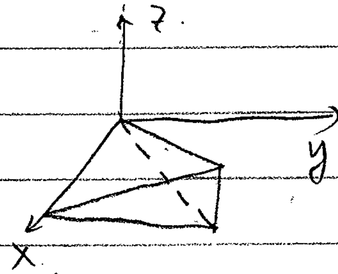
13. $\int_0^1 \int_0^x \int_0^y f(x,y,z)\, dz\, dy\, dx$

$\Rightarrow \quad 0 \le z \le y, \quad 0 \le y \le x, \quad 0 \le x \le 1$

The region is bounded by planes $y=z$. $y=x$



$\rightsquigarrow$ (To be more clear)

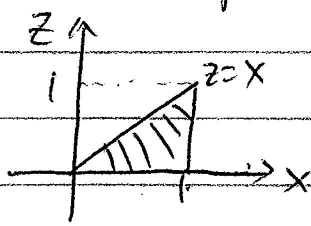


change order to $dy\, dx\, dz$.

① For y. ~~For~~ from Left surface to Right surface

$\Rightarrow \quad z \le y \le x$

② For other two variables: project the region onto x-z plane it looks like:



$\Rightarrow \quad z \le x \le 1$

$0 \le z \le 1$

$\Rightarrow \int_0^1\int_0^x\int_0^y f\, dz\, dy\, dx = \int_0^1\int_z^1\int_z^x f(x,y,z)\, dy\, dx\, dz.$