Notes on Stable Homotopy Theory

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April 27, 2022

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TALK 1

Introduction

Speaker: Joseph Helfer

The goal of this talk is to say something about the stable homotopy theory, also known as the homotopy category of spectra Ho(**Spectra**) = S, Quillen's results on complex cobordism, and recent applications in symplectic geometry, which is one of the motivations of this seminar. We start by looking at the homotopy category of topological spaces Ho(Top) and the derived category $D(R)$ of a given ring R. Invariants in topology, e.g. homology and homotopy groups, are functors on Top which passes to the corresponding homotopy category Ho(Top). $D(R)$ is the homotopy category of the category of chain complexes of R-modules Ch(R). The homotopy category of spectra is in some sense a category lying between the above two known examples.

The category of spectra is, roughly speaking, something kind of like both Top and Ch(R). The object of S are "stable spaces", i.e. $\Sigma^\infty X$ where Σ is the suspension operation. On the other hand, the objects of S are "generalized cohomology theories". We can use geometric constructions and categorical constructions to produce interesting cohomology theories. Complex cobordism is one kind of such generalized cohomology theory.

(1a) Stabilization Let X be a topological space. The **suspension** SX of X is the space $(X \times [0,1]/X \times \{0\})/X \times \{1\}$.

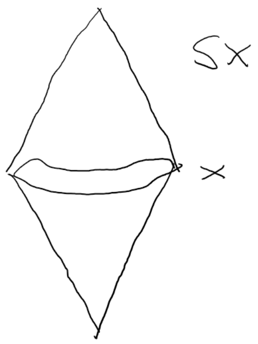

Figure 1.1: suspension

LEt (X, x_0) be a pointed space. The **reduced suspension** ΣX of X is the space $\Sigma X = SX/\{x_0\} \times I$.

1.1 Example. $S\mathbb{S}^n \simeq \mathbb{S}^{n+1}$, $\Rightarrow \mathbb{S}^n \simeq S^n\mathbb{S}^0$.

If (X, x_0) is well-pointed (e.g. x_0 is a vertex in a CW complex X, or X is a manifold), then $SX \to \Sigma X$ is a homotopy equivalence. (See [Hat00, Chapter 0] for a proof.) In fact, $\Sigma\mathbb{S}^n \simeq \mathbb{S}^{n+1}$, so $\Sigma^n\mathbb{S}^0 \simeq \mathbb{S}^n$.

There're some advantages of reduced suspension:

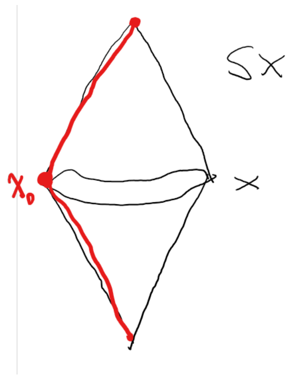

Figure 1.2: reduced suspension

- $\Sigma X = \mathbb{S}^1 \wedge X$, where $\wedge$ is the **smash product**, i.e. for pointed spaces Y and Z, the smash product $Y \wedge Z$ is defined to be $Y \wedge Z := Y \times Z / \{y_0\} \times Z \cup Y \times \{z_0\}$.
- Smash product is associative, so that $\Sigma^2 X \simeq \mathbb{S}^1 \wedge (\mathbb{S}^1 \wedge X) = (\mathbb{S}^1 \wedge \mathbb{S}^1) \wedge X = \mathbb{S}^2 \wedge X$, so in general, $\Sigma^n X = \mathbb{S}^n \wedge X$.
- There's an adjunction
$$\mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)) \simeq \mathrm{Map}_*(X \wedge Y, Z),$$
hence $\mathrm{Map}_*(\Sigma X, Y) \simeq \mathrm{Map}_*(X, \Omega Y)$ where $\Omega Y == \mathrm{Map}_*(\mathbb{S}^1, Y)$.

(1b) Properties of Reduced Suspension

1.2 Theorem. $H_n(X; G) \cong H_{n+1}(\Sigma X; G)$ and $H^n(X; G) \cong H^{n+1}(\Sigma X; G)$

Proof. Write SX as $CS \cup_X CX$, then use Mayer-Vietoris. □

1.3 Theorem (Freudenthal Suspension Theorem)**.** $\pi_n(X) \cong \pi_{n+1}(\Sigma X)$ for n large enough.

The isomorphism comes from the following: $\pi_n(X) = [\mathbb{S}^n, X] \to [\Sigma \mathbb{S}^n, \Sigma X] \simeq [\mathbb{S}^{n+1}, \Sigma X]$. i.e. the sequence of groups
$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$
stabilizes. For a proof, see [Hat00, Section 4.2]. More generally, for any finite CW complex Y, $[\Sigma^k Y, \Sigma^k X]$ stabilizes.

This is the first stable phenomena, and now we'll define

1.4 Definition. The **nth stable homotopy group** is given by $\pi_n^s(X) := \mathrm{colim}_k\, \pi_{n+k}(\Sigma^k X)$.

1.5 Remark. *$\pi_n(\mathbb{S}^m)$ are notoriously difficult to compute, but $\pi_n^s := \pi_n^s(\mathbb{S}^0)$ is somewhat easier, and much of stable homotopy theory is dedicated to this.*

The "stabilized spaces" "$\Sigma^\infty X$" should have well-defined H_*, H^*, π_*^s.

1.6 Definition. The **S-category** has

- Objects finite CW complexes;
- $\mathrm{Hom}(X, Y) := \mathrm{colim}_n [\Sigma^n X, \Sigma^n Y]$.

This is a first approximation to the homotopy category

1.7 Remark. *1) This category is additive. For any X, Y, $[\Sigma X, Y] \simeq [X, \Omega Y]$ is a group (for the same reason $\pi_1(Y)$ is), and if we suspend twice, then $[\Sigma^2 X, Y] \simeq [X, \Omega^2 Y]$ is an abelian group (as $\pi_2(Y)$ is, also $[\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y]$ is a homomorphism). Hence $\mathrm{Hom}(X, Y)$ is an abelian group and $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z)$ is bilinear.*

2) *Also, it's* **graded**: *we have groups*

$$\mathrm{Hom}(X,Y)_n := \mathrm{Hom}(\Sigma^n X, Y)$$

s.t. $\mathrm{Hom}(X,Y) = \mathrm{Hom}(X,Y)_0$ *and* $\mathrm{Hom}_*(X,Y) \otimes \mathrm{Hom}_*(Y,Z) \to \mathrm{Hom}(X,Z)$ *is a graded morphism.*

3) *The original motivation of introducing this category, due to Spanier and Whitehead, is a notion of "duality": objects in the S-category have a "dual" DX. This recovers the Alexander duality theorem $\tilde{H}_k(\mathbb{S}^n \setminus K) \cong \tilde{H}^{n-k-1}(K)$ for "good" compact $K \subseteq \mathbb{S}^n$([SW55]) and the Poincaré duality([Ati61]).*

If we define a "stable object" $\boldsymbol{X}$ to be a sequence of pointed spaces $X_n \in \mathrm{Top}_*$ with maps $\Sigma X_n \to X_{n+1}$, we can define

$$\pi_n^s(\boldsymbol{X}) := \operatorname*{colim}_k \pi_{n+k}(X_k),$$

where $\pi_n(X_0) = [\mathbb{S}^n, X_0] \to [\mathbb{S}^{n+1}, \Sigma X_1] \xrightarrow{f_1} [\mathbb{S}^{n+1}, X_1] \to \cdots$. This recovers $\pi_n^s(X)$ by taking

$$\boldsymbol{X} = \Sigma^\infty X = \{X, \Sigma X, \Sigma^2 X, \cdots\}$$

with $\Sigma X_n \xrightarrow{f_n = \mathrm{id}} X_{n+1}$.

(1c) Cohomology Theories Recall the Eilenberg-Steerod axioms for (reduced) cohomology theory: A cohomology theory is a sequence of contravariant functors $(h_n : \mathrm{Top}_* \to \mathrm{Ab}, \alpha_n)$ such that

- (homotopy invariance) h_n is invariant under homotopy equivalence, i.e. it defines a functor $h_n : \mathrm{Ho}\ (\mathrm{Top}_*) \to \mathrm{Ab}$;
- (suspension isomorphism) $h_n(-) \underset{\alpha_n}{\xrightarrow{\simeq}} h_{n+1}(\Sigma -)$;
- For a CW-pair (X,A), $h_n(A) \to h_n(X) \to h_n(X/A)$ is exact;
- (additivity) $h_n(\bigvee_{i\in I} X_i) \xrightarrow{\sim} \prod_{i \in I} h_n(X_i)$;
- (dimension axiom) $h_n(\mathrm{pt}) \simeq 0$ for all $n \neq 0$ and $h_0(\mathrm{pt}) \simeq G$ for some abelian group G.

1.8 Theorem. Any (h_n, α_n) satisfying these axioms is isomorphic to $H^*(-,G)$.

1.9 Definition. A **generalized(extraordinary) cohomology theory** is a datum (h_n, α_n) as above, satisfying everything except the dimension axiom.

There are corresponding axioms for homology, and definition of generalized homology theory.

1.10 Example.
- *The first one to be discovered is complex K-theory: $K^0(X) := \{$complex vector bundles $E \to X\}/\sim$ with $\oplus$ as addition and $\otimes$ as multiplication.*

  $K^{-2}(X) = K^0(\Sigma^2 X) \simeq K^0(X)$ by Bott periodicity theorem, which means $K^{-2n}(X) \simeq K^0(X)$, so now we can define $K^{2n}(X) \simeq K^0(X)$, and $K^{2n-1}(X) := K^{2n}(\Sigma X) = K^0(\Sigma X)$.

  1.11 Theorem. This is a generalized cohomology theory.

- *Given a space X, we define the* **bordism group** *of X, $\Omega_k(X)$, to be $\{M \to X | M$ a k-manifold$\}$/cobordism with $\coprod$ as addition. Here's a picture depicting this:*

  1.12 Theorem. This is a generalized homology theory.

  There's a corresponding cohomology theory as well.

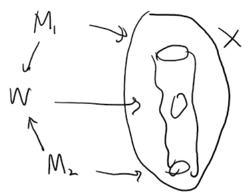

Figure 1.3: cobordism

- *Actually we have several different versions of bordism theorem. The above is unoriented cobordism, and we also have oriented bordism, framed bordism, and complex bordism. Each of them corresponds to different groups. The unoriented cobordism corresponds to the orthogonal group O, oriented bordism corresponds to SO, and complex bordism corresponds to U.*
- *There're lots of other generalized cohomology theories...*

1.13 Theorem (Brown representability theorem). For any cohomology theory $\{h^n\}$, there's a sequence of spaces $\{E_n\}$ such that

$$h^n \simeq [-, E_n] = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}_*)}(-, E_n) : \mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ab}.$$

<u>Observation:</u> The suspension isomorphism $h^n(-) \to h^{n+1}(\Sigma -)$ gives an isomorphism $[-, E_n] \to [\Sigma -, E_{n+1}] \simeq [-, \Omega E_n]$, hence by Yoneda lemma, we have homotopy equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$.(So $E_0 \simeq \Omega^n E_n$). Hence

$$\Sigma E_n \to E_{n+1}$$

and we have a Ω-spectrum $\{E_i\}_i$, which is exactly the data we need to form a stable object. Actually, every spectrum arises this way.

1.14 Example. *For ordinary cohomology $H^*(-, G)$, the space E_n is called $K(G, n)$, the* **Eilenberg-MacLane space**, *which has the special property that*

$$\pi_k(K(G,n)) \simeq [\mathbb{S}^k, K(G,n)] = H^n(\mathbb{S}^k; G) \simeq \begin{cases} G, & \text{if } k = n; \\ 0, & \text{otherwise.} \end{cases}$$

<u>Note:</u> the **Eilenberg-MacLane spectrum** $\boldsymbol{HG} = \{K(G,n)\}_n$ satisfies

$$\pi_n^s(\boldsymbol{HG}) = \begin{cases} G, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\boldsymbol{HG}$ acts like a discrete space.

(1d) Constructing the category. The objects of Ho(Spectra) are spectra or stable objects as above, and what about the morphisms? Note that for Ho(Top) and $D(R)$ there're two approaches:

1) Take nice objects (CW complexes or projective/injective complexes) and homotopy classes of maps between them. Recall from Whitehead theorem(see Hatcher) that all topological spaces are weakly equivalent to CW complexes.

2) Take all objects and invert weak equivalences/quasi-isomorphisms. From Whitehead's theorem we can see that these two approaches produce the same homotopy category.

1) F. Adams takes the first approach to contruct this category, in which the notion of map is very complicated. See [Ada74].

2) A more modern approach, as in [BR20], is the second one: $\boldsymbol{X} \to \boldsymbol{Y}$ is a **weak equivalence** if $\pi_n^s(\boldsymbol{X}) \to \pi_n^s(\boldsymbol{Y})$ is an isomorphism for all n. In this case we only need to invert the weak equivalences defined above. But the problem is we need to get some kind of handle on the result of inverting weak equivalences(localization), which leads to the notion of model categories. (There's another approach to the construction, using infinity categories.)

Since this categoryis complicated, B-R give "axioms": starting with a category S,

- Σ^∞: Ho(Top) $\to$ S;
- Hom sets in S are graded abelian groups;
- Each cohomology theory is represented by an object in S;
- etc.

(1e) More about stable homotopy. Given an additive and graded category S, for any $E \in$ S, we have a functor

$$\mathrm{Ho}(\mathrm{Top}_*) \to \mathrm{Ab}^{\mathbb{Z}}$$

which is a cohomology theory (We can check Eilenberg-Steenrod axioms) and a homology theory: $\pi_*(\Sigma^\infty - \wedge \boldsymbol{E})$: Ho(Top$_*$) $\to$ Ab$^{\mathbb{Z}}$. This satisfies the E-S axiom for homology again. Finally, there's an operation called "smash product" $\wedge$: S $\times$ S $\to$ S generalizing $\wedge$ on topological spaces. (It is to spectra what $\otimes$ is to abelian groups.) $\boldsymbol{S} := \Sigma^\infty \mathbb{S}^0$ is to spectra what $\mathbb{Z}$ to abelian groups.

1.15 Definition. A **ring spectrum** is a spectrum $\boldsymbol{E}$ with a morphism $\boldsymbol{E} \wedge \boldsymbol{E} \to \boldsymbol{E}$ in S satisfying unit, associativity(commutativity if we want commutative ring specra).

(1f) A bit more on model categories

Localization. Let $W \subseteq$ C be categories. The localization means a category C$[W^{-1}]$ with the universal property that

$$\begin{tikzcd} \mathrm{C} \arrow[r, "l"] \arrow[dr, "F"'] & \mathrm{C}[W^{-1}] \arrow[d, dashed] \\ & \mathrm{D} \end{tikzcd}$$

If F sends morphisms in W to isomorphisms in D.

Different models for the same homotopy theory. We can have different models for the same homotopy theory, for example, topological category and the category of simplicial sets. We say two categories are "Quillen equivalent" if they give the same homotopy theory. The sequential spectra, symmetric and orthogonal spectra we're going to talk about this semester are all Quillen equivalent.

Cohomology theories. Given $\boldsymbol{E} \in$ S, we can define $E^*(\boldsymbol{X}) := [\Sigma^\infty X, \boldsymbol{X}]_{-*}$. For X a CW complex this defines a cohomology theory. In general, given $\boldsymbol{X} \in$ S, $E^* X := [\boldsymbol{X}, \boldsymbol{E}]_{-n}$.

Homology Theories Given $E \in$ S, we define the generalized homology theory $E_*(\boldsymbol{X}) := [\boldsymbol{S} = \Sigma^\infty \mathbb{S}^0, X \wedge E]_* = \pi_*(X \wedge E)$. If X is a space, then we just let $E_*(X) = E_*(\Sigma^\infty X)$.

Closed Model Structure. Smash product on Top extends to a smash product Ho(Top) $\xrightarrow{\Sigma^\infty}$ S to a monoidal structure $\mathsf{S} \wedge \mathsf{S} \xrightarrow{\wedge} \mathsf{S}$ with **unit** $\boldsymbol{S}: \boldsymbol{S} \wedge \boldsymbol{E} \simeq \boldsymbol{E}$. For given $\boldsymbol{X}, \boldsymbol{Y} \in \mathsf{S}$, we have a **mapping spectrum** $\mathrm{Map}(\boldsymbol{X}, \boldsymbol{Y}) \in \mathsf{S}$ and $[\boldsymbol{X} \wedge \boldsymbol{Y}, \boldsymbol{Z}] \simeq [X, \mathrm{Map}(\boldsymbol{Y}, \boldsymbol{Z})]$.

Ring Spectra and Module Spectra. From the discussions above we know that S behaves in some sense similar to the category of abelian groups: tensor product of abelian groups correspond to smash products, $\mathrm{Hom}_{\mathbb{Z}}$ corresponds to mapping spectra, and S has a unit which is the sphere spectrum $\boldsymbol{S}$.

1.16 Definition. A **ring spectrum** is an object $\boldsymbol{R} \in \mathsf{S}$ with maps $\boldsymbol{R} \wedge \boldsymbol{R} \to \boldsymbol{R}$ and $\boldsymbol{S} \xrightarrow{e} \boldsymbol{R}$ such that the commutative diagrams in the homotopy category describing the associativity and units are satisfied.

One can also demand stronger associativity and commutativity conditions rather than "up to homotopy". For example, "A_∞", "E_∞", and "highly-structured ring spectra".

1.17 Definition. If $\boldsymbol{R}$ is a ring spectrum, then a **module spectrum** $\boldsymbol{M}$ is a spectrum with a map $\boldsymbol{R} \wedge \boldsymbol{M} \to \boldsymbol{M}$ with the condition similar to that of a module.

Note that any spectra is naturally a $\boldsymbol{S}$-module.

(1g) Thom Space Let $V \to X$ be a vector bundle over a topological space X, and assume V admits a norm, then we can construct the associated disk bundle $\mathbb{D}(V)$ of all vectors with norm ≤ 1 and the sphere bundle $\mathbb{S}(V)$ of all vectors with norm 1.

1.18 Definition. The **Thom space** is the quotient space $\mathrm{Th}(V) := \mathbb{D}(V)/\mathbb{S}(V)$.

Another way to describe is that the Thom space is the one-point compatification of each fibre and identify all the ∞s.

<u>Note:</u> If $V \simeq \mathbb{R}^n$, then $\mathrm{Th}(V) \simeq \Sigma^n X$.

There're very special Thom spaces, for example, BO_n, which completely classifies real vector bundles up to isomorphism, i.e. given any real vector bundle $V \to X$, there is a unique up to homotopy map $X \to BO_n$ and the universal vector bundle $\gamma_n \to BO_n$ such that we have a map of bundles $\Phi: V \to \gamma_n$ over $X \to BO_n$ with the pull-back diagram

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d] & \gamma_n \arrow[d] \\ X \arrow[r] & BO_n \end{tikzcd}$$

The same for BSO_n, which classifies oriented bundles, and BU_n, which classifies complex vector bundles.

MSO_n is the Thom space $\mathrm{Th}(\gamma_n \to BSO_n)$, and similarly MU_n is the Thom space $\mathrm{Th}(\gamma_n \to BU_n)$. MSO is a spectra, called a **Thom spectra**. First of all,

$$MSO = \{MSO_1, MSO_2, \cdots,\} \left| \begin{tikzcd} \gamma_n \oplus \mathbb{R} \arrow[r] \arrow[d] & \gamma_{n+1} \arrow[d] \\ BSO_n \arrow[r] & BSO_{n+1} \end{tikzcd} \right. \Rightarrow \Sigma MSO_n \simeq \mathrm{Th}(\gamma_n \oplus \mathbb{R}) \to MSO_{n+1},$$

and MU is similar, with a little twist that

$$MU = \{MU_1, \Sigma MU_1, MU_2, \Sigma MU_2, \cdots\}$$

(1h) Bordism and Cobordism Given $X \subseteq M$, we have the Pontrjagin-Thom construction: there is a map $X \to BO_n$ which classifies the bundle $N_M X \to X$, this map $N_M X \to \gamma_n$ of vector bundles then induces a map of Thom spaces $\mathrm{Th}(N_M X) \to MO_n$, but we then have the "collapse map" $M \to \mathrm{Th}(N_M X)$ by collapsing the complement of $N_M X$ to a point. So a codimension n submanifold is in one-to-one correspondence to maps from M to MO_n, and two submanifolds X and X' are cobordant if and only if the two maps from M to MO_n are stably homotopic, i.e.

$$\{\text{submanifolds of } M\}/\text{cobordism} \simeq \operatorname*{colim}_n [M, MO_n].$$

The outcome of this construction is that the homology theory represented by the spectra MO is the bordism: $MO_*(X) \simeq \Omega_*(X)$, and similarly, $MO^*(X)$ is called cobordism. If X is a manifold, then it's a duality, meaning that the bordism and cobordism groups are isomorphic. Similarly, MSO_* and MSO^* are oriented bordisms and cobordisms.

MU_* and MU^* are complex bordisms and cobordisms.

Observe that $\pi_*(MO) \simeq MO_*(\mathrm{pt}) \simeq \Omega_*$, which is the "cobordism ring".

(1i) MU and complex orientations Suppose we have a vector bundle $V \to X$, then $H^*(\mathrm{Th}(V)) \to H^*(\mathrm{Th}(V_x)) \simeq \mathbb{S}^n$. Then there's a theorem of Thom saying that there is a "Thom class" $u \in H^n(\mathrm{Th}(V))$ which goes to $\pm 1 \in H^n(\mathbb{S}^n)$. The orientation is the same as the existence of such a Thom class.

1.19 Definition. $\boldsymbol{E} \in \mathsf{S}$ is complex oriented if for each $\boldsymbol{X} \in \mathsf{Top}$ and $V \to X$ complex vector bundle, there exists a class $u \in H^{2n}(\mathrm{Th}(v)) = E^*(\mathrm{Th}(V))$. ($H\mathbb{Z}$, K and MU are complex oriented)

<u>Fact:</u> MU is the universal complex oriented cobordism theory: if $\boldsymbol{E}$ is complex oriented theory, then there exists a map $MU \to \boldsymbol{E}$ inducing that complex orientation.

(1j) Formal Group Laws. If E is complex oriented, we can see by a spectral sequence argument that $E^*(\mathbb{C}P^\infty) \cong E_*[[t]]$, where $E_* = \pi_*(\boldsymbol{E})$. Moreover, $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E_*[[u,v]]$. We can then find a universal class $f(u,v)$, which is a formal group over E_*, which means that it satisfies the following conditions:

- $f(a,0) = f(0,a)$;
- $f(f(a,b),c) = f(a, f(b,c))$(inverses are free);

<u>Fact 2:</u> $\pi_*(MU)$ is the "Lazard ring".

There're two interesting theories, the Brown-Peterson and Morava K-theory, which is obtained from MU. Finally, there's a theorem by Abouzaid-McLean-Smith:

1.20 Theorem. Assume Y is a projective variety, $Y \to \mathbb{C}P^1$ holomorphic submersion with fiber X, then $H^*(Y;\mathbb{Z}) \cong H^*(X;\mathbb{Z}) \otimes H^*(\mathbb{S}^2;\mathbb{Z})$.

This theorem was know over $\mathbb{Q}$, and the statement involves nothing about homotopy theory. The proof is to first replace $\mathbb{Z}$ by any complex oriented cohomology theory. To do this, they first prove this for MU, and then for BP, and finally for all "$K(n)$-local" cohomology theories. The reason why these all have to do with symplectic geometry is that the Morava K-theory are well-behaved with respect to orbifolds. In symplectic geometry, we have the moduli space of pseudo-holomorphic curves which are orbifolds.

TALK 2

Basics of Model Categories

Speaker: Suraj Yadav

The notion of model category allows us to do abstract homotopy theory.

2.1 Definition. A **model category** is a category C with three classes of morphisms $\mathcal{W}$, the class of weak equivalences, $\mathcal{C}$, the class of cofibrations, and $\mathcal{F}$, the class of fibrations, with the following properties:

1) C is closed under finite limits and colimits;

2) (2 out of 3) given three objects X, Y, Z and a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd},$$

   if any two morphisms are in $\mathcal{W}$, then so is the third.

3) (retracts) The retract of any morphismin $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ is again in $\mathcal{W}$, $\mathcal{C}$ or $\mathcal{F}$ respectively. Here we say a morphism $X \xrightarrow{f} Y$ is a **retract** of $U \to V$ if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & U \arrow[r] \arrow[d, "g"] & X \arrow[d] \\ Y \arrow[r] & V \arrow[r] & Y \end{tikzcd}$$

   so that the composition of the upper and lower rows are identities.

4) (lifting property) Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

   such that

   a) $f \in \mathcal{W} \cap \mathcal{C}$, $g \in \mathcal{F}$ implies there exists a lifting $H : B \to X$;

   b) If $f \in \mathcal{C}$, $g \in \mathcal{W} \cap \mathcal{F}$, then there exists a lifting $H : B \to X$;

5) (factorization) For any morphism $f: X \to Y$, there are factorizations

$$X \xrightarrow{f_1} Z \xrightarrow{f_2} Y$$

$$X \xrightarrow{f^1} Z \xrightarrow{f^2} Y$$

of f, where $f_1 \in \mathcal{W} \cap \mathcal{C}$, $f_2 \in \mathcal{F}$, $f^1 \in \mathcal{C}$ and $f^2 \in \mathcal{F} \cap \mathcal{W}$.

2.2 Definition. Let C be a model category. An object $X \in$ C is **fibrant** if $X \to *$ is a fibration; $Y \in$ C is **cofibrant** if $f: * \to Y$ is a cofibration.

For any object $X \in$ C, we have a unique morphism $\varnothing \to X$ which factors through a cofibrant object Y such that $Y \to X$ is the trivial fibration. (Here we say a fibration is **trivial** if it's both a fibration and a weak equivalence.) We want to get a cofibrant and fibrant object Z which is weakly equivalent to X, so we consider the morphism $Y \to *$, and consider the factorization $Y \to Z \to *$ where $Y \to Z$ is the trivial cofibration and $Z \to *$ is a fibration, so Z is both cofibrant and fibrant. Now we want to show that Z is weakly equivalent to X.

2.3 Example. *For the category of topological spaces, we have two kinds of model structures. One of them is called the* **Serre model structure***, where we make weak equivalences to be weak homotopy equivalences, fibrations to be Serre fibrations, i.e. we have the lifting property for all maps $A \to A \times [0,1]$ where A is a CW complex.*

Another structure is the **Hurewicz moel structure***, weak equivalences are homotopy equivalences, and fibrations have lifting properties with respect to maps $A \to A \times [0,1]$ where A is any topological space.*

2.4 Example. *The category of simplicial sets also admits a model structure. Let $\triangle$ be the cosimplex category whose objects are $[n] = \{0, 1, \cdots, n\}$ the set of natural numbers, and morphisms order-preserving maps $[n] \to [m]$. We have a class of special morphisms $d^i: [n] \to [n+1]$ defined by $d^i(k) = k$ if $k < i$, and $d^i(k) = k+1$ if $k \geq i$, and $s^j: [n+1] \to [n]$ given by $s^j(k) = k$ for $k < j$ and $s^k(k) = k-1$ for $k \geq j$.*

2.5 Definition. A **simplicial set** X is a functor

$$X: \triangle^{op} \to \text{Set}.$$

This means that a simplicial set is a data $[n] \mapsto X_n$ with maps $x_{n+1} \to X_n \to X_{n-1}$ with the given compatibility conditions. Simplicial sets are combinatorial data of topological spaces. Simplicial sets are representable functors with representation $\Delta^n := \text{Hom}(-, [n])$. Let sSet be the category of simplicial sets, then we have a **geometric realization** functor

$$|-|: \text{sSet} \to \text{Top}$$

which is adjoint to the singular functor $\text{Sing}_*: \text{Top} \to \text{sSet}$. For the standard n-simplex Δ^n, $|\Delta^n|$ is just the standard n-simplex $\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \Sigma x_i = 1,\ x_i \geq 0\}$.

For any topoogical space Y, we define $(\text{Sing}_* Y)_n = \text{Hom}_{\text{Top}}(|\triangle^n|, Y)$ and we can check that this actually defines a simplicial set. Although these two categorys sSet and Top are not equivalent, their homotopy categories are equivalent.

The model structure on sSet is given as follows: $X \to Y$ is a weak equivalence of simplicial sets if $|X| \to |Y|$ is a weak homotopy equivalence of topological spaces, $X \to Y$ is a cofibration if $X_n \to Y_n$ is a monomorphism for any n, and $X \to Y$ is a fibration if it has lifting property with respect to all cofibrations.

2.6 Example. *For any ring R, the category of chain complexes $\mathsf{Ch}(R)$ is the category with objects chain complexes $\cdots \to C_2 \to C_1 \to C_0 \to 0$. This category admits natural model structures: a morphism $f_\bullet : C_\bullet \to D_\bullet$ is a weak equivalence if the induced map on homology $f_* : H_*C \to H_*D$ is an isomorphism. $f_\bullet$ is a cofibration if $f_n : C_n \to D_n$ is injective with projective cokernel. $f_\bullet$ is a fibration if $f_n : C_n \twoheadrightarrow D_n$ is surjective. This is the projective model structure on $\mathsf{Ch}(R)$, since the cofibrant objects in this structure are chain complexes of projective modules, and the cofibrant replacement is just the same as taking projective resolutions.*

Another model structure is the so-called **injective model structure**, *where fibrations are degreewise surjective maps with injective kernels and cofibrations degreewise injective maps. Similarly, fibrant replacements in this category are injective resolutions.*

Now we proceed to define homotopy category of a model category.

2.7 Definition. Consider the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & X \sqcup X \end{tikzcd}$$

The identity map $X \xrightarrow{\text{id}} X$ gives a natural map $\tau : X \sqcup X \to X$. A **cylindrical object** is the following data

$$X \sqcup X \xrightarrow{C} \mathsf{Cyl}(X) \xrightarrow{W_F} X$$

where C is a cofibration and W_F is the trivial fibration.

The motivation of this cylinder object is the usual cylinder $A \times [0,1]$ for a given topological space A.

2.8 Lemma. Suppose we have a map $X \xrightarrow{f} Y$ which is a weak equivalence, then we have a natural induced map $\mathsf{Cyl}(X) \to \mathsf{Cyl}(Y)$ which is also a weak equivalence fitting into the commutative diagram

$$\begin{tikzcd} X \sqcup X \arrow[r] \arrow[d] & Y \sqcup Y \arrow[r, "C"] & Y \sqcup Y \arrow[d, "W_F"] \\ \mathsf{Cyl}(X) \arrow[r, "W_F"] \arrow[urr, dashed, "W_F"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

2.9 Definition. Two morphisms $f, g : X \to Y$ are **left homotopic** if there exists a morphism $H : \mathsf{Cyl}(X) \to Y$ such that $H_0 i_0 = f$ and $Hi_1 = g$. Here $(i_0, i_1) : X \sqcup X \to \mathsf{Cyl}(X)$ are the two inclusion maps of X into $\mathsf{Cyl}(X)$.

The problem is, in a general model category, the notion of homotopy equivalence is not an equivalence relation. Now we give a dual construction.

2.10 Definition. Taking any object $Y \in \mathsf{C}$, the **path object** of Y is the factorization of $Y \xrightarrow{\Delta} Y \times Y$

$$Y \xrightarrow{W_C} PY \xrightarrow{(e_0, e_1)} Y \times Y$$

where $Y \to PY$ is the trivial cofibration and $PY \to Y \times Y$ is the fibration.

2.11 Definition. $f, g : X \to Y$ are **right homotopic** if there exists a morphism $H : X \to PY$ such that $e_0 H = f$ and $e_1 H = g$.

Now we can define the homotopy category of a given model category C. Given $X, Y \in \mathsf{C}$, we consider the cofibrant-fibrant replacement of both X and Y, i.e. X^{cf} and Y^{cf}, and consider the set of morphisms $\mathrm{Hom}_{\mathsf{C}}(X^{cf}, Y^{cf})$. We use the following fact:

(a) If X is cofibrant, then left homotopy is an equivalence relation on $\mathsf{C}(X, Y)$;

(b) If Y is fibrant, then right homotopy is an equivalence relation on $\mathsf{C}(X, Y)$;

(c) If X is cofibrant and Y is fibrant, then $f, g: X \to Y$ are left homotopic if and only if they are right homotopic.

Therefore we can define the **homotopy category** $\mathrm{Ho}(\mathsf{C})$ of C to be the category with objects those objects in C and morphism sets $\mathrm{Hom}_{\mathsf{C}}(X^c, Y^f)/\sim$ where $f \sim g$ if and only if they are left or right homotopic.

In this homotopy category, we know that if $f: X \to Y$is a weak equivalence with X, Y cofibrant-fibrant, then f is a homotopy equivalence.

2.12 Lemma. $f: X \to Y$ is an isomorphism in $\mathrm{Ho}(\mathsf{C})$ if and only if f is a weak equivalence in C.

Therefore the notion of "localization at $\mathcal{W}$" in C is the same as the homotopy category of C.

Finally, we define the Quillen functors:

2.13 Definition. $F: \mathsf{C} \to \mathsf{D}$ is called **left Quillen** if it preserves cofibrations and trivial cofibrations, and **right Quillen** if it preserves fibrations and trivial fibrations.

2.14 Definition. $F: \mathsf{C} \rightleftarrows \mathsf{D}: G$ a pair of functors. We say they are **Quillen adjunction** if they are adjunctions and one of the following conditions hold:

1) F and G have to be left Quillen and right Quillen respectively;

2) F is left Quillen;

3) G is right Quillen;

4) F preserves trivial cofibrations and cofibrations between cofibrant objects.

5) G preserves trivial fibrations and fibrations between fibrant objects.

2.15 Example. $|-|: \mathsf{sSets} \rightleftarrows \mathsf{Top}_*: \mathsf{Sing}_*$ *are Quillen adjunct.*

2.16 Definition. Let $F: \mathsf{C} \to \mathsf{D}$ be a left Quillen functor, then $LF: \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{D})$ is given by $LF(X) := F(X^c)$. Similarly, if $G: \mathsf{D} \to \mathsf{C}$ is right Quillen, then we can define $RF: \mathrm{Ho}(\mathsf{D}) \to \mathrm{Ho}(\mathsf{C})$ by $RG(X) := G(X^f)$.

2.17 Example. *The sheaf cohomology $H^*(X, -)$ is the example of RG for G the global section functor.*

So LF and RF are generalizations of left and right derived functors in the model category.

2.18 Definition. A Quillen adjunction

$$F: \mathsf{C} \rightleftarrows \mathsf{D} \quad : G$$

is a **Quillen equivalence** if

$$LF: \mathrm{Ho}(\mathsf{C}) \rightleftarrows \mathrm{Ho}(\mathsf{D}) \quad : RG$$

is an equivalence of categories.

For example, the projective and injective model structures on $\mathrm{Ch}(R)$ are Quillen equivalent.

TALK 3

Basics of Homotopy Theory

Tianle Liu

Today we'll talk about basics of homotopy theory, following the last talk about model category.

(3a) Cofibrations and Fibrations. We have introduced cofibrations and fibrations in a general model category, and now let's see how they're defined in the category of topological spaces.

3.1 Definition. $i: A \to X$ is a **cofibration** if it satisfies **homotopy extension property**: for any continuous maps $f: X \to Y$ and $h: A \times I$ making the diagram commutative,

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "j"] \arrow[dd] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

there exists a map $X \times I \to Y$ filling in the commutative diagram.

With the notion of mapping cylinder, we can make things simpler:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A \times I \arrow[d] \\
X \arrow[r] & Mi
\end{tikzcd}$$

Here Mi is the mapping cylinder of i.

Dually we have the notion of fibration:

3.2 Definition. A surjective map $p: E \to B$ is called a **fibration** if it satisfies the **covering lifting property**:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & E \arrow[d] \\
Y \times I \arrow[r, "h"] \arrow[ur, dashed] & B
\end{tikzcd}$$

With the notion of path space, it's equivalent to

$$\begin{tikzcd}
E \arrow[d] & & E^I \arrow[ll] \arrow[d] \\
& Y \arrow[ul] \arrow[ur] \arrow[dr] & \\
B & & B^I \arrow[ll]
\end{tikzcd}$$

where $B^I = \mathrm{Maps}(I, B)$. With the notion of path object Np of p(the pull-back), we have

$$\begin{tikzcd}
E \arrow[d, "p"] \arrow[dr] & & E^I \arrow[ll] \arrow[d] \\
& Np \arrow[ur, dashed] \arrow[dr] & \\
B & & B^I \arrow[ll]
\end{tikzcd}$$

These are the Hurewicz fibrations and cofibrations as mentioned last week.

Recall from last week that any map can be decomposed into a composition of a weak equivalence followed by a cofibration or a fibration followed by a weak equivalence. Now we make this decomposition precise in the topological category. Given $f : X \to Y$, we can factor f via the mapping cylinder

$$X \xrightarrow{\text{cof}} Mf \xrightarrow{\sim} Y,$$

where $Mf \to Y$ is a weak homotopy equivalence. Dually, we can decompose f as

$$X \xrightarrow{\sim} Nf \xrightarrow{\text{fib}} Y.$$

(3b) Suspension and Loop Construction. Consider the category of pointed topological spaces Top_*, i.e. we choose a base point for each topological space X, and we take the homotopy pull-back of the diagram

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

To see why this is the usual loop object, notice that this is the homotopy pull-back, so we can replace maps by the fibrant or cofibrant objects. For example, we can replace $* \to X$ by the path fibration $PX \to X$, then we get

$$\begin{tikzcd}
& * \arrow[d] \\
PX \arrow[r] & X
\end{tikzcd}.$$

The suspension is given by the homotopy pushout of the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \\
* &
\end{tikzcd}$$

and we can replace both $*$ by the mapping cylinder Mi, then the pushout is exactly the suspension ΣX.

(3c) Fiber and Cofiber Sequences.

3.3 Definition. We say a sequence $Z \to X \to Y$ is a **fiber sequence** if the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

is a homotopy pullback. In this case, we say $Z = \mathrm{fib}(f)$ is the fiber of f.

If we look at the pull-back square

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{Fib}(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ * \arrow[r] & Y \end{tikzcd}$$

then the top left corner should be the loop space of Y. If we repeat this procedure, we would get the loop space ΩX. Repeated this process, we would get a long exact sequence

$$\cdots \Omega^2 Y \to \Omega \mathrm{Fib}(f) \to \Omega X \to \Omega Y \to \mathrm{Fib}(f) \to X \to Y.$$

in the sense that each consecutive three arrows are fiber sequences. Similarly, we can get the **cofiber sequence** $X \to Y \to \mathrm{cob}(f)$ if the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & \mathrm{cob}(f) \end{tikzcd}$$

is a homotopy pushout square. With the similar construction, we get a long exact sequence

$$X \to Y \to \mathrm{cob}(f) \to \Sigma X \to \Sigma Y \to \Sigma \mathrm{cob}(f) \to \cdots$$

Write $\langle -, - \rangle$ for the homotopy mapping space(i.e. the mapping space modulo homotopy), and given any topological space Z, we get a sequence of spaces

$$\cdots \to \langle Z, \Omega^2 X \rangle \to \langle Z, \Omega^2 Y \rangle \to \langle Z, \Omega \mathrm{fib}(f) \rangle \to \langle Z, \Omega X \rangle \to \cdots$$

3.4 Definition. For any space X, define $\pi_0(X) = [\mathbb{S}^0, X]$ the space of connected components of X.

If we apply π_0 to the sequence above, we get long exact sequence of sets

$$\cdots \to \pi_0 \langle Z, \Omega^2 X \rangle \to \pi_0 \langle Z, \Omega^2 Y \rangle \to \pi_0 \langle Z, \Omega \mathrm{fib}(f) \rangle \to \pi_0 \langle Z, \Omega X \rangle \to \cdots$$

and similarly for $\langle -, Z \rangle$.

3.5 Definition. We define the **n-th homotopy group** of X to be $\pi_n(X) = \pi_0(\Omega^n X)$.

So we know that $\pi_1(X) = [\mathbb{S}^0, \Omega X] = [\Sigma \mathbb{S}^0, X] = [\mathbb{S}^1, X]$ using the isomorphism $[\Sigma X, Y] = [X, \Omega Y]$, then $\pi_2(X) = [\mathbb{S}^0, \Omega^2 X] = [\Sigma \mathbb{S}^0, \Omega X]$ is abelian, so we get $\pi_1(X)$ is a group and $\pi_n(X)$ is an abelian group for $n \geq 2$.

Now in the sequence above, if we choose $Z = \mathbb{S}^0$, then we have an exact sequence

$$\cdots \to [\mathbb{S}^0, \Omega Y] = \pi_1(Y) \to [\mathbb{S}^0, \Omega X] = \pi_1(X) \to \pi_0(\mathrm{fib}(F)) \to \pi_0(X) \to \pi_0(Y)$$

of homotopy groups, with $f : X \to Y$ a fibration. This is the usual long exact sequence of a homotopy group under the condition that $f : X \to Y$ is a fibration.

(3d) CW Complexes. The first theorem here is the CW approximation:

3.6 Definition. $f: X \to Y$ is a **weak homotopy equivalence** if $\pi_i(f): \pi_i(X) \to \pi_i(Y)$ is an isomorphism of homotopy groups.

3.7 Theorem (CW Approximation)**.** For any topological space X, there exists a CW complex Y and a morphism $X \to Y$ such that f is a weak equivalence.

Recall from last time that we have a Quillen equivalence $|-|: \mathrm{sSets} \rightleftarrows \mathrm{Top}_*: \mathrm{Sing}_*$, which tells us that for each topological space X, the counit map $|\mathrm{Sing}(X)| \to X$ is a weak equivalence w.r.t. Serre model structure, and is hence a weak homotopy equivalence.

Another thing about CW approximation is the cellular approximation theorem.

3.8 Definition. Let $f: X \to Y$ be a map between two CW complexes, then we say f is **cellular** if $f(X^n) \subseteq Y^n$, i.e. the image of the n-skeleton of X is contained in the n-skeleton of Y.

3.9 Theorem (Cellular Approximation Theorem)**.** Any continuous map $f: X \to Y$ between topological spaces X and Y is homotopic to a cellular map.

The last theorem here is Whitehead's theorem, which says that

3.10 Theorem (Whitehead Theorem)**.** Let X, Y be CW complexes, and $f: X \to Y$ a weak homotopy equivalence, then f is a homotopy equivalence.

Proof. Again we consider the Quillen model category $\mathrm{Top}_{\mathrm{Quillen}}$. Note that in this model structure, all the CW complexes are fibrant and cofibrant. There's a theorem stated last time that weak equivalences between fibrant and cofibrant objects are actually homotopy equivalences. □

(3e) Freudenthal Suspension Theorem Now we state a theorem which is important in stable homotopy theory. The idea is that we want to study the suspension map

$$\Sigma: [X, Y] \to [\Sigma X, \Sigma Y],$$

and because of the adjunction between suspensions and loops, we have $[\Sigma X, \Sigma Y] = [X, \Omega\Sigma Y]$, so we only need to study the map $X \to \Omega\Sigma X$ induced from the identity map. Note that $\Omega\Sigma X$ is a topological group up to homotopy, and we can actually make it into a real topological monoid called **Moore space**, and if we take the free monoid $J(X)$ generated by X, and take the map $J(X) \to \Omega\Sigma X$,($J(X)$ is called the **James construction**. Explicitly, we take $J_n(X)$ to be the n-th Cartesian product X^n quotient by the relations $(x_1, \cdots, x_{k-1}, e, x_k, \cdots, x_{m-1}) \sim (x_1, \cdots, x_{k-1}, x_k, \cdots, x_{m-1})$. For more information about the James construction, see [Hat00, Section 4.J].

3.11 Theorem. $J(X) \simeq \Omega\Sigma X$.

Now the problem reduces to considering the natural mapping space $[X, J(X)]$. If X is $(n-1)$-connected CW complex(by CW approximation, it always suffices to consider CW complexes), i.e. $\pi_i(X) = 0$ for $i \le n-1$, then we can regard $X^{(n-1)}$ to be a point homotopically, then we can intuitively imagine that $J(X) \setminus X$ has cells of dimension at least $2n$, and we can conclude by this argument that $X \to J(X)$ is $(2n-1)$-connected. This means that $X \to \Omega\Sigma X$ is $(2n-1)$-connected, so if $\dim Y < 2n-1$, then $[Y, X] \xrightarrow{\sim} [Y, \Omega\Sigma X] = [\Sigma Y, \Sigma X]$ and if $\dim Y = 2n-1$, then $[Y, X] \twoheadrightarrow [\Sigma Y, \Sigma X]$, which can be proved via long exact sequence and cellular approximations. That is, for $\dim Y < 2n-1$, the map is injective by fiber sequence, and for $\dim Y \le 2n-1$, the map is surjective by cellular approximation. Let's conclude the Freudenthal suspension theorem:

3.12 Theorem (Freudenthal)**.** Let X, Y be topological spaces with X

(3f) Hurewicz Theorem Finally we talk about Hurewicz theorem. Firstly we give an alternative definition of homology theory: for any topological space X, we define $H_n(X) = \pi_n(\mathrm{sp}(X))$, where

$$\mathrm{sp}(X) = \varinjlim X^n/\sigma_n$$

where $\sigma_n \curvearrowright X^n$ acts by permutation. Then we get a natural map $f : X \to \mathrm{sp}(X)$ which gives a map

$$\pi_n(f) : \pi_n(X) \to \pi_n(\mathrm{sp}(X)) = H_n(X).$$

3.13 Theorem (Hurewicz). If X is $(n-1)$-connected, then the map $X \to \mathrm{sp}(X)$ is $(n+1)$-connected.

As a Corollary, we have

3.14 Corollary. $\pi_i(X) \to H_i(X)$ is an isomorphism if $i = n$, and is surjective if $i = n+1$.

(3g) Cohomology. Finally, we just quickly review the construction of cohomology theory. Similar to homology theory, we can compute our cohomology group $H^n(X; G)$ via homotopy groups

$$H^n(X; G) \cong [X, K(G, n)],$$

where $K(G, n)$ is the **Eilenberg-MacLane space**. Then we say the functor $H^n(-; G)$ is representable with representation $K(G, n)$. Actually we have a summary of this phenomena:

3.15 Theorem (Brown Representability). A functor $F : \mathrm{Ho}(\mathrm{Top}_*)^{op} \to \mathrm{Sets}_*$ is representable if and only if F is a **Brown functor**, i.e.

1) It takes coproducts to products;

2) It takes homotopy pushouts to weak pullbacks(we don't need the uniqueness for the pull-back).

One example is the cohomology functor, and another example is the functor $\mathrm{Bun}_G(X) = \{G - \text{bundles on } X\}/\sim$, and it's represented by some $BG \in \mathrm{Ho}(\mathrm{Top})$, which is called the **classifying space** of G.

A final thing is that if F happens to be a cohomological functor, i.e. it satisfies the cohomological axiom, then what space it should represent? Let E^n be the cohomological functor and L^n the spaces they represented, then we have $E^n(X) = [X, L^n]$ and $E^{n+1}(\Sigma X) = E^n(X) = [\Sigma X, L^{n+1}] = [X, \Omega L^{n+1}]$, so we should have $L^n \simeq \Omega L^{n+1}$. Now it goes into the notion of spectrum. We call $\{L^n\}$ an **Ω-spectrum**.

TALK 4

Basic of Stable Homotopy Theory

Haoyang Liu

We start with a review of some result Tianle talked about last time. In today's talk, when I talk about the category of topological spaces, it refers to the category of CW complexes, and pointed category of topological spaces refers to pointed CW complexes. When we talk about pointed space (X, x_0), we say X has a **non-degenerate base point** if the inclusion $x_0 \to X$ is an h-cofibration in Top.

4.1 Definition. We say (X, x_0) is **k-connected** if it is path connected and $\pi_i(X, x_0)$ for all $1 \leq i \leq k$.

A pointed map $f : X \to Y$ is a **k-equivalence** if for all $x_0 \in X$, $\pi_k(X, x_0) \xrightarrow{\pi_k(f)} (Y, f(x_0))$ is an isomorphism for $0 \leq n < k$, and surjective when $n = k$.

As a convention, every pointed topological space is (-1)-connected. Now we recall the Freudenthal suspension theorem from last time:

4.2 Theorem (Freudenthal Suspension Theorem)**.** Let $k \in \mathbb{N}$ and X k-connected with non-degenerate basepoint, then the map

$$\pi_n(X) = [\mathbb{S}^n, X] \xrightarrow{\Sigma} [\Sigma\mathbb{S}^n, \Sigma X] = \pi_{n+1}(\Sigma X)$$

is an isomorphism if $n < 2k + 1$ and surjection if $n = 2k + 1$.

4.3 Example. *Note that the degree in this theorem is really sharp. For example, if we look at the map*

$$\pi_2(\mathbb{S}^1) \to \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$$

where $\pi_2(\mathbb{S}^1) = 0$, $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ and $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$. The theorem above tells us the theorem holds only in degree 0, and we see directly from the sequence that $\pi_2(\mathbb{S}^1) \to \pi_3(\mathbb{S}^2)$ fails to be a surjection, and if we look at $\pi_2(\mathbb{S}^3) \to \pi_3(\mathbb{S}^4)$, the theorem tells us that this map is actually a surjection.

We can get a slightly different form of the suspension theorem, which leads us to the so-called "stable phenomenon":

4.4 Corollary (Freudenthal Suspension Theorem, Restated)**.** Assume X is a topological space with a non-degenerate base point x_0, $a, b \in \mathbb{N}$ with $b < a - 1$, then the suspension map

$$\pi_{a+b}(\Sigma^a X) \to \pi_{a+b+1}(\Sigma^{a+1} X)$$

is an isomorphism.

4.5 Remark. *If we fix b and let $a > b + 1$, then the map in Corollary 4.4 is an isomorphism for all such a, which gives us an idea why we call this a stable phenomena, and we can define the* **stable homotopy group** *as*

4.6 Definition. For X a pointed CW complex and $n \in \mathbb{N}$, we write the **stable homotopy group** of X as

$$\pi_n^{\text{stable}}(X) = \operatorname*{colim}_a \pi_{n+a}(\Sigma^a X).$$

When we pick $a > n+1$, then we can see that every morphism in this directed system is an isomorphism, so the system is terminal and we get

$$\operatorname{colim}_a \pi_{n+a}(\Sigma^a X) = \pi_{2n+2}(\Sigma^{n+2}(X)).$$

This is the first stable phenomena we have seen here.

4.7 Theorem. Let X, Y be pointed CW complexes with Y k-connected, then the suspension map

$$\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$$

is surjective if X is of diemension $2k+1$ and bijective if X has dimension $< 2k+1$.

This is something like the generalization of the Freudenthal theorem. Just like how we define the stable homotopy group, we can define

4.8 Definition. The set of **stable homotopy class** of pointed maps $X \to Y$ is

$$[X, Y]^s := \operatorname{colim}_a [\Sigma^a X, \Sigma^a Y].$$

Reduced Cohomology Theory The construction of reduced cohomology theory also leads us to think about the objects called spectra.

4.9 Definition. **Reduced homology theory** is a functor $\tilde{E}_*$ from pointed CW complexes to graded abelian groups Ab satisfying the following axioms:

(1) If $f \simeq g$, then $f_* = g_*$;

(2) For a CW pair (X, A) we have a boundary map $\partial_* : \tilde{E}_*(X/A) \to \tilde{E}_{*-1}(A)$;

(3) Let $i : A \hookrightarrow X$, $q : X \to X/A$ and ∂_* together gives a long exact sequence;

(4) Given a family of spaces X_α and $i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ induces an isomorphism $\bigoplus_\alpha \tilde{E}_*(X_\alpha) \to \tilde{E}_*(\bigvee_\alpha X_\alpha)$.

Another thing we can say is once we have this long exact sequence in definition 4.9, we can consider the CW pair (CX, X) and get

4.10 Lemma. $\tilde{E}_*(\Sigma X) \cong \tilde{E}_*(X)$.

4.11 Example. *A standard example for reduced homology theory is the reduced singular homology $\tilde{H}_*(X)$; The stable homotopy groups $\pi_n^{\text{stable}}(X)$ also defines a reduced homology theory. The reason is that, firstly, $\pi_n^{\text{stable}}(X)$ is an abelian group by the construction, and we can verify the axioms of a reduced homology theory.*

Note that $H_*(D^2, S^1) \cong H_*(D^2/S^1, *)$. However, the unstable homotopy group does not have this property. For example, $\pi_3(D^2, S^1) = 0$, but $\pi_3(D^2/S^1, *) \neq 0$.

We have also a dual definition for reduced cohomology theory, but since it's almost the same as homology theory, we just omit the formal definitions here, and we have the lemma

4.12 Lemma. $\tilde{E}^{*+1}(\Sigma X) \cong \tilde{E}^*(X)$.

4.13 Remark. *If we have two cohomology theories $\widehat{E}^*$, $\widetilde{E}^*$, we say they're isomorphic if we have a bijective natural transformation $\widehat{E}^* \to \widetilde{E}^*$ that is compatible with coboundary maps.*

Now we can introduce the notion of spectra. We start with the famous theorem mentioned by Helfer:

4.14 Theorem (Brown Representability Theorem). $\tilde{E}^*$ is represented by $\{K_n\}_{n\in\mathbb{Z}}$, which implies $\tilde{E}^n(X) = [X, K_n]$.

This theorem gives us a way to try to think about the cohomology theory by some set of topological spaces. When we look at the axioms of reduced cohomology theory, we would have more relations between these topological spaces $\{K_n\}$:

4.15 Corollary. $\tilde{E}^*(X) \cong \tilde{E}^{*+1}(\Sigma X)$.

This means that $[X, K_n] \cong [\Sigma X, K_{n+1}]$. By the suspension-loop duality we have $[\Sigma X, K_{n+1}] = [X, \Omega K_{n+1}]$, so we get **structure maps**

$$\alpha_n : K_n \to \Omega K_{n+1}.$$

which are weak homotopy equivalences. Brown's representability theorem tells us that for each given reduced cohomology theory $\tilde{E}^*$, we get a class $\{K_n\}$. Conversely, given a class of spaces $\{K_n\}$ with the above structure maps, we can recover the reduced cohomology theory $\tilde{E}^*$. In fact, they determine each other.

4.16 Example. *There're some examples of these kinds of sequences of spaces:*

(1) The **Eilenberg-MacLane Spaces** *$K(G, n)$, for G an abelian group and $n \in \mathbb{N}$;*

(2) $\{K_n\}$, which represents the complex K-theory: when n is even, $K_n = BU \times \mathbb{Z}$, and when n is odd, it's just U.

Now we introduce two attempts to construct the "stable model category", which does not fit into our requirements. The first one is the **Spanier-Whitehead category** SW, where objects are finite CW complexes(we can also add infinite-dimensional CW complexes and written $\widehat{\mathrm{Sw}}$), and morphisms are stable homotopy classes $[X, Y]^s := \mathrm{colim}_a[\Sigma^a X, \Sigma^a Y]^s$. But this construction has drawbacks: it does not have enough objects. One example is that it doesn't have countable coproducts. In section 1.1.4 of [BR20], they listed satisfactory 12 properties a stable homotopy category should have.

Now we define the notion of spectra.

4.17 Definition. A **spectrum** is a sequence of topological spaces $\{X_n\}$ with **structure maps** $\sigma_n^X : \Sigma X_n \to X_{n+1}$ which are weak homotopy equivalences.

An **Ω-spectrum** is a sequence of topological spaces $\{X_n\}$ with structure maps $\tilde{\sigma}_n^Z : Z_n \to \Omega Z_{n+1}$ which are weak homotopy equivalences.

The draw back for the category of spectra is that we do not have enough morphisms. One example is that

4.18 Example. *You can find two spectra representing the same cohomology theory but they're not homotopy equivalent to each other.*

To summarize, our goal is to find a good category that can represent all the reduced cohomology theories. Here we present some attempts but failed, and we'll see some constructions that finally resolve this issue.

TALK 5

K-theory and Bott Periodicity

Haosen Wu

We will assume some knowledge about K-theory throughout this talk and focus mainly on Bott periodicity. We'll present a Morse-theoretic proof which is originally due to Bott.

5.1 Theorem (Bott). Consider $U = \operatorname{colim} U(n)$, then we have $U \simeq \Omega^2 U$. Let BU be the classifying space of U, then we have $BU \times \mathbb{Z} \simeq \Omega^2(BU \times \mathbb{Z})$.

(5a) K-theory. K-theory is sort of a "cohomological theory". We know for vector bundles we have invariants like Chern classes or Stiefel-Whitney classes, and we know they can be subtracted from the universal bundle $EG \to BG$ over the classifying space BG by pulling back certain classes of the universal vector bundle. But these classes all lie in the vector bundles themselves, we would like to simply consider the vector bundles themselves.

Operations on Vector Bundles. We have several operations on vector bundles.

5.2 Definition. Given two bundles E and E' over the same base space B, we can take the pull-back of the diagonal map $\Delta : B \to B \times B$ which gives the direct sum bundle $E \oplus E' = \Delta^*(E \times E')$.

5.3 Definition. We can define the tensor product of two vector bundles E, E' over the same base B as follows:

$$E \otimes E' = \coprod_{b \in B} p^{-1}(b) \otimes p'^{-1}(b) \to B.$$

These two operations serve as the core ingredients for the K-group. Recall that given a commutative monoid $(M, +, 0)$, then we can form an abelian group $G(M)$ associated to the monoid M defined by the following universal property: we have a natural map $\iota : M \to G(M)$ and for any homomorphisms of monoids $i : M \to G$, there exists a unique group homomorphism $\phi : G(M) \to G$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "i"] & G(M) \arrow[dl, "\phi"] \\ G & \end{tikzcd}$$

commutes. An explicit construction of $G(M)$ goes as follows:

$$G(M) = \{(m, n) \in M \times M\} / \sim$$

where $(m, n) \sim (m', n')$ iff there exists $a \in M$ with $m + n' + a = m' + n + a$.

5.4 Definition (K_0-group). Let X be a paracompact topological space, then

$$K(X) = \{E - E' : E, E' \in \mathrm{Vect}(X)\},$$

where $\mathrm{Vect}(X)$ is the set of all vector bundles over X.

With this definition, we know $E - E' = F - F'$ iff there exists a vector bundle A such that $E \oplus F' \oplus A \cong E' \oplus F \oplus A$ as vector bundles, and we say $E = F$ if there exists a bundle A with $E \oplus A \cong F \oplus A$. If we can find a bundle A' with $A \oplus A' \cong \underline{\mathbb{R}}^N$, the trivial bundle of rank N.

5.5 Definition. We say two vector bundles E and F are **stably isomorphic** if there exists an N with $E \oplus \underline{\mathbb{R}}^N \cong F \oplus \underline{\mathbb{R}}^N$, and we write $[E]$ for the class of stable isomorphic classes.

Then we know that $K(X)$ consists of stable isomorphic classes of vector bundles. Note that stable isomorphism classes have cancellation property, i.e. if $A \oplus B \cong A' \oplus B$, then $[A] = [A']$. The reason is that, we can find a vector bundle B' such that $B \oplus B' \cong \underline{\mathbb{R}}^N$, so we get that $A \oplus \underline{\mathbb{R}}^N \cong A' \oplus \underline{\mathbb{R}}^N$, and hence $[A] = [A']$. This also verifies that $K(X)$ is an abelian group.

Now we want to define the reduced K-group. This group depends on the choose of a base point $*$. Consider an inclusion map $i\colon A \hookrightarrow X$ where $A \subseteq X$ is a closed subspace of X, then this inclusion induces a morphism $i^*K(X) \to K(A)$ defined by $E - \underline{\mathbb{R}}^N \mapsto E|_A - \underline{\mathbb{R}}^N|_A$. Pick $A = \{*\} \subseteq X$, then $K(A) = K(\{*\}) \cong \mathbb{Z}$.

5.6 Definition. We define the **reduced K-group** of $(X, *)$ to be $\tilde{K}(X, *) = \ker i^*$ where i^* is defined above.

The K-group is actually a contravariant functor $K\colon \text{Top}^{op} \to \text{Ab}$, which is representable. How can we get a representation for K? This is motivated by how we play with classifying spaces. If we can get a map $f \in [X, BG]$, then we are expected to get a unique G-bundle $P \to X$ over X. Assume we have a diagram of categories

$$D\colon \text{Vect}^0(X) \xrightarrow{\iota_0} \text{Vect}^1(X) \xrightarrow{\iota_1} \cdots \to \text{Vect}^n(X) \to \cdots$$

with $\iota_{n,m}\colon \text{Vect}^n(X) \to \text{Vect}^{n+m}(X)$ given by $E \mapsto E \oplus \underline{\mathbb{R}}^m$, then we can take the colimit of D, $\operatorname{colim} D$, then we'll get the reduced K-group $\tilde{K}(X) = \operatorname{colim} D$, and since each $\text{Vect}^i(X)$ can be represented by $BU(i)$, we get that

$$K(X) \cong [X, BU] \oplus \mathbb{Z} \quad \text{and} \quad \tilde{X} \cong [X, BU \times \mathbb{Z}]_0.$$

Assuming Theorem 5.1, we readily get

5.7 Corollary. $\tilde{K}(\Sigma^2 X) \xrightarrow{\sim} \tilde{K}(X)$.

Proof. Applying the adjunction $\text{Hom}_0(\Sigma X, Y) \cong \text{Hom}_0(X, \Omega Y)$, we get

$$\tilde{K}(\Sigma^2 X) \cong [\Sigma^2 X, BU]_0 \cong [X, \Omega^2(BU \times \mathbb{Z})]_0 \cong [X, BU \times \mathbb{Z}]_0 \cong \tilde{K}(X). \qquad \square$$

Another result from Bott periodicity is that

5.8 Corollary. $\tilde{K}(\mathbb{S}^{2n}) \cong \mathbb{Z}$ and $\tilde{K}(\mathbb{S}^{2n+1}) = 0$.

Proof. Note that $\tilde{K}(\mathbb{S}^k) \cong [\mathbb{S}^k, BU \times \mathbb{Z}]_0 = \pi_k(BU) = \pi_{k+1}(U)$. To compute the homotopy group of U, we need Bott periodicity again. By Bott periodicity, we just need to compute the first two homotopy groups of U:

$$\pi_1(U) \cong \pi_1(U(1)) \cong \pi_1(\mathbb{S}^1) = \mathbb{Z};$$
$$\pi_2(U) \cong \pi_2(U(2)) \cong \pi_2(SU(2)) = 0. \qquad \square$$

(5b) Proof of Bott Periodicity. For the time issue, let's just outline the proof of the Bott periodicity theorem 5.1. Let's just focus on the first half. In order to show $U \simeq \Omega^2 U$, we want the isomorphism $\pi_i(U) \cong \pi_{i+2}(U)$, and to achieve this, we study the space $P(U; p, q)$ of all paths connecting p and q, and the Morse theory tells us that $\pi_i BU \cong \pi_i \Omega SU$.

Note that each $U(n)$ has the homotopy type of a CW complex, so the colimit $U = \operatorname{colim} U(n)$ is also homotopic to some CW complex, and by the Whitehead theorem, all weak homotopy equivalences are homotopy equivalences. We can also express BU as the colimit $BU = \operatorname{colim} \text{Gr}_n(2n)$, where $\text{Gr}_n(2n)$ also has the homotopy type of finite dimensional CW complexes, so is BU.

5.9 Theorem. The loop space of U and BU are also homotopy equivalent to some CW complexes.

Now we consider the following diagram

$$\begin{tikzcd} \mathrm{Gr}_n(2n) \arrow[r] \arrow[d] & \Omega SU(2n) \arrow[d] \\ \mathrm{Gr}_{n+1}(2n+2) \arrow[r] & \Omega SU(2n+2) \end{tikzcd}$$

which induces a map $BU \to \Omega SU$. But from the result in Morse theory, this is indeed a weak equivalence, and by Whitehead theorem, this is a homotopy equivalence. Then we further consider the map $\Omega SU \to \Omega U$ giving a map $j : BU \to \Omega U$, and we define a map $BU \times \mathbb{Z} \to \Omega U$ by $(x, r) \mapsto j_r(x)$. This gives a corresponding map on homotopy groups $\pi_i(BU, (x, r)) \to \pi_i(\Omega U, j_r(x))$ which gives a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega U$.

The second part of the proof is $U \simeq \Omega BU$. If we achieve this, then we would get $U \simeq \Omega BU = \Omega(BU \times Z) = \Omega^2 U$, which proves theorem 5.1. □

To achieve this, recall that we have

5.10 Theorem. Given a fibre bundle $U \to E \to X$. If X is paracompact and E is contractible, then $U \simeq \Omega X$.

If we let $X = BU$, then we just need to construct a bundle E which is contractible. The construction is just given by

$$E = \text{colim}\{\text{Principal bundle } V_n | V_n \to BU(n)\}.$$

Now the remaining part is to show that $\pi_i BU \cong \pi_i \Omega SU$. The proof relies on some path analysis on the path space $\Omega(M; p, q)$.

5.11 Theorem. $\pi_{i+1}(SU(2n)) \cong \pi_i(\mathrm{Gr}_n(\mathbb{C}^{2n}))$.

Proof. Let I be the identity in $SU(2n)$, then we have

$$\pi_{i+1} SU(2n) \cong \pi_i(\Omega SU(2n, I, -I)) \cong \pi_i \Omega SU(2n).$$

Claim:$\Omega(M; p, p) \simeq \Omega(M; p, q)$.

We can prove this claim by construct the homotopy equivalence directly: for $\gamma \in \Omega(M; p, q)$, we can construct a path $\bar{\gamma}(t) = \gamma(1-t)$, and for all $\sigma \in \Omega(M; p, q)$, we get a map $\sigma \# \bar{\gamma} \in \Omega(M; p, p)$, and the inverse is given by $\sigma \mapsto \sigma \# \gamma$.

With this claim, we get the second isomorphism in the above sequence. Now we apply Morse theory to show that

$$\pi_i(\Omega SU(2n); I, -I) \cong \pi_i(\Omega_{\text{min}}) \cong \pi_i(\mathrm{Gr}_n(2n)),$$

where Ω_{min} is the smooth submanifold of Ω comsisting of minimal geodesics. Geodesics are exactly critical points of Morse functions, and we have

5.12 Theorem (Minimal Geodesic Index Theorem)**.** Consider the space of minimal geodesics connecting p, q. If non-minimal geodesics has Morse index $> \lambda_0$, then $\pi_i(\Omega_{\text{min}}, \Omega) = 0$ for all $i \leq \lambda_0$ and hence $\pi_i(\Omega_{\text{min}}) \cong \pi_i(\Omega SU(2n))$ for all $i < \lambda_0$.

The second isomorphism $\pi_i(\Omega_{\text{min}})$ follows from a detailed analysis on the minimal geodesics, which implies that $\Omega_{\text{min}} = \coprod_{k=0}^{n} \mathrm{Gr}_k(2n)$. □

TALK 6

Sequential Spectra

Siyang Liu

In this talk we're going to introduce the construction of stable homotopy category, following the ideas from Talk 4. We start with the objects and morphisms in this category.

6.1 Definition. A **sequential spectrum** X is a sequence of pointed topological spaces $\{X^i\}_{i\in\mathbb{N}}$ with **structure maps**

$$\sigma_X^i : \Sigma X^i \to X^{i+1}$$

or dually, the adjoint structure maps

$$\tilde{\sigma}_X^i : X^i \to \Omega X^{i+1}.$$

Here we do not require that the maps being weak homotopy equivalences. We call a spectrum X **Ω-spectrum** if the adjoint structure maps are weak homotopy equivalences.

6.2 Definition. Let X, Y be two sequential spectra, a **morphism** $f : X \to Y$ consists of a sequence of pointed maps $f^i : X^i \to Y^i$ compatible with structure maps, i.e. we have the commutative diagram

$$\begin{tikzcd} \Sigma X^i \arrow[r, "\Sigma f^i"] \arrow[d, "\sigma_X^i"] & \Sigma Y^i \arrow[d, "\sigma_Y^i"] \\ X^{i+1} \arrow[r, "f^{i+1}"] & Y^{i+1} \end{tikzcd}$$

for each i.

We then define the category of sequential spectra $\mathsf{S}^{\mathbb{N}}$ to be the category with objects and morphisms given above. In the category $\mathsf{S}^{\mathbb{N}}$, we define the functor $\Sigma : \mathsf{S}^{\mathbb{N}} \to \mathsf{S}^{\mathbb{N}}$ to be $(\Sigma X)_n = \Sigma X_n$, with structure maps given by suspensions of the corresponding structure maps. Similarly, we can define the loop functor $\Omega : \mathsf{S}^{\mathbb{N}} \to \mathsf{S}^{\mathbb{N}}$ by $(\Omega X)_n = \Omega X_n$.

Now we want a model structure, and furthermore a stable model structure on $\mathsf{S}^{\mathbb{N}}$. Let's make some obeservations on this category $\mathsf{S}^{\mathbb{N}}$ first.

6.3 Example. *There's a special kind of spectrum in $\mathsf{S}^{\mathbb{N}}$: the* **sphere spectrum** $\boldsymbol{S}$*, which is the sequence $\{\boldsymbol{S}_n = \mathbb{S}^n\}_{n\in\mathbb{N}}$ with structure map $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$. Note that $\sigma_{\boldsymbol{S}}^i$ are homeomorphisms for all i. Given $n \in \mathbb{N}$, we customly write $\boldsymbol{S}^n = \Sigma^n \boldsymbol{S}$ and $\boldsymbol{S}^{-n} = F_n^{\mathbb{N}}\mathbb{S}^0$.*

6.4 Example. *The functor $F_n^{\mathbb{N}}$ at the end of the above example is a functor* $\mathsf{Top}_* \to \mathsf{S}^{\mathbb{N}}$ *defined as follows: for each pointed space X, we define*

$$(F_d^{\mathbb{N}} X)_n = \begin{cases} \Sigma^{n-d} X, & \text{when } n \geq d; \\ *, & \text{when } n < d. \end{cases}$$

with structure maps $\sigma^n_{F^{\mathbb{N}}_d X} \equiv \mathrm{id}$ *for all* $n \neq d-1$ *and the canonical pointed map* $* \to X$ *for* $n = d-1$. *We call this spectrum the* **shifted suspension spectrum** *associated to the pointed space* X. *Conversely, given any spectrum* $X \in \mathsf{S}^{\mathbb{N}}$ *and any natural number* $d \in \mathbb{N}$, *we define* $\mathrm{Ev}^{\mathbb{N}}_d(X) = X_d$. *This gives a functor* $\mathrm{Ev}^{\mathbb{N}}_d : \mathsf{S}^{\mathbb{N}} \to \mathsf{Top}_*$. *We customly write* Σ^∞ *for* $F^{\mathbb{N}}_0$.

Moreover, the two functors

$$F^{\mathbb{N}}_d : \mathsf{Top}_* \rightleftarrows \mathsf{S}^{\mathbb{N}} : \mathrm{Ev}^{\mathbb{N}}_d$$

are adjoint to each other. This means that

6.5 Proposition. For all pointed space X and spectrum Y, we have

$$\mathrm{Hom}_{\mathsf{Top}_*}(X, \mathrm{Ev}^{\mathbb{N}}_d(Y)) \cong \mathrm{Hom}_{\mathsf{S}^{\mathbb{N}}}(F^{\mathbb{N}}_d(X), Y).$$

and we obtain the initial and final object in this category, which is $\Sigma^\infty\{*\} := *$.

This adjunction is only categorical, and we want something more: we want this adjunction to be a Quillen adjunction. Since morphisms of the category $\mathsf{S}^{\mathbb{N}}$ is defined levelwise, limits and colimits in this category can be constructed levelwise. That is, given a diagram of sequential spectra $\{X^{(i)}, \alpha_{i,j} : X^{(i)} \to X^{(j)}\}$, the limit of this diagram is the spectrum $\{\varprojlim X^{(i)}\}$ with dual structure maps

$$\tilde{\sigma}^k_{\varprojlim X^{(i)}} = \varprojlim_i \tilde{\sigma}^k_{X^{(i)}} : \varprojlim_i X^{(i)}_k \to \varprojlim_i \Omega X^{(i)}_{k+1}.$$

and similarly we can get the colimit spectrum $\{\varinjlim X^{(i)}\}$ with structure maps

$$\sigma^k_{\varinjlim X^{(i)}} = \varinjlim_i \sigma^k_{X^{(i)}} : \varinjlim_i \Sigma X^{(i)}_k \to \varinjlim_i X^{(i)}_{k+1}.$$

These constructions tell us that the category $\mathsf{S}^{\mathbb{N}}$ has all small limits and colimits. This tells us that $\mathsf{S}^{\mathbb{N}}$ satisfies the first half of condition 4 in [BR20, section 1.1.4].

6.6 Example. *Given a pointed topological space* A *and a spectrum* $X \in \mathsf{S}^{\mathbb{N}}$, *we can define the spectrum* $A \wedge X$ *to be the spectrum consisting of topological spaces* $\{A \wedge X_n\}_{n \in \mathbb{N}}$ *and structure maps*

$$\sigma^k_{A \wedge X} : \Sigma(A \wedge X_k) \cong A \wedge \Sigma X_k \xrightarrow{\mathrm{id}_A \wedge \sigma^k_X} A \wedge X_{k+1},$$

since the wedge sum is defined by wedge sum by $\mathbb{S}^1$, *and the first homeomorphism follows by the commutativity of the wedge product* $\wedge$. *Dually, we can define a spectrum* $\mathsf{Top}_*(A, X)$ *by the sequence of topological spaces* $\{\mathsf{Top}_*(A, X_n)\}_n$ *with dual structure maps*

$$\tilde{\sigma}^k_{\mathsf{Top}_*(A,X)} : \mathsf{Top}_*(A, X_k) \xrightarrow{\mathsf{Top}_*(A, \tilde{\sigma}^k_X)} \mathsf{Top}_*(A, \Omega X_{k+1}) \cong \Omega\, \mathsf{Top}_*(A, X_{k+1}).$$

Since morphism spaces are defined levelwise, we get an isomorphism of sets

$$\mathsf{S}^{\mathbb{N}}(A \wedge X, Y) \cong \mathsf{S}^{\mathbb{N}}(X, \mathsf{Top}_*(A, Y))$$

which gives us property 5 in [BR20, section 1.1.4].

Now we discuss the model structure on $\mathsf{S}^{\mathbb{N}}$. We call this model structure the **levelwise model structure**. Before going into the definition and proofs, let's make a digression into abstract model theory.

(6a) Cofibrantly generated model categories. A model structure on a given cateogry C can in general very difficult to describe, and we want some smaller classes of fibrations, cofibrations and weak equivalences that can generate the whole model structure. This leads to the notion of cofibrantly generated model categories.

6.7 Definition. Let C be a category with all small colimits, and I a set of morphisms in C. We write $I-$inj to be the set of morphisms in C that have the right lifting property with respect to all elements in I, and $I-$cof the class of morphisms in C with the left lifting property w.r.t. all elements in $I-$inj.

We write $I-$cell to be the set of all sequential colimits of pushouts of elements in I. This means that a map $f : A \to B$ is in $I-$cell if and only if there exists a sequence of morphisms

$$A = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

such that for each $f_i : X_i \to X_{i+1}$, there exists an indexed set of morphisms $\{i_\alpha : C_\alpha \to D_\alpha\}$ and a commutative diagram

$$\begin{tikzcd} \coprod_\alpha C_\alpha \arrow[r] \arrow[d, "\coprod_\alpha i_\alpha"] & X_i \arrow[d, "f_i"] \\ \coprod_\alpha D_\alpha \arrow[r] & X_{i+1} \end{tikzcd}$$

which is a pushout square, and that the colimit $A = X_0 \to \varinjlim X_i$ is the morphism f. Write $*$ for the initial object of C, then we say an object X is an **$I-$cell complex** if the canonical morphism $* \to X$ is in $I-$cell.

Observe that by definition, we have

6.8 Lemma. $I-$cell $\subseteq I-$cof.

6.9 Example. *Consider the category of topological spaces* Top*, and let*

$$I = \{\mathbb{S}^{n-1} \to \mathbb{D}^n | n \in \mathbb{N}\},$$

then $I-$inj is exactly the set of all Serre fibrations(See e.g. [Hat00, Section 4.2]), and both $I-$cof and $I-$cell are the set of q-cofibrations. $I-$cell is exactly the class of all CW complexes.

Now we define in an abstract category with a given class of morphisms I the notion of "compact objects", which would give us compact subsets when looking at Top.

6.10 Definition. An object $Z \in$ C is said to be **small** with respect to I if for all morphisms $i : A \to \operatorname{colim} X_n = X$ in $I-$cell, we have an isomorphism

$$\operatorname*{colim}_n \mathrm{C}(Z, X_n) \xrightarrow{\cong} \mathrm{C}(Z, X).$$

6.11 Example. *Obviously if $Z \in$ Top is a compact space, then Z is small with resepct to the I given in example 6.9. (See [Lee11, Chapter 5], for example) Conversely, I'm not clear if all such I-small spaces are small.*

6.12 Lemma. Small objects are preserved by push-outs. That is, if we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

consisting of I-small objects whose pushout is P, then P is also I-small.

Now we can define the notion of "cofibrantly generated model categories":

6.13 Definition. A model category C is **cofibrantly generated** if there are sets I and J such that the following hold.

- The domains of I are I-small;
- The domains of J are J-small;
- Fibrations in C are precisely $J-$inj;
- The acyclic fibrations in C are precisely $I-$inj.

6.14 Example. Top *with Quillen model structure is cofibrantly generated by classes*

$$I = \{\mathbb{S}^{n-1} \to \mathbb{D}^n\}_{n\in\mathbb{N}}$$

and

$$J = \{\mathbb{D}^n \to \mathbb{D}^n \times I\}_{n\in\mathbb{N}}.$$

Here we include the number 0 *and write* $\mathbb{S}^{-1} = \varnothing$*. One can verify that* $I-$inj *is exactly the set of acyclic Serre fibrations and* $J-$inj *exactly the set of Serre fibrations.*

6.15 Example. *The category of simplicial sets* sSet *with model structure defined in chapter 2 is also cofibrantly generated. For* $0 \le r \le n$*, we define the* ***r-horn*** $\Lambda^r[n]$ *to be a functor* $\triangle^n \to$ Set *sending* $[k]$ *to the order-preserving injections* $[k] \to [n]$ *excluding both the identity* $[n] \to [n]$ *and the map* $d^r : [n-1] \to [n]$ *which avoids* r*. We then let*

$$I = \{\partial\Delta[n] \to \Delta[n] | n \in \mathbb{N}\}$$

and

$$J = \{\Lambda^r[n] \to \Delta[n] | n \in \mathbb{N}\}$$

to be the corresponding generating sets for cofibrations and acyclic cofibrations.

We end this discussion with a criterion for morphism sets I and J cofibrantly generating a model structure:

6.16 Theorem (Recognition Theorem)**.** Let C be a category with all small limits and colimits. Let $\mathcal{W}$ be a class of morphisms closed under composition and contains all identity morphisms. Further, let I and J be the sets of morphisms in C. Assume that

- $\mathcal{W}$ satisfies 2-out-of-3 property,
- the domains of I are small with respect to I,
- the domains of J are small with respect to J,
- $J-$cell $\subseteq \mathcal{W} \cap I-$cof,
- $I-$inj $\subseteq \mathcal{W} \cap J-$inj,
- either $\mathcal{W} \cap I-$cof $\subseteq J-$cof or $\mathcal{W} \cap J-$inj $\subseteq I-$inj.

Then C can be given a cofibrantly generated model structure with $\mathcal{W}$ being the weak equivalences, I the set of generating cofibrations and J the set of generating acyclic cofibrations.

Now we go back to $\mathsf{S}^{\mathbb{N}}$.

6.17 Theorem. There is a levelwise model structure defined on $\mathsf{S}^{\mathbb{N}}$, where the weak equivalences are levelwise weak homotopy equivalences of pointed topological spaces. The fibrations are the class of levelwise Serre fibrations of pointed spaces. The cofibrations are generated canonically, and we call then **q-cofibrations**.

Moreover, the levelwise model structure is cofibrantly generated with generating sets given by

$$I^{\mathbb{N}}_{level} = \{F^{\mathbb{N}}_d \mathbb{S}^{n-1}_+ \to F^{\mathbb{N}}_d \mathbb{D}^n_+ | n, d \in \mathbb{N}\}$$
$$J^{\mathbb{N}}_{level} = \{F^{\mathbb{N}}_d \mathbb{D}^n_+ \to F^{\mathbb{N}}_d (\mathbb{D}^n \times [0,1])_+ | n, d \in \mathbb{N}\}.$$

In particular, the q-cofibrations are levelwise q-cofibrations of pointed topological spaces.

Here we use the notation convention that given $X \in \mathsf{Top}$, we have a functor $(-)_+ : \mathsf{Top} \to \mathsf{Top}_*$ where $X_+ = (X \sqcup *, *)$.

(6b) The stable model structure. Although we have defined a model structure on $\mathsf{S}^{\mathbb{N}}$, what we really want is a **stable model structure** on $\mathsf{S}^{\mathbb{N}}$, which are supposed to give us the correct "stable homotopy theory". We first state the definition of stable model structure. This is very similar to the "stable infinity category" as mentioned in [Lur17]:

6.18 Definition. We say a model category C is **stable** if we have a pair of functors (Ω, Σ) called loop functor and suspension functor, who give mutually inverse equivalences of categories from Ho(C) to itself.

In infinity category, we can say they are "homotopy equivalences", but here without the higher structures we do not have the notion of "homotopy equivalence", hence we can only say they should give a category equivalence when passing to the homotopy category Ho(C). To achieve this, we need to somewhat modify the levelwise model structure by slightly changing the class of fibrations, cofibrations and weak equivalences.

First of all, we construct the loop and suspension functors on $\mathsf{S}^{\mathbb{N}}$. These are defined simply using the action of Top_* on $\mathsf{S}^{\mathbb{N}}$:

6.19 Definition. For $X \in \mathsf{S}^{\mathbb{N}}$, we define $\Sigma X = \mathbb{S}^1 \wedge X$ and $\Omega X = \mathsf{Top}_*(\mathbb{S}^1_+, X)$.

These two functors Σ and Ω are not necessarily equivalences when passing to the homotopy category.

We need to modify the class of weak equivalences as follows:

6.20 Definition. Let X be a spectra, we define the **k-th homotopy group** of X to be the class of morphisms $[\Sigma^k \mathbb{S}, X]$, where $[\Sigma^k \mathbb{S}, X]$ is the quotient of $\mathsf{S}^{\mathbb{N}}(\Sigma^k \mathbb{S}, X)$ by homotopy.

By definition of the levelwise model structure, the map $[\Sigma^k \mathbb{S}, X]$ is exactly the colimit $\operatorname{colim}_n \pi_{n+k}(X_k)$.

6.21 Definition. We say a morphism $f : X \to Y$ in $\mathsf{S}^{\mathbb{N}}$ is a **π_*-isomorphism** if it induces isomorphisms on all homotopy groups.

We then define the class $\mathcal{W}'$ for the stable model structure on $\mathsf{S}^{\mathbb{N}}$ to be the class of π_*-isomorphisms. For fibrations, we need some more constructions. Let $f : X \to Y$ be a q-cofibration of spectra, $p : P \to Q$ a levelwise fibration, and $i : A \to B$ a q-cofibration of topological spaces, then

6.22 Proposition. The induced map of spectra

$$\hom_{\square}(f, p) : \mathsf{S}^{\mathbb{N}}(Y, P) \to \mathsf{S}^{\mathbb{N}}(X, P) \bigvee_{\mathsf{S}^{\mathbb{N}}(X,P)} \mathsf{S}^{\mathbb{N}}(Y, Q)$$

is a fibration of pointed spaces, and if f or p is a levelwise weak equivalence, then $\hom_{\square}(f, p)$ is also a weak homotopy equivalence.

$$f \square i : Y \wedge A \bigvee_{X \wedge A} X \wedge B \to Y \wedge B$$

is a q-cofibration of spectra, and if i is a weak homotopy equivalence or f is a levelwise weak equivalence, then $f\square i$ is a levelwise weak equivalence.

$$\mathrm{hom}_{\square}(i,p): \ \mathrm{Top}_*(B,P) \to \mathrm{Top}_*(A,P) \times_{\mathrm{Top}_*(A,Q)} \mathrm{Top}_*(B,Q)$$

is a levelwise fibration of spectra, and if i is a weak homotopy equivalence or p is a levelwise weak equivalence, then $\mathrm{hom}_{\square}(i,p)$ is a levelwise weak equivalence.

Here we enrich the category $\mathsf{S}^{\mathbb{N}}$ by giving $\mathsf{S}^{\mathbb{N}}(X,Y)$ a topological structure which makes it into a subspace of $\prod_{i\in\mathbb{N}} \mathrm{Top}_*(X_i, Y_i)$. With this enrichment, we get a duality

$$\mathsf{S}^{\mathbb{N}}(A \wedge X, Y) \cong \mathsf{S}^{\mathbb{N}}(X, \mathrm{Top}_*(A,Y)) \cong \mathrm{Top}_*(A, \mathsf{S}^{\mathbb{N}}(X,Y)).$$

This duality gives the proof of the above proposition.

Let $\lambda_n : F^{\mathbb{N}}_{n+1}\mathbb{S}^1 \to F^{\mathbb{N}}_n\mathbb{S}^0$ be the map corresponding to the identity map $\mathbb{S}^1 \to \mathrm{Ev}^{\mathbb{N}}_{n+1} F^{\mathbb{N}}_n \mathbb{S}^0 \cong \mathbb{S}^1$, and let $M\lambda_n$ be the mapping cylinder of λ_n, then we have the pushout square

$$\begin{tikzcd}
F^{\mathbb{N}}_{n+1}\mathbb{S}^1 \arrow[r, "\lambda_n"] \arrow[d, "i_1"] & F^{\mathbb{N}}_n \arrow[d] \\
F^{\mathbb{N}}_{n+1} \wedge [0,1]_+ \arrow[r, "t_n"] & M\lambda_n
\end{tikzcd}$$

and we write k_n to be the composition $t_n \circ i_0$. This map is a q-cofibration and a π_*-isomorhism.

6.23 Definition. We define the **stable model structure** on $\mathsf{S}^{\mathbb{N}}$ to be the model structure cofibrantly generated by the classes

$$I^{\mathbb{N}}_{stable} = I^{\mathbb{N}}_{level};$$
$$I^{\mathbb{N}}_{stable} = J^{\mathbb{N}}_{level} \cup \{k_n \square (\mathbb{S}^{a-1}_+ \to \mathbb{D}^a_+) | a, n \in \mathbb{N}\}$$

With this model structure, we get that

6.24 Proposition. A map of spectra $f : X \to Y$ has the right lifting property with respect to $J^{\mathbb{N}}_{stable}$ if and only if f is a levelwise fibration of spaces and for each $n \in \mathbb{N}$, the map

$$X_n \to Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$$

induced by $\tilde{\sigma}^X_n$ and f is a weak homotopy equivalence. In particular, if $Y = *$, then X has the right lifting property if and only if X is a Ω-spectrum.

We call the fibrations described in this Proposition **stable fibrations**. This implies that fibrant objects in this model category are Ω-spectra.

6.25 Theorem. The stable model structure on sequential sepctra is defined by the three classes below:

- The weak equivalences are the π_*-isomorphisms.
- The cofibrations are the q-cofibrations.
- The fibrations are given by Proposition 2.3.10 and are called the **stable fibrations**.

In this case, the loop functor Ω and the suspension functor Σ are Quillen equivalences if $\mathsf{S}^{\mathbb{N}}$ is equipped with the stable model structure, and therefore they induce categorical equivalences when passing to homotopy.

6.26 Definition. We define the **stable homotopy category** to be the homotopy category of $\mathsf{S}^{\mathbb{N}}$ with the stable model structure, i.e. we define

$$\mathsf{SHC} = \mathsf{Ho}(\mathsf{S}^{\mathbb{N}}).$$

TALK 7

Spectra in a General Model Category

Jishnu Bose

The goal of this talk is to define the suspension and loop functors in the model category and see what we get from these functors.

7.1 Definition. A model category C is **pointed** if the unique map from the initial and final object is an isomorphism. We denote this object by $*$, and this is called the **basepoint**.

With this object, we call the composition $A \to * \to B$ a **zero map**.

7.2 Definition. A **cylinder object** of $X \in \mathsf{C}$ is a factorization of the fold map $X \coprod X \to X$,

$$X \coprod X \xrightarrow{(i_0, i_1)} \mathrm{Cyl}(X) \underset{\sim}{\xrightarrow{r}} X$$

We can choose $\mathrm{Cyl}(X)$ such that i_0 and i_1 are also weak equivalences, (i_0, i_1) a cofibration, and r an acyclic fibration.

Similarly, we can define

7.3 Definition. A **path object** for $X \in \mathsf{C}$ is a factorization of $X \xrightarrow{\Delta} X \times X$,

$$X \underset{\sim}{\xrightarrow{s}} PX \xrightarrow{(p_0, p_1)} X \times X$$

One can choose PX such that p_0, p_1 are weak equivalences.

Such a cylinder object and path object are called "very good".

7.4 Definition (Suspension). Let C be a pointed model category and $X \in \mathsf{C}$ cofibrant, then we have the diagram

$$\begin{tikzcd} X \coprod X \arrow[r] \arrow[d] & * \arrow[d, dashed] \\ \mathrm{Cyl}(X) \arrow[r, dashed] & \Sigma X. \end{tikzcd}$$

and we write the **suspension** of X to be the pushout of this diagram.

Dually, for $Y \in \mathsf{C}$ fibrant, the **loop** of Y, ΩY, is given by

$$\begin{tikzcd} Y \times Y & * \arrow[l] \\ PY \arrow[u] & \Omega Y \arrow[l, dashed] \arrow[u, dashed] \end{tikzcd} .$$

Thus we get two functors Σ and Ω.

7.5 Proposition. Σ and Ω are functors $\mathrm{Ho}(\mathrm{C}) \to \mathrm{Ho}(\mathrm{C})$.

Proof. Two cylinder objects $\mathrm{Cyl}(X)$ and $\mathrm{Cyl}(X)'$ give ΣX and $\Sigma X'$ respectively, and we get mutually homotopically inverse maps $c_1 : \mathrm{Cyl}(X) \rightleftarrows \mathrm{Cyl}(X)' : c_2$, then we have and in $\mathrm{Ho}(\mathrm{C})$, we get $F \circ G \simeq \mathrm{id}$,

$$\begin{tikzcd}
X \sqcup X \arrow[r] \arrow[d] & * \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Cyl}(X) \arrow[r, "p_1"] \arrow[drr, bend right, "p_2 \circ c_1"'] & \Sigma X \arrow[dr, dashed, "F"] & \\
& & \Sigma X'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \sqcup X \arrow[r] \arrow[d] & * \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{Cyl}(X') \arrow[r, "p_2"] \arrow[drr, bend right, "p_1 \circ c_2"'] & \Sigma X' \arrow[dr, dashed, "G"] & \\
& & \Sigma X
\end{tikzcd}$$

and by inverting the objects, we get F and G are mutually homotopy inverses, and therefore $\Sigma X \simeq \Sigma X'$. This tells us that Σ is well-defined in the homotopy category. $\square$

Now given a morphism $f : A \to B$, assume A, B are both fibrant and cofibrant, then we have the commutative diagram

$$\begin{tikzcd}
* \arrow[d] & A \coprod A \arrow[l] \arrow[r] \arrow[d, "f \sqcup f"] & \mathrm{Cyl}(A) \arrow[d, "\mathrm{Cyl}(f)"] \\
* & B \sqcup B \arrow[l] \arrow[r] & \mathrm{Cyl}(B)
\end{tikzcd}$$

Since in $\mathrm{Ho}(\mathrm{C})$, $\mathrm{Cyl}(f)$ is unique by the universal property of pushouts, we get the map $\Sigma f : \Sigma A \to \Sigma B$ in $\mathrm{Ho}(\mathrm{C})$.

7.6 Example (Top_*). *Assume X is a (cofibrant) pointed topological space, then $X \sqcup X \simeq X \vee X$, $\mathrm{Cyl}(X) \simeq X \wedge [0,1]$ and the pushout diagram is*

$$\begin{tikzcd}
X \vee X \arrow[r] \arrow[d] & X \wedge [0,1] \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}$$

where the horizontal map of the first row is the inclusion at the ends of the cylinder. So ΣX is just the reduced suspension in this case, which is given explicitly by

$$\Sigma X = \frac{X \times [0,1]}{X \vee [0,1]}.$$

We can do the same for the category of spectra.

7.7 Example ($\mathrm{S}^{\mathbb{N}}$). *Let Z be cofibrant sequential spectrum, then we define*

$$(\Sigma Z)_n = \Sigma(Z_n),$$

and the structure map is given by

$$\sigma^k_{\Sigma Z} : \Sigma\Sigma Z_k \xrightarrow{\Sigma\sigma^n_Z} \Sigma Z_{n+1}.$$

Since we know that $\Sigma Z_n = \mathbb{S}^1 \wedge Z_n$, the spectrum ΣZ is just the spectrum $\mathbb{S}^1 \wedge Z$.

The last example is the chain complex of modules.

7.8 Example ($\mathsf{Ch}(R)$). *Let $A^\bullet \in \mathsf{Ch}(R)$ be a chain complex, then a choice of $\mathsf{Cyl}(A^\bullet)$ is just $\mathsf{Cyl}(A^\bullet) = A^n \oplus A^{n+1} \oplus A^n$ with differential given by*

$$\delta_{\mathsf{Cyl}} = \begin{pmatrix} d_A & \mathrm{id} & 0 \\ 0 & d_A & 0 \\ & -\mathrm{id} & d_A \end{pmatrix}$$

and the push-out diagram reads

$$\begin{tikzcd} A^\bullet \oplus A^\bullet \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\ \mathsf{Cyl}(A^\bullet) \arrow[r] & \Sigma A^\bullet \end{tikzcd}$$

so we have $\Sigma A^\bullet = \mathrm{coker}(i)$ where $(\Sigma A^n) = \mathrm{coker}(A^n \oplus A^n \to A^n \oplus A^{n+1} \oplus A^n) \cong A^{n+1}$. So $\Sigma A^\bullet = A^\bullet[1]$ and similarly, $\Omega A^\bullet = A^\bullet[-1]$.

In this case, the Σ and Ω are homotopy equivalences.

7.9 Proposition. Let C be a pointed model category, then

$$\Sigma\colon \mathsf{Ho}(\mathsf{C}) \rightleftarrows \mathsf{Ho}(\mathsf{C})\colon \Omega$$

define an adjunct pair, i.e. there exist natural isomorphisms $\rho_{A,B}\colon [\Sigma A, B] \to [A, \Omega B]$ for any $A, B \in \mathsf{C}$.

Sketch of Proof. Given any $[f] \in [\Sigma A, B]$ represented by $F\colon \mathsf{Cyl}(A) \to B$, we have the sequence

$$A \sqcup A \to \mathsf{Cyl}(A) \xrightarrow{F} B$$

where the composition is the zero map or F is a left homotopy between two copies of the zero maps $A \to B$. Similarly, for any $[g] \in [A, \Omega B]$ represented by $G\colon A \to PB$, then the composition $\Omega A \to PB \to B \times B$ should be a right homotopy between zero maps $A \to B$.

If we choose A, B to be both fibrant and cofibrant, then the left and right homotopies should coincide. $\square$

7.10 Definition. If Σ and Ω are equivalences in the homotopy category, then C is called a **stable model category**.

7.11 Example (Top_*). *The category of pointed topological spaces with Quillen model structure is not stable. For a functor Σ to be a categorical equivalence, it should be full, faithful and essentially surjective. Full means any function $[A, B] \to [\Sigma A, \Sigma B]$ should be surjective for any A, B, but this is not true if $A = \mathbb{S}^2$ and $B = \mathbb{S}^1$, since then $[A, B] = 0$ but $[\Sigma A, \Sigma B] = \mathbb{Z}$.*

7.12 Example. *The category $\mathsf{S}^{\mathbb{N}}$(with the stable model structure) is stable, and $\mathsf{Ch}(R)$ is also stable(with either injective or projective model structures).*

7.13 Definition. Let $f\colon A \to B$ be a morphism in C, then the **cofiber** of f is the pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & \mathrm{cof}(f) \end{tikzcd}$$

The **fiber** of f is the pull-back of the diagram

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] & * \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

7.14 Proposition. If $X, Y \in \mathsf{C}$, then $[\Sigma X, Y]$ and $[X, \Omega Y]$ are groups and the adjunction bijections $\rho_{X,Y}\colon [\Sigma X, Y] \to [X, \Omega Y]$ are natural group isomorphisms.

Now let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence with f a cofibration between cofibrant objects and $C = \mathrm{cof}(f)$. Let X be a fibrant object in C, we want to define an action of $[\Sigma A, X]$ on $[C, X]$. basically we want a map

$$[C, X] \times [\Sigma A, X] \to [C, X].$$

Pick an element on the group $[\Sigma A, X] \simeq [A, \Omega X]$, which is represented by a (right) homotopy $h\colon A \to PX$ bewteen two zero maps $A \to X$, and we want PX to be very good, i.e. $PX \xrightarrow{p_0} X$ is an acyclic fibration. Let $q\colon C \to X$, then we have a diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"] & PX \arrow[d, "p_0"] \\ B \arrow[r, "q \circ g"] & X. \end{tikzcd}$$

Since f is a cofibration, we get a lift $\varphi\colon B \to PX$ of this diagram, then $p_0 \circ \varphi = q \circ g$ and $p_1 \circ \varphi \circ f = p_1 \circ h = 0$. From the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ * \arrow[r] & C \end{tikzcd}$$

we get a map $w\colon C \to X$. Now our action is given by $[q] \circ [h] = [w]$. This is well-defined and is natural in X. Dually, we can define an action for a fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration between fibrant objects an action $[X, F] \times [X, \Omega B] \to [X, F]$.

7.15 Example. *If $X = \mathbb{S}^0$, then the action is $\pi_0(F) \times \pi_0(\Omega B) \to \pi_0(F)$, which is just the usual action of $\pi_1(B)$ on fibers of the fibration.*

Now let $A \xrightarrow{f} B \xrightarrow{q} C$ be a cofiber sequence. If f is a cofibration between cofibrant objects, then we have

$$[C \sqcup \Sigma A, C \sqcup \Sigma A] \simeq [C, C \sqcup \Sigma A] \times [\Sigma A, C \sqcup \Sigma A] \to [C, C \sqcup \Sigma A].$$

The identity map $\mathrm{id} \in [C \sqcup \Sigma A, C \sqcup \Sigma A]$ induces a map $\odot\colon C \to C \sqcup \Sigma A$.

7.16 Definition. A **cofiber sequence** in Ho(C) is a diagram $X \to Y \to Z$ which is isomorphic to a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ where f is a cofibration between cofibrant objects and $C = \mathrm{cof}(f)$.

This diagram is equipped with a right coaction in Ho(C) $Z \to Z \sqcup \Sigma X$ isomorphic to the coaction described before.

Dually, we define $X \to Y \to Z$ to be a **fiber sequence** if it's isomorphic to $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration between fibrant objects and $F = \mathrm{fib}(p)$. This admits a right action $X \times \Omega Z \to X$.

7.17 Definition. Let $X \to Y \to Z$ be a cofiber sequence in Ho(C). Then we can define a **boundary map** $\partial\colon Z \xrightarrow{\odot} Z \sqcup \Sigma X \xrightarrow{(0,\mathrm{id})} \Sigma X$.

Similarly, for fiber sequences, we can define

$$\partial\colon \Omega Z \to X \times \Omega Z \xrightarrow{\odot} X$$

Note: The boundary map can recover the data of the action/coaction by

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\simeq"] & Y \arrow[r] \arrow[d, "\simeq"] & Z \arrow[r, "\partial"] \arrow[d, "\simeq"] & \Sigma X \arrow[d, "\simeq"] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & \Sigma A \end{tikzcd}$$

where the first row is a cofiber sequence, then $A \to B \to C$ is also a cofiber sequence, and $C \to \Sigma A$ is the boundary map.

7.18 Proposition. Assume $X \to Y \to Z$ is a fiber sequence, then so is $\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y$ with the action

$$\Omega Z \times \Omega Y \to \Omega Z$$

where $\Omega Z \times \Omega Y \xrightarrow{\mathrm{id} \times \Omega g} \Omega Z \times \Omega Z \xrightarrow{(\mathrm{id}, -\mathrm{id})} \Omega Z \times \Omega Z \xrightarrow{*} \Omega Z$.

7.19 Corollary. $\Omega Y \xrightarrow{-\Omega g} \Omega Z \to X$ is also a fiber sequence.

We also have dual results for Σ, namely,

7.20 Proposition. $X \xrightarrow{f} Y \xrightarrow{g} Z$ a cofiber sequence, then $Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$ is a cofiber sequence, and similarly we can get a coaction

$$\odot \colon \Sigma X \to \Sigma X \sqcup \Sigma Y.$$

7.21 Corollary. $Z \xrightarrow{\partial} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$ is a cofiber sequence.

(7a) Long Exact Puppe Sequence.

7.22 Theorem. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a fiber sequence in Ho(C), and $\partial \colon \Omega Z \to X$ a boundary map, $A \in$ Ho(C), then we have the long exact sequence

$$\cdots \to [A, \Omega^{n+1} Z] \xrightarrow{(-1)^n(\Omega^n \partial)_*} [A, \Omega^n X] \xrightarrow{(-1)^{n-1}(\Omega^{n-1} g)_*} \cdots \to [A, \Omega Z] \xrightarrow{\partial_*} [A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z].$$

where all but the several terms at the end of this sequence are group homomorphisms, and the maps at the end of this sequence are maps of pointed sets.

Proof. Suppose $(X \to Y \to Z) \simeq (F \xrightarrow{i} E \xrightarrow{p} B)$ where p is a fiber of fibrant objects, then it's enough to check $[A, F] \xrightarrow{i_*} [A, E] \xrightarrow{p_*} [A, B]$ is exact. Since $p \circ i = 0$, we get $p_* \circ i_* = 0$, so $\mathrm{im}(i_*) \subseteq \ker(p_*)$. Let $u \colon A \to E$ be such that $[p \circ u] = 0$, i.e. there is $h \colon \mathrm{Cyl}(A) \to B$ with Cyl(A) very good, so that $h \circ i_0 = p \circ u$ and $h \circ i_1 = 0$. Then we have a diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & E \arrow[d, "p"] \\ \mathrm{Cyl}(A) \arrow[r, "h"] & B \end{tikzcd}$$

Since p is a fibration, we have a lift $H \colon \mathrm{Cyl}(A) \to E$ such that $H \circ i_0 = u$ and $p \circ H \circ i_1 = h \circ i_1 = 0$, then $H \circ i_1$ lifts over F, so we get $v \colon A \to F$ with $i \circ v = H \circ i_1$. So then $i_*[v] = [u]$. $\square$

Finally, we have a dual Puppe sequence for cofiber sequences:

7.23 Theorem. Let $X \to Y \to Z$ be a cofiber sequence in Ho(C) and $\partial \colon \Sigma Z \to X$, $A \in$ Ho(C), then we have the long exact sequence

$$\cdots \to [\Sigma^{n+1} X, A] \xrightarrow{(-1)^n(\Sigma^n \partial)^*} [\Sigma^n X, A] \to \cdots \to [Z, A] \xrightarrow{g^*} [Y, A] \xrightarrow{f^*} [X, A].$$

In Top_*, $A = \mathbb{S}^0$, then we recover the long exact sequence of homotopy groups.

TALK 8

Triangulated Structures

Siyang Liu

In this talk we're going to prove the following theorem:

8.1 Theorem. If C is a stable model category, then Ho(C) is triangulated.

Firstly, we recall the definition of the triangulated structure.

(8a) Triangulated Category. Let T be an additive category with an additive self-equivalence $\Sigma : \mathsf{T} \to \mathsf{T}$[1], then

8.2 Definition. A **triangle** in T is a sequence of morphisms $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z \xrightarrow{f_3} \Sigma X$. A **morphism** of triangles from $X \to Y \to Z \to \Sigma X$ to $X' \to Y' \to Z' \to \Sigma X'$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "\phi_1"] & Y \arrow[r, "f_2"] \arrow[d, "\phi_2"] & Z \arrow[r, "f_3"] \arrow[d, "\phi_3"] & \Sigma X \arrow[d, "\Sigma\phi_1"] \\ X' \arrow[r, "f_1'"] & Y' \arrow[r, "f_2'"] & Z' \arrow[r, "f_3'"] & \Sigma X'. \end{tikzcd}$$

Roughly speaking, a triangulated category is such an additive category T with a class of triangles named "distinguished triangles" satisfying some axioms, which can be thought of as an analogue of long exact sequences of homology groups of modules.

8.3 Definition. A **triangulated category** T is an additive category with an additive self-equivalence Σ, often called the **shift functor**, with a class of **distinguished triangles**, or **exact triangles**, satisfying the following axioms:

(T1) The triangle

$$* \longrightarrow X = X \longrightarrow *$$

is exact for every $X \in \mathsf{T}$. A triangle isomorphic to an exact triangle is exact. Every morphism $f : X \to Y$ fits into some exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{u} \Sigma X.$$

(T2) The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{u} \Sigma X$$

[1] We also use the notation $[1]$ to denote such an equivalence, but here we use Σ in order to be consistent with the corresponding functor in stable homotopy category

is exact if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{u} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is exact.

(T3) Let

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "\phi_1"] & Y \arrow[r, "f_2"] \arrow[d, "\phi_2"] & Z \arrow[r, "f_3"] & \Sigma X \arrow[d, "\Sigma\phi_1"] \\ X' \arrow[r, "f_1'"] & Y \arrow[r, "f_2'"] & Z' \arrow[r, "f_3'"] & \Sigma X' \end{tikzcd}$$

be a diagram such that the two rows are exact triangles and the left square commutes. Then one can add a morphism $\phi_3 : Z \to Z'$ to this diagram such that the resulting second and third square commute.

(T4) Let

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[r, "f_2"] \arrow[d, "u_1"] & Z \arrow[r, "f_3"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "g_1"] & U \arrow[r, "g_2"] \arrow[d, "u_2"] & V \arrow[r, "g_3"] & \Sigma X \\ & W \arrow[d, "u_3"] & & \\ & \Sigma Y & & \end{tikzcd}$$

be a commutative diagram such that the column and two rows are exact triangles. Then there is an exact triangle

$$Z \xrightarrow{v_1} V \xrightarrow{v_2} W \xrightarrow{v_3} \Sigma Z$$

that can be added to the first diagram to obtain the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[r, "f_2"] \arrow[d, "u_1"] & Z \arrow[r, "f_3"] \arrow[d, "v_1"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "g_1"] & U \arrow[r, "g_2"] \arrow[d, "u_2"] & V \arrow[r, "g_3"] \arrow[d, "v_2"] & \Sigma X \\ & W \arrow[r, equal] \arrow[d, "u_3"] & W \arrow[d, "v_3"] & \\ & \Sigma Y \arrow[r, "\Sigma f_2"] & \Sigma Z. & \end{tikzcd}$$

Axiom (T4) is also known as the **octahedron axiom**. If we regard $Z = Y/X$ and $V = U/X$, then the octahedron axiom is the same as saying that $\dfrac{Y/X}{U/X} \simeq \dfrac{Y}{U}$. This is an intuitive way to understand this complicated axiom.

Now we turn to the proof of theorem 8.1. We choose the shift functor in Ho(C) to be the suspension functor Σ, and the class of distinguished triangles to be the cofiber sequences $X \to Y \to Z$ with the map $Z \to \Sigma X$ given by the boundary map, then (T1) is almost trivial.

Proof of (T1). It remains to show that given any map $f: X \to Y$, we can complete f to a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. This is done via cofibrant replacement: for any $X \in \mathsf{C}$, we have a cofibrant object $X' \in \mathsf{C}$ with a fibration $X' \to X$. Let X', Y' be the cofibrant replacement of X, Y, respectively, then the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & Y' \arrow[d, "j"] \\ X' \arrow[r, "f\circ i"] & Y \end{tikzcd}$$

gives us a lifting $f': X' \to Y'$ which satisfies $jf' = fi$. 2 out of 3 axiom tells us that f' is a cofibration, hence we get a cofibration between cofibrant objects $X' \xrightarrow{f'} Y'$. Cofiber $Y' \xrightarrow{g'} Z'$ of f' gives a cofiber sequence $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$. In the homotopy category Ho(C), since all weak equivalences are isomorphisms, by inverting j we get the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g'j^{-1}} Z'$. □

Next we prove (T3) and use (T3) to prove (T2).

8.4 Lemma. Let C be a stable model category. Assume that we have a commutative diagram in Ho(C)

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\phi_1"] & Y' \arrow[r, "g'"] \arrow[d, "\phi_2"] & Z' \arrow[r, "u'"] & \Sigma X' \arrow[d, "\Sigma\phi_1"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "u"] & \Sigma X, \end{tikzcd}$$

where the two rows are cofiber sequences with respective boundary maps, then there is a map $\phi_3: Z' \to Z$ making the resulting second and third square commute.

Proof. Since the two rows are cofiber sequences, by universal property of pushout, there exists a unique morphism $\phi_3: Z' \to Z$ making the diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "\phi_2"] & Z' \arrow[d, "\phi_3"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutative. Now we need to verify that the diagram

$$\begin{tikzcd} Z' \arrow[r, "u'"] \arrow[d, "\phi_3"] & \Sigma X' \arrow[d, "\Sigma\phi_1"] \\ Z \arrow[r, "u"] & \Sigma X \end{tikzcd} \tag{8.1}$$

commutes. Note that the horizontal arrows are boundary maps of cofiber sequences, i.e. they are given by the composition

$$Z \xrightarrow{\odot} Z \sqcup \Sigma X \xrightarrow{0 \sqcup \mathrm{id}} \Sigma X,$$

so we firstly show that the diagram

$$\begin{tikzcd} Z \arrow[r, "\odot"] \arrow[d, "\phi_3"] & Z \sqcup \Sigma X \arrow[d, "\phi_3 \sqcup \Sigma\phi_1"] \\ Z' \arrow[r, "\odot"] & Z' \sqcup \Sigma X' \end{tikzcd} \tag{8.2}$$

commutes. Pick any $A \in \mathsf{C}$, let's look at the diagram

$$\begin{tikzcd}
[Z,A] & {[Z \sqcup \Sigma X, A]} \arrow[l] \\
{[Z',A]} \arrow[u] & {[Z' \sqcup \Sigma X', A]} \arrow[l] \arrow[u]
\end{tikzcd}.$$

Pick $(\alpha, \beta) \in [Z' \sqcup \Sigma X', A]$ where $\alpha\colon Z' \to A$ and $\beta\colon X' \to PA$ are representatives of equivalence classes, then we have the commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "\beta"] \arrow[d, tail, "f'"] & PA \arrow[d, two heads, "p_0"] \\
Y' \arrow[r, "\alpha g'"] & A
\end{tikzcd}$$

where f' is a cofibration while p_0 is an acyclic fibration, thus we have a lifting $\varphi\colon Y' \to PA$ making the diagram commutative. This implies $p_0\varphi = \alpha g'$ and we get a diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g'"] \arrow[ddr, bend left, "p_1\varphi"] & \\
* \arrow[r] \arrow[drr, bend right] & Z' \arrow[dr, dashed, "w"] & \\
 & & A
\end{tikzcd}$$

where $p_1\varphi f' = p_1\beta = 0$ since $\beta\colon X' \to PA$ comes from the map $\Sigma X' \to A$, i.e. β is a homotopy between zero maps. The universal property of pushout then gives us a morphism $w\colon Z' \to A$, which is exactly the morphism $[\alpha] \odot [\beta]$, and the image of this morphism under the map $[\phi_3, A]$ is the composition $[w \circ \phi_3]$.

On the other hand, via the image $[\phi_3 \sqcup \Sigma\phi_1, A]$, the morphism (α, β) is sent to the pair $(\alpha \circ \phi_3, \beta \circ \phi_1)$, and we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\beta\phi_1"] \arrow[d, tail, "f"] & PA \arrow[d, two heads, "p_0"] \\
Y \arrow[r, "\alpha g'\phi_2"] & A.
\end{tikzcd}$$

The morphism $\varphi\phi_2\colon Y \to Y' \to PA$ fits into this diagram as a lifting, and in the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \arrow[ddr, bend left, "p_1\varphi\phi_2"] & \\
* \arrow[r] \arrow[drr, bend right] & Z \arrow[dr, "w\phi_3"] & \\
 & & A
\end{tikzcd}$$

The morphism $w\phi_3$ makes the diagram commutative, hence we get that $[w\phi_3] = [w]$, which implies that the diagram (8.2) is commutative for all A, and hence the diagram (8.1) is commutative. This proves the lemma. □

There's a Corollary from the proof of axiom T3:

8.5 Corollary. Given a commutative diagram in Ho(C)

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\phi_1"] & Y' \arrow[r, "g'"] \arrow[d, "\phi_2"] & Z' \arrow[r, "u'"] \arrow[d, "\phi_3"] & \Sigma X' \arrow[d, "\Sigma\phi_1"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "u"] & \Sigma X, \end{tikzcd}$$

where the top row is a cofiber sequence with boundary map u' and the vertical arrows are isomorphisms(i.e. weak equivalences), then the bottom row is also a cofiber sequence with boundary map u.

Now we prove axiom (T2) using the axiom (T3). In a stable model category C, we have the suspension-loop adjunction, which gives for all $X \in$ C the unit and counit maps

$$\eta_X : X \to \Omega\Sigma X,$$
$$\epsilon_X : \Sigma\Omega X \to X,$$

which are isomorphisms in Ho(C).

8.6 Lemma. Let C be a stable model category, then

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence in Ho(C) with boundary map $u : Z \to \Sigma X$ if and only if

$$\Omega Z \xrightarrow{-\eta_X^{-1} \circ \Omega u} X \xrightarrow{f} Y$$

is a cofiber sequence with boundary map $\epsilon_Z^{-1} \circ g : Y \to \Sigma(\Omega Z)$.

Proof. In the previous talk we have stated that Σ preserves cofiber sequences, i.e. if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence with boundary map $\partial : Z \to \Sigma X$, then $Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$ is a cofiber sequence with boundary map $\Sigma X \xrightarrow{-\Sigma f} \Sigma Y$. This tells us that in a stable model category C, the functor Σ sends cofiber sequences to cofiber sequences(via a tricky isomorphism).

Now we turn to the proof. Assume $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in Ho(C), then using the counit map ϵ_X, we have the following commutative diagram

$$\begin{tikzcd} \Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\epsilon_X"] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] \arrow[d, "\epsilon_Y"] & \Sigma\Omega Z \arrow[r, "\Sigma\epsilon_X^{-1} \circ \Sigma\eta_X^{-1} \circ \Sigma\Omega u"] \arrow[d, "\epsilon_Z"] & \Sigma^2\Omega X \arrow[d, "\Sigma\epsilon_X"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "u"] & \Sigma X, \end{tikzcd}$$

where the vertical arrows are all isomorphisms. Therefore we get that the first row of this sequence is also a cofiber sequence. Now we look at the sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\Omega g} \Omega Z.$$

Note that Ωf automatically fills in a cofiber sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\alpha} W \xrightarrow{\beta} \Sigma\Omega X.$$

Applying Σ to this sequence and comparing with the sequence $\Sigma\Omega X \to \Sigma\Omega Y \to \Sigma\Omega Z \to \Sigma^2\Omega X$, we get a commutative diagram

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, equal] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "\Sigma\epsilon_X^{-1}\circ\Sigma\eta_X^{-1}\circ\Sigma\Omega u"] & \Sigma^2\Omega X \arrow[d, equal] \\
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] & \Sigma\Omega Y \arrow[r, "\Sigma\alpha"] & \Sigma W \arrow[r, "\Sigma\beta"] & \Sigma^2\Omega X.
\end{tikzcd}$$

By lemma 8.4, there exists a morphism $t\colon \Sigma\Omega Z \to \Sigma W$ filling the above commutative diagram, and by 5-lemma, this is an isomorphism, but then $t = \Sigma t'$ for some isomorphism $t'\colon \Omega Z \to W$ which gives the commutative diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] \arrow[d, equal] & \Omega Y \arrow[r, "\alpha"] \arrow[d, equal] & W \arrow[r, "\beta"] & \Sigma\Omega X \arrow[d, equal] \\
\Omega X \arrow[r, "\Omega f"] & \Omega Y \arrow[r, "\Omega g"] & \Omega Z \arrow[r, "\epsilon_X^{-1}\eta_X^{-1}\Omega u"] \arrow[u, "t'"] & \Sigma\Omega X
\end{tikzcd}$$

with vertical arrows isomorphisms, hence the sequence $\Omega X \to \Omega Y \to \Omega Z \to \Sigma\Omega X$ is a cofiber sequence with boundary map $\Omega Z \to \Sigma\Omega X$. Now via the counit map, we get a commutative diagram

$$\begin{tikzcd}
\Omega Z \arrow[r, "\epsilon_X^{-1}\eta_X^{-1}\Omega u"] \arrow[d, equal] & \Sigma\Omega X \arrow[r, "-\Sigma\Omega f"] \arrow[d, "\epsilon_X"] & \Sigma\Omega Y \arrow[r, "-\Sigma\Omega g"] \arrow[d, "\epsilon_Y"] & \Sigma\Omega Z \arrow[d, equal] \\
\Omega Z \arrow[r, "\eta_X^{-1}\Omega u"] & X \arrow[r, "-f"] & Y \arrow[r, "-\epsilon_Z^{-1}g"] & \Sigma\Omega Z,
\end{tikzcd}$$

thus the sequence $\Omega Z \xrightarrow{-\eta_X^{-1}\Omega u} X \xrightarrow{f} Y$ is a cofiber sequence with boundary map $\epsilon_Z^{-1}g\colon Y \to \Sigma\Omega Z$ by the previous Corollary. This proves one side of the lemma.

On the other side, assume $\Omega Z \xrightarrow{-\eta_X^{-1}\Omega u} X \xrightarrow{f} Y$ is a cofiber sequence, then we know that

$$X \xrightarrow{f} Y \xrightarrow{\epsilon_Z^{-1}\circ g} \Sigma\Omega Z$$

is also a cofiber sequence with boundary map $\Sigma\Omega Z \xrightarrow{\Sigma(\eta_X^{-1}\Omega u)} \Sigma X$. By use of the counit map, we get a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\epsilon_Z^{-1}g"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "\Sigma(\eta_X^{-1}\Omega u)"] \arrow[d, "\epsilon_Z"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "u"] & \Sigma X
\end{tikzcd}$$

with the first and second square commutative, and the commutativity of the third square follows from the adjunction of Σ and Ω. Since vertical arrows are all isomorphisms, it follows that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{u} \Sigma X$ is a cofiber sequence with $u\colon Z \to \Sigma X$ the corresponding boundary map. □

The last step in the proof is to verify the octahedron axiom.

8.7 Lemma. Let C be a pointed model category and suppose we have cofiber sequences

$$X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$$

$$X \xrightarrow{g_1} U \xrightarrow{g_2} V$$

$$Y \xrightarrow{u_1} U \xrightarrow{u_2} W$$

in Ho(C) with $g_1 = u_1 f_1$. Then there are maps $v_1 : Z \to V$, $v_2 : V \to W$ and $v_3 : W \to \Sigma Z$ making the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[r, "f_2"] \arrow[d, "u_1"] & Z \arrow[r, "f_3"] \arrow[d, "v_1"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "g_1"] & U \arrow[r, "g_2"] \arrow[d, "u_2"] & V \arrow[r, "g_3"] \arrow[d, "v_2"] & \Sigma X \\
 & W \arrow[r, equal] \arrow[d, "u_3"] & W \arrow[d, "v_3"] & \\
 & \Sigma Y \arrow[r, "\Sigma f_2"] & \Sigma Z. &
\end{tikzcd}$$

Furthermore,

- $Z \xrightarrow{v_1} V \xrightarrow{v_2} W$ is a cofiber sequence with boundary map v_3,
- the coaction of ΣZ on W is given by

$$W \xrightarrow{\odot} W \sqcup \Sigma Y \xrightarrow{\mathrm{id} \sqcup \Sigma f_2} W \sqcup \Sigma Z,$$

  where the first map is the coaction of ΣY on W from the third cofiber sequence.

To prove this lemma, we need to categorical facts:

8.8 Lemma (The Patching Lemma). Let

$$\begin{tikzcd}
B \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & C \arrow[d] \\
Y & X \arrow[l] \arrow[r] & Z
\end{tikzcd}$$

be a commutative diagram such that $C \to Z$ and $B \sqcup_A X \to Y$ are cofibrations(respectively acyclic cofibrations), then the map $B \sqcup_A C \to Y \sqcup_X Z$ is a cofibration(respectively acyclic cofibration).

8.9 Lemma ([Str11]). Assume that we have a 3×3 diagram

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] \arrow[r] & A_3 \\
B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] \\
C_1 & C_2 \arrow[l] \arrow[r] & C_3
\end{tikzcd}$$

with pushout of rows $A \leftarrow B \to C$ and of columns $X \leftarrow Y \to Z$, then the pushout of these two diagrams are isomorphic.

An immediate consequence is that, the cofiber of a map of pushouts is the pushout of a map of cofibers.

Proof of Lemma 8.7. We can write the three cofiber sequences in the following way:

$$\begin{tikzcd}
U & X \arrow[l, "g_1"'] \arrow[r] & * \\
Y \arrow[u] \arrow[d] & X \arrow[l, "f_1"'] \arrow[r] \arrow[u, equal] \arrow[d] & * \arrow[u, equal] \arrow[d, equal] \\
* \arrow[r, equal] & * \arrow[r, equal] & *.
\end{tikzcd}$$

The pushout of each row gives the diagram $* \leftarrow Z \xrightarrow{v_1} V$, and the pushout of columns give the diagram $W \leftarrow * \rightarrow *$, therefore the lemma tells us that the push-out of $* \leftarrow Z \xrightarrow{v_1} V$ is exactly W, with the map $v_2 : V \rightarrow W$. This tells us that $Z \xrightarrow{v_1} V \xrightarrow{v_2} W$ is a cofiber sequence.

Now we verify the coaction map $\bullet : W \rightarrow W \sqcup \Sigma Z$. Pick any object $A \in \mathsf{C}$ and let $([q],[h]) \in [W,A] \times [Z,PA]$ be a pair of representatives, then we have the diagram

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, tail, "v_1"'] & PA \arrow[d, two heads, "p_0"] \\ U \arrow[r, "qu_2"'] \arrow[ur, dashed, "\varphi"] & A, \end{tikzcd}$$

which gives a lifting $\varphi : U \rightarrow PA$, and the diagram

$$\begin{tikzcd} Z \arrow[r, "v_1"] \arrow[d] & V \arrow[r, "\varphi"] \arrow[d, "v_2"] & PA \arrow[d, "p_1"] \\ * \arrow[r] & W \arrow[r, dashed, "{[q]\bullet[h]}"] & A, \end{tikzcd}$$

inducing the map $[q] \bullet [h]$. Now by composing with f_2, we get a pair of representatives $([q],[h \circ f_2]) \in [W,A] \times [Y,PA]$, which gives a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "hf_2"] \arrow[d, tail, "u_1"'] & PA \arrow[d, two heads, "p_0"] \\ U \arrow[r, "fu_2"'] \arrow[ur, "\psi g_2"] & A, \end{tikzcd}$$

and hence ψg_2 gives the required lifting, and in the diagram

$$\begin{tikzcd} Y \arrow[r, "u_1"] \arrow[d] & U \arrow[r, "\psi g_2"] \arrow[d, "u_2"] & PA \arrow[d, "p_1"] \\ * \arrow[r] & W \arrow[r, "{[q]\bullet[h]}"] & A. \end{tikzcd}$$

$[q] \bullet [h]$ gives the required coaction. This tells us that the composition $W \xrightarrow{\odot} W \sqcup \Sigma Y \xrightarrow{\mathrm{id} \sqcup \Sigma f_2} W \sqcup \Sigma Z$ gives the coaction map, and hence $W \xrightarrow{\Sigma f \circ u_3} \Sigma Z$ gives the boundary map of the cofiber sequence $Z \rightarrow V \rightarrow W$. This proves the lemma. □

And we can conclude that $\mathrm{Ho}(\mathsf{C})$ is triangulated.

(8b) Some Consequences of Stability. We have seen that in a stable model category, we can extend a given cofiber sequence $X \rightarrow Y \rightarrow Z$ from both directions: in any model category, $Y \rightarrow Z \rightarrow \Sigma X$ is always a cofiber sequence, but here $\Omega Z \rightarrow X \rightarrow Y$ is also a cofiber sequence. Note that given a map $Y \rightarrow Z$, we can get a fiber sequence $X' \rightarrow Y \rightarrow Z$ in $\mathrm{Ho}(\mathsf{C})$ with coboundary map $\Omega Z \rightarrow X'$. A natural question is, is the cofiber and fiber sequence coincide in a stable model category? The answer is affirmative.

8.10 Lemma. Let C be a pointed model category, and let

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\alpha"] & B \arrow[r, "v"] \arrow[d, "\beta"] & C \arrow[r, "\partial'"] & \Sigma A \arrow[d, "-\epsilon_Z \Sigma\alpha"] \\ \Omega Z \arrow[r, "\partial"] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram in Ho(C), where the top arrow is a cofiber sequence and its boundary map and the bottom row is a fiber sequence with its boundary map. Then there is a fill-in map $\gamma\colon C \to Y$ making the resulting second and third square commutative.

Or dually, we have

8.11 Lemma. Let C be a pointed model category, and let

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "-\Omega\gamma\circ\eta_A"] & B \arrow[r, "v"] & C \arrow[r, "\partial'"] \arrow[d, "\beta"] & \Sigma A \arrow[d, "\gamma"] \\ \Omega Z \arrow[r, "\partial"] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a commutative diagram in Ho(C) with the top row cofiber sequence with its boundary map and the bottom row a fiber sequence with its boundary map. Then there is a fill-in map $\alpha\colon B \to X$ making the resulting diagram commutative.

This would then immediately give us the equivalence between fiber and cofiber sequence as follows:

8.12 Corollary. Let C be a stable model category. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence in Ho(C) with boundary map $\partial\colon \Omega Z \to X$, then

$$\Omega Z \xrightarrow{\partial} X \xrightarrow{f} Y$$

is a cofiber sequence with boundary map $-\epsilon_Z^{-1}\circ g\colon Y \to \Sigma\Omega Z$.

or equivalently,

8.13 Corollary. Let C be a stable model category. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in Ho(C) with boundary map $\partial\colon Z \to \Sigma X$, then

$$Y \xrightarrow{g} Z \xrightarrow{u} \Sigma X$$

is a fiber sequence with boundary map $f\circ(-\eta_A^{-1})\colon \Omega\Sigma X \to Y$.

Therefore,

8.14 Proposition. Let $f\colon X \to Y$ be a map in a stable model category C. There is a weak equivalence $Ff \to \Omega Cf$ between the homotopy fiber of f and loops of the homotopy cofiber of Cf.

In a general triangulated category T, by axiom we have exact triangles $X = X \longrightarrow * \longrightarrow \Sigma X$ and $* \longrightarrow Y = Y \longrightarrow *$. Taking their product and coproduct, we get the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X\coprod Y \arrow[r] & Y \arrow[r] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\ X \arrow[r] & X\prod Y \arrow[r] & Y \arrow[r] & \Sigma X, \end{tikzcd}$$

and by axiom (T2), we get a map $X\coprod Y \xrightarrow{\chi_{X,Y}} Y\prod X$, which is an isomorphism by construction. Therefore we get

8.15 Proposition. Let C be a stable model category, then for fibrant and cofibrant objects X and Y, the canonical map $\chi_{X,Y}\colon X\coprod Y \to X\prod Y$ is a weak equivalence.

By use of the fold map $X\coprod X \to X$ and the diagonal map $X \to X\prod X$, we could then get the addition operation of $[X,Y]$ in a stable model category C in the following two ways:

$$X \xrightarrow{\Delta} X\prod X \xrightarrow{f\prod g} Y\prod Y \xrightarrow{\chi_{Y,Y}^{-1}} Y\coprod Y \xrightarrow{\text{fold}} Y$$

$$X \xrightarrow{\Delta} X\prod X \xrightarrow{\chi_{X,X}^{-1}} X\coprod X \xrightarrow{f\coprod g} Y\coprod Y \xrightarrow{\text{fold}} Y.$$

(8c) Exact Functors and Quillen Functors. Given two triangulated categories T and T′, we say a functor $F\colon \mathsf{T} \to \mathsf{T}'$ is **exact** if for any exact triangle $A \to B \to C \to \Sigma A$, the image $F(A) \to F(B) \to F(C) \to \Sigma F(A)$ is an exact triangle in T′. Now let C and D be stable model categories. Recall that

8.16 Definition. A functor $F\colon \mathsf{C} \to \mathsf{D}$ is **left Quillen** if F preserves cofibrations and acyclic cofibrations. A functor $G\colon \mathsf{D} \to \mathsf{C}$ is **right Quillen** if it preserves fibrations and acyclic fibrations. An adjunction of functors

$$F\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon G$$

is a **Quillen adjunction** if F is left Quillen and G is right Quillen.

If $F\colon \mathsf{C} \to \mathsf{D}$ is left Quillen, we define its **left derived functor** $\mathbb{L}F\colon \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{D})$ to be the functor $\mathbb{L}F(X) := F(X^{cof})$, and if $G\colon \mathsf{D} \to \mathsf{C}$ is right Quillen, we define its **right derived functor** $\mathbb{R}G\colon \mathrm{Ho}(\mathsf{D}) \to \mathrm{Ho}(\mathsf{C})$ to be the functor $\mathbb{R}G(Y) := G(Y^{fib})$. The result here is that

8.17 Theorem. Let C and D be stable model categories and

$$F\colon \mathsf{C} \leftrightarrows \mathsf{D}\colon G$$

be a Quillen adjunction. Then the derived functors $\mathbb{L}F\colon \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{D})$ and $\mathbb{R}G\colon \mathrm{Ho}(\mathsf{D}) \to \mathrm{Ho}(\mathsf{C})$ are exact functors.

In particular, the derived functors of Σ and Ω are exact.

TALK 9

Steenrod Algebra and Adams Spectral Sequence

Fan Yang

(9a) Construction of Steenrod Operator. We just consider the coefficient in $\mathbb{Z}/2\mathbb{Z}$. Consider the composition of maps $X \xrightarrow{\Delta} X \times X \to K(\mathbb{Z}/2, 2n)$ which represents the cross product $\alpha \times \alpha$ for $\alpha \in H^n(X, \mathbb{Z}/2)$. Recall that there is a bijection $[X, K(G, n)] \cong H^n(X, G)$ for G an abelian group.

The cup product and cross product are all commutative since we are working over $\mathbb{Z}/2$, so we can consider the permutation map $T : X \times X \to X \times X$ by $T(x_1, x_2) = (x_2, x_1)$ where $\alpha \times \alpha = T^*(\alpha \times \alpha)$. We can also view $\alpha \times \alpha$ as the map $X \times X \to K(\mathbb{Z}/2, 2n)$, and we will get a homotopy $f_t : \alpha \times \alpha \to (\alpha \times \alpha)T \Rightarrow f_t \circ T : (\alpha \times \alpha)T \to (\alpha \times \alpha)T^2 = \alpha \times \alpha$, and so if we compose these two homotopies, we would end up with a loop $f_t T \circ f_T : \alpha \times \alpha \to (\alpha \times \alpha)T \to (\alpha \times \alpha)T^2 = \alpha \times \alpha$ of maps $X \times X \to K(\mathbb{Z}/2, 2n)$, hence a map $\mathbb{S}^1 \times X \times X \to K(\mathbb{Z}/2, 2n)$. This means we can choose an appropriate homotopy to make the loop of maps null-homotopically extend to a map $\mathbb{D}^2 \times X \times X \to K(\mathbb{Z}/2, 2n)$. The unit disk can be viewed as the upper/lower half disk of $\mathbb{S}^2$, and similarly we would get a map $\mathbb{S}^2 \times X \times X \to K(\mathbb{Z}/2, 2n)$ by composing with T again. Repeating this process, we would get a map

$$\mathbb{S}^\infty \times X \times X \xrightarrow{\varphi} K(\mathbb{Z}/2, 2n)$$

where $\varphi(s, x_1, x_2) = \varphi(-s, x_2, x_1)$. Now going back to the original composition

$$\begin{tikzcd} X \arrow[r] & X \times X \arrow[r] \arrow[d] & K(\mathbb{Z}/2, 2n) \\ & (\mathbb{S}^\infty \times X \times X)/T \arrow[ur] & \end{tikzcd}$$

Note that $(\mathbb{S}^\infty \times X \times X)/T \cong X \times \mathbb{R}P^\infty$. Künneth theorem implies that $H^*(X \times \mathbb{R}P^\infty) \cong H^*(X) \otimes H^*(\mathbb{R}P^\infty)$ where $H^n(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1^n]$.

This gives us a cohomology class α in $H^{2n}(X \times \mathbb{R}P^\infty)$ which can be written as $\sum_i w^{n-i} \times a^i$ where $a^i \in H^{n+i}(X, \mathbb{Z}/2)$.

9.1 Definition. We define the **Steenrod operation** $\mathrm{Sq}^i(\alpha) = a^i$.

(9b) Steenrod Squares and Steenrod Algebra.

9.2 Definition. For X a topological space, **Steenrod squares** are maps of the form $\mathrm{Sq}^i : H^k(X, \mathbb{Z}/2) \to H^{k+i}(X; \mathbb{Z}/2)$ satisfying

(1) $\mathrm{Sq}^i(f^*\alpha) = f^*(\mathrm{Sq}^i(\alpha))$ for $f : X \to Y$ continuous;

(2) $\mathrm{Sq}^i(\alpha+\beta) = \mathrm{Sq}^i(\alpha) + \mathrm{Sq}^i(\beta)$;

(3) $\mathrm{Sq}^i(\alpha \cup \beta) = \sum_j \mathrm{Sq}^j(\alpha) \cup \mathrm{Sq}^{i-j}(\beta)$;

(4) $\sigma : H^n(X;\mathbb{Z}/2) \to H^{n+1}(\Sigma X;\mathbb{Z}/2)$ where the suspension isomorphism given by reduced crossed product with a generator of $H^1(\mathbb{S}^1;\mathbb{Z}/2)$, i.e. $\mathrm{Sq}^i(\sigma(\alpha)) = \sigma(\mathrm{Sq}^i(\alpha))$;

(5) $\mathrm{Sq}^i(\alpha) = \alpha \cup \alpha$ if $\alpha \in H^i(X;\mathbb{Z}/2)$ and $\mathrm{Sq}^j(\alpha) = 0$ if $j > 1$;

(6) $\mathrm{Sq}^0 = \mathrm{id}$ is the identity;

(7) Sq^1 is the Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

These axioms give us properties of the Steenrod squares.

9.3 Definition. $\mathrm{Sq}^a \mathrm{Sq}^n = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$ for $a < 2b$ is called the **Adem Relation**.

9.4 Definition. The **Steenrod Algebra** $\mathcal{A}_2$ is generated by additive homomorphisms $\mathrm{Sq}^n : H^m(X;\mathbb{Z}/2) \to H^{n+m}(X,\mathbb{Z}/2)$ where Sq^n satisfies the Steenrod squares properties and the Adem Relation.

Alternatively, we can define the Steenrod algebra $\mathcal{A}_2$ as an algebra of cohomology operations.

9.5 Definition. The mod 2 **Steenrod algebra** $\mathcal{A}_2$ is the $\mathbb{Z}/2$-graded algebra of cohomology operations $H^*(-,\mathbb{Z}/2) \to H^*(-,\mathbb{Z}/2)$.

9.6 Theorem. The two definitions above are equivalent.

(9c) Structure of Steenrod Algebra. To discuss the structure of Steenrod algebra, we need

9.7 Definition. A monomial $\mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ in $\mathcal{A}_2$ is **admissible** if $i_k \geq 1$ and $i_{r-1} \geq 2i_r$ for $k \geq r \geq 2$.

9.8 Theorem. As a module over $\mathbb{Z}/2$, the admissible monomials form a basis of the Steenrod algebra $\mathcal{A}_2$.

Sketch of Proof. **Step 1:** to show arbitrary monomial $\mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ can be uniquely written as a sum of admissible monomials.

Let $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ be a monomial that is not admissible, then there is at least one pair, say $\mathrm{Sq}^{i_r} \mathrm{Sq}^{i_{r+1}}$, that Adem Relation can be applied. Define the moment of Sq^I to be $m(I) = \sum_{s=1}^{k} s i_s$ and assume by induction that Sq^I for $m(I) < m$ can be written as a combination of monomials, and we look at the case $m(I) = m$, then

$$\begin{aligned}\mathrm{Sq}^I &= \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r} \mathrm{Sq}^{i_{r+1}} \cdots \mathrm{Sq}^{i_k} \\ &= \sum_{j=0}^{\lfloor \frac{i_r}{2} \rfloor} a_j \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_r+i_{r+1}-j} \mathrm{Sq}^j \mathrm{Sq}^{i_r+2} \cdots \mathrm{Sq}^{i_k},\end{aligned}$$

but if we look at $m(I)$, we have

$$m = \sum_{s=1}^{r-1} s i_s + r(i_r + i_{r+1} - j) + (r+1)j + \sum_{s=r+2}^{k} s i_s$$

with $r i_r + (r+i) i_{r+1} > r(i_r + i_{r+1} - j) + (n+1)j$ when $j \leq \frac{i_r}{2} < i_{r+1}$. This proves our first claim.

Step 2: we want to show that the admissible monomials are linearly independent. Evaluating admissible monomials on $u^{\otimes n} \in H^n((\mathbb{R}P^\infty_+)^{\wedge n}, \mathbb{Z}/2)$, consider $\sum_I a_I \operatorname{Sq}^I(u^{\otimes n}) = 0$, where Sq^I is admissible and has degree $\leq n$. (Here $\operatorname{Sq}^I = \operatorname{Sq}^{i_1}\operatorname{Sq}^{i_2}\cdots\operatorname{Sq}^{i_k}$ with $I = (i_1, i_2, \cdots, i_k)$ where the length of I is k and the degree is $i_1 + \cdots + i_k$. If we can show $a_I = 0$ for all I, then we are done. Assume if $\sum a_I \operatorname{Sq}^I(u^{\otimes n-1}) = 0$ for all monomials of degree $\leq n-1$, then $a_I = 0$. Also assume that $a_I = 0$ if the length of $I \geq 0$. By Künneth theorem, we can do factorizations like $\operatorname{Sq}^I(u^{\otimes n}) \in \bigoplus_r H^r(\mathbb{R}P^\infty_+, \mathbb{Z}/2) \otimes H^{d+n-r}((\mathbb{R}P^\infty_+)^{\wedge n-1}, \mathbb{Z}/2)$. By Cartan formula, $\operatorname{Sq}^I(u^{\otimes n}) = \sum_{J\subseteq I} \operatorname{Sq}^J(u) \otimes \operatorname{Sq}^{I-J}(u^{\otimes n-1})$. Consider the case $r = 2^m$, then we have

$$\operatorname{Sq}^i(u^{2^k}) = \begin{cases} u^{2^k}, & i = 0; \\ u^{2^{k+1}}, & i = 2^k; \\ 0, & \text{otherwise.} \end{cases}$$

where $\mathbb{Z}/2[w] \cong H^i(\mathbb{R}P^\infty_+; \mathbb{Z}/2)$ and $|u| = 1$. This implies that

$$\operatorname{Sq}^J(u) = \begin{cases} u^{2^k} & \text{if } J = J_k = (2^{k-1}, 2^{k-2}, \cdots, 2, 1) \\ 0, & \text{otherwise.} \end{cases}$$

so Sq^{J_m} is the only non-trivial action. Consider the projection

$$\operatorname{pr}(\operatorname{Sq}^I(u^{\otimes n})) = \begin{cases} u^{2^m} \otimes \operatorname{Sq}^{I-J_m}(u^{\otimes n-1}), & \text{if the length of } I \text{ is } m; \\ 0, & \text{otherwise.} \end{cases}$$

then

$$(\sim_I a_I \operatorname{Sq}^I(u^{\otimes n})) = \sum_{\text{length}=m} a_I \operatorname{pr}(\operatorname{Sq}^I(u^{\otimes n})) + \sum_{\text{length}\leq n} a_I \operatorname{pr}(\operatorname{Sq}^I(u^{\otimes n})) = u^{2^m} \otimes \sum_{\text{length}=m} a_I \operatorname{Sq}^{I-J_m}(u^{\otimes n-1}).$$

So admissible monomials of the form $I - J_m$ is the same as admissible monomials of length m or less and degree $d - 2^m + 1$. By assumption, we have $a_I = 0$ for all I, and this proves the linear independence. □

(9d) The Adams Spectral Sequence. The motivation of this spectral sequence is that, we have two spectra X, Y, and if we apply the mod-p singular cohomology H^* to them, we get

$$[X, Y] \xrightarrow{H^*} \operatorname{Hom}_{\mathcal{A}_p}(H^*(Y), H^*(X)),$$

where $\operatorname{Hom}_{\mathcal{A}_p}(-,-)$ is the morphism in the category of modules over $\mathcal{A}_p$. We want to see the inverse to this morphism.

9.9 Theorem. For X and Y spaces of finite type, with Y finite-dimensional CW complex, there is a spectral sequence, converging to ${}_{(p)}\{Y, X\}_*$ with the second page

$$E_2^{s,t} \cong \operatorname{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X; \mathbb{F}_p), H^*(Y, \mathbb{F}_p)),$$

and differentials d_r of bidegree $(r, r-1)$.

Recall that $\{Y, X\}_k = \varinjlim_k [\Sigma^{n+k} Y, \Sigma^n X]$ and ${}_{(p)}G := G/\{\text{element of finite order prime to } p\}$.

9.10 Definition. A **differential bigraded module** over a ring R is a collection of R-modules $\{E^{p,q}\}$ where p, q are integers together with an R-linear map $d : E^{*,*} \to E^{*,*}$ which is called the differential of bidegree $(s, 1-s)$ or $(-s, s-1)$ for some integer s s.t. $d \circ d = 0$.

9.11 Remark. *We can define homology and cohomology of differential bigraded module* $H^{p,q}(E^{*,*},d) = \ker(d^s : E^{p,q} \to E^{p+s,q-s+1})/\mathrm{im}(d^s : E^{p-s,q+s-1} \to E^{p,q})$.

9.12 Definition. A **spectral sequence** is a collection of differential bigraded R-module $\{E_r^{*,*}, d_r\}$ where $r = 1, 2, \cdots$ and the differentials are either of bidegree $(-r, r-1)$ or $(r, 1-r)$. For all of r, q and p, $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$.

Now we still need to specify the notion Ext. We write the category of left Γ-modules as ${}_\Gamma\mathrm{Mod}$, and for M, N, ${}_\Gamma\mathrm{Mod}(M, N)$ is the Γ-linear map between the left Γ-module. Consider the suspension functor $M \in_\Gamma \mathrm{Mod}$, sM the graded vector space $(sM)_n = M_{n-1}$, then the action of Γ is given for $r \in \Gamma$, $r.(sx) = (-1)^{\deg r} s(r \cdot x)$. When $x \in M_n$, sx is the corresponding element in $(sM)_{n+1}$. Write $s^1 = s$ and $s \cdot s^{n-1} = s^n$. For M, N, we define $\mathrm{Hom}_\Gamma^n(M, N) =_\Gamma \mathrm{Mod}(M, s^n N)$. Suppose we have a long exact sequence

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

of modules with P_i projective, fix $N \in_\Gamma \mathrm{Mod}$, then $\mathrm{Hom}_\Gamma^*(-, N)$ gives us the complex

$$0 \to \mathrm{Hom}_\Gamma(P_0, N) \to \mathrm{Hom}_\Gamma(P_1, N) \to \cdots$$

and we define $\mathrm{Ext}^{*,*}(M, N)$ to be the homology of this complex.

9.13 Theorem. If L, M, N are Γ-modules, then we have the product map $\mathrm{Ext}_\Gamma^{p,t}(L, M) \otimes \mathrm{Ext}_\Gamma^{p',t'}(M, N) \to \mathrm{Ext}_\Gamma^{p+p',t+t'}(L, N)$, which is called the composition product.

9.14 Theorem. There are operations Sq^i on $\mathrm{Ext}_\Gamma^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ satisfying the Steenrod squares property.

TALK 10

Modern Categories of Spectra

Qiyu Zhang

TALK 11

Other Categories of Spectra

Jonnathan Michala

(11a) Spectra of Simplicial Sets

11.1 Definition. A **sequential spectrum** of simplicial sets X is a sequence of pointed simplicial sets X_n, $n \in \mathbb{N}$ and structure maps $\sigma_n^X : \Sigma X_n \to X_{n+1}$. A **morphism** $f : X \to Y$ of spectra is a sequence of pointed maps $f_n : X_n \to Y_n$ such that the following diagram commutes:

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n^X"] & \Sigma Y_n \arrow[d, "\sigma_n Y"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1}. \end{tikzcd}$$

We denote this category by $\mathrm{S}^{\mathbb{N}}(\mathrm{sSet}_*)$. More details can be found in (Hovey).

11.2 Theorem. $\mathrm{S}^{\mathbb{N}}(\mathrm{sSet}_*)$ has a stable model structure.

11.3 Theorem. The geometric realization $|\bullet|$ and the singular complex functor Sing are Quillen equivalent:

$$|\bullet| : \mathrm{S}^{\mathbb{N}}(\mathrm{sSet}_*) \leftrightarrows \mathrm{S}^{\mathbb{N}}(\mathrm{Top}_*) : \mathrm{Sing}\,.$$

11.4 Definition. We then obtain the symmetric spectrum X in simplicial sets, which is a sequence of pointed simplicial sets $\{X_n\}$ such that

1. X_n admits an action of $\mathfrak{S}_n$ which fixes the base-point;
2. there are maps $\sigma_n^X : \mathbb{S}^1 \wedge X_n \to X_{n+1}$;
3. $\mathbb{S}^k \wedge X_n \xrightarrow{\mathrm{id} \wedge \sigma_n} \mathbb{S}^{k-1} \wedge X_{n+1} \xrightarrow{\mathrm{id} \wedge \sigma_{n+1}} \cdots \xrightarrow{\sigma_{n+k-1}} X_{n+k}$ is compatible with the $\mathfrak{S}_k \times \mathfrak{S}_n$-actions on the domain $\mathbb{S}^k \wedge X_n$ and $\mathfrak{S}_{n+k}$-action on the target X_{n+k}.

A morphism $f : X \to Y$ is just a sequence of $\mathfrak{S}_n$-equivariant maps $f_n : X_n \to Y_n$ so that the diagram

$$\begin{tikzcd} \mathbb{S}^1 \wedge X_n \arrow[r, "\mathrm{id} \wedge f_n"] \arrow[d, "\sigma_n^X"] & \mathbb{S}^1 \wedge Y_n \arrow[d, "\sigma_n^Y"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

and we get the category of symmetric spectra $\mathrm{S}^{\Sigma}(\mathrm{sSet}_*)$.

11.5 Theorem. $\mathrm{S}^{\Sigma}(\mathrm{sSet}_*)$ has a stable model structure.

We have a forgetful functor $\mathbb{U}_{\mathbb{N}}^{\Sigma}$ from symmetric spectra to sequential spectra, which is right adjoint to $\mathbb{P}_{\mathbb{N}}^{\Sigma}$.

11.6 Theorem. The functors $\mathbb{U}_{\mathbb{N}}^{\Sigma}$ and $\mathbb{P}_{\mathbb{N}}^{\Sigma}$ are Quillen equivalent between simplicial sets $\mathsf{S}^{\mathbb{N}}(\mathsf{sSet}_*)$ and $\mathsf{S}^{\Sigma}(\mathsf{sSet}_*)$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{S}^{\mathbb{N}}(\mathsf{sSet}_*) \arrow[r] \arrow[d] & \mathsf{S}^{\mathbb{N}}(\mathsf{Top}_*) \arrow[d] \\
\mathsf{S}^{\Sigma}(\mathsf{sSet}_*) \arrow[r] & \mathsf{S}^{\Sigma}(\mathsf{Top}_*).
\end{tikzcd}$$

(11b) Diagram Spectra.

11.7 Definition. Let $\mathsf{W} \subseteq \mathsf{Top}_*$ be a full subcategory of all pointed topological spaces isomorphic to finite CW complexes, then we define the **W-spectrum** to be a functor $\mathsf{W} \to \mathsf{Top}_*$ enriched over Top_*, and morphisms are just natural transformations. We write the corresponding category by S^{W}.

11.8 Theorem. S^{W} has a stable model structure.

The functor $\mathbb{U}_{\mathbb{N}}^{\mathsf{W}} : \mathsf{S}^{\mathsf{W}} \to \mathsf{S}^{\mathbb{N}}$ has a left adjoint, and this adjunction is a Quillen equivalence with respect to the corresponding model structure, where $(\mathbb{U}_{\mathbb{N}}^{\mathsf{W}} F)_n = F(\mathbb{S}^n)$.

If F is enriched, then we get $F_{A,B} : \mathsf{W}(A,B) \to \mathsf{Top}_*(F(A), F(B))$ so that $\tilde{F}_{A,B} : F(A) \wedge \mathsf{W}(A,B) \to F(B)$, and we also get $\alpha_{A,B} : B \cong \mathsf{W}(\mathbb{S}^0, B) \xrightarrow{A\wedge -} \mathsf{W}(A, A\wedge B)$, and hence we get $F(A) \wedge B \xrightarrow{\mathrm{id}\wedge \alpha_{A,B}} F(A) \wedge \mathsf{W}(A, A\wedge B) \xrightarrow{\tilde{F}_{A,A\wedge B}} F(A \wedge B)$. This means we get $\Sigma F(\mathbb{S}^n) \cong F(\mathbb{S}^n) \wedge \mathbb{S}^1 \to F(\mathbb{S}^{n+1})$. This gives the structure map.

(11c) More Spectra. Given any model category C with a left Quillen functor $T : \mathsf{C} \to \mathsf{C}$ with right adjoint U, then we can form the category $\mathsf{S}^{\mathbb{N}}(\mathsf{C}, T)$ with object X where $X_n \in \mathsf{C}$, $n \geq 0$ and structure maps $\sigma_n : TX_n \to X_{n+1}$. $f : X \to Y$ are just sequences of maps $f_n : X_n \to Y_n$. Fibrant objects are just U-spectra, i.e. $\tilde{\sigma}_n^X : X_n \to UX_{n+1}$ are weak equivalences.

11.9 Theorem. As long as C is a left proper and cellular model category, with a Quillen adjunction (T, U) between C and itself, the functor T extends to a Quillen equivalence $T : \mathsf{S}^{\mathbb{N}}(\mathsf{C}, T) \to \mathsf{S}^{\mathbb{N}}(\mathsf{C}, T)$ using the stable model structure.

If C is a symmetric monoidal model category and $T = K \otimes -$ for some cofibrant objct K, then we have a similar result for the symmetric spectra.

Let G be a compact topological group, consider the category of all G-spaces. Weak equivalences of G-spaces are $f : X \to Y$ such that for each closed subgroup H of G, $f^H : X^H \to Y^H$(induced map on fixed points) is a weak homotopy equivalence. The G-spectrum is indexed over real representations V, which is called the **G-universe**. A **representation sphere** $\mathbb{S}^V$ is just a one-point compatification of V, with g acts on the infinity as fixed points.

11.10 Definition. A **G-spectrum** consists of the following data:

- A G-space $X(V)$ for each representation V in the G-universe;
- Structure maps $\mathbb{S}^{W-V} \wedge X(V) \to X(W)$ where the wedge sum is equipped with the diagonal action of G.

Morphsims between G-spectra are just G-maps between G-spaces, commuting with the structure maps. A morphism $f : X \to Y$ is a weak equivalence if $\pi_*^H(f) : \pi_*^H(X) \xrightarrow{\cong} \pi_*^H(Y)$ for all subgroup H

of G, where π_*^H is given by

$$\pi_q^H = \begin{cases} \mathrm{colim}_V[G/H_+ \wedge \mathbb{S}^{V\oplus\mathbb{R}^q}, X(V)]^G, & q \geq 0; \\ \mathrm{colim}_{V\supseteq\mathbb{R}^{-q}}[G/H_+ \wedge \mathbb{S}^{V-\mathbb{R}^{-q}}, X(V)]^G, & q < 0/ \end{cases}$$

Where $[-,-]^G$ is the homotopy in the category of G-spaces.

This category has a model structure.

(11d) Compact Objects. We start with some definitions.

11.11 Definition. Let T be a triangulated category with all small coproducts. $\mathcal{G} = \{X_i | i \in I\}$ objects in T. We say $\mathcal{G}$ is a **set of generators** for T if the only full triangulated subcategory of T containing $\mathcal{G}$ is T itself.

11.12 Definition. Let T be the triangulated category as before. An object $A \in$ T is **compact** if the functor $\mathrm{T}(A,-)_*$ commutes with arbitrary coproducts. I.e. $\mathrm{T}(A, \sqcup X_i) \cong \sqcup \mathrm{T}(A, X_i)$.

This relates to compactness in topological spaces, because for A compact, the functor $[A,-]$ commutes with arbitrary coproducts in the category of topological spaces.

11.13 Lemma. The class of compact objects is closed under finite coproducts, suspension and desuspension.

11.14 Lemma. Let $\{X_i\}$ be a set of compact objects in T, then TFAE:

- $\mathcal{G} = \{X_i\}$ generates T;
- A morphism $f: A \to B$ is an isomorphism if and only if $\mathrm{T}(X_i, f): \mathrm{T}(X_i, A)_* \to \mathrm{T}(X_i, B)_*$ is an isomorphism for all X_i;
- $Z \cong 0$ for $Z \in$ T is equivalent to $\mathrm{T}(X_i, Z) = 0$ for all $X_i \in \mathcal{G}$.

11.15 Proposition. Let X be a sequential spectrum, then for $k \in \mathbb{N}$, there exists natural isomorphisms of abelian groups

$$\pi_k(X) \cong [\Sigma^\infty \mathbb{S}^k, X] \quad \text{and} \quad \pi_{-k}(X) \cong [F_k^{\mathbb{N}} \mathbb{S}^0, X].$$

11.16 Corollary. $[\mathbb{S}, X]_* = 0$ if and only if $X \simeq *$.

So the stable homotopy category SHC is generated by the single sphere spectrum $\mathbb{S}$.

Finally, we state a result claiming the rigidity of spectra:

11.17 Theorem (Schwede). Let C be a stable model category. If there's an equivalence of triangulated categories $\psi:$ SHC $\to$ Ho(C), then C is Quillen equivalent to $\mathrm{S}^{\mathbb{N}}$.

Because of this theorem, we say SHc is rigid. All the model categories constructed are automatically isomorphic using this equivalence.

The proof is reduced to

11.18 Theorem. $F:$ SHC $\to$ SHC be an exact functor sending $\mathbb{S}$ to itself, then F is an equivalence of categories.

The proof of this theorem just use the fact that $F: [\mathbb{S}, \mathbb{S}]_n \to [\mathbb{S}, \mathbb{S}]_n$ is an isomorphism for all n and the lemma where for an exact functor $F:$ T $\to$ T commuting with arbitrary colimits and $\{F(X_i)\}$ generates T for $\{X_i\}$ compact generators of T, and $F: \mathrm{T}(X_i, X_j) \cong \mathrm{T}(F(X_i), F(X_j))$ for all i, j, then F is an equivalence of categories.

TALK 12

Monoidal Structures

Haosen Wu

(12a) Monoidal Model Category.

12.1 Definition. A **symmetric monoidal category** is a category C with a functor

$$- \otimes - : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$$

called **monoidal product**, a **monoidal unit** 1, and isomorphisms

- (Associativity) $(X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$
- (Unit) $1 \otimes X \xrightarrow{\cong} X$
- (Symmetry) $X \otimes Y \xrightarrow{\cong} Y \otimes X$

satisfying the coherence diagrams given below:

(a) (fourfold associativity is coherent)

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "a"] \arrow[d, "a\otimes \mathrm{id}"] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "a"] & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "\mathrm{id}\otimes a"']
\end{tikzcd}$$

(b) (symmetry is self-inverse)

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\tau"] \arrow[rr, bend right, "\mathrm{id}"'] & Y \otimes X \arrow[r, "\tau"] & X \otimes Y
\end{tikzcd}$$

(c) (symmetry and associativity are compatible)

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a"] \arrow[d, "\tau"] & X \otimes (Y \otimes Z) \arrow[d, "\tau"] \\
Z \otimes (X \otimes Y) \arrow[d, "\mathrm{id}\otimes\tau"] & (Y \otimes Z) \otimes X \arrow[d, "\tau\otimes\mathrm{id}"] \\
Z \otimes (Y \otimes X) & (Z \otimes Y) \otimes X \arrow[l, "a"]
\end{tikzcd}$$

(d) (compatibility of unit, symmetry, and associativity)

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[r, "a"] \arrow[d, "\tau\otimes\mathrm{id}"] & X \otimes (1 \otimes Y) \arrow[d, "\mathrm{id}\otimes u"] \\
(1 \otimes X) \otimes Y \arrow[r, "u\otimes\mathrm{id}"] & X \otimes Y
\end{tikzcd}$$

12.2 Example. *In the category of sets* Sets*, the Cartesion product* $-\times-$ *gives the tensor product. For the category of simplicial sets* sSets*, we define* $(A\times B)_n = A_n \times B_n$ *where the face and degeneracy maps are just product of the face and degeneracy maps of* A *and* B*.*

12.3 Example. *In the category of topological spaces* Top*, the Cartesian product* $-\times-$ *with Kelly product topology gives us the monoidal structure on* Top*, with unit a one-point space. The monoidal structure on* Top_* *is given by the smash product* $-\wedge-$*, with unit* $(\mathbb{S}^0, 1)$*.*

We want closed symmetric monoidal categories, where closed means the monoidal product is associated with a function object.

12.4 Definition. Let $(\mathsf{C}, \otimes, 1)$ be a symmetric monoidal category. Given maps $f: A \to B$ and $g: X \to Y$ in C, their **pushout product** is the natural map $f \square g: B\otimes X \coprod_{A\otimes X} A\otimes Y \to B\otimes Y$.

(a) Now let $g: X\to Y$ be a map in C and A is an object of C, then $\mathrm{id}_A \square g = \mathrm{id}_{A\otimes Y}: A\otimes Y \to A\otimes Y$

(b) For $\varnothing \in \mathsf{C}$ where C is a closed monoidal category, $\varnothing \otimes X = \varnothing$ for all $X\in\mathsf{C}$. This might not be true if C is not closed. Let $i_A: \varnothing \to A$ be the unique morphism, then we have $i_A \square g: A\otimes X \to A\otimes Y$

12.5 Definition. For $f: A\to B$ and $g: X\to Y$, we have the **termwise product** $f\otimes g: A\otimes X \to B\otimes Y$.

12.6 Definition. A monoidal category C is **closed** if there is a functor $\mathrm{Hom}: \mathsf{C}^{op}\times\mathsf{C}\to\mathsf{C}$ and natural isomorphisms $\phi: \mathsf{C}(A\otimes B, C) \to \mathsf{C}(A, \mathrm{Hom}(B,C))$. We call this functor **internal Hom**.

We call the quadruple $(\mathsf{C}, \otimes, 1, \mathrm{Hom})$ the closed symmetric monoidal category. Now let $A, B, C \in \mathsf{C}$, then we have

$$\mathsf{C}^{op}(\mathrm{Hom}(A,B), C) \cong \mathsf{C}(C, \mathrm{Hom}(A,B)) \cong \mathsf{C}(C\otimes A, B) \cong \mathsf{C}(A, \mathrm{Hom}(C,B)).$$

these isomorphisms imply that there is an adjunction

$$\mathrm{Hom}(-,B): \mathsf{C} \leftrightarrows \mathsf{C}^{op}: \mathrm{Hom}(-,B).$$

12.7 Example. *The categories* Sets, sSets *and* Top *are closed monoidal categories.*

(12b) Monoidal Structures on Model Categories. We want the tensor product structure so that it can pass to the homotopy category of the given stable model category. We require the adjunction

$$A\otimes -: \mathsf{C} \leftrightarrows \mathsf{C}: \mathrm{Hom}(A,-)$$

which is Quillen whenever A is cofibrant, and conversely, the pair of adjunction functors $\mathrm{Hom}(-,B), \mathrm{Hom}(-,B)$, should also be a Quillen adjunction.

12.8 Definition. The **pushout product axiom** is the following:

1) For some cofibrant replacement of 1 and any $A\in\mathsf{C}$, we have

$$1^{cof}\otimes A \xrightarrow{w.e.} g\otimes A \cong A.$$

2) For any cofibrations f, g, $f\square g$ is a cofibration.

3) Given $f, g \in \mathsf{C}$, if f/g is weak equivalence, then $f\square g$ is an acyclic cofibration.

12.9 Lemma. Let $(\mathsf{C}, \otimes, 1, \mathrm{Hom})$ be a symmetric monoidal category with a model structure satisfying the pushout product axiom, then TFAE:

1) Let $f: A\to B$ and $g: X\to Y$. If f or g is an acyclic cofibration, then $f\square g$ is an acyclic cofibration.

2) Let $f: A \to B$ and $h: P \to Q$ and consider the map

$$\mathrm{Hom}_{\square}(f,h): \ \mathrm{Hom}(B,P) \to \mathrm{Hom}(B,Q) \times_{\mathrm{Hom}(A,Q)} \mathrm{Hom}(A,P),$$

then if f or g is acyclic cofibration, then $\mathrm{Hom}_{\square}(f,h)$ is acyclic cofibration.

12.10 Definition. $(\mathsf{C}, \otimes, 1, \mathrm{Hom})$ with a given model structure is a **closed symmetric monoidal model category** if it further satisfies the pushout product axiom.

12.11 Lemma. Given a closed symmetric monoidal model category C, A cofibration and B fibration, then the $\otimes$-Hom adjunction is Quillen.

12.12 Theorem. Let $(\mathsf{C}, \otimes, 1, \mathrm{Hom})$ be the closed symmetric monoidal model category, then the homotopy category $(\mathrm{Ho}(\mathsf{C}), \otimes^{\mathbb{L}}, 1, \mathbb{R}\mathrm{Hom})$ is a closed symmetric monoidal category.

Sketch of Proof. Given A and Y, we have functors $A \otimes -$ and $- \otimes B$, which are right exact, and hence we can pass to the homotopy category to get the corresponding derived functors. $\square$

12.13 Example. sSet, Top *and* $\mathrm{Ch}(R)$ *with projective model structure are closed symmetric monoidal model categories.*

12.14 Theorem. Let C be the closed symmetric monoidal model category, then $\mathrm{Ho}(\mathsf{C})$ is triangulated satisfying the following properties:

- There is a map $e_{X,Y}: (\Sigma X) \otimes^{\mathbb{L}} Y \xrightarrow{\sim} \Sigma(X \otimes^{\mathbb{L}} Y)$;
- $- \otimes^{\mathbb{L}} A$ is exact;
- $\mathbb{R}\mathrm{Hom}(A, -)$ is exact;
- $\mathbb{R}\mathrm{Hom}(-, A)$ sends exact triangle $X \to Y \to Z \to X[1]$ to the exact triangle $\mathrm{Hom}(Z,A) \to \mathrm{Hom}(Y,A) \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(Z,A)[-1]$.
- Let $a, b \in \mathbb{Z}$ and (-1) be the additive inverse of $\mathrm{id}_{[1,1]}$, then we have the commutative diagram

$$\begin{tikzcd}
\Sigma^0 1 \otimes^{\mathbb{L}} \Sigma^b 1 \arrow[r, "\simeq"] \arrow[d] & \Sigma^{a+b} 1 \arrow[d, "(-1)^{ab}"] \\
\Sigma^b 1 \otimes^{\mathbb{L}} \Sigma^a 1 \arrow[r, "\simeq"] & \Sigma^{b+a} 1;
\end{tikzcd}$$

- The fourfold associativity diagram descends to the homotopy category $\mathrm{Ho}(\mathsf{C})$.
- The diagram

$$\begin{tikzcd}
\Sigma X \otimes^{\mathbb{L}} 1 \arrow[r, "u"] \arrow[d, "e_{X,1}"] & \Sigma X \\
\Sigma(X \otimes^{\mathbb{L}} 1) \arrow[ur, "\Sigma u"] &
\end{tikzcd}$$

is commutative.

(12c) The Smash Product. Note that we have the functor $\Sigma^\infty :$ Ho(Top$_*$) $\to$ SHC. If we endow SHC with a monoidal structure, then we should expect Σ^∞ to be symmetric monoidal, i.e. $\Sigma^\infty(A \wedge B) \cong \Sigma^\infty A \wedge^{\mathbb{L}} \Sigma^\infty B$, and $\Sigma^\infty \mathbb{S}^0 = \mathbb{S}$. Naively, we can define

$$(X \wedge_{naive} Y)_k = \begin{matrix} X_n \wedge Y_n, & k = 2n; \\ X_{n+1} \wedge Y_n, & k = 2n+1 \end{matrix}$$

with structure maps $\Sigma(X_n \wedge Y_n) \cong (\Sigma X_n) \wedge Y_n \xrightarrow{\sigma_n \wedge Y_n} X_{n+1} \wedge Y_n$ and $\Sigma(X_{n+1} \wedge Y_n) = \mathbb{S}^1 \wedge X_{n+1} \wedge Y_n \xrightarrow{\tau_{\mathbb{S}^1, X_{n+1}} \wedge \mathrm{id}} X_{n+1} \wedge \mathbb{S}^1 \wedge Y_n \xrightarrow{\mathrm{id} \wedge \sigma_n^Y} X_{n+1} \wedge Y_{n+1}$. This actually gives us the "tensor product" on SHC, but it's not associative, so we must take all the choices, which means that we need to think about ends and coends.

We define

$$(X \otimes Y)_n = \bigvee_{a+b=n} X_a \wedge Y_b,$$

with unit the spectrum with $\mathbb{S}^0$ in degree 1 and 0 elsewhere. $\mathbb{S}$ acts on the sequential spectrum X by setting

$$\mu_{X,n}(\mathbb{S} \otimes X)_n = \bigvee_{a+b=n} \mathbb{S}^a \wedge X_b \to X_n,$$

and let $X \otimes_{\mathbb{S}} Y := \mathrm{coeq}(X \otimes \mathbb{S} \otimes Y \rightrightarrows X \otimes Y)$. This would be the actual product we're gonna use, but this is not symmetric. But this lack of symmetry would be resolved by the internal symmetry of orthogonal and symmetric spectra. The solution in the sequential spectra would be the enrichment of ends and coends. The formula is

$$(F \otimes G)_a = \int^{b,c \in \mathsf{N}} \mathsf{N}(b+c, a) \wedge F_b \wedge G_c \cong \bigvee_{b+c=a} F_b \wedge G_c.$$

TALK 13

Monoidal Structures, Continued

Haosen Wu

Today we'll continue the discussion of monoidal structures. Recall that last time we define the monoidal product to be $(X \otimes Y)_n = \bigvee_{a+b=n} X_a \wedge Y_b$, and today we'll construct this monoidal product explicitly.

(13a) Closed Monoidal Structure on Spectra. Let's define three enriched categories first.

13.1 Definition. The objects of these three Top_*-enriched categories are all natural numbers $\mathbb{N} = \{0, 1, 2, \cdots\}$ and the morphism spaces are given separately as:

(a) N has morphism space given by $\mathsf{N}(a,b) = \begin{cases} \mathbb{S}^0, & \text{if } a = b; \\ *, & \text{if } a \neq b. \end{cases}$

(b) $\Sigma(a,b) = \begin{cases} (\Sigma_a)_+, & \text{if } a = b; \\ *, & \text{if } a \neq b. \end{cases}$

(c) $O(a,b) = \begin{cases} O(a)_+, & \text{if } a = b; \\ *, & \text{if } a \neq b. \end{cases}$

There are functors between these categories $\mathsf{N} \to \Sigma \to O$ sending n to n, and the induced maps on morphisms are identity if $a \neq b$ and inclusion maps $* \mapsto \Sigma_a \mapsto O(a)$ if $a = b$.

Now we need more definitions to define embedded functors or natural transformations. Let E be a Top_*-enriched category, then a functor $F : \mathsf{E} \to \mathrm{Top}_*$ is a collection of maps $F(a,b) : \mathsf{E}(a,b) \to \mathrm{Top}_*(F(a), F(b))$ which are required to be associative and satisfy the coherence condition we defined last time. We can write $F(a,b) : \mathsf{E}(a,b) \wedge F_a \to F_b$ as the adjoint of the original map. Now we can think that $\mathsf{E}(a,b)$ acts on F.

13.2 Definition. A **sequential space** is a functor $F : \mathsf{N} \to \mathrm{Top}_*$, and similarly for **symmetric spaces** and **orthogonal spaces**.

Applying our previous observation, we get that there are actions of Σ_a acting on a symmetric space X_a and $O(a)$ on an orthogonal space Y_a.

13.3 Theorem. N, Σ, O have symmetric monoidal products denoted by $+$.

Objectwise, we send (a,b) to $a+b$, and on morphisms, $\mathsf{N}(a,b) \wedge \mathsf{N}(c,d) \to \mathsf{N}(a+c, b+d)$, which are either $* \to *$ or $\mathbb{S}^0 \wedge \mathbb{S}^0 \to \mathbb{S}^0$; for $\Sigma(a,b) \wedge \Sigma(c,d) \to \Sigma(a+b, c+d)$, this is nontrivial only if $a = b$, $c = d$ and the inducing map is the natural map $\Sigma_a \times \Sigma_C \to \Sigma_{a+c}$; for $O(a,b) \wedge O(c,d) \to O(a+c, b+d)$, this map is non-trivial only when $a = c$, $b = d$ and the map is the inclusion $O(a) \times O(c) \to O(a+c)$, sending the pair of matrices to the diagonal matrix.

Now we want to construct our tensor $\otimes$. To do this we need the concept of end and coend. We start with end.

13.4 Definition. The end of a (bi)functor $\mathsf{C}^{op} \times \mathsf{C} \to \mathsf{D}$, if C is small and D complete, is the equalizer

$$\int_c F(c,c) \xrightarrow{\exists!} \prod_{c\in\mathsf{C}} F(c,c) \rightrightarrows \prod_{c\to c'} F(c,c'),$$

and the coend of a bifunctor F is the coequalizer

$$\int^c F(c,c) \xleftarrow{\exists!} \coprod_{c\in\mathsf{C}} F(c,c) \leftleftarrows \coprod_{c\to c'} F(c,c').$$

13.5 Example. *Let A be a natural transform and $f,g\colon \mathsf{C}\to\mathsf{X}$ any two (bi)functors, then* $\mathrm{Hom}_X(F(-),G(-))\colon \mathsf{C}^{op}\times\mathsf{C}\to\mathsf{Set}$, *and we have the commutative diagram*

$$\begin{tikzcd}
\mathrm{Nat}(F,G) \arrow[r] \arrow[d] & \mathrm{Hom}(F(c),G(c)) \arrow[d] \\
\mathrm{Hom}(F(c'),G(c')) \arrow[r, "f"] & \mathrm{Hom}(F(c),G(c'))
\end{tikzcd}$$

so $\mathrm{Nat}(F,G)$ *is the end of the functor* $\mathrm{Hom}_X(F(-),G(-))$.

13.6 Definition (Extranatural Transformation). Let $F\colon \mathsf{A}\times\mathsf{B}^{op}\times\mathsf{B}\to\mathsf{D}$ and $G\colon \mathsf{A}\times\mathsf{C}^{op}\times\mathsf{C}\to\mathsf{D}$ be two functors. A family $\eta(a,b,c)\colon F(a,b,b)\to G(a,c,c)$ natural in a is said to be **extranatural** in b,c if the following holds:

1) $\eta(-,b,c)$ is natural;

2) for any $g\colon b\to b'$ and $\forall a\in\mathsf{A}$, $c\in\mathsf{C}$, the following diagram

$$\begin{tikzcd}
F(a,b',b) \arrow[r, "F(1,1,g)"] \arrow[d, "F(1,g,1)"] & F(a,b',b') \arrow[d, "\eta(a,b',c)"] \\
F(a,b,b) \arrow[r, "\eta(a,b,c)"] & G(a,c,c)
\end{tikzcd}$$

commutes.

13.7 Definition. The end of $F\colon \mathsf{C}\times\mathsf{C}^{op}\to\mathsf{D}$ is a universal extranatural transformation from object e of D to F. More precisely, it's a pair (e,w) where $e\in\mathrm{Ob}\mathsf{D}$ and $w\colon e\to S$ is an extranatural transformation such that for all extranatural transformation $\beta\colon x\to S$, there exists a unique morphism $h\colon x\to e$ in D such that $\beta_a = w_a\circ h$ for any object $a\in\mathsf{C}$.

13.8 Example. *Consider the geometric realization functor* $|-| = \varinjlim_{\Delta^n\uparrow C}\sigma^n$. *We also have the singular set functor* $S\colon \mathsf{Top}\to\mathsf{Set}^{\Delta^{op}}$. *Let Y be a space, then* $S(Y)\colon\Delta^{op}\to\mathsf{Set}$ *with value at $[n]$ given by* $S(Y)[n]=\mathrm{hom}_{\mathsf{Top}}(\sigma^n,Y)$. *The geometric realization functor is the left adjoint to S. Recall for any simplicial set X, the set of natural transformations $X\to S(Y)$ should be in bijection to the continuous maps $|X|\to Y$. This is realized as follows: we deform* $X([n])\to\mathrm{hom}_{\mathsf{Top}}(\sigma^n,Y)\rightsquigarrow\phi_n\colon X([n])\times\sigma^n\to Y$ *such that for any given morphism $f\colon[m]\to[n]$,* $\left(X([n])\times\sigma^m\xrightarrow{X([f])\times\mathrm{id}}X([m])\times\sigma_m\xrightarrow{\phi_m}Y\right) = \left(X([n])\times\sigma^m\xrightarrow{\mathrm{id}\times\sigma f}X([n])\times\sigma^n\xrightarrow{\phi_n}Y\right)$. *So we got the extranatural transformation* $X([n])\times\sigma^n\to|X|$. *Therefore the coend is exactly* $|X| = \int^n X([n])\times\sigma^n$.

13.9 Definition. Let $M : \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{D}$ where C, D are all V-enriched categories, with C small and D complete, then the **enriched coend**

$$int^{c\in\mathsf{C}} M(c,c) = \mathsf{coeq}\left(\coprod M(a,b) \otimes \mathsf{C}(b,a) \rightrightarrows \coprod_{c\in\mathsf{C}} M(c,c) \right),$$

and the **enriched end** is given as

$$\int_{c\in\mathsf{C}} M(c,c) = \mathsf{eq}\left(\prod_{c\in\mathsf{C}} M(c,c) \rightrightarrows \prod_{a,b\in\mathsf{C}} \mathsf{hom}(\mathsf{C}(a,b), M(a,b)) \right).$$

13.10 Lemma (Yoneda Lemma). Let C, D be V-enriched, and $F : \mathsf{C} \to \mathsf{D}$ any V-enriched functor and any $c \in \mathsf{C}$, we have

$$\mathsf{Nat}(\mathsf{C}(c,-), F) \cong F(c).$$

Written in terms of enriched ends, this is $F(c) \xrightarrow{\cong} \int_d \mathsf{hom}(\mathsf{C}(c,d), F(d))$.

We can also describe the Kan extension:

13.11 Lemma (Kan Extension). Let C, D, E be enriched, and $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{E}$ be functors, then the **left Kan extension** is given by

$$(\mathsf{Lan}_G F)_e = \int^c \mathsf{E}(G(c), e) \otimes F(c).$$

Now let's return to sequential, symmetric and orthogonal spaces.

13.12 Definition. Let E be some Top_*-enriched symmetric monoidal category, then we define the **convolution product** $F \otimes G$ of F and G from $\mathsf{E} \to \mathsf{Top}_*$ to be the left Kan extension of $\wedge \circ (F,G)$ along $+$. In diagrams, we have

$$\begin{tikzcd} \mathsf{E} \times \mathsf{E} \arrow[r, "{(F,G)}"] \arrow[d, "+"] & \mathsf{Top}_* \times \mathsf{Top}_* \arrow[r, "\wedge"] & \mathsf{Top}_* \\ \mathsf{E} \arrow[urr, dashed, "F\otimes G"] & & \end{tikzcd}$$

and we require the universal property

$$\mathsf{E}\,\mathsf{Top}_*(F \otimes G, H) \cong (\mathsf{E} \times \mathsf{E})\,\mathsf{Top}_*(\wedge \circ (F,G), H \circ +).$$

In terms of ends, it's given by $\int_a \mathsf{Top}_*((F \otimes G)_d H_a) \cong \int_{b,c} \mathsf{Top}_*(F_b \wedge G_c, H_{b+c})$.

By Kan extension, $(F \otimes G)_a = \int^{b,c} \mathsf{E}(b+c, a) \wedge F_b \wedge G_c$.

13.13 Lemma. The categories $\mathsf{N}\,\mathsf{Top}_*$, $\Sigma\,\mathsf{Top}_*$ and $O\,\mathsf{Top}_*$ are closed monoidal categories with

$$(F \otimes G)_a = \int^{b,c} \mathsf{N}(b+c, a) \wedge F_b \wedge G_c \cong \bigvee_{b+c=a} F_b \wedge G_c.$$

13.14 Definition. The **sphere spectrum** in N, Σ, O is given by:

$\mathsf{N}\,\mathsf{Top}_*$ $n \mapsto \mathbb{S}^n$;

$\Sigma\,\mathsf{Top}_*$ $\Sigma_n \times \mathbb{S}^n$.

13.15 Lemma. In $\Sigma\,\mathsf{Top}_*$ and $O\,\mathsf{Top}_*$, the sphere spectrum is commutative with respect to $\otimes$.

Proof. $(\mathbb{S}\otimes\mathbb{S})_a = \bigvee_{b+c=a}(\Sigma_a)_+ \wedge_{\Sigma_b\times\Sigma_c} \mathbb{S}^b \wedge \mathbb{S}^c = \bigvee_{b+c}(\Sigma_a)_+ \wedge_{\Sigma_b\times\Sigma_c} \mathbb{S}^{b+c} \xrightarrow{\text{ev}} \mathbb{S}^a$, and similarly for $O\,\mathsf{Top}_*$. □

This fails for $\mathsf{N}\,\mathsf{Top}_*$.

13.16 Theorem. The category of $\mathbb{S}$-module is a category of spectra. The category of $\mathbb{S}$-module in $\mathsf{N}\,\mathsf{Top}_*$ is equivalent to the category of sequential spectra, and similarly for $\Sigma\,\mathsf{Top}_*$ and for $O\,\mathsf{Top}_*$.

Finally, we can define the smash product $X\wedge Y = X\otimes_{\mathbb{S}} Y = \mathrm{coeq}(X\otimes\mathbb{S}\otimes Y \rightrightarrows X\otimes Y)$, and the tensor product $X\otimes Y$ is what we have defined before using end.

13.17 Corollary. S^{Σ} has a stable monoidal smash product $\wedge$: for X and Y in S^{Σ}, $X\wedge Y = X\otimes_{\mathbb{S}} Y$.

13.18 Theorem ($\wedge$ on S^{Σ}, S^{O}). (Σ) $(X\wedge Y)_a = \int^{b,c} \Sigma_{\mathbb{S}}(b+c,a)\wedge X_b\wedge X_c$;

(O) $(X\wedge Y)_a = \int^{b,c} O_{\mathbb{S}}(b+c,a)\wedge X_b\wedge X_c$,

here $\Sigma_{\mathbb{S}}$, $O_{\mathbb{S}}$ should be thought of as follows: $\Sigma_{\mathbb{S}}\,\mathsf{Top}_* \cong \mathsf{S}^{\Sigma}$, and $O_{\mathbb{S}}\,\mathsf{Top}_* \cong \mathsf{S}^{O}$.

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