Floer Theory on Cotangent Bundles

Siyang Liu

October 2020

Contents

Introduction

It's a note for a talk at the Symplectic Reading Seminar in USC. The goal of this talk is to introduce Oh's development of the Floer theory on cotangent bundles [Oh97]. The cotangent bundle is the first object coming into symplectic topology, and before Floer, there are two main approaches toward the Arnold conjecture on cotangent bundles. This paper summarizes the two approaches and unify them by considering the Floer theory on the cotangent bundle(for submanifolds of the zero-section). He then introduce a dynamical invariant, the so-called spectral invariant, which is related to the mini-max theory in dynamical systems.

Let (M, g) be an arbitrary compact Riemannian manifold of dimension m. One of the most classical example of symplectic manifold is the cotangent bundle T^*M associated to M, the bundle of 1-forms on M. There is a canonical form $\lambda = p \,\mathrm{d}\, q$, called the **Liouville form** or **tautological 1-form**. There is an alternative description of the Liouville form as follows: for each $(x, \xi) \in T^*M$, we have $\lambda|_{(x,\xi)} = \xi$. The differential $\mathrm{d}\,\lambda$ is then an exact(hence closed) 2-form on T^*M, which can be verified to be a symplectic form, and hence $(T^*M, \omega = \mathrm{d}\,\lambda)$ is a symplectic manifold. The zero-section $M \subseteq T^*M$ is naturally a Lagrangian submanifold, so are all the conormal bundles N^*S for all submanifolds $S \subseteq M$. Let $H\colon [0, 1] \times T^*M \to \mathbb{R}$ any smooth function, then we can associate a corresponding **Hamiltonian vector field** X_H and the **Hamiltonian flow** ϕ_H. Oh's work focused on the Lagrangian intersection problems for the zero-section M and some conormal bundle N^*S. These two submanifolds do not intersect transversely, so we should construct the Floer theory with the help of some Hamiltonian H. Once we have $\phi_H^1(L) \pitchfork N^*S$, we can construct the Floer theory for this pair (M, N^*S) as follows:

(1a) The Floer Equation. There is a natural **action functional** $\mathcal{A}_H$ associated to the Hamiltonian H, defined for any path $\gamma\colon [0, 1] \to T^*M$,

$$\mathcal{A}_H(\gamma) = \int_\gamma \lambda - \int_0^1 H(t, \gamma(t))\,\mathrm{d}\,t.$$

We can compute the differential of $\mathcal{A}_H$ using variation, and the result is the following

$$\mathrm{d}\,\mathcal{A}_H(\gamma)(\xi) = \int_0^1 -\omega(\xi, \dot{\gamma})\,\mathrm{d}\,t - \int_0^1 (X_H, \xi)\,\mathrm{d}\,t = \int_0^1 \omega(\dot{\gamma} - X_H, \xi)\,\mathrm{d}\,t.$$

Therefore we know that γ is a critical point of $\mathcal{A}_H$ if and only if γ satisfies the Hamiltonian equation

$$\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t} - X_H(t, \gamma(t)) = 0. \tag{1.1}$$

If we impose the condition that $\phi(M) \pitchfork N^*S$, then there are only finitely many solutions to this Hamiltonian equation under given boundary conditions. As in Floer's paper, we want to find the gradient flow equation for $\mathcal{A}_H$ with respect to the given metric induced by ω and some compactible almost complex structure J. Here the J can be chosen to be canonical: for each $(p, \xi) \in T^*M$, we construct J via the equation $\omega(v, Jw) = g(v, w)$ for all $v, w \in T_pM \subseteq T_{(p,\xi)}T^*M$. It's obviously well-defined, compactible with ω, and squares to be $-\mathrm{id}$. Then we could just extend the definition of g to a global Riemannian metric on T^*M induced from ω and J. Now we can introduce a Riemannian metric on the path space $\Omega(T^*M; M, N^*S) = \{\gamma\colon [0,1] \to M | \gamma(0) \in M, \gamma(1) \in N^*S\}$ with given boundary conditions. For any path γ, the Riemannian metric on the tangent space at γ is just

$$\langle \xi, \eta \rangle_\gamma = \int_0^1 g(\xi, \eta)\,\mathrm{d}\,t,$$

and therefore the gradient flow of $\mathcal{A}_H$ is just a map $u\colon [0,1] \times \mathbb{R} \to T^*M$ satisfying the following partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - X_H) = 0, & \forall (s,t) \in \mathbb{R} \times [0,1]; \\ u(s, 0) \in M, & \forall s \in \mathbb{R}; \\ u(s,1) \in N^*S, & \forall s \in \mathbb{R}. \end{cases} \tag{1.2}$$

This is the **Floer equation** in this setting. Here we have involved a non-trivial Hamiltonian H, so we also call this a **perturbed Floer equation**. We define the **energy** of such a solution u to be

$$E(u) = \int_{\mathbb{R}} \int_0^1 \left\| \frac{\partial u}{\partial s} \right\| \mathrm{d}\,t\,\mathrm{d}\,s,$$

and we only focus on those solutions with finite energy. We write $\mathcal{M}(T^*M, J, H)$ to be the space of all such solutions. The exponential estimate at infinity then gives

1.1 Proposition. When $s \to \pm\infty$, $u(s,t)$ tends exponentially to critical points of $\mathcal{A}_H$, i.e. $u(\pm\infty, t)$ are solutions to the Hamiltonian equation (1.1).

A systematic study of the asymptotic behaviour of such strips was done in Robbin and Salamon's paper [RS01]. So we have a natural decomposition

$$\mathcal{M}(M, N^*S; J, H) = \bigcup_{x,y \in \mathrm{Crit}\mathcal{A}_H} \mathcal{M}(x, y; J, H).$$

The first result is the following, which states that, if we impose some conditions on the Hamiltonian H and the almost complex structure J, then T^*M actually satisfies the "bounded geometry" condition. Given a symplectomorphism ϕ, we say ϕ is **Hamiltonian** if it's generated by some Hamiltonian function H. We say a Hamiltonian diffeomorphism ϕ is **weakly compactly supported** if the corresponding Hamiltonian vector field X_H is compactly supported.

1.2 Theorem. Assume that ϕ is a weakly compactly supported Hamiltonian diffeomorphism with Hamiltonian H and J is compactly supported. For any solution $u \in \mathcal{M}(x, y; J, H)$, if we write $u(x, \xi) = (v(x, \xi), p(x, \xi))$, with $v(x, \xi) \in M$ and $p(x, \xi) \in T_v^*M$, then there exists a positive constant $R_0 > 0$ such that $|p| \le R_0$, and the constant can be determined by $R_0 = \min\{R > 0 | T^*M \backslash D_R \subset T^*M \backslash (\operatorname{supp} H \cup \operatorname{supp} J)\}$. Here D_R is the disk bundle with radius R.

So in this case, we can still achieve Gromov compactness for the moduli space of pseudo-holomorphic strips. Moreover, both M and N^*S are exact Lagrangian submanifolds of T^*M, hence there would be no bubbles occuring as a limit of some sequence of pseudo-holomorphic strips, so everything just goes as in Floer's paper [Flo88] and we will obtain the Floer cohomology $HF(M, N^*M; J, H)$ for this two pairs of exact Lagrangian submanifolds. There're more information we can extract to obtain a Floer cohomology with better properties.

(1b) Genericness of Parameters Here in constructing the Floer theory, we have introduced several parameters. The first one is the additional Hamiltonian function $H \colon T^*M \to \mathbb{R}$, and the chosen almost complex structure J compactible with the Liouville form on T^*M.(Here we can change the chosen Riemannian metric on M) We can also deform the submanifold $S \subseteq M$ in M, so what we are concerning is the pair (H, S, J) consisting of these three data.

1.3 Definition. For a given closed submanifold $S_0 \subseteq M$, we write $\operatorname{Emb}(S_0; M)$ for the space of all embeddings from S_0 to M. For a given Hamiltonian H, we write $\operatorname{Emb}^H(S_0; M)$ to be the set of all embeddings from S_0 to M such that $\phi_H^1(M) \pitchfork N^*S_0$, and $\operatorname{Iso}^H(S_0; M)$ to be the connected component of S_0, i.e. the set of all embeddings in $\operatorname{Emb}^H(S_0; M)$ that is isotopic to S_0.

The transversality theorem then implies that for a given H, $\operatorname{Emb}^H(S_0; M)$ is dense in $\operatorname{Emb}(S_0; M)$. On the other hand, for a given embedding $S \in \operatorname{Emb}^H(S_0; M)$, we set $\mathcal{H}(S)$ to be the set of all asymptotically constant Hamiltonian functions such that $\phi_H^1(M) \pitchfork N^*S$. We could then consider the pair (H, S) where $H \in \mathcal{H}(S)$ and $S \in \operatorname{Emb}^H(S_0; M)$.

1.4 Definition. Note that for a given Riemannian metric g on M and the Liouville form θ with $\mathrm{d}\theta = \omega$ the canonical symplectic form on T^*M, there is a unique associated canonical almost complex structure J_g on T^*M. We define $\mathcal{J}^c$ to be the set of all almost complex structures J on T^*M such that J is compactible with ω and that $J \equiv J_g$ outside a compact neighbourhood of M in T^*M.

Now we must choose $J \in \mathcal{J}^c$ such that with the data (H, J, S), solutions to (1.2) are all **Floer-regular**, i.e. the linearization of the Cauchy-Riemann operator at u should be Fredholm. In this case, we say the pair (H, J, S) is **regular**. The main result concerning regularity is that

1.5 Theorem. There is a dense subset $\mathcal{J}^c_{H,S} \subseteq \mathcal{J}^c$ such that the pair (H, J, S) is regular.

In the next section we will define the $\frac{\mathbb{Z}}{2}$-graded **Floer chain group** with a fixed orientation σ in this case, to be $CF_\sigma^*(H, J, S; \mathbb{Z})$ associated to this given pair (H, J, S). As in Floer's paper [Flo88], we want the corresponding homology group $HF_\sigma^*(H, J, S; \mathbb{Z})$ to be independent of the choice of the pair (H, J, S) and only depends on the isotopy class $[(H, J, S)] := [S]$.

1.6 Theorem (Dependence on Parameters)**.** Given a path of parameters (H^a, J^a, S^a), $0 \le a \le 1$ there exists a chain map $\rho_\sigma \colon CF_\sigma^*(H^0, J^0, S^0; M) \to CF_\sigma^*(H^1, J^1, S^1; M)$ which preserves the degree. Moreover, this map is a quasi-isomorphism, i.e. it induces an isomorphism on the corresponding homology group.

Therefore we can write abbreviately $CF^*([S], M; \mathbb{Z})$ and remove the parameters.

(1c) Gradings and Orientation. The extra property we obtain is the grading and orientation for Floer chain complexes. In the original result of Floer, the grading is only relatively defined: we can define the difference of degrees of a given pair of intersection points (p, q), but it's not so obvious that we can give all the points canonical gradings, i.e. a function μ that assigns to each intersection point p a number $\mu(p)$ such that for any strip u connecting p to q, we have $\operatorname{Ind}(u) = \mu(q) - \mu(p)$. But here, in the special case of the cotangent bundle, we do have a canonical grading for each solutions.

1.7 Definition. We denote by

$$\operatorname{Chord}(H, J, S; M) = \{z \colon [0, 1] \to T^*M | z(0) \in M, z(1) \in N^*S, \dot{z}(t) = X_H(t, z)\}$$

the set of all solutions to the Hamiltonian equation with given boundary conditions.

$\operatorname{Chord}(H, J, S; M)$ is exactly the set of generators for the Floer chain group $CF^*_\sigma(H, J, S; M)$. But to construct this group, we need the information about the grading and the orientation σ. For the grading, [Oh97] states that

1.8 Theorem (Canonical Grading)**.** There exists a function $\mu_S \colon \operatorname{Chord}(H, J, S; M) \to \dfrac{\mathbb{Z}}{2}$ satisfying the following properties:

1. $\mu_S + \frac{1}{2}\dim S \in \mathbb{Z}$ and for each solution u of (1.2) such that $u(-\infty) = z^\alpha$ and $u(+\infty) = z^\beta$, the Fredholm index of u is given by $\operatorname{Ind}(u) = \mu_S(z^\alpha) - \mu_S(z^\beta)$.

2. For the time-independent Hamiltonian function $F = f \circ \pi$ where $f \colon M \to \mathbb{R}$ is a smooth function and $\pi \colon T^*M \to M$ the canonical projection map, pick $p \in \operatorname{Graph}(\mathrm{d}\, f) \cap N^*S$ which is the critical point of the function $\mathrm{d}\, f$ with $\pi(p) = x$, then the index of the path $z_x(t) = (x, t\, \mathrm{d}\, f(x))$ is given by

   $$\mu_S(z_x) = \mu_f^S(x) - \frac{1}{2}\dim S,$$

   where μ_f^S is the Morse index of $f|_S$ at x.

So the group $CF^*(H, J, S; M) = \mathbb{Z}[\operatorname{Chord}(H, J, S; M)]$ is $\frac{\mathbb{Z}}{2}$-graded. In order to make it into a graded complex, we need to define a differential $\partial \colon CF^*(H, J, S, M; \mathbb{Z}) \to CF^{*-1}(H, J, S, M; \mathbb{Z})$. Assuming transversality, Gromov compactness and the grading presented, the differential is defined by the counting of trajectories with index $\operatorname{Ind}(u) = 1$, since we are going to mod out the actions of translation by $\mathbb{R}$. The compactness then implies that there are only finitely many of them. What remains is to give each trajectory a coherent orientation, i.e. a plus or minus sign, so we can have $\partial^2 = 0$ holds in $\mathbb{Z}$-coefficients.(In [Flo88] we only consider $\frac{\mathbb{Z}}{2\mathbb{Z}}$-coefficients, s in that case we can avoid the discussion of orientation) The main result concerning this is that

1.9 Theorem (Coherent Orientation)**.**

1. Pick regular parameters (H, J, S) representing the isotopy class $[S]$, then for each $z^\alpha, z^\beta \in \operatorname{Chord}([S]; M)$, there exists an orientation of the moduli space $\mathcal{M}_J(z^\alpha, z^\beta)$, i.e. the determinant bundle $\operatorname{Det} \to \mathcal{M}_J(z^\alpha, z^\beta)$ with fibre at u the one-dimensional vector space

   $$\det(\mathrm{d}\, \bar{\partial}_{J,H}(u)) = \Lambda^{\max}(\ker \mathrm{d}\, \bar{\partial}_{J,H}(u)) \otimes \Lambda^{\max}(\operatorname{coker} \mathrm{d}\, \bar{\partial}_{J,H}(u))$$

   is trivial.

2. Furthermore, there exists a coherent orientation on the set of all $\mathcal{M}_J(H, S)$'s. We denote the set of such coherent orientations by $\operatorname{Or}([S]; M)$.

The term "coherent orientation", introduced by Floer and Hofer in [FH93], is to assign to each equivalent class of strips $[u]$ an orientation $\sigma([u])$ such that the orientation is compatible with gluing. With an orientation, we could then define the differential $\partial\colon CF^*_\sigma([S], M; \mathbb{Z}) \to CF^{*-1}_\sigma([S], M; \mathbb{Z})$ by signed count of trajectories and verified that $\partial^2 = 0$.

1.10 Definition. We define the **Floer cohomology** with orientation σ for the class $[S]$ to be

$$HF^*_\sigma([S]; \mathbb{Z}) = \ker \partial / \Im \partial.$$

Finally, with a similar way as in Floer's work [Flo89], we can show that

1.11 Theorem. For a generic choice of parameters (H, S, J), there exists a coherent orientation σ, which we call the **canonical coherent orientation**, such that there exists an isomorphism

$$F_{(H,S,J)}\colon H^*(S; \mathbb{Z}) \to HF^*_\sigma(H, S, J, M; \mathbb{Z}).$$

In particular, $HF^*_\sigma(H, J, S, M; \mathbb{Z})$ is non-trivial.

References

[FH93] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math. Z.*, 212(1):13–38, Jan 1993.

[Flo88] Andreas Floer. Morse theory for Lagrangian intersections. *Journal of Differential Geometry*, 28(3):513–547, 1988.

[Flo89] Andreas Floer. Witten's complex and infinite-dimensional Morse theory. *J. Differential Geom.*, 30(1):207–221, 1989.

[Oh97] Yong-Geun Oh. Symplectic topology as the geometry of action functional. I. Relative Floer theory on the cotangent bundle. *J. Differential Geom.*, 46(3):499–577, 1997.

[RS01] Joel W Robbin and Dietmar A Salamon. Asymptotic behaviour of holomorphic strips. *Annales de l'I.H.P. Analyse non linéaire*, 18(5):573–612, 2001.