Example: Find & classify the critical points of

$$f(x,y,z) = x^2 + y^2 + 7z^2 - xy - 3yz.$$

Finding critical points:

$$\frac{\partial f}{\partial x} = 2x - y \; ; \; \frac{\partial f}{\partial y} = 2y - x - 3z \, , \; \frac{\partial f}{\partial z} = 14z - 3y$$

For which (x_0, y_0, z_0) are these expressions all 0?

$$0 = 2x_0 - y_0$$
$$0 = 2y_0 - x_0 - 3z_0$$
$$0 = 14z_0 - 3y_0$$

$\Rightarrow$ only possible if $x_0 = y_0 = z_0 = 0$. (check this)

Compute the Hessian at (x_0, y_0, z_0):

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1 \qquad \frac{\partial^2 f}{\partial x \partial z} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial y \partial z} = -3$$

$$\frac{\partial^2 f}{\partial z^2} = 14$$

matrix of $\mathbb{H}f(0,0,0)$ is

$$\Rightarrow \quad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 14 \end{bmatrix}.$$

$B_1 = [2] \quad \Rightarrow \det(B_1) > 0.$

$B_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \Rightarrow \det(B_2) = 4 - (-1)^2 = 3 > 0$

$B_3 = B \quad \Rightarrow \det B = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 14 \end{vmatrix} \begin{matrix} 2 & -1 \\ -1 & 2 \\ 0 & -3 \end{matrix}$

$$2 \cdot 2 \cdot 14 + 3 \cdot 0 + 0 \cdot 3$$
$$- 0 \cdot 2 \cdot 0 - 3^2 \cdot 2 - 14$$
$$= 14(4-1) - 18 = 42 - 18 > 0.$$

so $\mathbb{H}f(0,0,0)$ is positive definite
& $(0,0,0)$ is a local minimum