Notes on Stable Homotopy Theory

Notes by Siyang Liu

February 26, 2022

Contents

TALK 1

Introduction

Speaker: Joseph Helfer

The goal of this talk is to say something about the stable homotopy theory, also known as the homotopy category of spectra $\mathsf{Ho}(\mathbf{Spectra}) = \mathsf{S}$, Quillen's results on complex cobordism, and recent applications in symplectic geometry, which is one of the motivations of this seminar. We start by looking at the homotopy category of topological spaces $\mathsf{Ho}(\mathsf{Top})$ and the derived category $D(R)$ of a given ring R. Invariants in topology, e.g. homology and homotopy groups, are functors on Top which passes to the corresponding homotopy category $\mathsf{Ho}(\mathsf{Top})$. $D(R)$ is the homotopy category of the category of chain complexes of R-modules $\mathsf{Ch}(R)$. The homotopy category of spectra is in some sense a category lying between the above two known examples.

The category of spectra is, roughly speaking, something kind of like both Top and $\mathsf{Ch}(R)$. The object of S are "stable spaces", i.e. $\Sigma^\infty X$ where Σ is the suspension operation. On the other hand, the objects of S are "generalized cohomology theories". We can use geometric constructions and categorical constructions to produce interesting cohomology theories. Complex cobordism is one kind of such generalized cohomology theory.

(1a) Stabilization Let X be a topological space. The **suspension** SX of X is the space $(X\times[0,1]/X\times\{0\})/X\times\{1\}$.

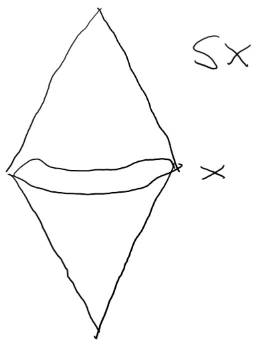

Figure 1.1: suspension

LEt (X, x_0) be a pointed space. The **reduced suspension** ΣX of X is the space $\Sigma X = SX/\{x_0\}\times I$.

1.1 Example. $S\mathbb{S}^n \simeq \mathbb{S}^{n+1}, \Rightarrow \mathbb{S}^n \simeq S^n\mathbb{S}^0$.

If (X, x_0) is well-pointed (e.g. x_0 is a vertex in a CW complex X, or X is a manifold), then $SX \to \Sigma X$ is a homotopy equivalence. (See [Hat00, Chapter 0] for a proof.) In fact, $\Sigma\mathbb{S}^n \simeq \mathbb{S}^{n+1}$, so $\Sigma^n\mathbb{S}^0 \simeq \mathbb{S}^n$.

There're some advantages of reduced suspension:

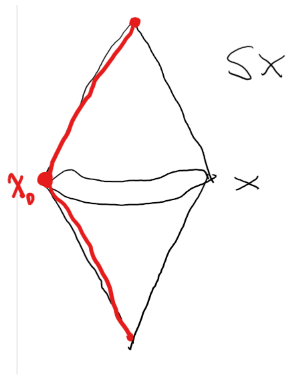

FIGURE 1.2: REDUCED SUSPENSION

- $\Sigma X = \mathbb{S}^1 \wedge X$, where $\wedge$ is the **smash product**, i.e. for pointed spaces Y and Z, the smash product $Y \wedge Z$ is defined to be $Y \wedge Z := Y \times Z / \{y_0\} \times Z \cup Y \times \{z_0\}$.
- Smash product is associative, so that $\Sigma^2 X \simeq \mathbb{S}^1 \wedge (\mathbb{S}^1 \wedge X) = (\mathbb{S}^1 \wedge \mathbb{S}^1) \wedge X = \mathbb{S}^2 \wedge X$, so in general, $\Sigma^n X = \mathbb{S}^n \wedge X$.
- There's an adjunction
$$\mathsf{Map}_*(X, \mathsf{Map}_*(Y, Z)) \simeq \mathsf{Map}_*(X \wedge Y, Z),$$
hence $\mathsf{Map}_*(\Sigma X, Y) \simeq \mathsf{Map}_*(X, \Omega Y)$ where $\Omega Y == \mathsf{Map}_*(\mathbb{S}^1, Y)$.

(1b) Properties of Reduced Suspension

1.2 Theorem. $H_n(X; G) \cong H_{n+1}(\Sigma X; G)$ and $H^n(X; G) \cong H^{n+1}(\Sigma X; G)$

Proof. Write SX as $CS \cup_X CX$, then use Mayer-Vietoris. □

1.3 Theorem (Freudenthal Suspension Theorem)**.** $\pi_n(X) \cong \pi_{n+1}(\Sigma X)$ for n large enough.

The isomorphism comes from the following: $\pi_n(X) = [\mathbb{S}^n, X] \to [\Sigma \mathbb{S}^n, \Sigma X] \simeq [\mathbb{S}^{n+1}, \Sigma X]$. i.e. the sequence of groups
$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$
stabilizes. For a proof, see [Hat00, Section 4.2]. More generally, for any finite CW complex Y, $[\Sigma^k Y, \Sigma^k X]$ stabilizes.

This is the first stable phenomena, and now we'll define

1.4 Definition. The ***n*th stable homotopy group** is given by $\pi_n^s(X) := \mathsf{colim}_k\, \pi_{n+k}(\Sigma^k X)$.

1.5 Remark. *$\pi_n(\mathbb{S}^m)$ are notoriously difficult to compute, but $\pi_n^s := \pi_n^s(\mathbb{S}^0)$ is somewhat easier, and much of stable homotopy theory is dedicated to this.*

The "stabilized spaces" "$\Sigma^\infty X$" should have well-defined H_*, H^*, π_*^s.

1.6 Definition. The ***S*-category** has

- Objects finite CW complexes;
- $\mathsf{Hom}(X, Y) := \mathsf{colim}_n[\Sigma^n X, \Sigma^n Y]$.

This is a first approximation to the homotopy category

1.7 Remark. *1) This category is additive. For any X, Y, $[\Sigma X, Y] \simeq [X, \Omega Y]$ is a group (for the same reason $\pi_1(Y)$ is), and if we suspend twice, then $[\Sigma^2 X, Y] \simeq [X, \Omega^2 Y]$ is an abelian group (as $\pi_2(Y)$ is, also $[\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y]$ is a homomorphism). Hence $\mathsf{Hom}(X, Y)$ is an abelian group and $\mathsf{Hom}(X, Y) \times \mathsf{Hom}(Y, Z)$ is bilinear.*

2) *Also, it's* **graded**: *we have groups*

$$\mathrm{Hom}(X,Y)_n := \mathrm{Hom}(\Sigma^n X, Y)$$

s.t. $\mathrm{Hom}(X,Y) = \mathrm{Hom}(X,Y)_0$ *and* $\mathrm{Hom}_*(X,Y) \otimes \mathrm{Hom}_*(Y,Z) \to \mathrm{Hom}(X,Z)$ *is a graded morphism.*

3) *The original motivation of introducing this category, due to Spanier and Whitehead, is a notion of "duality": objects in the* S*-category have a "dual"* DX*. This recovers the Alexander duality theorem* $\tilde{H}_k(\mathbb{S}^n \setminus K) \cong \tilde{H}^{n-k-1}(K)$ *for "good" compact* $K \subseteq \mathbb{S}^n$ *([SW55]) and the Poincaré duality([Ati61]).*

If we define a "stable object" $\boldsymbol{X}$ to be a sequence of pointed spaces $X_n \in \mathrm{Top}_*$ with maps $\Sigma X_n \to X_{n+1}$, we can define

$$\pi_n^s(\boldsymbol{X}) := \operatorname*{colim}_k \pi_{n+k}(X_k),$$

where $\pi_n(X_0) = [\mathbb{S}^n, X_0] \to [\mathbb{S}^{n+1}, \Sigma X_1] \xrightarrow{f_1} [\mathbb{S}^{n+1}, X_1] \to \cdots$. This recovers $\pi_n^s(X)$ by taking

$$\boldsymbol{X} = \Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \cdots\}$$

with $\Sigma X_n \xrightarrow{f_n = \mathrm{id}} X_{n+1}$.

(1c) Cohomology Theories Recall the Eilenberg-Steerod axioms for (reduced) cohomology theory: A cohomology theory is a sequence of contravariant functors $(h_n : \mathrm{Top}_* \to \mathrm{Ab}, \alpha_n)$ such that

- (homotopy invariance) h_n is invariant under homotopy equivalence, i.e. it defines a functor $h_n : \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ab}$;
- (suspension isomorphism) $h_n(-) \xrightarrow[\alpha_n]{\simeq} h_{n+1}(\Sigma -)$;
- For a CW-pair (X,A), $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact;
- (additivity) $h_n(\bigvee_{i\in I} X_i) \xrightarrow{\sim} \prod_{i\in I} h_n(X_i)$;
- (dimension axiom) $h_n(\mathrm{pt}) \simeq 0$ for all $n \neq 0$ and $h_0(\mathrm{pt}) \simeq G$ for some abelian group G.

1.8 Theorem. Any (h_n, α_n) satisfying these axioms is isomorphic to $H^*(-, G)$.

1.9 Definition. A **generalized(extraordinary) cohomology theory** is a datum (h_n, α_n) as above, satisfying everything except the dimension axiom.

There are corresponding axioms for homology, and definition of generalized homology theory.

1.10 Example.
- *The first one to be discovered is complex K-theory:* $K^0(X) := \{$*complex vector bundles* $E \to X\}/\sim$ *with* $\oplus$ *as addition and* $\otimes$ *as multiplication.*

  $K^{-2}(X) = K^0(\Sigma^2 X) \simeq K^0(X)$ *by Bott periodicity theorem, which means* $K^{-2n}(X) \simeq K^0(X)$*, so now we can define* $K^{2n}(X) \simeq K^0(X)$*, and* $K^{2n-1}(X) := K^{2n}(\Sigma X) = K^0(\Sigma X)$*.*

  1.11 Theorem. This is a generalized cohomology theory.

- *Given a space* X*, we define the* **bordism group** *of* X*,* $\Omega_k(X)$*, to be* $\{M \to X | M$ *a* k*-manifold*$\}/$*cobordism with* $\coprod$ *as addition. Here's a picture depicting this:*

  1.12 Theorem. This is a generalized homology theory.

  There's a corresponding cohomology theory as well.

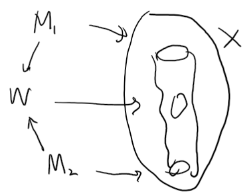

FIGURE 1.3: COBORDISM

- *Actually we have several different versions of bordism theorem. The above is unoriented cobordism, and we also have oriented bordism, framed bordism, and complex bordism. Each of them corresponds to different groups. The unoriented cobordism corresponds to the orthogonal group* O*, oriented bordism corresponds to* SO*, and complex bordism corresponds to* U*.*

- *There're lots of other generalized cohomology theories...*

1.13 Theorem (Brown representability theorem). For any cohomology theory $\{h^n\}$, there's a sequence of spaces $\{E_n\}$ such that

$$h^n \simeq [-, E_n] = \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Top}_*)}(-, E_n) : \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}.$$

<u>**Observation:**</u> The suspension isomorphism $h^n(-) \to h^{n+1}(\Sigma -)$ gives an isomorphism $[-, E_n] \to [\Sigma -, E_{n+1}] \simeq [-, \Omega E_n]$, hence by Yoneda lemma, we have homotopy equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$.(So $E_0 \simeq \Omega^n E_n$). Hence

$$\Sigma E_n \to E_{n+1}$$

and we have a Ω-spectrum $\{E_i\}_i$, which is exactly the data we need to form a stable object. Actually, every spectrum arises this way.

1.14 Example. *For ordinary cohomology* $H^*(-, G)$*, the space* E_n *is called* $K(G, n)$*, the* **Eilenberg-MacLane space***, which has the special property that*

$$\pi_k(K(G,n)) \simeq [\mathbb{S}^k, K(G,n)] = H^n(\mathbb{S}^k; G) \simeq \begin{cases} G, & \textit{if } k = n; \\ 0, & \textit{otherwise.} \end{cases}$$

<u>**Note:**</u> the **Eilenberg-MacLane spectrum** $\boldsymbol{HG} = \{K(G,n)\}_n$ satisfies

$$\pi_n^s(\boldsymbol{HG}) = \begin{cases} G, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\boldsymbol{HG}$ acts like a discrete space.

(1d) Constructing the category. The objects of $\mathsf{Ho}(\mathsf{Spectra})$ are spectra or stable objects as above, and what about the morphisms? Note that for $\mathsf{Ho}(\mathsf{Top})$ and $D(R)$ there're two approaches:

1) Take nice objects (CW complexes or projective/injective complexes) and homotopy classes of maps between them. Recall from Whitehead theorem(see Hatcher) that all topological spaces are weakly equivalent to CW complexes.

2) Take all objects and invert weak equivalences/quasi-isomorphisms. From Whitehead's theorem we can see that these two approaches produce the same homotopy category.

1) F. Adams takes the first approach to contruct this category, in which the notion of map is very complicated. See [Ada74].

2) A more modern approach, as in [BR20], is the second one: $\boldsymbol{X} \to \boldsymbol{Y}$ is a **weak equivalence** if $\pi_n^s(\boldsymbol{X}) \to \pi_n^s(\boldsymbol{Y})$ is an isomorphism for all n. In this case we only need to invert the weak equivalences defined above. But the problem is we need to get some kind of handle on the result of inverting weak equivalences(localization), which leads to the notion of model categories. (There's another approach to the construction, using infinity categories.)

Since this categoryis complicated, B-R give "axioms": starting with a category S,

- $\Sigma^\infty\colon \mathsf{Ho}(\mathsf{Top}) \to \mathsf{S}$;
- Hom sets in S are graded abelian groups;
- Each cohomology theory is represented by an object in S;
- etc.

(1e) More about stable homotopy.

Given an additive and graded category S, for any $E \in \mathsf{S}$, we have a functor

$$\mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}^{\mathbb{Z}}$$

which is a cohomology theory (We can check Eilenberg-Steenrod axioms) and a homology theory: $\pi_*(\Sigma^\infty - \wedge \boldsymbol{E})\colon \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}^{\mathbb{Z}}$. This satisfies the E-S axiom for homology again. Finally, there's an operation called "smash product" $\wedge\colon \mathsf{S} \times \mathsf{S} \to \mathsf{S}$ generalizing $\wedge$ on topological spaces. (It is to spectra what $\otimes$ is to abelian groups.) $\boldsymbol{S} := \Sigma^\infty \mathbb{S}^0$ is to spectra what $\mathbb{Z}$ to abelian groups.

1.15 Definition. A **ring spectrum** is a spectrum $\boldsymbol{E}$ with a morphism $\boldsymbol{E} \wedge \boldsymbol{E} \to \boldsymbol{E}$ in S satisfying unit, associativity(commutativity if we want commutative ring specra).

(1f) A bit more on model categories

Localization. Let $W \subseteq \mathsf{C}$ be categories. The localization means a category $\mathsf{C}[W^{-1}]$ with the universal property that

$$\begin{tikzcd} \mathsf{C} \arrow[r, "l"] \arrow[dr, "F"'] & \mathsf{C}[W^{-1}] \arrow[d, dashed] \\ & \mathsf{D} \end{tikzcd}$$

If F sends morphisms in W to isomorphisms in D.

Different models for the same homotopy theory. We can have different models for the same homotopy theory, for example, topological category and the category of simplicial sets. We say two categories are "Quillen equivalent" if they give the same homotopy theory. The sequential spectra, symmetric and orthogonal spectra we're going to talk about this semester are all Quillen equivalent.

Cohomology theories. Given $\boldsymbol{E} \in \mathsf{S}$, we can define $E^*(\boldsymbol{X}) := [\Sigma^\infty X, \boldsymbol{X}]_{-*}$. For X a CW complex this defines a cohomology theory. In general, given $\boldsymbol{X} \in \mathsf{S}$, $E^*X := [\boldsymbol{X}, \boldsymbol{E}]_{-n}$.

Homology Theories Given $E \in \mathsf{S}$, we define the generalized homology theory $E_*(\boldsymbol{X}) := [\boldsymbol{S} = \Sigma^\infty \mathbb{S}^0, X \wedge E]_* = \pi_*(X \wedge E)$. If X is a space, then we just let $E_*(X) = E_*(\Sigma^\infty X)$.

Closed Model Structure. Smash product on Top extends to a smash product $\mathsf{Ho(Top)} \xrightarrow{\Sigma^\infty} \mathsf{S}$ to a monoidal structure $\mathsf{S} \wedge \mathsf{S} \xrightarrow{\wedge} \mathsf{S}$ with **unit** $\boldsymbol{S}\colon \boldsymbol{S} \wedge \boldsymbol{E} \simeq \boldsymbol{E}$. For given $\boldsymbol{X}, \boldsymbol{Y} \in \mathsf{S}$, we have a **mapping spectrum** $\mathsf{Map}(\boldsymbol{X}, \boldsymbol{Y}) \in \mathsf{S}$ and $[\boldsymbol{X} \wedge \boldsymbol{Y}, \boldsymbol{Z}] \simeq [X, \mathsf{Map}(\boldsymbol{Y}, \boldsymbol{Z})]$.

Ring Spectra and Module Spectra. From the discussions above we know that S behaves in some sense similar to the category of abelian groups: tensor product of abelian groups correspond to smash products, $\mathsf{Hom}_{\mathbb{Z}}$ corresponds to mapping spectra, and S has a unit which is the sphere spectrum $\boldsymbol{S}$.

1.16 Definition. A **ring spectrum** is an object $\boldsymbol{R} \in \mathsf{S}$ with maps $\boldsymbol{R} \wedge \boldsymbol{R} \to \boldsymbol{R}$ and $\boldsymbol{S} \xrightarrow{e} \boldsymbol{R}$ such that the commutative diagrams in the homotopy category describing the associativity and units are satisfied.

One can also demand stronger associativity and commutativity conditions rather than "up to homotopy". For example, "A_∞", "E_∞", and "highly-structured ring spectra".

1.17 Definition. If $\boldsymbol{R}$ is a ring spectrum, then a **module spectrum** $\boldsymbol{M}$ is a spectrum with a map $\boldsymbol{R} \wedge \boldsymbol{M} \to \boldsymbol{M}$ with the condition similar to that of a module.

Note that any spectra is naturally a $\boldsymbol{S}$-module.

(1g) Thom Space Let $V \to X$ be a vector bundle over a topological space X, and assume V admits a norm, then we can construct the associated disk bundle $\mathbb{D}(V)$ of all vectors with norm ≤ 1 and the sphere bundle $\mathbb{S}(V)$ of all vectors with norm 1.

1.18 Definition. The **Thom space** is the quotient space $\mathsf{Th}(V) := \mathbb{D}(V)/\mathbb{S}(V)$.

Another way to describe is that the Thom space is the one-point compatification of each fibre and identify all the ∞s.

<u>**Note:**</u> If $V \simeq \mathbb{R}^n$, then $\mathsf{Th}(V) \simeq \Sigma^n X$.

There're very special Thom spaces, for example, BO_n, which completely classifies real vector bundles up to isomorphism, i.e. given any real vector bundle $V \to X$, there is a unique up to homotopy map $X \to BO_n$ and the universal vector bundle $\gamma_n \to BO_n$ such that we have a map of bundles $\Phi\colon V \to \gamma_n$ over $X \to BO_n$ with the pull-back diagram

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d] & \gamma_n \arrow[d] \\ X \arrow[r] & BO_n \end{tikzcd}$$

The same for BSO_n, which classifies oriented bundles, and BU_n, which classifies complex vector bundles.

MSO_n is the Thom space $\mathsf{Th}(\gamma_n \to BSO_n)$, and similarly MU_n is the Thom space $\mathsf{Th}(\gamma_n \to BU_n)$. MSO is a spectra, called a **Thom spectra**. First of all,

$$MSO = \{MSO_1, MSO_2, \cdots,\} \left| \begin{tikzcd} \gamma_n \oplus \mathbb{R} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] \\ BSO_n \arrow[r] & BSO_{n+1} \end{tikzcd} \right. \Rightarrow \Sigma MSO_n \simeq \mathsf{Th}(\gamma_n \oplus \mathbb{R}) \to MSO_{n+1},$$

and MU is similar, with a little twist that

$$MU = \{MU_1, \Sigma MU_1, MU_2, \Sigma MU_2, \cdots\}$$

(1h) Bordism and Cobordism Given $X \subseteq M$, we have the **Pontrjagin-Thom construction**: there is a map $X \to BO_n$ which classifies the bundle $N_M X \to X$, this map $N_M X \to \gamma_n$ of vector bundles then induces a map of Thom spaces $\mathsf{Th}(N_M X) \to MO_n$, but we then have the "collapse map" $M \to \mathsf{Th}(N_M X)$ by collapsing the complement of $N_M X$ to a point. So a codimension n submanifold is in one-to-one correspondence to maps from M to MO_n, and two submanifolds X and X' are cobordant if and only if the two maps from M to MO_n are **stably homotopic**, i.e.

$$\{\text{submanifolds of } M\}/\text{cobordism} \simeq \operatorname*{colim}_n [M, MO_n].$$

The outcome of this construction is that the homology theory represented by the spectra MO is the **bordism**: $MO_*(X) \simeq \Omega_*(X)$, and similarly, $MO^*(X)$ is called **cobordism**. If X is a manifold, then it's a duality, meaning that the bordism and cobordism groups are isomorphic. Similarly, MSO_* and MSO^* are oriented bordisms and cobordisms.

MU_* and MU^* are complex bordisms and cobordisms.

Observe that $\pi_*(MO) \simeq MO_*(\mathsf{pt}) \simeq \Omega_*$, which is the "cobordism ring".

(1i) MU and complex orientations Suppose we have a vector bundle $V \to X$, then $H^*(\mathsf{Th}(V)) \to H^*(\mathsf{Th}(V_x)) \simeq \mathbb{S}^n$. Then there's a theorem of Thom saying that there is a "Thom class" $u \in H^n(\mathsf{Th}(V))$ which goes to $\pm 1 \in H^n(\mathbb{S}^n)$. The orientation is the same as the existence of such a Thom class.

1.19 Definition. $\boldsymbol{E} \in \mathsf{S}$ is **complex oriented** if for each $\boldsymbol{X} \in \mathsf{Top}$ and $V \to X$ complex vector bundle, there exists a class $u \in H^{2n}(\mathsf{Th}(v)) = E^*(\mathsf{Th}(V))$. ($H\mathbb{Z}$, K and MU are complex oriented)

<u>**Fact:**</u> MU is the universal complex oriented cobordism theory: if $\boldsymbol{E}$ is complex oriented theory, then there exists a map $MU \to \boldsymbol{E}$ inducing that complex orientation.

(1j) Formal Group Laws. If E is complex oriented, we can see by a spectral sequence argument that $E^*(\mathbb{C}P^\infty) \cong E_*[[t]]$, where $E_* = \pi_*(\boldsymbol{E})$. Moreover, $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E_*[[u,v]]$. We can then find a universal class $f(u,v)$, which is a **formal group** over E_*, which means that it satisfies the following conditions:

- $f(a,0) = f(0,a)$;
- $f(f(a,b),c) = f(a,f(b,c))$(inverses are free);

<u>**Fact 2:**</u> $\pi_*(MU)$ is the "Lazard ring".

There're two interesting theories, the Brown-Peterson and Morava K-theory, which is obtained from MU. Finally, there's a theorem by Abouzaid-McLean-Smith:

1.20 Theorem. Assume Y is a projective variety, $Y \to \mathbb{C}P^1$ holomorphic submersion with fiber X, then $H^*(Y;\mathbb{Z}) \cong H^*(X;\mathbb{Z}) \otimes H^*(\mathbb{S}^2;\mathbb{Z})$.

This theorem was know over $\mathbb{Q}$, and the statement involves nothing about homotopy theory. The proof is to first replace $\mathbb{Z}$ by any complex oriented cohomology theory. To do this, they first prove this for MU, and then for BP, and finally for all "$K(n)$-local" cohomology theories. The reason why these all have to do with symplectic geometry is that the Morava K-theory are well-behaved with respect to orbifolds. In symplectic geometry, we have the moduli space of pseudo-holomorphic curves which are orbifolds.

TALK 2

Basics of Model Categories

Speaker: Suraj Yadav

The notion of model category allows us to do abstract homotopy theory.

2.1 Definition. A **model category** is a category C with three classes of morphisms $\mathcal{W}$, the class of weak equivalences, $\mathcal{C}$, the class of cofibrations, and $\mathcal{F}$, the class of fibrations, with the following properties:

1) C is closed under finite limits and colimits;

2) (2 out of 3) given three objects X, Y, Z and a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd},$$

   if any two morphisms are in $\mathcal{W}$, then so is the third.

3) (retracts) The retract of any morphismin $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ is again in $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ respectively. Here we say a morphism $X \xrightarrow{f} Y$ is a **retract** of $U \to V$ if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & U \arrow[r] \arrow[d, "g"] & X \arrow[d] \\ Y \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

   so that the composition of the upper and lower rows are identities.

4) (lifting property) Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

   such that

   a) $f \in \mathcal{W} \cap \mathcal{C}$, $g \in \mathcal{F}$ implies there exists a lifting $H : B \to X$;
   b) If $f \in \mathcal{C}$, $g \in \mathcal{W} \cap \mathcal{F}$, then there exists a lifting $H : B \to X$;

5) (factorization) For any morphism $f : X \to Y$, there are factorizations

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y$$

$$X \xrightarrow{f^1} Z \xrightarrow{f^2} Y$$

of f, where $f_1 \in \mathcal{W} \cap \mathcal{C}$, $f_2 \in \mathcal{F}$, $f^1 \in \mathcal{C}$ and $f^2 \in \mathcal{F} \cap \mathcal{W}$.

2.2 Definition. Let C be a model category. An object $X \in$ C is **fibrant** if $X \to *$ is a fibration; $Y \in$ C is **cofibrant** if $f : * \to Y$ is a cofibration.

For any object $X \in$ C, we have a unique morphism $\varnothing \to X$ which factors through a cofibrant object Y such that $Y \to X$ is the trivial fibration. (Here we say a fibration is **trivial** if it's both a fibration and a weak equivalence.) We want to get a cofibrant and fibrant object Z which is weakly equivalent to X, so we consider the morphism $Y \to *$, and consider the factorization $Y \to Z \to *$ where $Y \to Z$ is the trivial cofibration and $Z \to *$ is a fibration, so Z is both cofibrant and fibrant. Now we want to show that Z is weakly equivalent to X.

2.3 Example. *For the category of topological spaces, we have two kinds of model structures. One of them is called the* **Serre model structure***, where we make weak equivalences to be weak homotopy equivalences, fibrations to be Serre fibrations, i.e. we have the lifting property for all maps $A \to A \times [0, 1]$ where A is a CW complex.*

Another structure is the **Hurewicz moel structure***, weak equivalences are homotopy equivalences, and fibrations have lifting properties with respect to maps $A \to A \times [0, 1]$ where A is any topological space.*

2.4 Example. *The category of simplicial sets also admits a model structure. Let $\triangle$ be the cosimplex category whose objects are $[n] = \{0, 1, \cdots, n\}$ the set of natural numbers, and morphisms order-preserving maps $[n] \to [m]$. We have a class of special morphisms $d^i : [n] \to [n+1]$ defined by $d^i(k) = k$ if $k < i$, and $d^i(k) = k + 1$ if $k \geq i$, and $s^j : [n+1] \to [n]$ given by $s^j(k) = k$ for $k < j$ and $s^k(k) = k - 1$ for $k \geq j$.*

2.5 Definition. A **simplicial set** X is a functor

$$X : \triangle^{op} \to \mathrm{Set}.$$

This means that a simplicial set is a data $[n] \mapsto X_n$ with maps $x_{n+1} \to X_n \to X_{n-1}$ with the given compatibility conditions. Simplicial sets are combinatorial data of topological spaces. Simplicial sets are representable functors with representation $\Delta^n := \mathsf{Hom}(-, [n])$. Let sSet be the category of simplicial sets, then we have a **geometric realization** functor

$$|-| : \mathrm{sSet} \to \mathrm{Top}$$

which is adjoint to the singular functor $\mathsf{Sing}_* : \mathrm{Top} \to \mathrm{sSet}$. For the standard n-simplex Δ^n, $|\Delta^n|$ is just the standard n-simplex $\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \Sigma x_i = 1,\ x_i \geq 0\}$.

For any topoogical space Y, we define $(\mathsf{Sing}_* Y)_n = \mathsf{Hom}_{\mathrm{Top}}(|\triangle^n|, Y)$ and we can check that this actually defines a simplicial set. Although these two categorys sSet and Top are not equivalent, their homotopy categories are equivalent.

The model structure on sSet is given as follows: $X \to Y$ is a weak equivalence of simplicial sets if $|X| \to |Y|$ is a weak homotopy equivalence of topological spaces, $X \to Y$ is a cofibration if $X_n \to Y_n$ is a monomorphism for any n, and $X \to Y$ is a fibration if it has lifting property with respect to all cofibrations.

2.6 Example. *For any ring R, the category of chain complexes $\mathsf{Ch}(R)$ is the category with objects chain complexes $\cdots \to C_2 \to C_1 \to C_0 \to 0$. This category admits natural model structures: a morphism $f_\bullet : C_\bullet \to D_\bullet$ is a weak equivalence if the induced map on homology $f_* : H_*C \to H_*D$ is an isomorphism. $f_\bullet$ is a cofibration if $f_n : C_n \to D_n$ is injective with projective cokernel. $f_\bullet$ is a fibration if $f_n : C_n \twoheadrightarrow D_n$ is surjective.*

This is the projective model structure on $\mathsf{Ch}(R)$*, since the cofibrant objects in this structure are chain complexes of projective modules, and the cofibrant replacement is just the same as taking projective resolutions.*

Another model structure is the so-called **injective model structure***, where fibrations are degreewise surjective maps with injective kernels and cofibrations degreewise injective maps. Similarly, fibrant replacements in this category are injective resolutions.*

Now we proceed to define homotopy category of a model category.

2.7 Definition. Consider the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X \sqcup X \end{tikzcd}$$

The identity map $X \xrightarrow{\mathsf{id}} X$ gives a natural map $\tau : X \sqcup X \to X$. A **cylindrical object** is the following data

$$X \sqcup X \xrightarrow{C} \mathsf{Cyl}(X) \xrightarrow{W_F} X$$

where C is a cofibration and W_F is the trivial fibration.

The motivation of this cylinder object is the usual cylinder $A \times [0,1]$ for a given topological space A.

2.8 Lemma. *Suppose we have a map* $X \xrightarrow{f} Y$ *which is a weak equivalence, then we have a natural induced map* $\mathsf{Cyl}(X) \to \mathsf{Cyl}(Y)$ *which is also a weak equivalence fitting into the commutative diagram*

$$\begin{tikzcd} X \sqcup X \arrow[r] \arrow[d] & Y \sqcup Y \arrow[r, "C"] & Y \sqcup Y \arrow[d, "W_F"] \\ \mathsf{Cyl}(X) \arrow[r, "W_F"] \arrow[rru, dashed] & X \arrow[r, "f"] & Y \end{tikzcd}$$

2.9 Definition. Two morphisms $f, g : X \to Y$ are **left homotopic** if there exists a morphism $H : \mathsf{Cyl}(X) \to Y$ such that $H_0 i_0 = f$ and $H i_1 = g$. Here $(i_0, i_1) : X \sqcup X \to \mathsf{Cyl}(X)$ are the two inclusion maps of X into $\mathsf{Cyl}(X)$.

The problem is, in a general model category, the notion of homotopy equivalence is not an equivalence relation. Now we give a dual construction.

2.10 Definition. Taking any object $Y \in \mathsf{C}$, the **path object** of Y is the factorization of $Y \xrightarrow{\Delta} Y \times Y$

$$Y \xrightarrow{W_C} PY \xrightarrow{(e_0, e_1)} Y \times Y$$

where $Y \to PY$ is the trivial cofibration and $PY \to Y \times Y$ is the fibration.

2.11 Definition. $f, g : X \to Y$ are **right homotopic** if there exists a morphism $H : X \to PY$ such that $e_0 H = f$ and $e_1 H = g$.

Now we can define the homotopy category of a given model category C. Given $X, Y \in \mathsf{C}$, we consider the cofibrant-fibrant replacement of both X and Y, i.e. X^{cf} and Y^{cf}, and consider the set of morphisms $\mathsf{Hom}_{\mathsf{C}}(X^{cf}, Y^{cf})$. We use the following fact:

(a) If X is cofibrant, then left homotopy is an equivalence relation on $\mathsf{C}(X, Y)$;

(b) If Y is fibrant, then right homotopy is an equivalence relation on $\mathsf{C}(X, Y)$;

(c) If X is cofibrant and Y is fibrant, then $f, g : X \to Y$ are left homotopic if and only if they are right homotopic.

Therefore we can define the **homotopy category** $\mathsf{Ho}(\mathsf{C})$ of C to be the category with objects those objects in C and morphism sets $\mathsf{Hom}_{\mathsf{C}}(X^c, Y^f)/\sim$ where $f \sim g$ if and only if they are left or right homotopic.

In this homotopy category, we know that if $f : X \to Y$is a weak equivalence with X, Y cofibrant-fibrant, then f is a homotopy equivalence.

2.12 Lemma. *$f : X \to Y$ is an isomorphism in $\mathsf{Ho}(\mathsf{C})$ if and only if f is a weak equivalence in C.*

Therefore the notion of "localization at $\mathcal{W}$" in C is the same as the homotopy category of C.

Finally, we define the Quillen functors:

2.13 Definition. $F : \mathsf{C} \to \mathsf{D}$ is called **left Quillen** if it preserves cofibrations and trivial cofibrations, and **right Quillen** if it preserves fibrations and trivial fibrations.

2.14 Definition. $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ a pair of functors. We say they are **Quillen adjunction** if they are adjunctions and one of the following conditions hold:

1) F and G have to be left Quillen and right Quillen respectively;

2) F is left Quillen;

3) G is right Quillen;

4) F preserves trivial cofibrations and cofibrations between cofibrant objects.

5) G preserves trivial fibrations and fibrations between fibrant objects.

2.15 Example. *$|-| : \mathsf{sSets} \rightleftarrows \mathsf{Top}_* : \mathsf{Sing}_*$ are Quillen adjunct.*

2.16 Definition. Let $F : \mathsf{C} \to \mathsf{D}$ be a left Quillen functor, then $LF : \mathsf{Ho}(\mathsf{C}) \to \mathsf{Ho}(\mathsf{D})$ is given by $LF(X) := F(X^c)$. Similarly, if $G : \mathsf{D} \to \mathsf{C}$ is right Quillen, then we can define $RF : \mathsf{Ho}(\mathsf{D}) \to \mathsf{Ho}(\mathsf{C})$ by $RG(X) := G(X^f)$.

2.17 Example. *The sheaf cohomology $H^*(X, -)$ is the example of RG for G the global section functor.*

So LF and RF are generalizations of left and right derived functors in the model category.

2.18 Definition. A Quillen adjunction

$$F : \mathsf{C} \rightleftarrows \mathsf{D} \ : G$$

is a **Quillen equivalence** if

$$LF : \mathsf{Ho}(\mathsf{C}) \rightleftarrows \mathsf{Ho}(\mathsf{D}) \ : RG$$

is an equivalence of categories.

For example, the projective and injective model structures on $\mathsf{Ch}(R)$ are Quillen equivalent.

TALK 3

Basics of Homotopy Theory

Tianle Liu

Today we'll talk about basics of homotopy theory, following the last talk about model category.

(3a) Cofibrations and Fibrations. We have introduced cofibrations and fibrations in a general model category, and now let's see how they're defined in the category of topological spaces.

3.1 Definition. $i\colon A \to X$ is a **cofibration** if it satisfies **homotopy extension property**: for any continuous maps $f\colon X \to Y$ and $h\colon A \times I$ making the diagram commutative,

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "j"] \arrow[dd] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

there exists a map $X \times I \to Y$ filling in the commutative diagram.

With the notion of mapping cylinder, we can make things simpler:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A \times I \arrow[d] \\
X \arrow[r] & Mi
\end{tikzcd}$$

Here Mi is the mapping cylinder of i.

Dually we have the notion of fibration:

3.2 Definition. A surjective map $p\colon E \to B$ is called a **fibration** if it satisfies the **covering lifting property**:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & E \arrow[d] \\
Y \times I \arrow[r, "h"] \arrow[ur, dashed] & B
\end{tikzcd}$$

With the notion of path space, it's equivalent to

$$\begin{tikzcd}
E \arrow[dd] & & E^I \arrow[ll] \arrow[dd] \\
& Y \arrow[ul] \arrow[ur] \arrow[dr] & \\
B & & B^I \arrow[ll]
\end{tikzcd}$$

where $B^I = \mathsf{Maps}(I, B)$. With the notion of path object Np of p(the pull-back), we have

$$\begin{tikzcd}
E \arrow[dd, "p"] \arrow[dr] & & E^I \arrow[ll] \arrow[dd] \\
& Np \arrow[ur, dashed] \arrow[dr] & \\
B & & B^I \arrow[ll]
\end{tikzcd}$$

These are the Hurewicz fibrations and cofibrations as mentioned last week.

Recall from last week that any map can be decomposed into a composition of a weak equivalence followed by a cofibration or a fibration followed by a weak equivalence. Now we make this decomposition precise in the topological category. Given $f : X \to Y$, we can factor f via the mapping cylinder

$$X \xrightarrow{\mathsf{cof}} Mf \xrightarrow{\sim} Y,$$

where $Mf \to Y$ is a weak homotopy equivalence. Dually, we can decompose f as

$$X \xrightarrow{\sim} Nf \xrightarrow{\mathsf{fib}} Y.$$

(3b) Suspension and Loop Construction. Consider the category of pointed topological spaces Top_*, i.e. we choose a base point for each topological space X, and we take the homotopy pull-back of the diagram

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

To see why this is the usual loop object, notice that this is the homotopy pull-back, so we can replace maps by the fibrant or cofibrant objects. For example, we can replace $* \to X$ by the path fibration $PX \to X$, then we get

$$\begin{tikzcd}
& * \arrow[d] \\
PX \arrow[r] & X
\end{tikzcd}.$$

The suspension is given by the homotopy pushout of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \\
* &
\end{tikzcd}$$

and we can replace both $*$ by the mapping cylinder Mi, then the pushout is exactly the suspension ΣX.

(3c) Fiber and Cofiber Sequences.

3.3 Definition. We say a sequence $Z \to X \to Y$ is a **fiber sequence** if the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

is a homotopy pullback. In this case, we say $Z = \mathsf{fib}(f)$ is the fiber of f.

If we look at the pull-back square

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & * \arrow[d] \\ \mathsf{Fib}(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd}$$

then the top left corner should be the loop space of Y. If we repeat this procedure, we would get the loop space ΩX. Repeated this process, we would get a long exact sequence

$$\cdots \Omega^2 Y \to \Omega \mathsf{Fib}(f) \to \Omega X \to \Omega Y \to \mathsf{Fib}(f) \to X \to Y.$$

in the sense that each consecutive three arrows are fiber sequences. Similarly, we can get the **cofiber sequence** $X \to Y \to \mathsf{cob}(f)$ if the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & \mathsf{cob}(f) \end{tikzcd}$$

is a homotopy pushout square. With the similar construction, we get a long exact sequence

$$X \to Y \to \mathsf{cob}(f) \to \Sigma X \to \Sigma Y \to \Sigma \mathsf{cob}(f) \to \cdots$$

Write $\langle -, - \rangle$ for the homotopy mapping space(i.e. the mapping space modulo homotopy), and given any topological space Z, we get a sequence of spaces

$$\cdots \to \langle Z, \Omega^2 X \rangle \to \langle Z, \Omega^2 Y \rangle \to \langle Z, \Omega \mathsf{fib}(f) \rangle \to \langle Z, \Omega X \rangle \to \cdots$$

3.4 Definition. For any space X, define $\pi_0(X) = [\mathbb{S}^0, X]$ the space of connected components of X.

If we apply π_0 to the sequence above, we get long exact sequence of sets

$$\cdots \to \pi_0 \langle Z, \Omega^2 X \rangle \to \pi_0 \langle Z, \Omega^2 Y \rangle \to \pi_0 \langle Z, \Omega \mathsf{fib}(f) \rangle \to \pi_0 \langle Z, \Omega X \rangle \to \cdots$$

and similarly for $\langle -, Z \rangle$.

3.5 Definition. We define the ***n*-th homotopy group** of X to be $\pi_n(X) = \pi_0(\Omega^n X)$.

So we know that $\pi_1(X) = [\mathbb{S}^0, \Omega X] = [\Sigma \mathbb{S}^0, X] = [\mathbb{S}^1, X]$ using the isomorphism $[\Sigma X, Y] = [X, \Omega Y]$, then $\pi_2(X) = [\mathbb{S}^0, \Omega^2 X] = [\Sigma \mathbb{S}^0, \Omega X]$ is abelian, so we get $\pi_1(X)$ is a group and $\pi_n(X)$ is an abelian group for $n \geq 2$.

Now in the sequence above, if we choose $Z = \mathbb{S}^0$, then we have an exact sequence

$$\cdots \to [\mathbb{S}^0, \Omega Y] = \pi_1(Y) \to [\mathbb{S}^0, \Omega X] = \pi_1(X) \to \pi_0(\mathsf{fib}(F)) \to \pi_0(X) \to \pi_0(Y)$$

of homotopy groups, with $f : X \to Y$ a fibration. This is the usual long exact sequence of a homotopy group under the condition that $f : X \to Y$ is a fibration.

(3d) CW Complexes. The first theorem here is the CW approximation:

3.6 Definition. $f: X \to Y$ is a **weak homotopy equivalence** if $\pi_i(f): \pi_i(X) \to \pi_i(Y)$ is an isomorphism of homotopy groups.

3.7 Theorem (CW Approximation)**.** For any topological space X, there exists a CW complex Y and a morphism $X \to Y$ such that f is a weak equivalence.

Recall from last time that we have a Quillen equivalence $|-|: \mathsf{sSets} \rightleftarrows \mathsf{Top}_* : \mathsf{Sing}_*$, which tells us that for each topological space X, the counit map $|\mathsf{Sing}(X)| \to X$ is a weak equivalence w.r.t. Serre model structure, and is hence a weak homotopy equivalence.

Another thing about CW approximation is the cellular approximation theorem.

3.8 Definition. Let $f: X \to Y$ be a map between two CW complexes, then we say f is **cellular** if $f(X^n) \subseteq Y^n$, i.e. the image of the n-skeleton of X is contained in the n-skeleton of Y.

3.9 Theorem (Cellular Approximation Theorem)**.** Any continuous map $f: X \to Y$ between topological spaces X and Y is homotopic to a cellular map.

The last theorem here is Whitehead's theorem, which says that

3.10 Theorem (Whitehead Theorem)**.** Let X, Y be CW complexes, and $f: X \to Y$ a weak homotopy equivalence, then f is a homotopy equivalence.

Proof. Again we consider the Quillen model category $\mathsf{Top}_{\mathsf{Quillen}}$. Note that in this model structure, all the CW complexes are fibrant and cofibrant. There's a theorem stated last time that weak equivalences between fibrant and cofibrant objects are actually homotopy equivalences. □

(3e) Freudenthal Suspension Theorem Now we state a theorem which is important in stable homotopy theory. The idea is that we want to study the suspension map

$$\Sigma: [X, Y] \to [\Sigma X, \Sigma Y],$$

and because of the adjunction between suspensions and loops, we have $[\Sigma X, \Sigma Y] = [X, \Omega\Sigma Y]$, so we only need to study the map $X \to \Omega\Sigma X$ induced from the identity map. Note that $\Omega\Sigma X$ is a topological group up to homotopy, and we can actually make it into a real topological monoid called **Moore space**, and if we take the free monoid $J(X)$ generated by X, and take the map $J(X) \to \Omega\Sigma X$,($J(X)$ is called the **James construction**. Explicitly, we take $J_n(X)$ to be the n-th Cartesian product X^n quotient by the relations $(x_1, \cdots, x_{k-1}, e, x_k, \cdots, x_{m-1}) \sim (x_1, \cdots, x_{k-1}, x_k, \cdots, x_{m-1})$. For more information about the James construction, see [Hat00, Section 4.J].

3.11 Theorem. $J(X) \simeq \Omega\Sigma X$.

Now the problem reduces to considering the natural mapping space $[X, J(X)]$. If X is $(n-1)$-connected CW complex(by CW approximation, it always suffices to consider CW complexes), i.e. $\pi_i(X) = 0$ for $i \leq n-1$, then we can regard $X^{(n-1)}$ to be a point homotopically, then we can intuitively imagine that $J(X) \setminus X$ has cells of dimension at least $2n$, and we can conclude by this argument that $X \to J(X)$ is $(2n-1)$-connected. This means that $X \to \Omega\Sigma X$ is $(2n-1)$-connected, so if $\dim Y < 2n-1$, then $[Y, X] \xrightarrow{\sim} [Y, \Omega\Sigma X] = [\Sigma Y, \Sigma X]$ and if $\dim Y = 2n-1$, then $[Y, X] \twoheadrightarrow [\Sigma Y, \Sigma X]$, which can be proved via long exact sequence and cellular approximations. That is, for $\dim Y < 2n-1$, the map is injective by fiber sequence, and for $\dim Y \leq 2n-1$, the map is surjective by cellular approximation. Let's conclude the Freudenthal suspension theorem:

3.12 Theorem (Freudenthal)**.** Let X, Y be topological spaces with X

(3f) Hurewicz Theorem Finally we talk about Hurewicz theorem. Firstly we give an alternative definition of homology theory: for any topological space X, we define $H_n(X) = \pi_n(\mathsf{sp}(X))$, where

$$\mathsf{sp}(X) = \varinjlim X^n/\sigma_n$$

where $\sigma_n \curvearrowright X^n$ acts by permutation. Then we get a natural map $f: X \to \mathsf{sp}(X)$ which gives a map

$$\pi_n(f): \pi_n(X) \to \pi_n(\mathsf{sp}(X)) = H_n(X).$$

3.13 Theorem (Hurewicz). If X is $(n-1)$-connected, then the map $X \to \mathsf{sp}(X)$ is $(n+1)$-connected.

As a Corollary, we have

3.14 Corollary. $\pi_i(X) \to H_i(X)$ is an isomorphism if $i = n$, and is surjective if $i = n+1$.

(3g) Cohomology. Finally, we just quickly review the construction of cohomology theory. Similar to homology theory, we can compute our cohomology group $H^n(X; G)$ via homotopy groups

$$H^n(X; G) \cong [X, K(G, n)],$$

where $K(G, n)$ is the **Eilenberg-MacLane space**. Then we say the functor $H^n(-; G)$ is representable with representation $K(G, n)$. Actually we have a summary of this phenomena:

3.15 Theorem (Brown Representability). A functor $F: \mathsf{Ho}(\mathsf{Top}_*)^{op} \to \mathsf{Sets}_*$ is representable if and only if F is a **Brown functor**, i.e.

1) It takes coproducts to products;

2) It takes homotopy pushouts to weak pullbacks(we don't need the uniqueness for the pull-back).

One example is the cohomology functor, and another example is the functor $\mathsf{Bun}_G(X) = \{G - \text{bundles on } X\}/\sim$, and it's represented by some $BG \in \mathsf{Ho}(\mathsf{Top})$, which is called the **classifying space** of G.

A final thing is that if F happens to be a cohomological functor, i.e. it satisfies the cohomological axiom, then what space it should represent? Let E^n be the cohomological functor and L^n the spaces they represented, then we have $E^n(X) = [X, L^n]$ and $E^{n+1}(\Sigma X) = E^n(X) = [\Sigma X, L^{n+1}] = [X, \Omega L^{n+1}]$, so we should have $L^n \simeq \Omega L^{n+1}$. Now it goes into the notion of spectrum. We call $\{L^n\}$ an **Ω-spectrum**.

TALK 4

Basic of Stable Homotopy Theory

Haoyang Liu

We start with a review of some result Tianle talked about last time. In today's talk, when I talk about the category of topological spaces, it refers to the category of CW complexes, and pointed category of topological spaces refers to pointed CW complexes. When we talk about pointed space (X, x_0), we say X has a **non-degenerate base point** if the inclusion $x_0 \to X$ is an h-cofibration in Top.

4.1 Definition. We say (X, x_0) is **k-connected** if it is path connected and $\pi_i(X, x_0)$ for all $1 \le i \le k$.

A pointed map $f : X \to Y$ is a **k-equivalence** if for all $x_0 \in X$, $\pi_k(X, x_0) \xrightarrow{\pi_k(f)} (Y, f(x_0))$ is an isomorphism for $0 \le n < k$, and surjective when $n = k$.

As a convention, every pointed topological space is (-1)-connected. Now we recall the Freudenthal suspension theorem from last time:

4.2 Theorem (Freudenthal Suspension Theorem)**.** Let $k \in \mathbb{N}$ and X k-connected with non-degenerate basepoint, then the map

$$\pi_n(X) = [\mathbb{S}^n, X] \xrightarrow{\Sigma} [\Sigma\mathbb{S}^n, \Sigma X] = \pi_{n+1}(\Sigma X)$$

is an isomorphism if $n < 2k + 1$ and surjection if $n = 2k + 1$.

4.3 Example. *Note that the degree in this theorem is really sharp. For example, if we look at the map*

$$\pi_2(\mathbb{S}^1) \to \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$$

where $\pi_2(\mathbb{S}^1) = 0$, $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ and $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. The theorem above tells us the theorem holds only in degree 0, and we see directly from the sequence that $\pi_2(\mathbb{S}^1) \to \pi_3(\mathbb{S}^2)$ fails to be a surjection, and if we look at $\pi_2(\mathbb{S}^3) \to \pi_3(\mathbb{S}^4)$, the theorem tells us that this map is actually a surjection.

We can get a slightly different form of the suspension theorem, which leads us to the so-called "stable phenomenon":

4.4 Corollary (Freudenthal Suspension Theorem, Restated)**.** Assume X is a topological space with a non-degenerate base point x_0, $a, b \in \mathbb{N}$ with $b < a - 1$, then the suspension map

$$\pi_{a+b}(\Sigma^a X) \to \pi_{a+b+1}(\Sigma^{a+1} X)$$

is an isomorphism.

4.5 Remark. *If we fix b and let $a > b + 1$, then the map in Corollary 4.4 is an isomorphism for all such a, which gives us an idea why we call this a stable phenomena, and we can define the* **stable homotopy group** *as*

4.6 Definition. For X a pointed CW complex and $n \in \mathbb{N}$, we write the **stable homotopy group** of X as

$$\pi_n^{\mathsf{stable}}(X) = \operatorname*{colim}_a \pi_{n+a}(\Sigma^a X).$$

When we pick $a > n+1$, then we can see that every morphism in this directed system is an isomorphism, so the system is terminal and we get

$$\operatorname*{colim}_a \pi_{n+a}(\Sigma^a X) = \pi_{2n+2}(\Sigma^{n+2}(X)).$$

This is the first stable phenomena we have seen here.

4.7 Theorem. Let X, Y be pointed CW complexes with Y k-connected, then the suspension map

$$\Sigma\colon [X,Y] \to [\Sigma X, \Sigma Y]$$

is surjective if X is of diemension $2k+1$ and bijective if X has dimension $< 2k+1$.

This is something like the generalization of the Freudenthal theorem. Just like how we define the stable homotopy group, we can define

4.8 Definition. The set of **stable homotopy class** of pointed maps $X \to Y$ is

$$[X,Y]^s := \operatorname*{colim}_a [\Sigma^a X, \Sigma^a Y].$$

Reduced Cohomology Theory The construction of reduced cohomology theory also leads us to think about the objects called spectra.

4.9 Definition. **Reduced homology theory** is a functor $\tilde{E}_*$ from pointed CW complexes to graded abelian groups Ab satisfying the following axioms:

(1) If $f \simeq g$, then $f_* = g_*$;

(2) For a CW pair (X, A) we have a boundary map $\partial_*\colon \tilde{E}_*(X/A) \to \tilde{E}_{*-1}(A)$;

(3) Let $i\colon A \hookrightarrow X$, $q\colon X \to X/A$ and ∂_* together gives a long exact sequence;

(4) Given a family of spaces X_α and $i_\alpha\colon X_\alpha \to \bigvee_\alpha X_\alpha$ induces an isomorphism $\bigoplus\limits_\alpha \tilde{E}_*(X_\alpha) \to \tilde{E}_*(\bigvee\limits_\alpha X_\alpha)$.

Another thing we can say is once we have this long exact sequence in definition 4.9, we can consider the CW pair (CX, X) and get

4.10 Lemma. $\tilde{E}_*(\Sigma X) \cong \tilde{E}_*(X)$.

4.11 Example. *A standard example for reduced homology theory is the reduced singular homology $\tilde{H}_*(X)$; The stable homotopy groups $\pi_n^{\mathrm{stable}}(X)$ also defines a reduced homology theory. The reason is that, firstly, $\pi_n^{\mathrm{stable}}(X)$ is an abelian group by the construction, and we can verify the axioms of a reduced homology theory.*

Note that $H_*(D^2, S^1) \cong H_*(D^2/S^1, *)$. However, the unstable homotopy group does not have this property. For example, $\pi_3(D^2, S^1) = 0$, but $\pi_3(D^2/S^1, *) \neq 0$.

We have also a dual definition for reduced cohomology theory, but since it's almost the same as homology theory, we just omit the formal definitions here, and we have the lemma

4.12 Lemma. $\tilde{E}^{*+1}(\Sigma X) \cong \tilde{E}^*(X)$.

4.13 Remark. *If we have two cohomology theories $\widehat{E}^*$, $\widetilde{E}^*$, we say they're isomorphic if we have a bijective natural transformation $\widehat{E}^* \to \widetilde{E}^*$ that is compatible with coboundary maps.*

Now we can introduce the notion of spectra. We start with the famous theorem mentioned by Helfer:

4.14 Theorem (Brown Representability Theorem)**.** $\tilde{E}^*$ is represented by $\{K_n\}_{n\in\mathbb{Z}}$, which implies $\tilde{E}^n(X) = [X, K_n]$.

This theorem gives us a way to try to think about the cohomology theory by some set of topological spaces. When we look at the axioms of reduced cohomology theory, we would have more relations between these topological spaces $\{K_n\}$:

4.15 Corollary. $\tilde{E}^*(X) \cong \tilde{E}^{*+1}(\Sigma X)$.

This means that $[X, K_n] \cong [\Sigma X, K_{n+1}]$. By the suspension-loop duality we have $[\Sigma X, K_{n+1}] = [X, \Omega K_{n+1}]$, so we get **structure maps**

$$\alpha_n : K_n \to \Omega K_{n+1}.$$

which are weak homotopy equivalences. Brown's representability theorem tells us that for each given reduced cohomology theory $\tilde{E}^*$, we get a class $\{K_n\}$. Conversely, given a class of spaces $\{K_n\}$ with the above structure maps, we can recover the reduced cohomology theory $\tilde{E}^*$. In fact, they determine each other.

4.16 Example. *There're some examples of these kinds of sequences of spaces:*

(1) The **Eilenberg-MacLane Spaces** $K(G, n)$*, for* G *an abelian group and* $n \in \mathbb{N}$*;*

(2) $\{K_n\}$*, which represents the complex* K*-theory: when* n *is even,* $K_n = BU \times \mathbb{Z}$*, and when* n *is odd, it's just* U*.*

Now we introduce two attempts to construct the "stable model category", which does not fit into our requirements. The first one is the **Spanier-Whitehead category** SW, where objects are finite CW complexes(we can also add infinite-dimensional CW complexes and written $\widehat{\mathrm{Sw}}$), and morphisms are stable homotopy classes $[X, Y]^s := \mathrm{colim}_a[\Sigma^a X, \Sigma^a Y]^s$. But this construction has drawbacks: it does not have enough objects. One example is that it doesn't have countable coproducts. In section 1.1.4 of [BR20], they listed satisfactory 12 properties a stable homotopy category should have.

Now we define the notion of spectra.

4.17 Definition. A **spectrum** is a sequence of topological spaces $\{X_n\}$ with **structure maps** $\sigma_n^X : \Sigma X_n \to X_{n+1}$ which are weak homotopy equivalences.

An **Ω-spectrum** is a sequence of topological spaces $\{X_n\}$ with structure maps $\tilde{\sigma}_n^Z : Z_n \to \Omega Z_{n+1}$ which are weak homotopy equivalences.

The draw back for the category of spectra is that we do not have enough morphisms. One example is that

4.18 Example. *You can find two spectra representing the same cohomology theory but they're not homotopy equivalent to each other.*

To summarize, our goal is to find a good category that can represent all the reduced cohomology theories. Here we present some attempts but failed, and we'll see some constructions that finally resolve this issue.

TALK 5

K-theory and Bott Periodicity

Haosen Wu

We will assume some knowledge about K-theory throughout this talk and focus mainly on Bott periodicity. We'll present a Morse-theoretic proof which is originally due to Bott.

5.1 Theorem (Bott)**.** Consider $U = \operatorname{colim} U(n)$, then we have $U \simeq \Omega^2 U$. Let BU be the classifying space of U, then we have $BU \times \mathbb{Z} \simeq \Omega^2(BU \times \mathbb{Z})$.

(5a) K-theory. K-theory is sort of a "cohomological theory". We know for vector bundles we have invariants like Chern classes or Stiefel-Whitney classes, and we know they can be subtracted from the universal bundle $EG \to BG$ over the classifying space BG by pulling back certain classes of the universal vector bundle. But these classes all lie in the vector bundles themselves, we would like to simply consider the vector bundles themselves.

Operations on Vector Bundles. We have several operations on vector bundles.

5.2 Definition. Given two bundles E and E' over the same base space B, we can take the pull-back of the diagonal map $\Delta\colon B \to B \times B$ which gives the direct sum bundle $E \oplus E' = \Delta^*(E \times E')$.

5.3 Definition. We can define the **tensor product** of two vector bundles E, E' over the same base B as follows:

$$E \otimes E' = \coprod_{b \in B} p^{-1}(b) \otimes p'^{-1}(b) \to B.$$

These two operations serve as the core ingredients for the K-group. Recall that given a commutative monoid $(M, +, 0)$, then we can form an abelian group $G(M)$ associated to the monoid M defined by the following universal property: we have a natural map $\iota\colon M \to G(M)$ and for any homomorphisms of monoids $i\colon M \to G$, there exists a unique group homomorphism $\phi\colon G(M) \to G$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "i"] & G(M) \arrow[dl, "\phi"] \\ G & \end{tikzcd}$$

commutes. An explicit construction of $G(M)$ goes as follows:

$$G(M) = \{(m, n) \in M \times M\}/\sim$$

where $(m, n) \sim (m', n')$ iff there exists $a \in M$ with $m + n' + a = m' + n + a$.

5.4 Definition (K_0-group)**.** Let X be a paracompact topological space, then

$$K(X) = \{E - E' : E, E' \in \operatorname{Vect}(X)\},$$

where $\operatorname{Vect}(X)$ is the set of all vector bundles over X.

With this definition, we know $E-E' = F-F'$ iff there exists a vector bundle A such that $E\oplus F'\oplus A \cong E'\oplus F\oplus A$ as vector bundles, and we say $E = F$ if there exists a bundle A with $E\oplus A \cong F\oplus A$. If we can find a bundle A' with $A\oplus A' \cong \underline{\mathbb{R}}^N$, the trivial bundle of rank N.

5.5 Definition. We say two vector bundles E and F are **stably isomorphic** if there exists an N with $E\oplus\underline{\mathbb{R}}^N \cong F\oplus\underline{\mathbb{R}}^N$, and we write $[E]$ for the class of stable isomorphic classes.

Then we know that $K(X)$ consists of stable isomorphic classes of vector bundles. Note that stable isomorphism classes have cancellation property, i.e. if $A\oplus B\cong A'\oplus B$, then $[A]=[A']$. The reason is that, we can find a vector bundle B' such that $B\oplus B'\cong\underline{\mathbb{R}}^N$, so we get that $A\oplus\underline{\mathbb{R}}^N\cong A'\oplus\underline{\mathbb{R}}^N$, and hence $[A]=[A']$. This also verifies that $K(X)$ is an abelian group.

Now we want to define the reduced K-group. This group depends on the choose of a base point $*$. Consider an inclusion map $i\colon A\hookrightarrow X$ where $A\subseteq X$ is a closed subspace of X, then this inclusion induces a morphism $i^*K(X)\to K(A)$ defined by $E-\underline{\mathbb{R}}^N\mapsto E|_A-\underline{\mathbb{R}}^N|_A$. Pick $A=\{*\}\subseteq X$, then $K(A)=K(\{*\})\cong\mathbb{Z}$.

5.6 Definition. We define the **reduced K-group** of $(X,*)$ to be $\tilde{K}(X,*) = \ker i^*$ where i^* is defined above.

The K-group is actually a contravariant functor $K\colon \mathsf{Top}^{op}\to\mathsf{Ab}$, which is representable. How can we get a representation for K? This is motivated by how we play with classifying spaces. If we can get a map $f\in[X,BG]$, then we are expected to get a unique G-bundle $P\to X$ over X. Assume we have a diagram of categories

$$D\colon\ \mathsf{Vect}^0(X)\xrightarrow{\iota_0}\mathsf{Vect}^1(X)\xrightarrow{\iota_1}\cdots\to\mathsf{Vect}^n(X)\to\cdots$$

with $\iota_{n,m}\colon\mathsf{Vect}^n(X)\to\mathsf{Vect}^{n+m}(X)$ given by $E\mapsto E\oplus\underline{\mathbb{R}}^m$, then we can take the colimit of D, $\operatorname{colim} D$, then we'll get the reduced K-group $\tilde{K}(X)=\operatorname{colim} D$, and since each $\mathsf{Vect}^i(X)$ can be represented by $BU(i)$, we get that

$$K(X)\cong[X,BU]\oplus\mathbb{Z}\quad\text{and}\quad \tilde{X}\cong[X,BU\times\mathbb{Z}]_0.$$

Assuming Theorem 5.1, we readily get

5.7 Corollary. $\tilde{K}(\Sigma^2X)\xrightarrow{\sim}\tilde{K}(X)$.

Proof. Applying the adjunction $\mathrm{Hom}_0(\Sigma X,Y)\cong\mathrm{Hom}_0(X,\Omega Y)$, we get

$$\tilde{K}(\Sigma^2X)\cong[\Sigma^2X,BU]_0\cong[X,\Omega^2(BU\times\mathbb{Z})]_0\cong[X,BU\times\mathbb{Z}]_0\cong\tilde{K}(X).\qquad\square$$

Another result from Bott periodicity is that

5.8 Corollary. $\tilde{K}(\mathbb{S}^{2n})\cong\mathbb{Z}$ and $\tilde{K}(\mathbb{S}^{2n+1})=0$.

Proof. Note that $\tilde{K}(\mathbb{S}^k)\cong[\mathbb{S}^k,BU\times\mathbb{Z}]_0=\pi_k(BU)=\pi_{k+1}(U)$. To compute the homotopy group of U, we need Bott periodicity again. By Bott periodicity, we just need to compute the first two homotopy groups of U:

$$\pi_1(U)\cong\pi_1(U(1))\cong\pi_1(\mathbb{S}^1)=\mathbb{Z};$$
$$\pi_2(U)\cong\pi_2(U(2))\cong\pi_2(SU(2))=0.\qquad\square$$

(5b) Proof of Bott Periodicity. For the time issue, let's just outline the proof of the Bott periodicity theorem 5.1. Let's just focus on the first half. In order to show $U\simeq\Omega^2U$, we want the isomorphism $\pi_i(U)\cong\pi_{i+2}(U)$, and to achieve this, we study the space $P(U;p,q)$ of all paths connecting p and q, and the Morse theory tells us that $\pi_iBU\cong\pi_i\Omega SU$.

Note that each $U(n)$ has the homotopy type of a CW complex, so the colimit $U=\operatorname{colim}U(n)$ is also homotopic to some CW complex, and by the Whitehead theorem, all weak homotopy equivalences are homotopy equivalences. We can also express BU as the colimit $BU=\operatorname{colim}\mathrm{Gr}_n(2n)$, where $\mathrm{Gr}_n(2n)$ also has the homotopy type of finite dimensional CW complexes, so is BU.

5.9 Theorem. The loop space of U and BU are also homotopy equivalent to some CW complexes.

Now we consider the following diagram

$$\begin{tikzcd} \mathsf{Gr}_n(2n) \arrow[r] \arrow[d] & \Omega SU(2n) \arrow[d] \\ \mathsf{Gr}_{n+1}(2n+2) \arrow[r] & \Omega SU(2n+2) \end{tikzcd}$$

which induces a map $BU \to \Omega SU$. But from the result in Morse theory, this is indeed a weak equivalence, and by Whitehead theorem, this is a homotopy equivalence. Then we further consider the map $\Omega SU \to \Omega U$ giving a map $j: BU \to \Omega U$, and we define a map $BU \times \mathbb{Z} \to \Omega U$ by $(x,r) \mapsto j_r(x)$. This gives a corresponding map on homotopy groups $\pi_i(BU,(x,r)) \to \pi_i(\Omega U, j_r(x))$ which gives a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega U$.

The second part of the proof is $U \simeq \Omega BU$. If we achieve this, then we would get $U \simeq \Omega BU = \Omega(BU \times Z) = \Omega^2 U$, which proves theorem 5.1. □

To achieve this, recall that we have

5.10 Theorem. Given a fibre bundle $U \to E \to X$. If X is paracompact and E is contractible, then $U \simeq \Omega X$.

If we let $X = BU$, then we just need to construct a bundle E which is contractible. The construction is just given by

$$E = \mathsf{colim}\{\text{Principal bundle } V_n | V_n \to BU(n)\}.$$

Now the remaining part is to show that $\pi_i BU \cong \pi_i \Omega SU$. The proof relies on some path analysis on the path space $\Omega(M; p, q)$.

5.11 Theorem. $\pi_{i+1}(SU(2n)) \cong \pi_i(\mathsf{Gr}_n(\mathbb{C}^{2n}))$.

Proof. Let I be the identity in $SU(2n)$, then we have

$$\pi_{i+1} SU(2n) \cong \pi_i(\Omega SU(2n, I, -I)) \cong \pi_i \Omega SU(2n).$$

Claim:$\Omega(M; p, p) \simeq \Omega(M; p, q)$.

We can prove this claim by construct the homotopy equivalence directly: for $\gamma \in \Omega(M; p, q)$, we can construct a path $\bar{\gamma}(t) = \gamma(1-t)$, and for all $\sigma \in \Omega(M; p, q)$, we get a map $\sigma \# \bar{\gamma} \in \Omega(M; p, p)$, and the inverse is given by $\sigma \mapsto \sigma \# \gamma$.

With this claim, we get the second isomorphism in the above sequence. Now we apply Morse theory to show that

$$\pi_i(\Omega SU(2n); I, -I) \cong \pi_i(\Omega_{\mathsf{min}}) \cong \pi_i(\mathsf{Gr}_n(2n)),$$

where Ω_{min} is the smooth submanifold of Ω comsisting of minimal geodesics. Geodesics are exactly critical points of Morse functions, and we have

5.12 Theorem (Minimal Geodesic Index Theorem)**.** Consider the space of minimal geodesics connecting p, q. If non-minimal geodesics has Morse index $> \lambda_0$, then $\pi_i(\Omega_{\mathsf{min}}, \Omega) = 0$ for all $i \leq \lambda_0$ and hence $\pi_i(\Omega_{\mathsf{min}}) \cong \pi_i(\Omega SU(2n))$ for all $i < \lambda_0$.

The second isomorphism $\pi_i(\Omega_{\mathsf{min}})$ follows from a detailed analysis on the minimal geodesics, which implies that $\Omega_{\mathsf{min}} = \coprod_{k=0}^{n} \mathsf{Gr}_k(2n)$. □

TALK 6

Sequential Spectra

Siyang Liu

In this talk we're going to introduce the construction of stable homotopy category, following the ideas from Talk 4. We start with the objects and morphisms in this category.

6.1 Definition. A **sequential spectrum** X is a sequence of pointed topological spaces $\{X^i\}_{i\in\mathbb{N}}$ with **structure maps**

$$\sigma_X^i : \Sigma X^i \to X^{i+1}$$

or dually, the adjoint structure maps

$$\tilde{\sigma}_X^i : X^i \to \Omega X^{i+1}.$$

Here we do not require that the maps being weak homotopy equivalences. We call a spectrum X **Ω-spectrum** if the adjoint structure maps are weak homotopy equivalences.

6.2 Definition. Let X, Y be two sequential spectra, a **morphism** $f : X \to Y$ consists of a sequence of pointed maps $f^i : X^i \to Y^i$ compatible with structure maps, i.e. we have the commutative diagram

$$\begin{tikzcd}
\Sigma X^i \arrow[r, "\Sigma f^i"] \arrow[d, "\sigma_X^i"] & \Sigma Y^i \arrow[d, "\sigma_Y^i"] \\
X^{i+1} \arrow[r, "f^{i+1}"] & Y^{i+1}
\end{tikzcd}$$

for each i.

We then define the category of sequential spectra $\mathrm{S}^{\mathbb{N}}$ to be the category with objects and morphisms given above. In the category $\mathrm{S}^{\mathbb{N}}$, we define the functor $\Sigma : \mathrm{S}^{\mathbb{N}} \to \mathrm{S}^{\mathbb{N}}$ to be $(\Sigma X)_n = \Sigma X_n$, with structure maps given by suspensions of the corresponding structure maps. Similarly, we can define the loop functor $\Omega : \mathrm{S}^{\mathbb{N}} \to \mathrm{S}^{\mathbb{N}}$ by $(\Omega X)_n = \Omega X_n$.

Now we want a model structure, and furthermore a stable model structure on $\mathrm{S}^{\mathbb{N}}$. Let's make some obeservations on this category $\mathrm{S}^{\mathbb{N}}$ first.

6.3 Example. *There's a special kind of spectrum in $\mathrm{S}^{\mathbb{N}}$: the* **sphere spectrum** $\boldsymbol{S}$*, which is the sequence $\{\boldsymbol{S}_n = \mathbb{S}^n\}_{n\in\mathbb{N}}$ with structure map $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. Note that $\sigma_{\boldsymbol{S}}^i$ are homeomorphisms for all i. Given $n \in \mathbb{N}$, we customly write $\boldsymbol{S}^n = \Sigma^n \boldsymbol{S}$ and $\boldsymbol{S}^{-n} = F_n^{\mathbb{N}}\mathbb{S}^0$.*

6.4 Example. *The functor $F_n^{\mathbb{N}}$ at the end of the above example is a functor $\mathrm{Top}_* \to \mathrm{S}^{\mathbb{N}}$ defined as follows: for each pointed space X, we define*

$$(F_d^{\mathbb{N}}X)_n = \begin{cases} \Sigma^{n-d}X, & \text{when } n \geq d; \\ *, & \text{when } n < d. \end{cases}$$

with structure maps $\sigma^n_{F^{\mathbb{N}}_d X} \equiv \mathrm{id}$ *for all* $n \neq d-1$ *and the canonical pointed map* $* \to X$ *for* $n = d-1$. *We call this spectrum the* **shifted suspension spectrum** *associated to the pointed space* X. *Conversely, given any spectrum* $X \in \mathrm{S}^{\mathbb{N}}$ *and any natural number* $d \in \mathbb{N}$, *we define* $\mathrm{Ev}^{\mathbb{N}}_d(X) = X_d$. *This gives a functor* $\mathrm{Ev}^{\mathbb{N}}_d : \mathrm{S}^{\mathbb{N}} \to \mathrm{Top}_*$. *We customly write* Σ^∞ *for* $F^{\mathbb{N}}_0$.

Moreover, the two functors

$$F^{\mathbb{N}}_d : \mathrm{Top}_* \rightleftarrows \mathrm{S}^{\mathbb{N}} : \mathrm{Ev}^{\mathbb{N}}_d$$

are adjoint to each other. This means that

6.5 Proposition. For all pointed space X and spectrum Y, we have

$$\mathrm{Hom}_{\mathrm{Top}_*}(X, \mathrm{Ev}^{\mathbb{N}}_d(Y)) \cong \mathrm{Hom}_{\mathrm{S}^{\mathbb{N}}}(F^{\mathbb{N}}_d(X), Y).$$

and we obtain the initial and final object in this category, which is $\Sigma^\infty\{*\} := *$.

This adjunction is only categorical, and we want something more: we want this adjunction to be a Quillen adjunction. Since morphisms of the category $\mathrm{S}^{\mathbb{N}}$ is defined levelwise, limits and colimits in this category can be constructed levelwise. That is, given a diagram of sequential spectra $\{X^{(i)}, \alpha_{i,j} : X^{(i)} \to X^{(j)}\}$, the limit of this diagram is the spectrum $\{\varprojlim X^{(i)}\}$ with dual structure maps

$$\tilde{\sigma}^k_{\varprojlim X^{(i)}} = \varprojlim_i \tilde{\sigma}^k_{X^{(i)}} : \varprojlim_i X^{(i)}_k \to \varprojlim_i \Omega X^{(i)}_{k+1}.$$

and similarly we can get the colimit spectrum $\{\varinjlim X^{(i)}\}$ with structure maps

$$\sigma^k_{\varinjlim X^{(i)}} = \varinjlim_i \sigma^k_{X^{(i)}} : \varinjlim_i \Sigma X^{(i)}_k \to \varinjlim_i X^{(i)}_{k+1}.$$

These constructions tell us that the category $\mathrm{S}^{\mathbb{N}}$ has all small limits and colimits. This tells us that $\mathrm{S}^{\mathbb{N}}$ satisfies the first half of condition 4 in [BR20, section 1.1.4].

6.6 Example. *Given a pointed topological space* A *and a spectrum* $X \in \mathrm{S}^{\mathbb{N}}$, *we can define the spectrum* $A \wedge X$ *to be the spectrum consisting of topological spaces* $\{A \wedge X_n\}_{n \in \mathbb{N}}$ *and structure maps*

$$\sigma^k_{A \wedge X} : \Sigma(A \wedge X_k) \cong A \wedge \Sigma X_k \xrightarrow{\mathrm{id}_A \wedge \sigma^k_X} A \wedge X_{k+1},$$

since the wedge sum is defined by wedge sum by $\mathbb{S}^1$, *and the first homeomorphism follows by the commutativity of the wedge product* $\wedge$. *Dually, we can define a spectrum* $\mathrm{Top}_*(A, X)$ *by the sequence of topological spaces* $\{\mathrm{Top}_*(A, X_n)\}_n$ *with dual structure maps*

$$\tilde{\sigma}^k_{\mathrm{Top}_*(A,X)} : \mathrm{Top}_*(A, X_k) \xrightarrow{\mathrm{Top}_*(A, \tilde{\sigma}^k_X)} \mathrm{Top}_*(A, \Omega X_{k+1}) \cong \Omega\, \mathrm{Top}_*(A, X_{k+1}).$$

Since morphism spaces are defined levelwise, we get an isomorphism of sets

$$\mathrm{S}^{\mathbb{N}}(A \wedge X, Y) \cong \mathrm{S}^{\mathbb{N}}(X, \mathrm{Top}_*(A, Y))$$

which gives us property 5 in [BR20, section 1.1.4].

Now we discuss the model structure on $\mathrm{S}^{\mathbb{N}}$. We call this model structure the **levelwise model structure**. Before going into the definition and proofs, let's make a digression into abstract model theory.

(6a) Cofibrantly generated model categories. A model structure on a given cateogry C can in general very difficult to describe, and we want some smaller classes of fibrations, cofibrations and weak equivalences that can generate the whole model structure. This leads to the notion of cofibrantly generated model categories.

6.7 Definition. Let C be a category with all small colimits, and I a set of morphisms in C. We write $I-\mathrm{inj}$ to be the set of morphisms in C that have the right lifting property with respect to all elements in I, and $I-\mathrm{cof}$ the class of morphisms in C with the left lifting property w.r.t. all elements in $I-\mathrm{inj}$.

We write $I-\mathrm{cell}$ to be the set of all sequential colimits of pushouts of elements in I. This means that a map $f : A \to B$ is in $I-\mathrm{cell}$ if and only if there exists a sequence of morphisms

$$A = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

such that for each $f_i : X_i \to X_{i+1}$, there exists an indexed set of morphisms $\{i_\alpha : C_\alpha \to D_\alpha\}$ and a commutative diagram

$$\begin{tikzcd} \coprod_\alpha C_\alpha \arrow[r] \arrow[d, "\coprod_\alpha i_\alpha"] & X_i \arrow[d, "f_i"] \\ \coprod_\alpha D_\alpha \arrow[r] & X_{i+1} \end{tikzcd}$$

which is a pushout square, and that the colimit $A = X_0 \to \varinjlim X_i$ is the morphism f. Write $*$ for the initial object of C, then we say an object X is an **$I-\mathrm{cell}$ complex** if the canonical morphism $* \to X$ is in $I-\mathrm{cell}$.

Observe that by definition, we have

6.8 Lemma. $I-\mathrm{cell} \subseteq I-\mathrm{cof}$.

6.9 Example. *Consider the category of topological spaces* Top*, and let*

$$I = \{\mathbb{S}^{n-1} \to \mathbb{D}^n | n \in \mathbb{N}\},$$

then $I-\mathrm{inj}$ is exactly the set of all Serre fibrations(See e.g. [Hat00, Section 4.2]), and both $I-\mathrm{cof}$ and $I-\mathrm{cell}$ are the set of q-cofibrations. $I-\mathrm{cell}$ is exactly the class of all CW complexes.

Now we define in an abstract category with a given class of morphisms I the notion of "compact objects", which would give us compact subsets when looking at Top.

6.10 Definition. An object $Z \in$ C is said to be **small** with respect to I if for all morphisms $i : A \to \mathrm{colim}\, X_n = X$ in $I-\mathrm{cell}$, we have an isomorphism

$$\operatorname*{colim}_n \mathrm{C}(Z, X_n) \xrightarrow{\cong} \mathrm{C}(Z, X).$$

6.11 Example. *Obviously if $Z \in$ Top is a compact space, then Z is small with resepct to the I given in example 6.9. (See [Lee11, Chapter 5], for example) Conversely, I'm not clear if all such I-small spaces are small.*

6.12 Lemma. *Small objects are preserved by push-outs. That is, if we have a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

consisting of I-small objects whose pushout is P, then P is also I-small.

Now we can define the notion of "cofibrantly generated model categories":

6.13 Definition. A model category C is **cofibrantly generated** if there are sets I and J such that the following hold.

- The domains of I are I-small;
- The domains of J are J-small;
- Fibrations in C are precisely $J-\mathsf{inj}$;
- The acyclic fibrations in C are precisely $I-\mathsf{inj}$.

6.14 Example. Top *with Quillen model structure is cofibrantly generated by classes*

$$I = \{\mathbb{S}^{n-1} \to \mathbb{D}^n\}_{n\in\mathbb{N}}$$

and

$$J = \{\mathbb{D}^n \to \mathbb{D}^n \times I\}_{n\in\mathbb{N}}.$$

Here we include the number 0 *and write* $\mathbb{S}^{-1} = \varnothing$. *One can verify that* $I-\mathsf{inj}$ *is exactly the set of acyclic Serre fibrations and* $J-\mathsf{inj}$ *exactly the set of Serre fibrations.*

6.15 Example. *The category of simplicial sets* sSet *with model structure defined in chapter 2 is also cofibrantly generated. For* $0 \leq r \leq n$, *we define the* **r-horn** $\Lambda^r[n]$ *to be a functor* $\triangle^n \to \mathsf{Set}$ *sending* $[k]$ *to the order-preserving injections* $[k] \to [n]$ *excluding both the identity* $[n] \to [n]$ *and the map* $d^r : [n-1] \to [n]$ *which avoids* r. *We then let*

$$I = \{\partial\Delta[n] \to \Delta[n] | n \in \mathbb{N}\}$$

and

$$J = \{\Lambda^r[n] \to \Delta[n] | n \in \mathbb{N}\}$$

to be the corresponding generating sets for cofibrations and acyclic cofibrations.

We end this discussion with a criterion for morphism sets I and J cofibrantly generating a model structure:

6.16 Theorem (Recognition Theorem)**.** Let C be a category with all small limits and colimits. Let $\mathcal{W}$ be a class of morphisms closed under composition and contains all identity morphisms. Further, let I and J be the sets of morphisms in C. Assume that

- $\mathcal{W}$ satisfies 2-out-of-3 property,
- the domains of I are small with respect to I,
- the domains of J are small with respect to J,
- $J-\mathsf{cell} \subseteq \mathcal{W} \cap I-\mathsf{cof}$,
- $I-\mathsf{inj} \subseteq \mathcal{W} \cap J-\mathsf{inj}$,
- either $\mathcal{W} \cap I-\mathsf{cof} \subseteq J-\mathsf{cof}$ or $\mathcal{W} \cap J-\mathsf{inj} \subseteq I-\mathsf{inj}$.

Then C can be given a cofibrantly generated model structure with $\mathcal{W}$ being the weak equivalences, I the set of generating cofibrations and J the set of generating acyclic cofibrations.

Now we go back to $\mathsf{S}^{\mathbb{N}}$.

6.17 Theorem. There is a levelwise model structure defined on $\mathsf{S}^{\mathbb{N}}$, where the weak equivalences are levelwise weak homotopy equivalences of pointed topological spaces. The fibrations are the class of levelwise Serre fibrations of pointed spaces. The cofibrations are generated canonically, and we call then **q-cofibrations**.

Moreover, the levelwise model structure is cofibrantly generated with generating sets given by

$$I^{\mathbb{N}}_{level} = \{F^{\mathbb{N}}_d \mathbb{S}^{n-1}_+ \to F^{\mathbb{N}}_d \mathbb{D}^n_+ | n, d \in \mathbb{N}\}$$
$$J^{\mathbb{N}}_{level} = \{F^{\mathbb{N}}_d \mathbb{D}^n_+ \to F^{\mathbb{N}}_d (\mathbb{D}^n \times [0,1])_+ | n, d \in \mathbb{N}\}.$$

In particular, the q-cofibrations are levelwise q-cofibrations of pointed topological spaces.

Here we use the notation convention that given $X \in \mathsf{Top}$, we have a functor $(-)_+ : \mathsf{Top} \to \mathsf{Top}_*$ where $X_+ = (X \sqcup *, *)$.

(6b) The stable model structure. Although we have defined a model structure on $\mathsf{S}^{\mathbb{N}}$, what we really want is a **stable model structure** on $\mathsf{S}^{\mathbb{N}}$, which are supposed to give us the correct "stable homotopy theory". We first state the definition of stable model structure. This is very similar to the "stable infinity category" as mentioned in [Lur17]:

6.18 Definition. We say a model category C is **stable** if we have a pair of functors (Ω, Σ) called loop functor and suspension functor, who give mutually inverse equivalences of categories from $\mathsf{Ho}(\mathsf{C})$ to itself.

In infinity category, we can say they are "homotopy equivalences", but here without the higher structures we do not have the notion of "homotopy equivalence", hence we can only say they should give a category equivalence when passing to the homotopy category $\mathsf{Ho}(\mathsf{C})$. To achieve this, we need to somewhat modify the levelwise model structure by slightly changing the class of fibrations, cofibrations and weak equivalences.

First of all, we construct the loop and suspension functors on $\mathsf{S}^{\mathbb{N}}$. These are defined simply using the action of Top_* on $\mathsf{S}^{\mathbb{N}}$:

6.19 Definition. For $X \in \mathsf{S}^{\mathbb{N}}$, we define $\Sigma X = \mathbb{S}^1 \wedge X$ and $\Omega X = \mathsf{Top}_*(\mathbb{S}^1_+, X)$.

These two functors Σ and Ω are not necessarily equivalences when passing to the homotopy category. We need to modify the class of weak equivalences as follows:

6.20 Definition. Let X be a spectra, we define the **k-th homotopy group** of X to be the class of morphisms $[\Sigma^k \mathbb{S}, X]$, where $[\Sigma^k \mathbb{S}, X]$ is the quotient of $\mathsf{S}^{\mathbb{N}}(\Sigma^k \mathbb{S}, X)$ by homotopy.

By definition of the levelwise model structure, the map $[\Sigma^k \mathbb{S}, X]$ is exactly the colimit $\operatorname{colim}_n \pi_{n+k}(X_k)$.

6.21 Definition. We say a morphism $f : X \to Y$ in $\mathsf{S}^{\mathbb{N}}$ is a **π_*-isomorphism** if it induces isomorphisms on all homotopy groups.

We then define the class $\mathcal{W}'$ for the stable model structure on $\mathsf{S}^{\mathbb{N}}$ to be the class of π_*-isomorphisms. For fibrations, we need some more constructions. Let $f : X \to Y$ be a q-cofibration of spectra, $p : P \to Q$ a levelwise fibration, and $i : A \to B$ a q-cofibration of topological spaces, then

6.22 Proposition. The induced map of spectra

$$\mathsf{hom}_\square(f, p) : \mathsf{S}^{\mathbb{N}}(Y, P) \to \mathsf{S}^{\mathbb{N}}(X, P) \bigvee_{\mathsf{S}^{\mathbb{N}}(X,P)} \mathsf{S}^{\mathbb{N}}(Y, Q)$$

is a fibration of pointed spaces, and if f or p is a levelwise weak equivalence, then $\mathsf{hom}_\square(f, p)$ is also a weak homotopy equivalence.

$$f \square i : Y \wedge A \bigvee_{X \wedge A} X \wedge B \to Y \wedge B$$

is a q-cofibration of spectra, and if i is a weak homotopy equivalence or f is a levelwise weak equivalence, then $f\square i$ is a levelwise weak equivalence.

$$\mathsf{hom}_{\square}(i,p)\colon\ \mathsf{Top}_*(B,P) \to \mathsf{Top}_*(A,P) \times_{\mathsf{Top}_*(A,Q)} \mathsf{Top}_*(B,Q)$$

is a levelwise fibration of spectra, and if i is a weak homotopy equivalence or p is a levelwise weak equivalence, then $\mathsf{hom}_{\square}(i,p)$ is a levelwise weak equivalence.

Here we enrich the category $\mathsf{S}^{\mathbb{N}}$ by giving $\mathsf{S}^{\mathbb{N}}(X,Y)$ a topological structure which makes it into a subspace of $\prod\limits_{i\in\mathbb{N}} \mathsf{Top}_*(X_i, Y_i)$. With this enrichment, we get a duality

$$\mathsf{S}^{\mathbb{N}}(A\wedge X, Y) \cong \mathsf{S}^{\mathbb{N}}(X, \mathsf{Top}_*(A,Y)) \cong \mathsf{Top}_*(A, \mathsf{S}^{\mathbb{N}}(X,Y)).$$

This duality gives the proof of the above proposition.

Let $\lambda_n\colon F^{\mathbb{N}}_{n+1}\mathbb{S}^1 \to F^{\mathbb{N}}_n\mathbb{S}^0$ be the map corresponding to the identity map $\mathbb{S}^1 \to \mathsf{Ev}^{\mathbb{N}}_{n+1} F^{\mathbb{N}}_n \mathbb{S}^0 \cong \mathbb{S}^1$, and let $M\lambda_n$ be the mapping cylinder of λ_n, then we have the pushout square

$$\begin{tikzcd} F^{\mathbb{N}}_{n+1}\mathbb{S}^1 \arrow[r, "\lambda_n"] \arrow[d, "i_1"] & F^{\mathbb{N}}_n \arrow[d] \\ F^{\mathbb{N}}_{n+1}\wedge[0,1]_+ \arrow[r, "t_n"] & M\lambda_n \end{tikzcd}$$

and we write k_n to be the composition $t_n\circ i_0$. This map is a q-cofibration and a π_*-isomorhism.

6.23 Definition. We define the **stable model structure** on $\mathsf{S}^{\mathbb{N}}$ to be the model structure cofibrantly generated by the classes

$$\begin{aligned} I^{\mathbb{N}}_{stable} &= I^{\mathbb{N}}_{level}; \\ I^{\mathbb{N}}_{stable} &= J^{\mathbb{N}}_{level} \cup \{k_n\square(\mathbb{S}^{a-1}_+ \to \mathbb{D}^a_+) | a,n\in\mathbb{N}\} \end{aligned}$$

With this model structure, we get that

6.24 Proposition. A map of spectra $f\colon X\to Y$ has the right lifting property with respect to $J^{\mathbb{N}}_{stable}$ if and only if f is a levelwise fibration of spaces and for each $n\in\mathbb{N}$, the map

$$X_n \to Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$$

induced by $\tilde{\sigma}^X_n$ and f is a weak homotopy equivalence. In particular, if $Y=*$, then X has the right lifting property if and only if X is a Ω-spectrum.

We call the fibrations described in this Proposition **stable fibrations**. This implies that fibrant objects in this model category are Ω-spectra.

6.25 Theorem. The stable model structure on sequential sepctra is defined by the three classes below:

- The weak equivalences are the π_*-isomorphisms.
- The cofibrations are the q-cofibrations.
- The fibrations are given by Proposition 2.3.10 and are called the **stable fibrations**.

In this case, the loop functor Ω and the suspension functor Σ are Quillen equivalences if $\mathsf{S}^{\mathbb{N}}$ is equipped with the stable model structure, and therefore they induce categorical equivalences when passing to homotopy.

6.26 Definition. We define the **stable homotopy category** to be the homotopy category of $\mathsf{S}^{\mathbb{N}}$ with the stable model structure, i.e. we define

$$\mathsf{SHC} = \mathsf{Ho}(\mathsf{S}^{\mathbb{N}}).$$

Bibliography

[Ada74] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago Press, Chicago, IL, USA, 1974.

[Ati61] M. F. Atiyah. Thom Complexes. *Proc. London Math. Soc.*, s3-11(1):291–310, Jan 1961.

[BR20] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory.* Cambridge University Press, Cambridge, England, UK, Mar 2020.

[Hat00] Allen Hatcher. *Algebraic topology.* Cambridge Univ. Press, Cambridge, 2000.

[Lee11] John M. Lee. *Introduction to Topological Manifolds.* Springer, New York, NY, New York, NY, USA, 2011.

[Lur17] Jacob Lurie. Higher algebra, Sep 2017. [Online; accessed 20. Feb. 2022].

[SW55] E. H. Spanier and J. H. C. Whitehead. Duality in homotopy theory. *Mathematika*, 2(1):56–80, Jun 1955.