II. Invariance

AS.150.498: Modal Logic and Its Applications
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Now that we have a syntax and semantics for the basic sentential modal language $\mathcal{L}$, let us ask: when are two Kripke models effectively the same with respect to $\mathcal{L}$?

1 Bisimulation

Definition 2.1. Pointed models $\mathcal{M}, w$ and $\mathcal{N}, v$ are **modally equivalent**, $\mathcal{M}, w \leftrightsquigarrow \mathcal{N}, v$, provided that for every $\varphi \in S_{\mathcal{L}}$, $[\![\varphi]\!]^w_{\mathcal{M}} = [\![\varphi]\!]^v_{\mathcal{N}}$.

That is, $\mathcal{L}$ cannot tell modally equivalent pointed models apart.

It is sufficient for modal equivalence that a special kind of relation holds between pointed models:

Definition 2.2. Given $\mathcal{M} = \langle \mathcal{W}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, \mathcal{V}^{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle \mathcal{W}^{\mathcal{N}}, \mathcal{R}^{\mathcal{N}}, \mathcal{V}^{\mathcal{N}} \rangle$, a **bisimulation** between $\mathcal{M}, w$ and $\mathcal{N}, v$ is a binary relation $\mathcal{Z} \subseteq \mathcal{W}^{\mathcal{M}} \times \mathcal{W}^{\mathcal{N}}$ such that $w\mathcal{Z}v$ and for all worlds $x \in \mathcal{W}^{\mathcal{M}}$ and $y \in \mathcal{W}^{\mathcal{N}}$, if $x\mathcal{Z}y$ then:

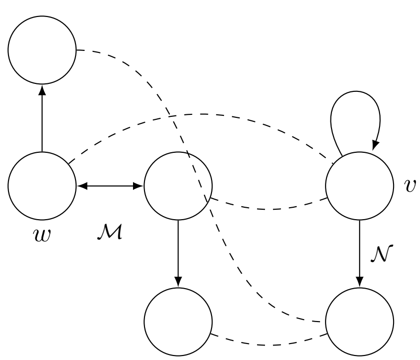
(**atomic harmony**) For all $p \in At_{\mathcal{L}}$, $\mathcal{V}^{\mathcal{M}}(p, x) = \mathcal{V}^{\mathcal{N}}(p, y)$.

(**zig**) If $x\mathcal{R}^{\mathcal{M}}z$, then there exists $z' \in \mathcal{W}^{\mathcal{N}}$ such that $y\mathcal{R}^{\mathcal{N}}z'$ and $z\mathcal{Z}z'$.

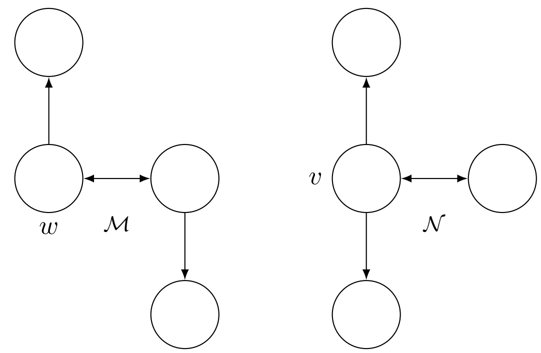
(**zag**) If $y\mathcal{R}^{\mathcal{N}}z'$, then there exists $z \in \mathcal{W}^{\mathcal{M}}$ such that $x\mathcal{R}^{\mathcal{M}}z$ and $z'\mathcal{Z}z$.

We say that $\mathcal{M}, w$ and $\mathcal{N}, v$ are **bisimilar**: $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{N}, v$.

For example, these pointed models are bisimilar:



But these pointed models are not:



Lemma 2.1. (**Invariance Lemma**) $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{N}, v$ only if $\mathcal{M}, w \leftrightsquigarrow \mathcal{N}, v$.

The proof is a straightforward induction on the complexity of sentences in $\mathcal{S}_{\mathcal{L}}$.

To establish that $\mathcal{M}, w \not\underline{\leftrightarrow} \mathcal{N}, v$, it suffices to find some sentence $\varphi \in S_{\mathcal{L}}$ such that $[\![\varphi]\!]^w_{\mathcal{M}} \neq [\![\varphi]\!]^v_{\mathcal{N}}$. For the above non-bisimilar pointed models, $[\![\Box(\Box\bot \vee \Diamond\Box\bot)]\!]^w_{\mathcal{M}} = T$ but $[\![\Box(\Box\bot \vee \Diamond\Box\bot)]\!]^v_{\mathcal{N}} = F$.

The converse of the Invariance Lemma also holds when the pointed models are *finite*—that is, when $|\mathcal{W}^{\mathcal{M}}| = m$ and $|\mathcal{W}^{\mathcal{N}}| = n$ for $m, n \in \mathbb{N}$:[1]

Lemma 2.2. For finite pointed models $\mathcal{M}, w$ and $\mathcal{N}, v$, $\mathcal{M}, w \leftrightsquigarrow \mathcal{N}, v$ only if $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{N}, v$.

For the proof, let $x\mathcal{Z}y$ when $\mathcal{M}, x \leftrightsquigarrow \mathcal{N}, y$.

2 Smaller Models

Since $\mathcal{L}$ cannot tell bisimilar pointed models apart, it is sometimes useful to thin a pointed model by finding a bisimilar pointed submodel. Let us consider two ways to do this.

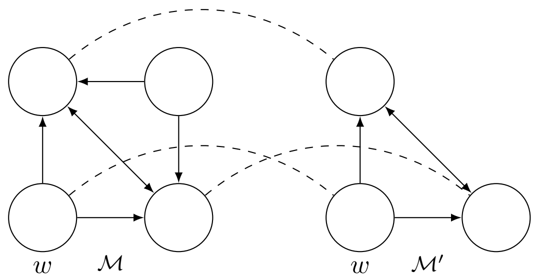
Definition 2.3. Given $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \mathcal{V}' \rangle$ is a **submodel** of $\mathcal{M}$ if and only if:

$\mathcal{W}' \subseteq \mathcal{W}$
$\mathcal{R}'$ is the restriction of $\mathcal{R}$ to $\mathcal{W}'$
$\mathcal{V}'$ is the restriction of $\mathcal{V}$ to $\mathcal{W}'$

[1] The converse of the Invariance Lemma would hold for *all* pointed models if $\mathcal{L}$ allowed for infinite conjunctions.

Definition 2.4. Given $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ and $w \in \mathcal{W}$, the **submodel generated from** w is the submodel $\mathcal{M}'$ where $\mathcal{W}'$ is the set of worlds reachable from w in 0 or more steps along $\mathcal{R}$.

Fact 2.1. If $\mathcal{M}'$ is the submodel of $\mathcal{M}$ generated from w, $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w$. The identity relation is a bisimulation.



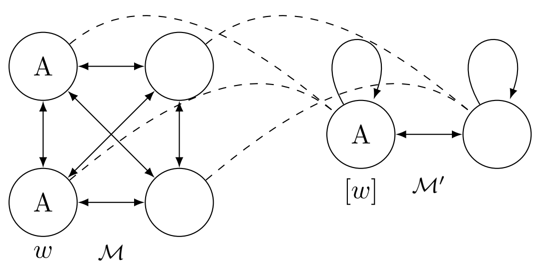
For the second kind of submodel, consider the set $\mathcal{Z}_\mathcal{M}$ of bisimulations between a model $\mathcal{M}$ and any of its worlds and this same model $\mathcal{M}$ and any of its worlds (the *autobisimulations* of $\mathcal{M}$). Note that $\mathcal{Z}_\mathcal{M}$ is nonempty since it includes the identity relation. Now consider the union $\cup\mathcal{Z}_\mathcal{M}$ of all the bisimulations in $\mathcal{Z}_\mathcal{M}$. It is easy to verify that $\cup\mathcal{Z}_\mathcal{M}$ is both a bisimulation and an equivalence relation on $\mathcal{W}$.

For $w \in \mathcal{W}$, let $[w] = \{v \in \mathcal{W} : w \cup \mathcal{Z}_\mathcal{M} v\}$.

Definition 2.5. Given $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, the **bisimulation contraction** of $\mathcal{M}$ is the model $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \mathcal{V}' \rangle$ where:

$\mathcal{W}' = \{[w] : w \in \mathcal{W}\}$
$\mathcal{R}' = \{\langle [w], [v] \rangle : \text{there is } x \in [w] \text{ and } y \in [v] \text{ such that } x\mathcal{R}y\}$
$\mathcal{V}'(p, [w]) = \mathcal{V}(p, w)$

Fact 2.2. If $\mathcal{M}'$ is the bisimulation contraction of $\mathcal{M}$, $\mathcal{M}, w \leftrightarroweq \mathcal{M}', [w]$. The relation sending $x \in \mathcal{W}$ to $[x] \in \mathcal{W}'$ is a bisimulation.



Note that in the above figure, the submodel of $\mathcal{M}$ generated from w is just $\mathcal{M}$ itself. So the two kinds of bisimilar submodels that we have been considering differ.

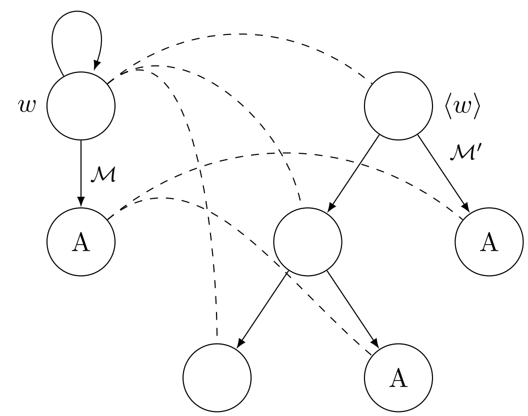
3 Bigger Models

It is also sometimes useful to transform a pointed model into a larger bisimilar pointed model.

Definition 2.6. Given $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ that is generated from w, the **tree unraveling** of $\mathcal{M}$ around w is the model $\mathcal{M}' = \langle \mathcal{W}', \mathcal{R}', \mathcal{V}' \rangle$ where:

$\mathcal{W}' = \{\langle w, ..., w_n \rangle : w, ..., w_n \in \mathcal{W} \text{ and } w\mathcal{R}w_2...w_{n-1}\mathcal{R}w_n\}$ (note: $w_1 = w$)
$\mathcal{R}' = \{\langle \langle x_1, ..., x_n \rangle, \langle y_1, ..., y_m \rangle \rangle : \langle y_1, ..., y_m \rangle = \langle x_1, ..., x_n, z \rangle \text{ for } z \in \mathcal{W}\}$
$\mathcal{V}'(p, \langle w, ..., w_n \rangle) = \mathcal{V}(p, w_n)$

Fact 2.3. If $\mathcal{M}'$ is the tree unraveling of $\mathcal{M}$ around w, $\mathcal{M}, w \leftrightarroweq \mathcal{M}', \langle w \rangle$. The relation sending $x \in \mathcal{W}$ to all worlds $\langle w, ..., w_n \rangle \in \mathcal{W}'$ where $w_n = x$ is a bisimulation.



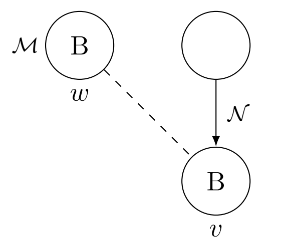
4 Application: Undefinability of Operators

A nice application of the Invariance Lemma is that we can use it to prove that certain operators are undefinable in the sentential modal language $\mathcal{L}$.

Fact 2.4. The following universal operator $\mathcal{A}$ is undefinable in $\mathcal{L}$:

$$[\![\mathcal{A}\varphi]\!]^w_\mathcal{M} = T \quad \text{iff} \quad \forall v \in \mathcal{W}([\![\varphi]\!]^v_\mathcal{M} = T)$$

Proof: Suppose that $\mathcal{A}$ is definable in $\mathcal{L}$—that is, there is a basic modal formula $\alpha(\cdot)$ in $\mathcal{L}$ such that $[\![\mathcal{A}\varphi]\!]^w_{\mathcal{M}} = [\![\alpha(\varphi)]\!]^w_{\mathcal{M}}$. Now consider the bisimilar pointed models below. $[\![\mathcal{A}B]\!]^w_{\mathcal{M}} = T$ so $[\![\alpha(B)]\!]^w_{\mathcal{M}} = T$ where $\alpha(B) \in S_{\mathcal{L}}$. But since $\mathcal{M}, w \leftrightarrows \mathcal{N}, v$, $\mathcal{M}, w \leftrightsquigarrow \mathcal{N}, v$, so $[\![\alpha(B)]\!]^v_{\mathcal{N}} = [\![\mathcal{A}B]\!]^v_{\mathcal{N}} = T$ which is a contradiction.



Fact 2.5. The following 'exists two' operator $\Diamond_2$ is undefinable in $\mathcal{L}$:

$$[\![\Diamond_2\varphi]\!]^w_{\mathcal{M}} = T \quad \text{iff} \quad \exists v_1, v_2 \in \{v : w\mathcal{R}v\}(v_1 \neq v_2 \wedge [\![\varphi]\!]^{v_1}_{\mathcal{M}} = [\![\varphi]\!]^{v_2}_{\mathcal{M}} = T)$$

Proof: Suppose that $\Diamond_2$ is definable in $\mathcal{L}$—that is, there is a basic modal formula $\alpha(\cdot)$ such that $[\![\Diamond_2\varphi]\!]^w_{\mathcal{M}} = [\![\alpha(\varphi)]\!]^w_{\mathcal{M}}$. Now consider the bisimilar pointed models below. $[\![\Diamond_2 B]\!]^w_{\mathcal{M}} = T$ so $[\![\alpha(B)]\!]^w_{\mathcal{M}} = T$ where $\alpha(B) \in S_{\mathcal{L}}$. But since $\mathcal{M}, w \leftrightarrows \mathcal{N}, v$, $\mathcal{M}, w \leftrightsquigarrow \mathcal{N}, v$, so $[\![\alpha(B)]\!]^v_{\mathcal{N}} = [\![\Diamond_2 B]\!]^v_{\mathcal{N}} = T$ which is a contradiction.