Lecture 2 : 1 Feb 2017

Last time:

- Introduced vector notation for points in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^n$.
- Introduced addition & scalar multiplication of vectors.

"To give a line in $\mathbb{R}^2$ or $\mathbb{R}^3$, it's enough to give a point & a direction."

Vector equation of a line: point & direction form

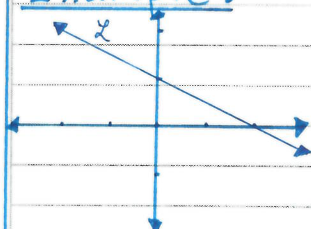
$\mathcal{L}(t) = \vec{a} + t\vec{v}$ where $\vec{a}$ = a point on our line
$\vec{v}$ = vector giving the direction of the line
t = real parameter.

In $\mathbb{R}^3$ case: If $\vec{a} = (a_1, a_2, a_3)$
$\vec{v} = (v_1, v_2, v_3)$

then $\mathcal{L}(t) = \vec{a} + t\vec{v} = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$

$\mathbb{R}^2$ case: similar.

Example:



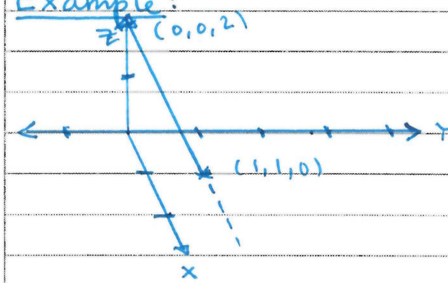
$\mathcal{L}(t) = (0,1) + t(2,-1) = (2t, 1-t)$

"Note that different $\vec{a}$ and $\vec{v}$ can give the same line." (with a different parametrization).

Note: Above example is also $\mathcal{L}(t) = (2,0) + t(-2,1)$.

"Let's look at an example in $\mathbb{R}^3$."

Example:



$$\mathcal{L}(t) = \vec{a} + t\vec{v}$$

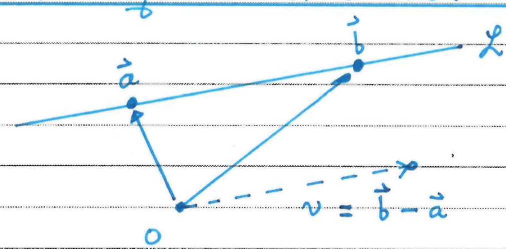
$$\vec{a} = (0,0,2)$$

$$\vec{v} = (1,1,0) - (0,0,2) = (1,1$$

$$\mathcal{L}(t) = (0,0,2) + t(1,1,-2) = (t, t, 2-2t).$$

" Recall that a line is also determined by any two distinct points on it."

vector equation of a line: point-point form.



$$\mathcal{L}(t) = \vec{a} + t(\vec{b} - \vec{a}) = (1-t)\vec{a} + t\vec{b}$$

Note: $\mathcal{L}(0) = (1-0)\vec{a} + 0\vec{b} = \vec{a}$

$\mathcal{L}(1) = (1-1)\vec{a} + 1.\vec{b} = \vec{b}$

The line segment between $\vec{a}$ & $\vec{b}$ is $\mathcal{L}(t)$ with $0 \le t \le 1$.

Example: If $\vec{a} = -\vec{i} + \vec{j}$, $\vec{b} = \vec{k}$

$\Rightarrow \mathcal{L}(t) = (1-t)\vec{a} + t\vec{b}$

$= (1-t)(-\vec{i} + \vec{j}) + t\vec{k}$

$= (-1+t)\vec{i} + (1-t)\vec{j} + t\vec{k}$

$= (-1+t, 1-t, t).$

Note: If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

then $\mathcal{L}(t) = (1-t)\vec{a} + t\vec{b} = ((1-t)a_1, (1-t)a_2, (1-t)a_3)$

$+ (tb_1, tb_2, tb_3)$

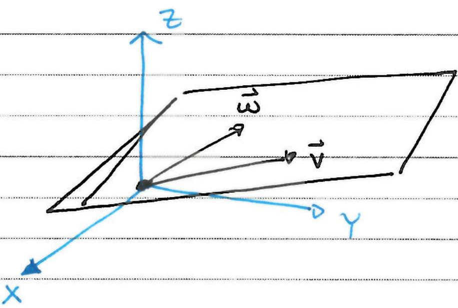
$= (a_1 + (b_1 - a_1)t, a_2 + (b_2 - a_2)t, a_3 + ($

Planes in $\mathbb{R}^3$ passing through $(0,0,0)$:

If $\vec{v}, \vec{w}$ vectors in $\mathbb{R}^3$, nonzero & not paralell, then

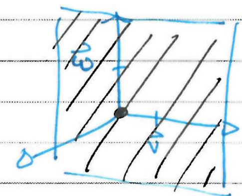
$$\lambda \vec{v} + \mu \vec{w} \qquad (\lambda, \mu \text{ any real \#'s})$$

describes a plane in $\mathbb{R}^3$ passing through $(0,0,0)$, $\vec{v}$, and $\vec{w}$.



~~If $0 \le \lambda \le 1$, $0 \le \mu \le 1$ then this~~

Example:



$\vec{v} = (0,1,0)$, $\vec{w} = (0,0,1)$

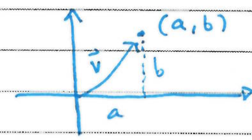
get the YZ-plane.

Length, distance, and inner product (§1.2)

- **Length in $\mathbb{R}^2$:**

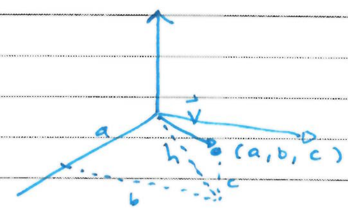
$(\text{length of } \vec{v})^2 = a^2 + b^2$ "Pyth thm"

~~(Pythagorean th)~~



Def: $\|\vec{v}\| = \sqrt{a^2+b^2}$.

- **length in $\mathbb{R}^3$:**



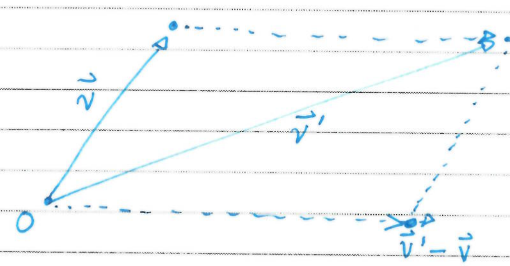
length of $\vec{v} = \|\vec{v}\|$

$\|\vec{v}\|^2 = h^2 + c^2$. by PT.

$h^2 = a^2 + b^2$ by PT

$\Rightarrow \|\vec{v}\| = \sqrt{a^2+b^2+c^2}$.

- **Distance between two vectors:**



distance between $\vec{v}'$ and $\vec{v}$ $= \|\vec{v}' - \vec{v}\|$

Example: $\vec{v} = (1,1,1)$, $\vec{v}' = (0,0,1)$.

$\|\vec{v}' - \vec{v}\| = \|(-1,-1,0)\| = \sqrt{(-1)^2+(-1)^2+0^2} = \sqrt{2}$.

Q: when are two vectors perpendicular/orthogonal?

A: Exactly when $\|\vec{v}' - \vec{v}\|^2 = \|\vec{v}'\|^2 + \|\vec{v}\|^2$

Example: $\vec{v} = (1,1,1) \Rightarrow \|\vec{v}\|^2 = 1^2+1^2+1^2 = 3$.

$\vec{v}' = (0,0,1) \Rightarrow \|\vec{v}'\|^2 = 0^2+0^2+1^2 = 1$.

$\|\vec{v}' - \vec{v}\|^2 = \|(-1,-1,0)\|^2 = (-1)^2+(-1)^2+0^2 = 2$

But $2 \neq 3+1$, so $\vec{v}$ and $\vec{v}'$ not perpendicular.

NOTE: If $\vec{v} = (a_1, a_2, a_3)$
$\vec{v}' = (b_1, b_2, b_3)$

A: $\|\vec{v}\|^2 + \|\vec{v}'\|^2 = (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2)$.

B: $\|\vec{v}' - \vec{v}\|^2 = \|(b_1 - a_1, b_2 - a_2, b_3 - a_3)\|^2$

$= (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$

$= (b_1^2 - 2b_1a_1 + a_1^2) + (b_2^2 - 2b_2a_2 + a_2^2) + (b_3^2 - 2b_3a_3 + a_3^2)$.

Note: $A = B \iff -2b_1a_1 - 2b_2a_2 - 2b_3a_3 = 0$

$\iff b_1a_1 + b_2a_2 + b_3a_3 = 0$.

Consequence: $\vec{v}$ and $\vec{v}'$ perpendicular $\iff a_1b_1 + a_2b_2 + a_3b_3 =$

Def The inner product of $\vec{v}$ and $\vec{v}'$ is the numbe
$\vec{v} \cdot \vec{v}' = a_1b_1 + a_2b_2 + a_3b_3$

Examples

① $\vec{v} = 3\vec{i} + \vec{j} - 2\vec{k} = (3, 1, -2)$
$\vec{v}' = \vec{i} - \vec{j} + \vec{k} = (1, -1, 1)$

$\vec{v} \cdot \vec{v}' = 3 \cdot 1 + 1 \cdot (-1) + (-2) \cdot 1 = 0$.

so $\vec{v}$ & $\vec{v}'$ are perpendicular.

② $\vec{v} = 2\vec{i} + \vec{j} - \vec{k} = (2, 1, -1)$.
$\vec{v}' = 3\vec{k} + (-2)\vec{j} = (0, -2, 3)$

$\vec{v} \cdot \vec{v}' = 2 \cdot 0 + 1 \cdot (-2) + (-1) \cdot 3 = -5$

$\vec{v}$ & $\vec{v}'$ not perpendicular.

We should also say how inner product interacts with addition & scalar multiplication of vectors.

Properties of inner product:

① $\vec{v} \cdot \vec{v} \geq 0$ with equality only when $\vec{v} = (0,0,0)$

(Since $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$).

② $(\lambda \vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (\lambda \vec{w})$ λ scalar

③ $\vec{v} \cdot (\vec{w} + \vec{w}') = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{w}'$.

④ $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
$= w_1 v_1 + w_2 v_2 + w_3 v_3 = \vec{w} \cdot \vec{v}$.

Let's check ③:

$\vec{v} \cdot (\vec{w} + \vec{w}') = (v_1, v_2, v_3) \cdot (w_1 + w_1', w_2 + w_2', w_3 + w_3')$

$= v_1(w_1 + w_1') + v_2(w_2 + w_2') + v_3(w_3 + w_3')$

$= (v_1 w_1 + v_2 w_2 + v_3 w_3) + (v_1 w_1' + v_2 w_2' + v_3 w_3')$

$= \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{w}'$.

SKIPPED! Say it next time

Unit vectors: $\vec{v}$ is called a unit vector if $\|\vec{v}\| = 1$

Non-examples

- $\vec{0} = (0,0,0)$ $\quad \|\vec{0}\| = \sqrt{0^2+0^2+0^2} = 0$.
- $\vec{v} = (1,1,1)$ $\quad \|\vec{v}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

Examples

- $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$.
- put $\vec{v} = \frac{1}{\sqrt{3}}(1,1,1) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$\|\vec{v}\| = \sqrt{(1/\sqrt{3})^2 + (1/\sqrt{3})^2 + (1/\sqrt{3})^2} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}$

$= \sqrt{3/3} = 1$.

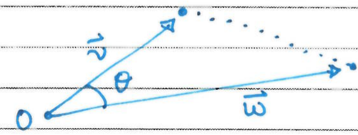
Note: If $\vec{v}$ is any nonzero vector, then

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} \text{ is a unit vector.}$$

"the normalization of $\vec{v}$."

Can also use inner product to describe the

Angle between two vectors.



Prop: If $\vec{v}, \vec{w}$ are both nonzero, then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta .$$

Consequence 1: $\theta = \cos^{-1}\left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right)$

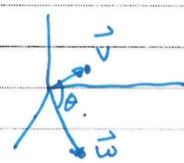
not easy to compute by hand.

Consequence 2:

$$|\vec{v}\cdot\vec{w}| = \underbrace{\|\vec{v}\|\,\|\vec{w}\|}_{\geq 0} \underbrace{|\cos\theta|}_{\leq 1} \leq \|\vec{v}\|\,\|\vec{w}\|$$

"Cauchy-Schwarz inequality"

Example: $\vec{v} = (1,1,1)$, $\vec{w} = (1,1,-1)$



$$\vec{v}\cdot\vec{w} = 1\cdot 1 + 1\cdot 1 + 1\cdot(-1) = 1$$

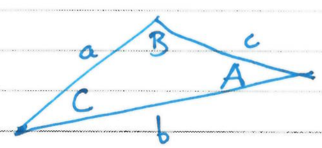
$\|\vec{v}\| = \sqrt{3}$ $\quad \|\vec{w}\| = \sqrt{3}$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23 \text{ radians} \approx 71^\circ .$$

why is prop true?

It's basically just a restatement of the law of cosines from trigonometry in the language of vectors & inner product.

Law of cosines



$\Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos(C).$