Notes on Stable Homotopy Theory

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CHAPTER 1

Introduction

Speaker: Joseph Helfer

The goal of this talk is to say something about the stable homotopy theory, also known as the homotopy category of spectra $\mathsf{Ho}(\mathbf{Spectra}) = \mathsf{S}$, Quillen's results on complex cobordism, and recent applications in symplectic geometry, which is one of the motivations of this seminar. We start by looking at the homotopy category of topological spaces $\mathsf{Ho}(\mathsf{Top})$ and the derived category $D(R)$ of a given ring R. Invariants in topology, e.g. homology and homotopy groups, are functors on Top which passes to the corresponding homotopy category $\mathsf{Ho}(\mathsf{Top})$. $D(R)$ is the homotopy category of the category of chain complexes of R-modules $\mathsf{Ch}(R)$. The homotopy category of spectra is in some sense a category lying between the above two known examples.

The category of spectra is, roughly speaking, something kind of like both Top and $\mathsf{Ch}(R)$. The object of S are "stable spaces", i.e. $\Sigma^\infty X$ where Σ is the suspension operation. On the other hand, the objects of S are "generalized cohomology theories". We can use geometric constructions and categorical constructions to produce interesting cohomology theories. Complex cobordism is one kind of such generalized cohomology theory.

(1a) Stabilization Let X be a topological space. The **suspension** SX of X is the space $(X\times[0,1]/X\times\{0\})/X\times\{1\}$.

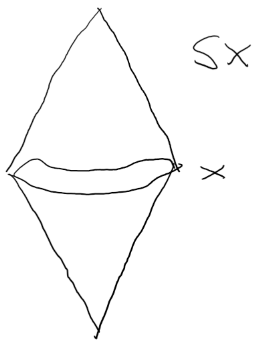

FIGURE 1.1: SUSPENSION

LEt (X, x_0) be a pointed space. The **reduced suspension** ΣX of X is the space $\Sigma X = SX/\{x_0\}\times I$.

1.1 Example. $S\mathbb{S}^n \simeq \mathbb{S}^{n+1}, \Rightarrow \mathbb{S}^n \simeq S^n\mathbb{S}^0$.

If (X, x_0) is well-pointed (e.g. x_0 is a vertex in a CW complex X, or X is a manifold), then $SX \to \Sigma X$ is a homotopy equivalence. (See [Hat00, Chapter 0] for a proof.) In fact, $\Sigma\mathbb{S}^n \simeq \mathbb{S}^{n+1}$, so $\Sigma^n\mathbb{S}^0 \simeq \mathbb{S}^n$.

There're some advantages of reduced suspension:

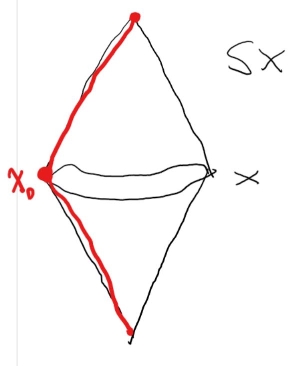

Figure 1.2: reduced suspension

- $\Sigma X = \mathbb{S}^1 \wedge X$, where $\wedge$ is the **smash product**, i.e. for pointed spaces Y and Z, the smash product $Y \wedge Z$ is defined to be $Y \wedge Z := Y \times Z / \{y_0\} \times Z \cup Y \times \{z_0\}$.

- Smash product is associative, so that $\Sigma^2 X \simeq \mathbb{S}^1 \wedge (\mathbb{S}^1 \wedge X) = (\mathbb{S}^1 \wedge \mathbb{S}^1) \wedge X = \mathbb{S}^2 \wedge X$, so in general, $\Sigma^n X = \mathbb{S}^n \wedge X$.

- There's an adjunction
$$\mathsf{Map}_*(X, \mathsf{Map}_*(Y, Z)) \simeq \mathsf{Map}_*(X \wedge Y, Z),$$
hence $\mathsf{Map}_*(\Sigma X, Y) \simeq \mathsf{Map}_*(X, \Omega Y)$ where $\Omega Y == \mathsf{Map}_*(\mathbb{S}^1, Y)$.

(1b) Properties of Reduced Suspension

1.2 Theorem. $H_n(X; G) \cong H_{n+1}(\Sigma X; G)$ and $H^n(X; G) \cong H^{n+1}(\Sigma X; G)$

Proof. Write SX as $CS \cup_X CX$, then use Mayer-Vietoris. □

1.3 Theorem (Freudenthal Suspension Theorem)**.** $\pi_n(X) \cong \pi_{n+1}(\Sigma X)$ for n large enough.

The isomorphism comes from the following: $\pi_n(X) = [\mathbb{S}^n, X] \to [\Sigma \mathbb{S}^n, \Sigma X] \simeq [\mathbb{S}^{n+1}, \Sigma X]$. i.e. the sequence of groups
$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$
stabilizes. For a proof, see [Hat00, Section 4.2]. More generally, for any finite CW complex Y, $[\Sigma^k Y, \Sigma^k X]$ stabilizes.

This is the first stable phenomena, and now we'll define

1.4 Definition. The ***n*th stable homotopy group** is given by $\pi_n^s(X) := \mathsf{colim}_k\, \pi_{n+k}(\Sigma^k X)$.

1.5 Remark. *$\pi_n(\mathbb{S}^m)$ are notoriously difficult to compute, but $\pi_n^s := \pi_n^s(\mathbb{S}^0)$ is somewhat easier, and much of stable homotopy theory is dedicated to this.*

The "stabilized spaces" "$\Sigma^\infty X$" should have well-defined H_*, H^*, π_*^s.

1.6 Definition. The **S-category** has

- Objects finite CW complexes;

- $\mathsf{Hom}(X, Y) := \mathsf{colim}_n [\Sigma^n X, \Sigma^n Y]$.

This is a first approximation to the homotopy category

1.7 Remark. *1) This category is additive. For any X, Y, $[\Sigma X, Y] \simeq [X, \Omega Y]$ is a group (for the same reason $\pi_1(Y)$ is), and if we suspend twice, then $[\Sigma^2 X, Y] \simeq [X, \Omega^2 Y]$ is an abelian group (as $\pi_2(Y)$ is, also $[\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y]$ is a homomorphism). Hence $\mathsf{Hom}(X, Y)$ is an abelian group and $\mathsf{Hom}(X, Y) \times \mathsf{Hom}(Y, Z)$ is bilinear.*

2) *Also, it's* **graded**: *we have groups*

$$\mathsf{Hom}(X,Y)_n := \mathsf{Hom}(\Sigma^n X, Y)$$

s.t. $\mathsf{Hom}(X,Y) = \mathsf{Hom}(X,Y)_0$ *and* $\mathsf{Hom}_*(X,Y) \otimes \mathsf{Hom}_*(Y,Z) \to \mathsf{Hom}(X,Z)$ *is a graded morphism.*

3) *The original motivation of introducing this category, due to Spanier and Whitehead, is a notion of "duality": objects in the* S*-category have a "dual"* DX*. This recovers the Alexander duality theorem* $\tilde{H}_k(\mathbb{S}^n \setminus K) \cong \tilde{H}^{n-k-1}(K)$ *for "good" compact* $K \subseteq \mathbb{S}^n$ *([SW55]) and the Poincaré duality([Ati61]).*

If we define a "stable object" $\boldsymbol{X}$ to be a sequence of pointed spaces $X_n \in \mathsf{Top}_*$ with maps $\Sigma X_n \to X_{n+1}$, we can define

$$\pi_n^s(\boldsymbol{X}) := \operatorname*{colim}_k \pi_{n+k}(X_k),$$

where $\pi_n(X_0) = [\mathbb{S}^n, X_0] \to [\mathbb{S}^{n+1}, \Sigma X_1] \xrightarrow{f_1} [\mathbb{S}^{n+1}, X_1] \to \cdots$. This recovers $\pi_n^s(X)$ by taking

$$\boldsymbol{X} = \Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \cdots\}$$

with $\Sigma X_n \xrightarrow{f_n = \mathrm{id}} X_{n+1}$.

(1c) Cohomology Theories Recall the Eilenberg-Steerod axioms for (reduced) cohomology theory: A cohomology theory is a sequence of contravariant functors $(h_n : \mathsf{Top}_* \to \mathsf{Ab}, \alpha_n)$ such that

- (homotopy invariance) h_n is invariant under homotopy equivalence, i.e. it defines a functor $h_n : \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}$;
- (suspension isomorphism) $h_n(-) \xrightarrow[\alpha_n]{\simeq} h_{n+1}(\Sigma -)$;
- For a CW-pair (X,A), $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact;
- (additivity) $h_n(\bigvee_{i\in I} X_i) \xrightarrow{\sim} \prod_{i\in I} h_n(X_i)$;
- (dimension axiom) $h_n(\mathrm{pt}) \simeq 0$ for all $n \neq 0$ and $h_0(\mathrm{pt}) \simeq G$ for some abelian group G.

1.8 Theorem. Any (h_n, α_n) satisfying these axioms is isomorphic to $H^*(-, G)$.

1.9 Definition. A **generalized(extraordinary) cohomology theory** is a datum (h_n, α_n) as above, satisfying everything except the dimension axiom.

There are corresponding axioms for homology, and definition of generalized homology theory.

1.10 Example.

- *The first one to be discovered is complex K-theory:* $K^0(X) := \{$*complex vector bundles* $E \to X\}/\sim$ *with* $\oplus$ *as addition and* $\otimes$ *as multiplication.*

  $K^{-2}(X) = K^0(\Sigma^2 X) \simeq K^0(X)$ *by Bott periodicity theorem, which means* $K^{-2n}(X) \simeq K^0(X)$*, so now we can define* $K^{2n}(X) \simeq K^0(X)$*, and* $K^{2n-1}(X) := K^{2n}(\Sigma X) = K^0(\Sigma X)$*.*

  1.11 Theorem. This is a generalized cohomology theory.

- *Given a space* X*, we define the* **bordism group** *of* X*,* $\Omega_k(X)$*, to be* $\{M \to X | M \text{ a } k\text{-manifold}\}/$*cobordism with* $\coprod$ *as addition. Here's a picture depicting this:*

  1.12 Theorem. This is a generalized homology theory.

  There's a corresponding cohomology theory as well.

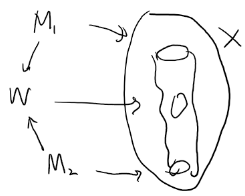

FIGURE 1.3: COBORDISM

- *Actually we have several different versions of bordism theorem. The above is unoriented cobordism, and we also have oriented bordism, framed bordism, and complex bordism. Each of them corresponds to different groups. The unoriented cobordism corresponds to the orthogonal group* O*, oriented bordism corresponds to* SO*, and complex bordism corresponds to* U*.*

- *There're lots of other generalized cohomology theories...*

1.13 Theorem (Brown representability theorem). For any cohomology theory $\{h^n\}$, there's a sequence of spaces $\{E_n\}$ such that

$$h^n \simeq [-, E_n] = \mathrm{Hom}_{\mathrm{Ho(Top_*)}}(-, E_n) : \mathrm{Ho(Top_*)} \to \mathrm{Ab}.$$

Observation: The suspension isomorphism $h^n(-) \to h^{n+1}(\Sigma -)$ gives an isomorphism $[-, E_n] \to [\Sigma -, E_{n+1}] \simeq [-, \Omega E_n]$, hence by Yoneda lemma, we have homotopy equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$.(So $E_0 \simeq \Omega^n E_n$). Hence

$$\Sigma E_n \to E_{n+1}$$

and we have a Ω-spectrum $\{E_i\}_i$, which is exactly the data we need to form a stable object. Actually, every spectrum arises this way.

1.14 Example. *For ordinary cohomology $H^*(-, G)$, the space E_n is called $K(G, n)$, the* **Eilenberg-MacLane space***, which has the special property that*

$$\pi_k(K(G, n)) \simeq [\mathbb{S}^k, K(G, n)] = H^n(\mathbb{S}^k; G) \simeq \begin{cases} G, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases}$$

Note: the **Eilenberg-MacLane spectrum $\boldsymbol{HG}$** $= \{K(G, n)\}_n$ satisfies

$$\pi_n^s(\boldsymbol{HG}) = \begin{cases} G, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\boldsymbol{HG}$ acts like a discrete space.

(1d) Constructing the category. The objects of Ho(Spectra) are spectra or stable objects as above, and what about the morphisms? Note that for Ho(Top) and $D(R)$ there're two approaches:

1) Take nice objects (CW complexes or projective/injective complexes) and homotopy classes of maps between them. Recall from Whitehead theorem(see Hatcher) that all topological spaces are weakly equivalent to CW complexes.

2) Take all objects and invert weak equivalences/quasi-isomorphisms. From Whitehead's theorem we can see that these two approaches produce the same homotopy category.

1) F. Adams takes the first approach to contruct this category, in which the notion of map is very complicated. See [AA74].

2) A more modern approach, as in [BR20], is the second one: $\boldsymbol{X} \to \boldsymbol{Y}$ is a **weak equivalence** if $\pi_n^s(\boldsymbol{X}) \to \pi_n^s(\boldsymbol{Y})$ is an isomorphism for all n. In this case we only need to invert the weak equivalences defined above. But the problem is we need to get some kind of handle on the result of inverting weak equivalences(localization), which leads to the notion of model categories. (There's another approach to the construction, using infinity categories.)

Since this categoryis complicated, B-R give ”axioms”: starting with a category S,

- Σ^∞: $\mathsf{Ho}(\mathsf{Top}) \to \mathsf{S}$;
- Hom sets in S are graded abelian groups;
- Each cohomology theory is represented by an object in S;
- etc.

(1e) More about stable homotopy. Given an additive and graded category S, for any $E \in \mathsf{S}$, we have a functor

$$\mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}^{\mathbb{Z}}$$

which is a cohomology theory (We can check Eilenberg-Steenrod axioms) and a homology theory: $\pi_*(\Sigma^\infty - \wedge \boldsymbol{E}) : \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}^{\mathbb{Z}}$. This satisfies the E-S axiom for homology again. Finally, there's an operation called ”smash product” $\wedge : \mathsf{S} \times \mathsf{S} \to \mathsf{S}$ generalizing $\wedge$ on topological spaces. (It is to spectra what $\otimes$ is to abelian groups.) $\boldsymbol{S} := \Sigma^\infty \mathbb{S}^0$ is to spectra what $\mathbb{Z}$ to abelian groups.

1.15 Definition. A **ring spectrum** is a spectrum $\boldsymbol{E}$ with a morphism $\boldsymbol{E} \wedge \boldsymbol{E} \to \boldsymbol{E}$ in S satisfying unit, associativity(commutativity if we want commutative ring specra).

CHAPTER 2

Basics of Model Categories

Speaker: Suraj Yadav

CHAPTER 3

Basics of Homotopy Theory

Speaker: Tianle Liu

CHAPTER 4

Basic of Stable Homotopy Theory

Speaker: Haoyang Liu

CHAPTER 5

K-theory and Bott Periodicity

Speaker: Haosen Wu

CHAPTER 6

Sequential Spectra

Speaker: Siyang Liu

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