Johns Hopkins University, Department of Mathematics
Calc III - Spring 2012
Final Exam

Instructions: This exam has 5 pages and is out of 80 points. No calculators, books or notes allowed. Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper. Be sure to clearly label each problem and attach them to the exam. You have **3 HOURS**.

Academic Honesty Certification

I certify that I have taken this exam without the aid of unauthorized people or objects.

Signature: [signature] Date: 5/9/12

(1 pt) Name: Nick Marshburn Section: D

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Bonus	
Total	

1. (14 pts, 2pts each) True/False. Write "True" or "False" below each of the following statements:

(a) In cylindrical coordinates, the equation $z = -2r$ describes a cone.

True

(b) For a two-variable function f, if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at a point, then f is continuous at that point.

False

(c) The first-order Taylor approximation to a C^1 function at a point is the same as the usual linear (i.e. tangent plane) approximation.

True

(d) The Implicit Function Theorem may be applied to show that a level set of a three-variable function $f(x, y, z)$ is globally the graph of a two-variable function $z = g(x, y)$.

False

(e) $\int_0^1 \int_2^4 f(x,y)dxdy = -\int_2^4 \int_0^1 f(x,y)dydx.$

False

(f) If F is a C^1 vector field on $\mathbb{R}^3$, then the surface integral of the curl of F over a sphere is 0.

True

(g) If F is a conservative vector field on $\mathbb{R}^3$, then for any path c in $\mathbb{R}^3$, $\int_c F \cdot d\vec{s} = 0$.

False

2. (15 pts, 5 pts each) Give an example of the following objects:

(a) Vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ such that the angle between $\vec{v}$ and $\vec{w}$ is $\frac{2\pi}{3}$ and $||\vec{v} \times \vec{w}|| = \sqrt{3}$.

$\vec{v} = (1, 0, 0)$

$\vec{w} = (-1, \sqrt{3}, 0) \qquad \cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{||\vec{v}||\,||\vec{w}||} = \dfrac{-1}{1\cdot 2} = \dfrac{-1}{2} \qquad \theta = \dfrac{2\pi}{3}$

$\vec{v}\times\vec{w} = \sqrt{3}\,k \qquad ||\vec{v}\times\vec{w}|| = \sqrt{3}$

(b) A C^∞ two-variable function f such that at the point $(x, y) = (2, 1)$, the direction of steepest decrease of f is $\vec{v} = (1, \sqrt{3})$.

$\nabla f(2,1) = (-1, -\sqrt{3})$

$f(x,y) = -x - \sqrt{3}\,y$

(c) A vector field F in $\mathbb{R}^3$ such that the path $c(t) = (t^3, 4t, t^2 - 1)$ is a flow line for F.

$F(c(t)) = c'(t) \qquad c'(t) = (3t^2, 4, 2t)$

$F(x,y,z) = (3z + 3, 4, y/2)$

3. (10 pts) Let $f(x, y, z) = x^2 - 4x + 3y^2 + z^2$.

(a) (5 pts) Let S be the level set of f with value $c = 5$. Find the tangent plane to S at $(4, \frac{1}{\sqrt{3}}, 2)$.

$$\nabla f = (2x - 4, 6y, 2z)$$

$$\nabla f(4, \tfrac{1}{\sqrt{3}}, 2) = (4, \cancel{\tfrac{6}{\sqrt{3}}}\ 2\sqrt{3}, 4)$$

$$4(x-4) + 2\sqrt{3}(y - \tfrac{1}{\sqrt{3}}) + 4(z-2) = 0$$

$$4x - 16 + 2\sqrt{3}y - 2 + 2z - \cancel{8} = 0$$

$$4x + 2\sqrt{3}y + 2z = 26$$

(b) (5 pts) Find and classify the critical points of f.

$\nabla f = 0$ when $(x,y,z) = (2,0,0)$

min

$f(x,y,z) = (x-2)^2 + 3y^2 + z^2 - 4$, which clearly has an absolute min at $(2,0,0)$

4. (10 pts) Use the ϵ-δ definition of limit to show $\lim_{x \to 1} [2(x-1)^3 + 3] = 3$.

Let $\epsilon > 0$. Set $\delta = \sqrt[3]{\frac{\epsilon}{2}}$.

$|x-1| < \delta = \sqrt[3]{\epsilon/2}$

$|x-1|^3 < \epsilon/2$

$2|x-1|^3 < \epsilon$

$|2(x-1)^3 + 3 - 3| < \epsilon$ ✓

5. (10 pts) Compute $\iiint_W z\,dxdydz$, where W is the region above the xy plane bounded by the ellipsoid $x^2+9y^2+z^2 = 1.$

$x = \rho\cos\theta\sin\phi$

$y = \frac{\rho}{3}\sin\theta\sin\phi$

$z = \rho\cos\phi$

$dx\,dy\,dz = \frac{1}{3}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho\cos\phi\cdot\frac{1}{3}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$= \int_0^{\pi/2}\int_0^{2\pi}\frac{1}{12}\sin\phi\cos\phi\,d\theta\,d\phi$

$= \int_0^{\pi/2}\frac{2\pi}{12}\sin\phi\cos\phi\,d\phi$

$= \frac{\pi}{12}\sin^2\phi\Big|_0^{\pi/2} = \frac{\pi}{12}$

6. (10 pts) Let S be the surface parametrized by $\Phi(u,v) = (2\cos u + 1, 2\sin u, v)$, where $0 \le u \le 2\pi$ and $0 \le v \le 2$.

(a) (5 pts) Find a vector orthogonal to S at the point $\Phi(\frac{\pi}{6}, 1)$.

$$\vec{T}_u = (-2\sin u, 2\cos u, 0)$$
$$\vec{T}_v = (0, 0, 1)$$
$$\vec{T}_u \times \vec{T}_v = (2\cos u, 2\sin u, 0)$$
$$\vec{T}_u \times \vec{T}_v \text{ @ } \Phi(\tfrac{\pi}{6}, 1) = (\sqrt{3}, 1, 0)$$

(b) (5 pts) Compute $\iint_S xz \, dS$.

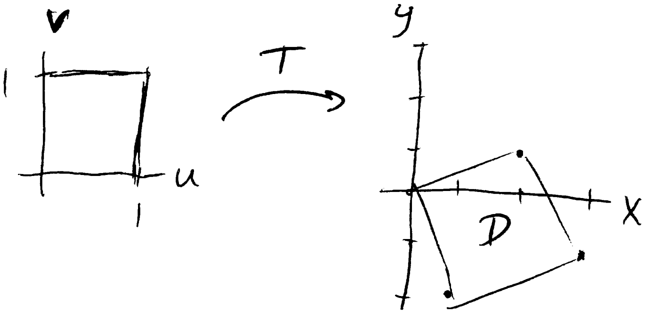
$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{4\cos^2 u + 4\sin^2 u + 0} = 2$$

$$= \int_0^2 \int_0^{2\pi} (2\cos u + 1)v \cdot 2\,du\,dv$$

$$= \int_0^2 (2\sin u + u)v \cdot 2\Big|_0^{2\pi} dv = \int_0^2 4\pi \cdot v\,dv = 2\pi v^2\Big|_0^2 = 8\pi$$

7. (10 pts) Let C be the boundary of the square in the plane with vertices (0,0), (2,1), (1,-2), and (3,-1), oriented counterclockwise. Compute $\int_C xy\,dx - y^2\,dy$.

$$\int_C xy\,dx - y^2\,dy \overset{\text{Green's}}{=} \iint_D -x\,dx\,dy$$



$$T(u,v) = (u + 2v, -2u + v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5$$

$$= \int_0^1\int_0^1 (-u - 2v)\cdot 5\,du\,dv$$

$$= \int_0^1 -\frac{5}{2}u^2 - 10uv\Big|_0^1 dv = \int_0^1 \left(-\frac{5}{2} - 10v\right) dv = -\frac{5}{2}v - 5v^2\Big|_0^1$$

$$= -\frac{5}{2} - 5 = -\frac{15}{2}$$

8. (3 pts, 1 pt each) Bonus!

(a) Give an example of a constant-speed path in $\mathbb{R}^4$ with nonzero acceleration.

$c(t) = (\cos t, \sin t, 0, 0)$

(b) Buns the rabbit loves only himself and adding numbers less than 27 in groups of four. What is Buns' favorite number?

$2 + 21 + 14 + 19 = 56$

B u N S

(c) Draw a picture involving (but not limited to!) a squirrel, a screw, and a sandcastle.