# Fall 2006 - Calculus III
# The Johns Hopkins University
**Instructors: Prof. Ha and Wilson**

**Solutions by Mike Limarzi**

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**Name:**

**<u>Midterm I</u>**

**(**1)(2 points) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x, y, z) = (x^2 - y, xy + z, y^2 - xz)$. Compute the divergence of $f$.

$$div f = \nabla \cdot f = \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3$$
$$\nabla \cdot f = \frac{\partial}{\partial x}(x^2 - y) + \frac{\partial}{\partial y}(xy + z) + \frac{\partial}{\partial z}(y^2 - xz)$$
$$\nabla \cdot f = 2x + x - x = 2x.$$

**(**2)(2 points) Compute the derivative of $f$ in problem #1.

To get the derivative, we use the derivative matrix $\mathbf{D}f(x, y, z)$.

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$

So, we get:

$$\mathbf{D}f(x,y,z) = \begin{pmatrix} 2x & -1 & 0 \\ y & x & 1 \\ -z & 2y & -x \end{pmatrix}$$

(3)(2 points) Compute the curl of $f$ in problem #1, $f(x,y,z) = (x^2 - y, xy + z, y^2 - xz)$.

We calculate curl of $f$.

$$\nabla \times f = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{pmatrix}$$

So, we get:

$$\nabla \times f = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & xy - z & y^2 - xz \end{pmatrix}$$

$$\nabla \times f = (2y - -1)\mathbf{i} - (-z)\mathbf{j} + (y - -1)\mathbf{k}$$
$$\nabla \times f = (2y + 1)\mathbf{i} + z\mathbf{j} + (y + 1)\mathbf{k}$$

(4)(2 points) Let $c : \mathbb{R} \to \mathbb{R}^3$ be the path given by $c(t) = (t^3 - 1, t^3 + 1, t^2 + 3)$. What is the velocity at $t$?

Velocity is the derivative of our curve, $c'(t)$. So we get:
$v(t) = c'(t) = (3t^2, 3t^2, 2t)$.

(5)(2 points) Let $c$ be the path in #4, $c(t) = (t^3 - 1, t^3 + 1, t^2 + 3)$. What is the speed at $t = 1$?

Speed is the norm of velocity, $||v(t)||$. So we get:
$s(t) = ||v(t)|| = \sqrt{(3t^2)^2 + (3t^2)^2 + (2t)^2}$.
$s(1) = ||v(1)|| = \sqrt{(3)^2 + (3)^2 + (2)^2} = \sqrt{22}$.

(6)(2 points) Let $c$ be the path in #4, What is the acceleration at $t = 1$?

Acceleration is the derivative of velocity, $v'(t) = c''(t)$. So we get:
$v(t) = c'(t) = (3t^2, 3t^2, 2t)$.
$a(t) = c''(t) = (6t, 6t, 2)$.
$a(1) = (6, 6, 2)$.

(7)(2 points) Let $c$ be the path in #4, $c(t) = (t^3 - 1, t^3 + 1, t^2 + 3)$. Set up the integral (but do not try to integrate) for the length of the curve $c(t)$ from $t = 1$ to $t = 3$.

The formula for arclength is:

$$L(c) = \int_{t_0}^{t_1} ||v(t)||dt$$

So, we get:

$$L(c) = \int_1^3 \sqrt{(3t^2)^2 + (3t^2)^2 + (2t)^2}dt = \int_1^3 t\sqrt{18t^2 + 4}dt$$

(8)(2 points) Let $c$ be the path in #4. Find an equation for the tangent line to $c(t)$ at $t = 1$.

An equation for a tangent line is:
$l(t) = c(t_0) + (t - t_0)c'(t_0)$.

Since $c(1) = (0, 2, 4)$ and $c'(1) = (3, 3, 2)$, we get:

$l(t) = (0, 2, 4) + (t - 1)(3, 3, 2))$.

(9)(2 points) Let $g : \mathbb{R}^3 \to \mathbb{R}$ be given by $g(x, y, z) = x^2y + yz + x - z$. Compute the Laplacian of $g$.

The Laplacian is defined as:

$$\Delta g = \nabla \cdot (\nabla g) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}$$

So, we get:

$$\Delta g = \nabla \cdot (2xy + 1, x^2 + z, y - 1) = 2y$$

(10)(2 points) Compute the gradient of the function $g$ of problem #9.

$$\begin{aligned} g &= x^2y + yz + x - z \\ \nabla g &= (2xy + 1, x^2 + z, y - 1). \end{aligned}$$

(11)(2 points) Find the directional derivative of the function $g$ of problem #9, $g(x, y, z) = x^2y + yz + x - z$ in the direction $\mathbf{j} = (0, 1, 0)$ at an arbitrary point $(x, y, z)$.

The directional derivative can be computed as $\nabla f \cdot \mathbf{v}$, where $\mathbf{v}$ is a unit vector. So, since $\mathbf{v} = (0, 1, 0)$ has norm $= 1$, we get:
$\nabla f \cdot \mathbf{v} = (2xy + 1, x^2 + z, y - 1) \cdot (0, 1, 0) = x^2 + z$.

(12)(2 points) Find the direction that the directional derivative of the function $g$ of problem #9 is maximal at the point (1, 1, 1).

The directional derivative is maximal when the gradient is in the same direction as the vector $\mathbf{v}$. So, we look at $\nabla f(1, 1, 1)$:
$\nabla f(1, 1, 1) = (2xy + 1, x^2 + z, y - 1)|_{(1,1,1)} = (3, 2, 0)$

So, we see that $\mathbf{v}$ should be in the direction of $(3, 2, 0)$, and making it of unit length, we get:

$$\mathbf{v} = (\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 0)$$

(13)(2 points) Find the Taylor expansion of the function $g$ of problem #9, $g(x, y, z) = x^2y+yz+x-z$, around $(0, 0, 0)$ up through the quadratic terms.

So, we compute the partial derivatives, and evaluate them at the point $(0, 0, 0)$:

$$g(0,0,0) = x^2y + yz + x - z|_{(0,0,0)} = 0$$
$$\frac{\partial g}{\partial x} = 2xy + 1|_{(0,0,0)} = 1$$
$$\frac{\partial g}{\partial y} = x^2 + z|_{(0,0,0)} = 0$$
$$\frac{\partial g}{\partial z} = y - 1|_{(0,0,0)} = -1$$

$$\frac{\partial^2 g}{\partial x^2} = 2y|_{(0,0,0)} = 0$$
$$\frac{\partial^2 g}{\partial y^2} = 0|_{(0,0,0)} = 0$$
$$\frac{\partial^2 g}{\partial z^2} = 0|_{(0,0,0)} = 0$$

$$\frac{\partial^2 g}{\partial y \partial x} = 2x|_{(0,0,0)} = 0$$
$$\frac{\partial^2 g}{\partial z \partial x} = 0|_{(0,0,0)} = 0$$
$$\frac{\partial^2 g}{\partial z \partial y} = 1|_{(0,0,0)} = 1$$

Now, we plug into our formula and obtain:

$$f(\mathbf{h}) = f(0,0,0) + \frac{\partial g}{\partial x}h_1 + \frac{\partial g}{\partial y}h_2 + \frac{\partial g}{\partial z}h_3$$
$$+\frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2}h_1^2+\frac{\partial^2 g}{\partial y^2}h_2^2+\frac{\partial^2 g}{\partial z^2}h_3^2+2\frac{\partial^2 g}{\partial y\partial x}h_1h_2+2\frac{\partial^2 g}{\partial z\partial x}h_1h_3+2\frac{\partial^2 g}{\partial z\partial y}h_2h_3\right)+R_2((h),(0,0,0))$$

$$f(\mathbf{h}) = h_1 - h_3 + h_2h_3 + R_2((h),(0,0,0))$$
$$\text{where } \frac{R_2((h),(0,0,0))}{||\mathbf{h}||^2} \to 0 \text{ as } \mathbf{h} \to \mathbf{0}.$$

(14)(3 points) Find all of the critical points of the function $g$ of problem # 9 (2 points) and decide, if you can, if they are max/min or saddle points (1 point). Give a reason for the latter.

$g = x^2y + yz + x - z$
To find critical points, we look at $\nabla g = 0$.
$\nabla g = (2xy + 1, x^2 + z, y - 1) = (0, 0, 0)$.
Solving, we get:
$y - 1 = 0 \quad \Rightarrow \quad y = 1$
$2x + 1 = 0 \quad \Rightarrow \quad x = -1/2$
$1/4 + z = 0 \quad \Rightarrow \quad z = -1/4$

So, the only point we get is $\mathbf{x}_0 = (-1/2, 1, -1/4)$. Now we must check to see if this is a max, min, or saddle point. We look at the Hessian and show it is positive definite. To do this, it suffices to look at the $1 \times 1$, $2 \times 2$ and $3 \times 3$ matrices as show these determinants are positive.

$$\mathbf{H}f(x,y,z) = \begin{pmatrix} 2y & 2x & 0 \\ 2x & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Looking at this at our point, we get:

$$\mathbf{H}f(x,y,z) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

So, we first check the $1 \times 1$ matrix, starting from the upper left, which is $2 > 0$, so it is positive. Next, we look at the determinant of the $2 \times 2$ matrix from the upper left:

$$\det \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

This determinant is $1 > 0$, so that is also positive. Now, we check the determinant of the $3 \times 3$ matrix:

$$\det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -2 < 0$$

This is less than 0, so our point is a saddle point.

(15)(3 points) Find the maximum and minimum values of the function $f(x,y) = x/3 + y/4$ subject to the constraint $x^2 + y^2 - 1 = 0$. (Use Calc III and show your work.)

With constraints, we use Lagrange multipliers.

$f = x/3 + y/4, \qquad g = x^2 + y^2$

$\nabla f = (1/3, 1/4)$

$\nabla g = (2x, 2y)$

$\nabla f = \lambda \nabla g$

$1/3 = \lambda 2x$

$1/4 = \lambda 2y$

$x^2 + y^2 - 1 = 0$

Since $\lambda \neq 0$, we get:

$x = 4y/3$

Plugging into our constraint, we get:

$x^2 + y^2 - 1 = 0$

$16y^2/9 + y^2 - 1 = 0$

$25y^2/9 = 1$

$y^2 = 9/25$

$y = \pm 3/5$

So, we get 2 points, $(4/5, 3/5)$ and $(-4/5, -3/5)$. Plugging into our original function, we get:

$f(4/5, 3/5) = 4/15 + 3/20 = 5/12$

$f(-4/5, -3/5) = -4/15 - 3/20 = -5/12$

So, we get a max of $5/12$ at $(4/5, 3/5)$ and a min of $-5/12$ at $(-4/5, -3/5)$.

(16)(3 points) Let $f(x,y) = x^2 + y^2$. Write an equation for the tangent plane to the graph at the point given by $(x,y) = (1,1)$.

We begin by writing $z$ as a function of $x$ and $y$, and then bring it all to one side.
$z = x^2 + y^2$
$g(x,y,z) = x^2 + y^2 - z = 0$

Now, we can take the gradient of this new function:
$\nabla g = (2x, 2y, -1)$

Since we are looking at the point $(1,1)$, in our $g$ this becomes the point $(1,1,2)$.
To find the tangent plane at the point $(1,1,2)$, we simply calculate:
$\nabla g(1,1,2) \cdot (x-1, y-1, z-2) = 0$

We get:

$(2,2,-1) \cdot (x-1, y-1, z-2)) = 2x - 2 + 2y - 2 - z + 2 = 0$
$2x + 2y - z = 2$

So, our tangent plane is $2x + 2y - z = 6$.

Alternatively, we can use the formula for a tangent plane for a function of two variables, given by:

$$z = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0)(y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)$$

Plugging in, we get:
$z = f(1,1) + (x-1)2(1) + (y-1)2(1)$
$z = 2 + 2x - 2 + 2y - 2$
$z = 2x + 2y - 2$,
And so we get the same equation for our tangent plane.

(17)(3 points) Let $f(x,y,z) = axy^2 + byz + cz^2x^3$. What must the constants $a$, $b$, and $c$ be so that the maximal rate of change of $f$ at the point $(1,2,-1)$, is 24 in the positive $z$-direction?

First, find the gradient of the function and evaluate at the point $(1,2,-1)$.

$$\nabla f(x,y,z) = (ay^2 + 3cz^2x^2, 2axy + bz, by + 2czx^3)$$

$$\nabla f(1,2,-1) = (4a + 3c, 4a - b, 2b - 2c)$$

Now, we know that if this is to give us the maximum value, it must be in the same direction as the $z$-axis, i.e. in the direction of our normalized vector $\overrightarrow{v} = (0,0,1)$. Furthermore, we need its dot product with the $\overrightarrow{v}$ to give us 24. So, we solve:

$4a + 3c = 0 \qquad 4a = -3c$
$4a - b = 0 \qquad b = 4a$
$2b - 2c = 24 \qquad -6c - 2c = 24 \Rightarrow -8c = 24$

So, we get:

$c = -3$
$b = 18$
$a = 9/2$

(18)(4 points)Let $z = f(x,y)$. Consider $z$ as a function of the polar coordinates$(r,\theta)$, (i.e. $(x,y) = (r\cos\theta, r\sin\theta)$). Compute $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$, and $\frac{\partial^2 z}{\partial r \partial \theta}$ in terms of the partials of $f$ with respect to $x$ and $y$.

We use the chain rule:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}.$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta.$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}.$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta).$$

Now, we must calculate the mixed partial. To do this, we first do a substitution. Let $g := \frac{\partial z}{\partial x}$ and $h := \frac{\partial z}{\partial y}$.

Now, we calculate the mixed partial.

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial r}\Big(g(-r\sin\theta) + h(r\cos\theta)\Big).$$

We now use product rule:

$$\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}(g) * (-r\sin\theta) + g * \frac{\partial}{\partial r}(-r\sin\theta) + \frac{\partial}{\partial r}(h) * (r\cos\theta) + h * \frac{\partial}{\partial r}(r\cos\theta)$$

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r}.$$

$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial r}.$$

So, we are left with:

$$\frac{\partial^2 z}{\partial r \partial \theta} = \left(\frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r}\right) * (-r\sin\theta) + g * \frac{\partial}{\partial r}(-r\sin\theta) + \left(\frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial r}\right) * (r\cos\theta) + h * \frac{\partial}{\partial r}(r\cos\theta)$$

Taking the derivatives of our known functions, we get:

$$\frac{\partial^2 z}{\partial r \partial \theta} = \left(\frac{\partial g}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial r}\right)*(-r\sin\theta)+g*(-\sin\theta)+\left(\frac{\partial h}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial h}{\partial y}\frac{\partial y}{\partial r}\right)*(r\cos\theta)+h*(\cos\theta)$$

Substituting back in for our $g$ and $h$, we get:

$$\frac{\partial^2 z}{\partial r \partial \theta} = \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial r}\right) * (-r\sin\theta) + \frac{\partial z}{\partial x} * (-\sin\theta)$$
$$+\left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right) * (r\cos\theta) + \frac{\partial z}{\partial y} * (\cos\theta)$$

Finally, we can replace the $\frac{\partial x}{\partial r}$ with $\cos\theta$ and $\frac{\partial y}{\partial r}$ with $\sin\theta$.

$$\frac{\partial^2 z}{\partial r \partial \theta} = \left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y \partial x}\sin\theta\right) * (-r\sin\theta) + \frac{\partial z}{\partial x} * (-\sin\theta)$$
$$+\left(\frac{\partial^2 z}{\partial x \partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\right) * (r\cos\theta) + \frac{\partial z}{\partial y} * (\cos\theta)$$

which is our answer.

(19)(6 points) Let $f(x, y, z) = x^2 + y^2 - z^2$. When this function is restricted to the ellipsoid, $x^2 + 4y^2 + 9z^2 = 16$ it has 2 points that are maximums, 2 that are minimums, and 2 that are saddle points. Find all 6 such points and say which they are.

We first take the gradients of our functions.
$\nabla f = (2x, 2y, -2z)$
$\nabla g = (2x, 8y, 18z)$

We note that the only time $\nabla f = 0$ is at the point $(0, 0, 0)$, which is not on the ellipsoid, so we don't need to worry about it. We can now use a Lagrange multiplier.

$\nabla f = \lambda \nabla g$
$2x = \lambda 2x$
$2y = \lambda 8y$
$-2z = \lambda 18z$
$x^2 + 4y^2 + 9z^2 = 16$

The $\lambda = 0$ case gives us the point $(0, 0, 0)$, which we have already dealt with. So, we can assume $\lambda \neq 0$. We look at the first equation:
$2x = \lambda 2x \qquad \Rightarrow \qquad 2x(\lambda - 1) = 0$
If $x \neq 0$, we get $\lambda = 1$, and plugging into our other equations gives us:
$2y = 8y \qquad \text{and} \qquad -2z = 18z$

This is only the case when $y = 0$ and $z = 0$
Plugging into our 4th equation we get:
$x^2 = 16 \qquad \Rightarrow \qquad x = \pm 4$

So, we get two points, $(4, 0, 0)$ and $(-4, 0, 0)$.

If $x = 0$, we move on to the second equation:
$2y = \lambda 8y \qquad \Rightarrow \qquad 2y(1 - 4\lambda) = 0$
If $y \neq 0$, we get $\lambda = 1/4$, and plugging into our other equation gives us:
$-2z = 9/2z$

This is only the case when $z = 0$
Plugging into our 4th equation we get:
$4y^2 = 16 \qquad \Rightarrow \qquad y = \pm 2$

So, we get two points, $(0, 2, 0)$ and $(0, -2, 0)$.

Finally if both $x = 0$ and $y = 0$, we look at our 3rd equation:
$-2z = \lambda 18z \qquad \Rightarrow \qquad 2z(9\lambda + 1) = 0$
We know $z \neq 0$, since we are assuming we are not the point $(0, 0, 0)$, so we get $\lambda = -1/9$, and plugging into our 4th equation we get:
$9z^2 = 16 \qquad \Rightarrow \qquad z = \pm 4/3$

So, we get two points, $(0, 0, 4/3)$ and $(0, 0, -4/3)$.

Now, we have all 6 points, we can plug into $f$.

$f(4,0,0) = 4^2 + 0^2 - 0^2 = 16$
$f(-4,0,0) = (-4)^2 + 0^2 - 0^2 = 16$
$f(0,2,0) = 0^2 + 2^2 - 0^2 = 4$
$f(0,-2,0) = 0^2 + (-2)^2 - 0^2 = 4$
$f(0,0,4/3) = 0^2 + 0^2 - (4/3)^2 = -16/9$
$f(0,0,4/3) = 0^2 + 0^2 - (-4/3)^2 = -16/9$

So, the points $(4,0,0)$ and $(-4,0,0)$ are maximums, $(0,2,0)$ and $(0,-2,0)$ are saddle points, and $(0,0,4/3)$ and $(0,0,-4/3)$ are minimums.