Notes on Stable Homotopy Theory

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CHAPTER 1

Introduction

Speaker: Joseph Helfer

The goal of this talk is to say something about the stable homotopy theory, also known as the homotopy category of spectra $\mathsf{Ho}(\mathbf{Spectra}) = \mathsf{S}$, Quillen's results on complex cobordism, and recent applications in symplectic geometry, which is one of the motivations of this seminar. We start by looking at the homotopy category of topological spaces $\mathsf{Ho}(\mathsf{Top})$ and the derived category $D(R)$ of a given ring R. Invariants in topology, e.g. homology and homotopy groups, are functors on Top which passes to the corresponding homotopy category $\mathsf{Ho}(\mathsf{Top})$. $D(R)$ is the homotopy category of the category of chain complexes of R-modules $\mathsf{Ch}(R)$. The homotopy category of spectra is in some sense a category lying between the above two known examples.

The category of spectra is, roughly speaking, something kind of like both Top and $\mathsf{Ch}(R)$. The object of S are "stable spaces", i.e. $\Sigma^\infty X$ where Σ is the suspension operation. On the other hand, the objects of S are "generalized cohomology theories". We can use geometric constructions and categorical constructions to produce interesting cohomology theories. Complex cobordism is one kind of such generalized cohomology theory.

(1a) Stabilization Let X be a topological space. The **suspension** SX of X is the space $(X\times[0,1]/X\times\{0\})/X\times\{1\}$.

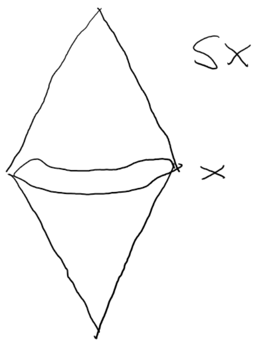

Figure 1.1: suspension

LEt (X, x_0) be a pointed space. The **reduced suspension** ΣX of X is the space $\Sigma X = SX/\{x_0\}\times I$.

1.1 Example. $S\mathbb{S}^n \simeq \mathbb{S}^{n+1}, \Rightarrow \mathbb{S}^n \simeq S^n\mathbb{S}^0$.

If (X, x_0) is well-pointed (e.g. x_0 is a vertex in a CW complex X, or X is a manifold), then $SX \to \Sigma X$ is a homotopy equivalence. (See [Hat00, Chapter 0] for a proof.) In fact, $\Sigma\mathbb{S}^n \simeq \mathbb{S}^{n+1}$, so $\Sigma^n\mathbb{S}^0 \simeq \mathbb{S}^n$.

There're some advantages of reduced suspension:

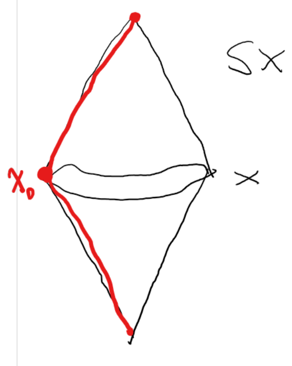

FIGURE 1.2: REDUCED SUSPENSION

- $\Sigma X = \mathbb{S}^1 \wedge X$, where $\wedge$ is the **smash product**, i.e. for pointed spaces Y and Z, the smash product $Y \wedge Z$ is defined to be $Y \wedge Z := Y \times Z / \{y_0\} \times Z \cup Y \times \{z_0\}$.

- Smash product is associative, so that $\Sigma^2 X \simeq \mathbb{S}^1 \wedge (\mathbb{S}^1 \wedge X) = (\mathbb{S}^1 \wedge \mathbb{S}^1) \wedge X = \mathbb{S}^2 \wedge X$, so in general, $\Sigma^n X = \mathbb{S}^n \wedge X$.

- There's an adjunction
$$\mathsf{Map}_*(X, \mathsf{Map}_*(Y, Z)) \simeq \mathsf{Map}_*(X \wedge Y, Z),$$
hence $\mathsf{Map}_*(\Sigma X, Y) \simeq \mathsf{Map}_*(X, \Omega Y)$ where $\Omega Y == \mathsf{Map}_*(\mathbb{S}^1, Y)$.

(1b) Properties of Reduced Suspension

1.2 Theorem. $H_n(X; G) \cong H_{n+1}(\Sigma X; G)$ and $H^n(X; G) \cong H^{n+1}(\Sigma X; G)$

Proof. Write SX as $CS \cup_X CX$, then use Mayer-Vietoris. □

1.3 Theorem (Freudenthal Suspension Theorem). $\pi_n(X) \cong \pi_{n+1}(\Sigma X)$ for n large enough.

The isomorphism comes from the following: $\pi_n(X) = [\mathbb{S}^n, X] \to [\Sigma \mathbb{S}^n, \Sigma X] \simeq [\mathbb{S}^{n+1}, \Sigma X]$. i.e. the sequence of groups
$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$
stabilizes. For a proof, see [Hat00, Section 4.2]. More generally, for any finite CW complex Y, $[\Sigma^k Y, \Sigma^k X]$ stabilizes.

This is the first stable phenomena, and now we'll define

1.4 Definition. The ***n*th stable homotopy group** is given by $\pi_n^s(X) := \mathsf{colim}_k\, \pi_{n+k}(\Sigma^k X)$.

1.5 Remark. *$\pi_n(\mathbb{S}^m)$ are notoriously difficult to compute, but $\pi_n^s := \pi_n^s(\mathbb{S}^0)$ is somewhat easier, and much of stable homotopy theory is dedicated to this.*

The "stabilized spaces" "$\Sigma^\infty X$" should have well-defined H_*, H^*, π_*^s.

1.6 Definition. The **S-category** has

- Objects finite CW complexes;
- $\mathsf{Hom}(X, Y) := \mathsf{colim}_n[\Sigma^n X, \Sigma^n Y]$.

This is a first approximation to the homotopy category

1.7 Remark. *1) This category is additive. For any X, Y, $[\Sigma X, Y] \simeq [X, \Omega Y]$ is a group (for the same reason $\pi_1(Y)$ is), and if we suspend twice, then $[\Sigma^2 X, Y] \simeq [X, \Omega^2 Y]$ is an abelian group (as $\pi_2(Y)$ is, also $[\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y]$ is a homomorphism). Hence $\mathsf{Hom}(X, Y)$ is an abelian group and $\mathsf{Hom}(X, Y) \times \mathsf{Hom}(Y, Z)$ is bilinear.*

2) *Also, it's* **graded**: *we have groups*

$$\mathrm{Hom}(X,Y)_n := \mathrm{Hom}(\Sigma^n X, Y)$$

s.t. $\mathrm{Hom}(X,Y) = \mathrm{Hom}(X,Y)_0$ *and* $\mathrm{Hom}_*(X,Y) \otimes \mathrm{Hom}_*(Y,Z) \to \mathrm{Hom}(X,Z)$ *is a graded morphism.*

3) *The original motivation of introducing this category, due to Spanier and Whitehead, is a notion of "duality": objects in the* S*-category have a "dual"* DX. *This recovers the Alexander duality theorem* $\tilde{H}_k(\mathbb{S}^n \setminus K) \cong \tilde{H}^{n-k-1}(K)$ *for "good" compact* $K \subseteq \mathbb{S}^n$ *([SW55]) and the Poincaré duality([Ati61]).*

If we define a "stable object" $\boldsymbol{X}$ to be a sequence of pointed spaces $X_n \in \mathrm{Top}_*$ with maps $\Sigma X_n \to X_{n+1}$, we can define

$$\pi_n^s(\boldsymbol{X}) := \operatorname*{colim}_k \pi_{n+k}(X_k),$$

where $\pi_n(X_0) = [\mathbb{S}^n, X_0] \to [\mathbb{S}^{n+1}, \Sigma X_1] \xrightarrow{f_1} [\mathbb{S}^{n+1}, X_1] \to \cdots$. This recovers $\pi_n^s(X)$ by taking

$$\boldsymbol{X} = \Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \cdots\}$$

with $\Sigma X_n \xrightarrow{f_n = \mathrm{id}} X_{n+1}$.

(1c) Cohomology Theories Recall the Eilenberg-Steerod axioms for (reduced) cohomology theory: A cohomology theory is a sequence of contravariant functors $(h_n : \mathrm{Top}_* \to \mathrm{Ab}, \alpha_n)$ such that

- (homotopy invariance) h_n is invariant under homotopy equivalence, i.e. it defines a functor $h_n : \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ab}$;
- (suspension isomorphism) $h_n(-) \xrightarrow[\alpha_n]{\simeq} h_{n+1}(\Sigma -)$;
- For a CW-pair (X,A), $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact;
- (additivity) $h_n(\bigvee_{i\in I} X_i) \xrightarrow{\sim} \prod_{i \in I} h_n(X_i)$;
- (dimension axiom) $h_n(\mathrm{pt}) \simeq 0$ for all $n \neq 0$ and $h_0(\mathrm{pt}) \simeq G$ for some abelian group G.

1.8 Theorem. Any (h_n, α_n) satisfying these axioms is isomorphic to $H^*(-, G)$.

1.9 Definition. A **generalized(extraordinary) cohomology theory** is a datum (h_n, α_n) as above, satisfying everything except the dimension axiom.

There are corresponding axioms for homology, and definition of generalized homology theory.

1.10 Example.
- *The first one to be discovered is complex K-theory:* $K^0(X) := \{$*complex vector bundles* $E \to X\}/\sim$ *with* $\oplus$ *as addition and* $\otimes$ *as multiplication.*

  $K^{-2}(X) = K^0(\Sigma^2 X) \simeq K^0(X)$ *by Bott periodicity theorem, which means* $K^{-2n}(X) \simeq K^0(X)$, *so now we can define* $K^{2n}(X) \simeq K^0(X)$, *and* $K^{2n-1}(X) := K^{2n}(\Sigma X) = K^0(\Sigma X)$.

  1.11 Theorem. This is a generalized cohomology theory.

- *Given a space* X, *we define the* **bordism group** *of* X, $\Omega_k(X)$, *to be* $\{M \to X | M$ *a* k*-manifold*$\}/$*cobordism with* $\coprod$ *as addition. Here's a picture depicting this:*

  1.12 Theorem. This is a generalized homology theory.

  There's a corresponding cohomology theory as well.

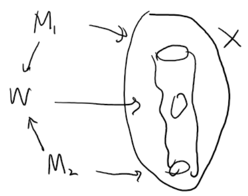

FIGURE 1.3: COBORDISM

- *Actually we have several different versions of bordism theorem. The above is unoriented cobordism, and we also have oriented bordism, framed bordism, and complex bordism. Each of them corresponds to different groups. The unoriented cobordism corresponds to the orthogonal group* O*, oriented bordism corresponds to* SO*, and complex bordism corresponds to* U*.*

- *There're lots of other generalized cohomology theories...*

1.13 Theorem (Brown representability theorem)**.** For any cohomology theory $\{h^n\}$, there's a sequence of spaces $\{E_n\}$ such that

$$h^n \simeq [-, E_n] = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}_*)}(-, E_n) : \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ab}.$$

<u>**Observation:**</u> The suspension isomorphism $h^n(-) \to h^{n+1}(\Sigma -)$ gives an isomorphism $[-, E_n] \to [\Sigma -, E_{n+1}] \simeq [-, \Omega E_n]$, hence by Yoneda lemma, we have homotopy equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$.(So $E_0 \simeq \Omega^n E_n$). Hence

$$\Sigma E_n \to E_{n+1}$$

and we have a Ω-spectrum $\{E_i\}_i$, which is exactly the data we need to form a stable object. Actually, every spectrum arises this way.

1.14 Example. *For ordinary cohomology* $H^*(-, G)$*, the space* E_n *is called* $K(G, n)$*, the* **Eilenberg-MacLane space***, which has the special property that*

$$\pi_k(K(G,n)) \simeq [\mathbb{S}^k, K(G,n)] = H^n(\mathbb{S}^k; G) \simeq \begin{cases} G, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases}$$

<u>**Note:**</u> the **Eilenberg-MacLane spectrum** $\boldsymbol{HG} = \{K(G,n)\}_n$ satisfies

$$\pi_n^s(\boldsymbol{HG}) = \begin{cases} G, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\boldsymbol{HG}$ acts like a discrete space.

(1d) Constructing the category. The objects of $\mathsf{Ho}(\mathsf{Spectra})$ are spectra or stable objects as above, and what about the morphisms? Note that for $\mathsf{Ho}(\mathrm{Top})$ and $D(R)$ there're two approaches:

1) Take nice objects (CW complexes or projective/injective complexes) and homotopy classes of maps between them. Recall from Whitehead theorem(see Hatcher) that all topological spaces are weakly equivalent to CW complexes.

2) Take all objects and invert weak equivalences/quasi-isomorphisms. From Whitehead's theorem we can see that these two approaches produce the same homotopy category.

1) F. Adams takes the first approach to contruct this category, in which the notion of map is very complicated. See [Ada74].

2) A more modern approach, as in [BR20], is the second one: $\boldsymbol{X} \to \boldsymbol{Y}$ is a **weak equivalence** if $\pi_n^s(\boldsymbol{X}) \to \pi_n^s(\boldsymbol{Y})$ is an isomorphism for all n. In this case we only need to invert the weak equivalences defined above. But the problem is we need to get some kind of handle on the result of inverting weak equivalences(localization), which leads to the notion of model categories. (There's another approach to the construction, using infinity categories.)

Since this categoryis complicated, B-R give "axioms": starting with a category S,

- Σ^∞: $\mathsf{Ho}(\mathsf{Top}) \to \mathsf{S}$;
- Hom sets in S are graded abelian groups;
- Each cohomology theory is represented by an object in S;
- etc.

(1e) More about stable homotopy. Given an additive and graded category S, for any $E \in \mathsf{S}$, we have a functor

$$\mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}^{\mathbb{Z}}$$

which is a cohomology theory (We can check Eilenberg-Steenrod axioms) and a homology theory: $\pi_*(\Sigma^\infty - \wedge \boldsymbol{E}) : \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}^{\mathbb{Z}}$. This satisfies the E-S axiom for homology again. Finally, there's an operation called "smash product" $\wedge : \mathsf{S} \times \mathsf{S} \to \mathsf{S}$ generalizing $\wedge$ on topological spaces. (It is to spectra what $\otimes$ is to abelian groups.) $\boldsymbol{S} := \Sigma^\infty \mathbb{S}^0$ is to spectra what $\mathbb{Z}$ to abelian groups.

1.15 Definition. A **ring spectrum** is a spectrum $\boldsymbol{E}$ with a morphism $\boldsymbol{E} \wedge \boldsymbol{E} \to \boldsymbol{E}$ in S satisfying unit, associativity(commutativity if we want commutative ring specra).

(1f) A bit more on model categories

Localization. Let $W \subseteq \mathsf{C}$ be categories. The localization means a category $\mathsf{C}[W^{-1}]$ with the universal property that

$$\begin{tikzcd} \mathsf{C} \arrow[r, "l"] \arrow[dr, "F"'] & \mathsf{C}[W^{-1}] \arrow[d, dashed] \\ & \mathsf{D} \end{tikzcd}$$

If F sends morphisms in W to isomorphisms in D.

Different models for the same homotopy theory. We can have different models for the same homotopy theory, for example, topological category and the category of simplicial sets. We say two categories are "Quillen equivalent" if they give the same homotopy theory. The sequential spectra, symmetric and orthogonal spectra we're going to talk about this semester are all Quillen equivalent.

Cohomology theories. Given $\boldsymbol{E} \in \mathsf{S}$, we can define $E^*(\boldsymbol{X}) := [\Sigma^\infty X, \boldsymbol{X}]_{-*}$. For X a CW complex this defines a cohomology theory. In general, given $\boldsymbol{X} \in \mathsf{S}$, $E^*X := [\boldsymbol{X}, \boldsymbol{E}]_{-n}$.

Homology Theories Given $E \in \mathsf{S}$, we define the generalized homology theory $E_*(\boldsymbol{X}) := [\boldsymbol{S} = \Sigma^\infty \mathbb{S}^0, X \wedge E]_* = \pi_*(X \wedge E)$. If X is a space, then we just let $E_*(X) = E_*(\Sigma^\infty X)$.

Closed Model Structure. Smash product on Top extends to a smash product $\mathsf{Ho}(\mathsf{Top}) \xrightarrow{\Sigma^\infty} \mathsf{S}$ to a monoidal structure $\mathsf{S} \wedge \mathsf{S} \xrightarrow{\wedge} \mathsf{S}$ with **unit** $\boldsymbol{S}\colon \boldsymbol{S} \wedge \boldsymbol{E} \simeq \boldsymbol{E}$. For given $\boldsymbol{X}, \boldsymbol{Y} \in \mathsf{S}$, we have a **mapping spectrum** $\mathsf{Map}(\boldsymbol{X}, \boldsymbol{Y}) \in \mathsf{S}$ and $[\boldsymbol{X} \wedge \boldsymbol{Y}, \boldsymbol{Z}] \simeq [X, \mathsf{Map}(\boldsymbol{Y}, \boldsymbol{Z})]$.

Ring Spectra and Module Spectra. From the discussions above we know that S behaves in some sense similar to the category of abelian groups: tensor product of abelian groups correspond to smash products, $\mathsf{Hom}_{\mathbb{Z}}$ corresponds to mapping spectra, and S has a unit which is the sphere spectrum $\boldsymbol{S}$.

1.16 Definition. A **ring spectrum** is an object $\boldsymbol{R} \in \mathsf{S}$ with maps $\boldsymbol{R} \wedge \boldsymbol{R} \to \boldsymbol{R}$ and $\boldsymbol{S} \xrightarrow{e} \boldsymbol{R}$ such that the commutative diagrams in the homotopy category describing the associativity and units are satisfied.

One can also demand stronger associativity and commutativity conditions rather than "up to homotopy". For example, "A_∞", "E_∞", and "highly-structured ring spectra".

1.17 Definition. If $\boldsymbol{R}$ is a ring spectrum, then a **module spectrum** $\boldsymbol{M}$ is a spectrum with a map $\boldsymbol{R} \wedge \boldsymbol{M} \to \boldsymbol{M}$ with the condition similar to that of a module.

Note that any spectra is naturally a $\boldsymbol{S}$-module.

(1g) Thom Space Let $V \to X$ be a vector bundle over a topological space X, and assume V admits a norm, then we can construct the associated disk bundle $\mathbb{D}(V)$ of all vectors with norm ≤ 1 and the sphere bundle $\mathbb{S}(V)$ of all vectors with norm 1.

1.18 Definition. The **Thom space** is the quotient space $\mathsf{Th}(V) := \mathbb{D}(V)/\mathbb{S}(V)$.

Another way to describe is that the Thom space is the one-point compatification of each fibre and identify all the ∞s.

<u>**Note:**</u> If $V \simeq \mathbb{R}^n$, then $\mathsf{Th}(V) \simeq \Sigma^n X$.

There're very special Thom spaces, for example, BO_n, which completely classifies real vector bundles up to isomorphism, i.e. given any real vector bundle $V \to X$, there is a unique up to homotopy map $X \to BO_n$ and the universal vector bundle $\gamma_n \to BO_n$ such that we have a map of bundles $\Phi\colon V \to \gamma_n$ over $X \to BO_n$ with the pull-back diagram

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d] & \gamma_n \arrow[d] \\ X \arrow[r] & BO_n \end{tikzcd}$$

The same for BSO_n, which classifies oriented bundles, and BU_n, which classifies complex vector bundles.

MSO_n is the Thom space $\mathsf{Th}(\gamma_n \to BSO_n)$, and similarly MU_n is the Thom space $\mathsf{Th}(\gamma_n \to BU_n)$. MSO is a spectra, called a **Thom spectra**. First of all,

$$MSO = \{MSO_1, MSO_2, \cdots,\} \left| \begin{tikzcd} \gamma_n \oplus \mathbb{R} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] \\ BSO_n \arrow[r] & BSO_{n+1} \end{tikzcd} \right. \Rightarrow \Sigma MSO_n \simeq \mathsf{Th}(\gamma_n \oplus \mathbb{R}) \to MSO_{n+1},$$

and MU is similar, with a little twist that

$$MU = \{MU_1, \Sigma MU_1, MU_2, \Sigma MU_2, \cdots\}$$

(1h) Bordism and Cobordism Given $X \subseteq M$, we have the **Pontrjagin-Thom construction**: there is a map $X \to BO_n$ which classifies the bundle $N_M X \to X$, this map $N_M X \to \gamma_n$ of vector bundles then induces a map of Thom spaces $\mathsf{Th}(N_M X) \to MO_n$, but we then have the "collapse map" $M \to \mathsf{Th}(N_M X)$ by collapsing the complement of $N_M X$ to a point. So a codimension n submanifold is in one-to-one correspondence to maps from M to MO_n, and two submanifolds X and X' are cobordant if and only if the two maps from M to MO_n are **stably homotopic**, i.e.

$$\{\text{submanifolds of } M\}/\text{cobordism} \simeq \operatorname*{colim}_n [M, MO_n].$$

The outcome of this construction is that the homology theory represented by the spectra MO is the **bordism**: $MO_*(X) \simeq \Omega_*(X)$, and similarly, $MO^*(X)$ is called **cobordism**. If X is a manifold, then it's a duality, meaning that the bordism and cobordism groups are isomorphic. Similarly, MSO_* and MSO^* are oriented bordisms and cobordisms.

MU_* and MU^* are complex bordisms and cobordisms.

Observe that $\pi_*(MO) \simeq MO_*(\mathsf{pt}) \simeq \Omega_*$, which is the "cobordism ring".

(1i) MU and complex orientations Suppose we have a vector bundle $V \to X$, then $H^*(\mathsf{Th}(V)) \to H^*(\mathsf{Th}(V_x)) \simeq \mathbb{S}^n$. Then there's a theorem of Thom saying that there is a "Thom class" $u \in H^n(\mathsf{Th}(V))$ which goes to $\pm 1 \in H^n(\mathbb{S}^n)$. The orientation is the same as the existence of such a Thom class.

1.19 Definition. $\boldsymbol{E} \in \mathsf{S}$ is **complex oriented** if for each $\boldsymbol{X} \in \mathsf{Top}$ and $V \to X$ complex vector bundle, there exists a class $u \in H^{2n}(\mathsf{Th}(v)) = E^*(\mathsf{Th}(V))$. ($H\mathbb{Z}$, K and MU are complex oriented)

<u>**Fact:**</u> MU is the universal complex oriented cobordism theory: if $\boldsymbol{E}$ is complex oriented theory, then there exists a map $MU \to \boldsymbol{E}$ inducing that complex orientation.

(1j) Formal Group Laws. If E is complex oriented, we can see by a spectral sequence argument that $E^*(\mathbb{C}P^\infty) \cong E_*[[t]]$, where $E_* = \pi_*(\boldsymbol{E})$. Moreover, $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E_*[[u,v]]$. We can then find a universal class $f(u,v)$, which is a **formal group** over E_*, which means that it satisfies the following conditions:

- $f(a,0) = f(0,a)$;
- $f(f(a,b),c) = f(a,f(b,c))$(inverses are free);

<u>**Fact 2:**</u> $\pi_*(MU)$ is the "Lazard ring".

There're two interesting theories, the Brown-Peterson and Morava K-theory, which is obtained from MU. Finally, there's a theorem by Abouzaid-McLean-Smith:

1.20 Theorem. Assume Y is a projective variety, $Y \to \mathbb{C}P^1$ holomorphic submersion with fiber X, then $H^*(Y;\mathbb{Z}) \cong H^*(X;\mathbb{Z}) \otimes H^*(\mathbb{S}^2;\mathbb{Z})$.

This theorem was know over $\mathbb{Q}$, and the statement involves nothing about homotopy theory. The proof is to first replace $\mathbb{Z}$ by any complex oriented cohomology theory. To do this, they first prove this for MU, and then for BP, and finally for all "$K(n)$-local" cohomology theories. The reason why these all have to do with symplectic geometry is that the Morava K-theory are well-behaved with respect to orbifolds. In symplectic geometry, we have the moduli space of pseudo-holomorphic curves which are orbifolds.

CHAPTER 2

Basics of Model Categories

Speaker: Suraj Yadav

The notion of model category allows us to do abstract homotopy theory.

2.1 Definition. A **model category** is a category C with three classes of morphisms $\mathcal{W}$, the class of weak equivalences, $\mathcal{C}$, the class of cofibrations, and $\mathcal{F}$, the class of fibrations, with the following properties:

1) C is closed under finite limits and colimits;

2) (2 out of 3) given three objects X, Y, Z and a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd},$$

   if any two morphisms are in $\mathcal{W}$, then so is the third.

3) (retracts) The retract of any morphismin $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ is again in $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ respectively. Here we say a morphism $X \xrightarrow{f} Y$ is a **retract** of $U \to V$ if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & U \arrow[r] \arrow[d, "g"] & X \arrow[d] \\ Y \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

   so that the composition of the upper and lower rows are identities.

4) (lifting property) Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

   such that

   a) $f \in \mathcal{W} \cap \mathcal{C}$, $g \in \mathcal{F}$ implies there exists a lifting $H\colon B \to X$;
   b) If $f \in \mathcal{C}$, $g \in \mathcal{W} \cap \mathcal{F}$, then there exists a lifting $H\colon B \to X$;

5) (factorization) For any morphism $f : X \to Y$, there are factorizations

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y$$

$$X \xrightarrow{f^1} Z \xrightarrow{f^2} Y$$

of f, where $f_1 \in \mathcal{W} \cap \mathcal{C}$, $f_2 \in \mathcal{F}$, $f^1 \in \mathcal{C}$ and $f^2 \in \mathcal{F} \cap \mathcal{W}$.

2.2 Definition. Let C be a model category. An object $X \in$ C is **fibrant** if $X \to *$ is a fibration; $Y \in$ C is **cofibrant** if $f : * \to Y$ is a cofibration.

For any object $X \in$ C, we have a unique morphism $\varnothing \to X$ which factors through a cofibrant object Y such that $Y \to X$ is the trivial fibration. (Here we say a fibration is **trivial** if it's both a fibration and a weak equivalence.) We want to get a cofibrant and fibrant object Z which is weakly equivalent to X, so we consider the morphism $Y \to *$, and consider the factorization $Y \to Z \to *$ where $Y \to Z$ is the trivial cofibration and $Z \to *$ is a fibration, so Z is both cofibrant and fibrant. Now we want to show that Z is weakly equivalent to X.

2.3 Example. *For the category of topological spaces, we have two kinds of model structures. One of them is called the* **Serre model structure***, where we make weak equivalences to be weak homotopy equivalences, fibrations to be Serre fibrations, i.e. we have the lifting property for all maps $A \to A \times [0,1]$ where A is a CW complex.*

Another structure is the **Hurewicz moel structure***, weak equivalences are homotopy equivalences, and fibrations have lifting properties with respect to maps $A \to A \times [0,1]$ where A is any topological space.*

2.4 Example. *The category of simplicial sets also admits a model structure. Let $\triangle$ be the cosimplex category whose objects are $[n] = \{0, 1, \cdots, n\}$ the set of natural numbers, and morphisms order-preserving maps $[n] \to [m]$. We have a class of special morphisms $d^i : [n] \to [n+1]$ defined by $d^i(k) = k$ if $k < i$, and $d^i(k) = k+1$ if $k \geq i$, and $s^j : [n+1] \to [n]$ given by $s^j(k) = k$ for $k < j$ and $s^k(k) = k-1$ for $k \geq j$.*

2.5 Definition. A **simplicial set** X is a functor

$$X : \triangle^{op} \to \mathrm{Set}.$$

This means that a simplicial set is a data $[n] \mapsto X_n$ with maps $x_{n+1} \to X_n \to X_{n-1}$ with the given compatibility conditions. Simplicial sets are combinatorial data of topological spaces. Simplicial sets are representable functors with representation $\Delta^n := \mathsf{Hom}(-, [n])$. Let sSet be the category of simplicial sets, then we have a **geometric realization** functor

$$|-| : \mathrm{sSet} \to \mathrm{Top}$$

which is adjoint to the singular functor $\mathsf{Sing}_* : \mathrm{Top} \to \mathrm{sSet}$. For the standard n-simplex Δ^n, $|\Delta^n|$ is just the standard n-simplex $\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \Sigma x_i = 1,\ x_i \geq 0\}$.

For any topoogical space Y, we define $(\mathsf{Sing}_* Y)_n = \mathsf{Hom}_{\mathsf{Top}}(|\triangle^n|, Y)$ and we can check that this actually defines a simplicial set. Although these two categorys sSet and Top are not equivalent, their homotopy categories are equivalent.

The model structure on sSet is given as follows: $X \to Y$ is a weak equivalence of simplicial sets if $|X| \to |Y|$ is a weak homotopy equivalence of topological spaces, $X \to Y$ is a cofibration if $X_n \to Y_n$ is a monomorphism for any n, and $X \to Y$ is a fibration if it has lifting property with respect to all cofibrations.

2.6 Example. *For any ring R, the category of chain complexes $\mathsf{Ch}(R)$ is the category with objects chain complexes $\cdots \to C_2 \to C_1 \to C_0 \to 0$. This category admits natural model structures: a morphism $f_\bullet : C_\bullet \to D_\bullet$ is a weak equivalence if the induced map on homology $f_* : H_* C \to H_* D$ is an isomorphism. $f_\bullet$ is a cofibration if $f_n : C_n \to D_n$ is injective with projective cokernel. $f_\bullet$ is a fibration if $f_n : C_n \twoheadrightarrow D_n$ is surjective.*

This is the projective model structure on $\mathsf{Ch}(R)$*, since the cofibrant objects in this structure are chain complexes of projective modules, and the cofibrant replacement is just the same as taking projective resolutions.*

Another model structure is the so-called **injective model structure***, where fibrations are degreewise surjective maps with injective kernels and cofibrations degreewise injective maps. Similarly, fibrant replacements in this category are injective resolutions.*

Now we proceed to define homotopy category of a model category.

2.7 Definition. Consider the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X \sqcup X \end{tikzcd}$$

The identity map $X \xrightarrow{\mathrm{id}} X$ gives a natural map $\tau : X \sqcup X \to X$. A **cylindrical object** is the following data

$$X \sqcup X \xrightarrow{C} \mathsf{Cyl}(X) \xrightarrow{W_F} X$$

where C is a cofibration and W_F is the trivial fibration.

The motivation of this cylinder object is the usual cylinder $A \times [0,1]$ for a given topological space A.

2.8 Lemma. *Suppose we have a map* $X \xrightarrow{f} Y$ *which is a weak equivalence, then we have a natural induced map* $\mathsf{Cyl}(X) \to \mathsf{Cyl}(Y)$ *which is also a weak equivalence fitting into the commutative diagram*

$$\begin{tikzcd} X \sqcup X \arrow[r] \arrow[d] & Y \sqcup Y \arrow[r, "C"] & Y \sqcup Y \arrow[d, "W_F"] \\ \mathsf{Cyl}(X) \arrow[r, "W_F"] \arrow[urr, dashed] & X \arrow[r, "f"] & Y \end{tikzcd}$$

2.9 Definition. Two morphisms $f, g : X \to Y$ are **left homotopic** if there exists a morphism $H : \mathsf{Cyl}(X) \to Y$ such that $H_0 i_0 = f$ and $Hi_1 = g$. Here $(i_0, i_1) : X \sqcup X \to \mathsf{Cyl}(X)$ are the two inclusion maps of X into $\mathsf{Cyl}(X)$.

The problem is, in a general model category, the notion of homotopy equivalence is not an equivalence relation. Now we give a dual construction.

2.10 Definition. Taking any object $Y \in \mathsf{C}$, the **path object** of Y is the factorization of $Y \xrightarrow{\Delta} Y \times Y$

$$Y \xrightarrow{W_C} PY \xrightarrow{(e_0, e_1)} Y \times Y$$

where $Y \to PY$ is the trivial cofibration and $PY \to Y \times Y$ is the fibration.

2.11 Definition. $f, g : X \to Y$ are **right homotopic** if there exists a morphism $H : X \to PY$ such that $e_0 H = f$ and $e_1 H = g$.

Now we can define the homotopy category of a given model category C. Given $X, Y \in \mathsf{C}$, we consider the cofibrant-fibrant replacement of both X and Y, i.e. X^{cf} and Y^{cf}, and consider the set of morphisms $\mathsf{Hom}_{\mathsf{C}}(X^{cf}, Y^{cf})$. We use the following fact:

(a) If X is cofibrant, then left homotopy is an equivalence relation on $\mathsf{C}(X, Y)$;

(b) If Y is fibrant, then right homotopy is an equivalence relation on $\mathsf{C}(X, Y)$;

(c) If X is cofibrant and Y is fibrant, then $f, g : X \to Y$ are left homotopic if and only if they are right homotopic.

Therefore we can define the **homotopy category** $\mathsf{Ho}(\mathsf{C})$ of C to be the category with objects those objects in C and morphism sets $\mathsf{Hom}_{\mathsf{C}}(X^c, Y^f)/\sim$ where $f \sim g$ if and only if they are left or right homotopic.

In this homotopy category, we know that if $f : X \to Y$is a weak equivalence with X, Y cofibrant-fibrant, then f is a homotopy equivalence.

2.12 Lemma. *$f : X \to Y$ is an isomorphism in $\mathsf{Ho}(\mathsf{C})$ if and only if f is a weak equivalence in C.*

Therefore the notion of "localization at $\mathcal{W}$" in C is the same as the homotopy category of C.

Finally, we define the Quillen functors:

2.13 Definition. $F : \mathsf{C} \to \mathsf{D}$ is called **left Quillen** if it preserves cofibrations and trivial cofibrations, and **right Quillen** if it preserves fibrations and trivial fibrations.

2.14 Definition. $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ a pair of functors. We say they are **Quillen adjunction** if they are adjunctions and one of the following conditions hold:

1) F and G have to be left Quillen and right Quillen respectively;

2) F is left Quillen;

3) G is right Quillen;

4) F preserves trivial cofibrations and cofibrations between cofibrant objects.

5) G preserves trivial fibrations and fibrations between fibrant objects.

2.15 Example. *$|-| : \mathsf{sSets} \rightleftarrows \mathsf{Top}_* : \mathsf{Sing}_*$ are Quillen adjunct.*

2.16 Definition. Let $F : \mathsf{C} \to \mathsf{D}$ be a left Quillen functor, then $LF : \mathsf{Ho}(\mathsf{C}) \to \mathsf{Ho}(\mathsf{D})$ is given by $LF(X) := F(X^c)$. Similarly, if $G : \mathsf{D} \to \mathsf{C}$ is right Quillen, then we can define $RF : \mathsf{Ho}(\mathsf{D}) \to \mathsf{Ho}(\mathsf{C})$ by $RG(X) := G(X^f)$.

2.17 Example. *The sheaf cohomology $H^*(X, -)$ is the example of RG for G the global section functor.*

So LF and RF are generalizations of left and right derived functors in the model category.

2.18 Definition. A Quillen adjunction

$$F : \mathsf{C} \rightleftarrows \mathsf{D} \ : G$$

is a **Quillen equivalence** if

$$LF : \mathsf{Ho}(\mathsf{C}) \rightleftarrows \mathsf{Ho}(\mathsf{D}) \ : RG$$

is an equivalence of categories.

For example, the projective and injective model structures on $\mathsf{Ch}(R)$ are Quillen equivalent.

CHAPTER 3

Basics of Homotopy Theory

Speaker: Tianle Liu

CHAPTER 4

Basic of Stable Homotopy Theory

Speaker: Haoyang Liu

CHAPTER 5

K-theory and Bott Periodicity

Speaker: Haosen Wu

CHAPTER 6

Sequential Spectra

Speaker: Siyang Liu

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