A Ticket to Topological K-Theory

Siyang Liu

January 23, 2020

Contents

This lecture, as its title, is only a ticket to topological K-theory, which means that it will cover only a few very basic topics in topological K, and readers who want to make further acquaintance with this guy should go to the references for help. Textbooks like [9], [3] and [6] are highly recommended.

1 Start Point: Vector Bundles.

Let us first fix a topological space X and write Ab for the category of abelian groups. Readers familiar with homological algebra will shout out immediately this is an abelian category. Let $\mathcal{T}$ be the topology of X, and an easy translation from the language of lattice to the language of categories tells us $\mathcal{T}$ gives a category denoted T.

1.1 Definition. *The so-called* **abelian presheaf** *is just a contravariant functor $F\colon \mathsf{T} \to \mathsf{Ab}$. In other words, it assigns any open subset $U \subset X$ an ablian group $F(U)$ such that for any inclusion $V \subset U$ in $\mathcal{T}$ we assign a homomorphism $F(U) \to F(V)$ and for three inclusions $W \subset V \subset U$ we have the commutative diagram*

$$\begin{tikzcd} F(U) \arrow[rr] \arrow[dr] & & F(W) \\ & F(V) \arrow[ur] & \end{tikzcd}$$

. We write r_{UW} for the "restriction" map in the definition from W to the smaller open set U.

The **abelian sheaf** F is wrongly defined as an abelian presheaf satisfying for each open subset $U \subset X$ an exact sequence

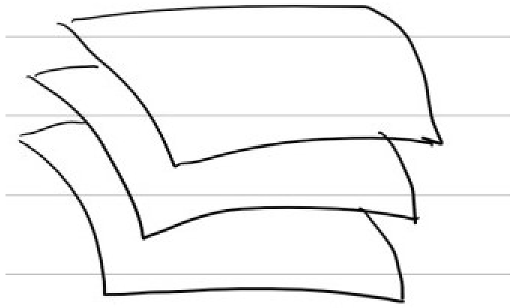
$$0 \to F(U) \to \left(\prod_{V \in \mathcal{T}(X)} F(U \cap V) \right) \oplus \left(\prod_{W \in \mathcal{T}(X)} F(U \cap W) \right) \xrightarrow{\cdot \, - \, \circ} \prod_{V, W \in \mathcal{T}(X)} F(V \cap W \cap U)$$

which requires some effort to explain, but the explanation is very easy: a sheaf is a presheaf which satisfies an additional requirement that for each $U \subset X$ the group $F(U)$ is actually the "group of sections" on U. We must go into some details about this. Sheaf was firstly introduced when people were imaging some "total space" over a manifold X such that at each point $x \in X$ we associate a "germ of functions" G_x, and glue these data together just as what we do with "vector bundle". Rigorously, it goes as follows: for each $x \in X$, we write

$$F_x = \operatorname*{colim}_{x \in U \in \mathcal{T}(X)} F(U)$$

as the **stalk** of F at $x \in X$. Then we obtain a family $\{F_x\}_{x \in X}$ of abelian groups(recall that the limit in some concrete abelian category always exists), and we could define a new abelian group $\Gamma(U; F)$ of sections, that is, a set of families of elements $\{s_x \in F_x\}_{x \in U}$ such that for any $y \in U$ there exists a neighbourhood V_y of y and an element $s \in F(V_y)$ such that the limit $\lim\limits_{x \in U \subset V_y} r_{UV_y}(s)$ is identical with s_y. We write for convenience r_{yV_y} for such a limit, and will use the notation s_y for the map r_{yV_y}, although it is abused in our above discussion.

We can then construct another related presheaf G given by assigning to each $U \subset X$ the abelian group $\Gamma(U; F)$. It is easy to verified that it is an abelian sheaf[1] which is not necessarily identical with the original presheaf F. We call this procedure a **sheafification**. Despite the algebraic definition, sheaf it self is endowed with a natural topology given as the picture shows:



Seriously speaking, the topology is given as follows: any open set in $\mathscr{F}$(the sheaf F endowed with given topology) is a union of the form $s(U)$ for each $U \subset X$ and $s \in \Gamma(U; F)$. Of course $\mathscr{F}$ itself is a set given by the disjoint union of the family $\{F_x\}_{x \in X}$. It can then be easily seen that any section s is an embedding from its domain.

1.1.1 Proposition. *The sheafification of an abelian sheaf F is isomorphic to itself.*

The **morphism** between sheaves is the natural transform between functors, and we say it is an isomorphism if the natural transform is a natural equivalence. We use $\mathrm{Hom}(F, G)$ to denote the set of natural transforms between functors F and G.[2] It has an obviously translation to the language of topology that any natural transforms are some special continuous functions from $\mathscr{F}$ to $\mathscr{G}$. In our given topology, the addition of elements that in the same fibre F_x is continuous, and we have natural direct sum, tensor product and quotient of sheaves[3] given by direct sum, tensor product and quotient on the presheaf level. It follows directly that

1.1.2 Proposition. *The category of abelian sheaves over X with morphisms morphisms between sheaves is an abelian category.*

Let X and Y be two topological spaces with a continuous map $f \colon X \to Y$, then we can pull any sheaf $\mathscr{F}$ over Y back to a sheaf $f^*\mathscr{F}$ over X given for any $x \in X$, $(f^*\mathscr{F})_x = \mathscr{F}_{f(x)}$.

Now we come from abelian sheaf to sheaf of rings and modules. A sheaf $\mathscr{R}$ of commutative rings is defined similarly as a contravariant functor and the sheaf $\mathscr{M}$ of $\mathscr{R}$-modules $\mathscr{M}$ is defined as a

[1] and here's the monograph [4] for those who really need some help.
[2] In this case, this is a set.
[3] For tensor product and quotient, we need to take a sheafification. See [1].

sheaf assigning to each open set $U \subset X$ an $R(U)$-module. Everything is defined similarly with some corrections, so we have for each x a ring $\mathscr{R}_x$ and a module $\mathscr{M}_x$, and we say a sheaf $\mathscr{F}$ is a **constant sheaf** if it is a sheafification of some constant presheaf $U \mapsto G$ for some abelian group/ring/module G. $\mathscr{F}$ is said to be **locally constant** if there exists an open covering $\{U_i\}$ of X such that for each U_i, the restriction $\mathscr{F}|_{U_i}$ is a constant sheaf. $\mathscr{M}$ is **locally free** if locally it is a free $\mathscr{R}$-module[4]([2]). There is a natural sheaf of rings: the sheaf $\mathscr{C}(X)$ of rings of continuous functions over X, and when X has smooth or analytic structure, we can define $\mathscr{C}^\infty(X)$ and $\mathscr{A}(X)$ to be the sheaf of smooth and analytic functions over X.

1.2 Definition. *A* **vector bundle** $\mathcal{E}$ *is defined to be a locally free sheaf* $\mathscr{E}$ *of* $\mathscr{C}(X)$*-modules over* X *which is isomorphic to* $\mathscr{C}(X)^r$ *for some positive integer* r*. Similarly, we could define smooth or holomorphic vector bundles.*

We have another obvious definition, which is stated as some fibre bundle over some bases with fibres vector spaces. The relation is easy in philosophy: locally free sheaves are sheaves of sections of the vector bundle. Given $x \in X$, the **germ** $\mathscr{C}(X)_x$ of infinitesimal functions at x is a local ring, and the ideal $\mathscr{M}(X)_x$ of infinitesimal functions at x which vanish at x is the maximal ideal of $\mathscr{C}(X)_x$, and hence the quotient $\mathscr{C}(X)_x/\mathscr{M}(X)_x$ is the field isomorphic to $\mathbb{R}$. Similarly for a vector bundle $\mathcal{E}$, we have the corresponding quotient $\mathcal{E}_x := \mathscr{E}_x \otimes \mathscr{C}(X)_x/\mathscr{M}(X)_x$ which is a vector space over $\mathbb{R}$ of dimension r. Consider the disjoint union $\mathcal{E} := \coprod_{x\in X} \mathcal{E}_x$ with a natural projection $\pi\colon \mathcal{E} \to X$, we can endow $\mathcal{E}$ with topology defined by the "locally product topology", since $\mathbb{R}^r$ has a natural Euclidean topology. Now we obtain the classical definition of a vector bundle.

1.2.1 Definition. *A real* **vector bundle** *over* X *of rank* r *is a topological space* $\mathcal{E}$ *with a continuous projection* $\pi\colon \mathcal{E} \to X$ *such that for each point* $x \in X$, $\pi^{-1}(x)$ *is a real vector space of dimension* r *and there exists an open subset* $U_x \subset X$ *such that there exists a homeomorphism* $\phi_U\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$.

We can then recover the locally free sheaf $\mathscr{E}$ by assigning to each open subset $U \subset X$ the module of sections $\Gamma(U;\mathcal{E})$ consisting of continuous sections $s\colon U \to \pi^{-1}(U)$ such that $\pi \circ s = \mathbb{1}$. It is obvious a locally free module over the structure sheaf $\mathscr{C}(X)$.

1.2.2 Proposition. *Assume* $\mathscr{E}$ *is a locally free* $\mathscr{C}(X)$*-module, then the corresponding* $\mathscr{C}(X)$*-module made up by taking sections of the vector bundle* $\mathcal{E}$ *coincides with* $\mathscr{E}$.

Proof. Write $\mathscr{E}_1$ as the corresponding module, then we want to show that any continuous sections on $\mathcal{E}$ is an element in $\mathscr{E}$. Let $f \in \Gamma(U;\mathscr{E})$, then since $\mathscr{E}$ is a locally free sheaf over $\mathscr{C}(X)$, we have for each $x \in U$ there is an open neighbourhood U_x of x such that $r_{U_xU}f \in \mathscr{E}(U_x)$ can be written as $(f_1, \cdots, f_r)$ where $f_i \in \mathscr{C}(X)(U_x)$, and hence if we consider the evaluation map $ev_y\colon f \to f(y)$, then ev_y is exactly the map that passes the germ $\mathscr{E}_x$ to its quotient $\mathbb{R}^r$, and therefore $r_{U_xU}f$ coincides with $\bar{f}|_{U_x} \in \Gamma(U_x;\mathcal{E})$. The converse is obvious. □

Having constructing this correspondence, it is easy to get the direct sum $\mathcal{E} \oplus \mathcal{F}$ of two vector bundles, called the **Whitney sum**, just as the product of two section sheaves and then translating to vector bundles. Similarly, we have the pull-back of a vector bundle, quotient bundle, dual bundle, and tensor product of vector bundles. We now use the normal uppercase to write a vector bundle.

1.2.3 Definition. *Assume* E *and* F *are two vector bundles over* X*, then a* **bundle map** *from* E *to* F *consists of continuous maps* $E \to F$ *satisfying the commutative diagram*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$

and for each x, $f_x\colon \pi_E^{-1}(x) \to \pi_F^{-1}(x)$ *is a linear map.*

[4]But not globally. We will see the global sections of a vector bundle corresponds to projective modules, according to a result by Swan.

Hence all vector bundles over X forms a category Vect with morphisms bundle maps. There are relationships between categories Vect and $\mathscr{C}(X)$-Mod_{LF}, the latter being categories of locally free $\mathscr{C}(X)$-modules. Note that any morphism between locally free $\mathscr{C}(X)$-modules $\mathscr{E}$ and $\mathscr{F}$ gives natural linear maps at each point $x \in X$, and hence glues to a map $E \to F$.

1.2.4 Proposition. *The induced map $E \to F$ is a bundle map.*

Proof. This also relies on the fact that local sections on a vector bundle describes completely the topology of this vector bundle. □

We now discuss additional structures over a vector bundle $E \to X$. Such structure is given by a global section b over the tensor product $E^* \otimes E^*$, and if we could descend to some quotient, then b will have additional properties. We discuss the case when b is **non-degenerate**, that is, when we write it pointwise in terms of matrices, then these matrices must be invertible, and this additional structure gives an isomorphism $E^* \xrightarrow{\simeq} E$, just as the case of vector spaces. This will help us to determine vector bundles in some special cases, just as if the base space X is a differential manifold, then we could immerse it into Euclidean space to get the "orthogonal complement" vector bundle, and apply characteristic classes to determine which vector bundle it is. What we need is some existence.

1.3 Proposition. *Assume X is a paracompact Hausdorff space, then any vector bundle over X admits a non-degenerate bilinear form.*

This is obvious since any paracompact Hausdorff space admits a partition of unity. Assuming a given non-degenerate symmetry bilinear form b on a vector bundle $E \to X$, and let $F \to X$ be a vector subbundle of E, then we can form the "orthogonal" complement of F in E, still a subbundle of E, written $F^{\perp}$, which is defined pointwise by assigning each $x \in X$ the vector space $F_x^{\perp}$ where the complement is defined with respect to E, and we call it the orthogonal complement bundle of F w.r.t. E.

1.3.1 Example. *Let M be a differential manifold of dimension n, then we have a natural immersion $M \to \mathbb{R}^{2n+k}$ into some Euclidean spaces. And if we acknowledge the result that any vector bundles over Euclidean spaces are trivial, the tangent bundle TM has a natural complement bundle NM defined as the complement of TM in the pull-back of $\mathbb{R}^{2n+k}$. This is called in general the* **normal bundle** *of M.*

In this example, we use some classification result of vector bundles:

1.3.2 Proposition. *Assume X is paracompact and contractible, then any vector bundle over X is trivial.*

Proof. By definition, any vector bundle $E \to X$ pulls back to the trivial bundle $E_1 \to \{\mathrm{pt}\}$, and via the natural map $X \to \mathrm{pt}$ back to a trivial bundle over X. Therefore our proof is a direct consequence of the following homotopical lemma. □

1.3.3 Proposition. *Assume that X is paracompact, $f, g\colon X \to Y$ homotopic continuous maps, and $E \to Y$ a vector bundle over Y, then f^*E and g^*E is isomorphic.*

Proof. We use Milnor's lemma 5.8 in his book [9] to get a locally finite countable covering $\{U_i\}$ of X to obtain a sequence of trivializations(use tube lema) $\{U_i \times I\}_{i=1}^{\infty}$ of h^*E, where I is the unit inteval and h be the homotopy from f to g. Now define $\{\phi_i\}$ to be the partition of unity subordinate to $\{U_i\}$ and let Φ_i be the graph of the partial sum of $\{\phi_i\}$, then the image of Φ_i is homeomorphic to X and since each adjacent graphs lie in the same trivialization. Hence it is reduced to the case when X is covered by two neighbourhoods $U_1 \cup U_2$, and let φ_{U_i}, $1 \leq i \leq 2$ to be the trivialization of $h^*E := \xi$, and let $\xi|_{X\times\{0\}} = \xi_0$, $\xi|_{X\times\{1\}} = \xi_2$, $\xi|_{\Phi_1(X)} = \xi_1$, and we have on U_1 a commutative diagram

$$\begin{tikzcd}
\xi_0 \arrow[d] \arrow[dr, "\varphi_{U_1\times I}|_{\xi_0}"] & \\
U_1 & U_1 \times \mathbb{R}^r \arrow[l] \\
\xi_1 \arrow[u] \arrow[ur, "\varphi_{U_1\times I}|_{\xi_1}"'] &
\end{tikzcd}$$

and let $\alpha = \varphi_{U_1\times I}|_{\xi_1}^{-1} \circ \varphi_{U_1\times I}|_{\xi_0}$. This gives an isomorphism from ξ_0 to ξ_1 on U_1, and since $X = U_1 \times U_2$ and ξ_1 and ξ_0 coincide on U_2, we have $\xi_0 \simeq \xi_1$. A similar procedure gives $\xi_1 \simeq \xi_2$. $\square$

2 A Stop: The Universal Bundle and Classification

The train starts and is traveling toward the first stop, which is a first classification of vector bundles over a paracompact topological space X. Here we will use Greek characters ξ, η, etc. to write a vector bundle, and use $\underline{\mathbb{R}}^k$ to denote a trivial bundle of rank k. The idea is that one could view a vector bundle as a r-plane field, that is, a function to the Grassmannian manifold $\mathrm{Gr}(n, \infty)$, the manifold of n-planes in the infinity Euclidean space $\mathbb{R}^\infty$. Note that

2.1 Example. *There are natural vector bundles over* $\mathrm{Gr}(n, k)$*, written* γ^n_{n+k}*, which assigns to each* $P \in \mathrm{Gr}(n, k)$ *the vector subspace* P *in the trivial bundle* $\underline{\mathbb{R}}^{n+k}$*. This is called the* **tautological vector bundle***, and we could show that this is actually a vector bundle over* $\mathrm{Gr}(n, k)$*. When* $n = 1$*, this is the real projective space* $\mathbb{RP}^k$*.*

Now we must prove that γ^n_{n+k} is actually a vector bundle over $\mathrm{Gr}(n, k)$. To see this, for each $P \in \mathrm{Gr}(n, k)$, we pick as in [9] the coordinate neighbourhood $U \cong \mathrm{Hom}(P, P^*)$, where P^* is the dual space of P, and we define the homeomorphism $h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ to be $h(V, v) = (V, v + T(V)v)$ where $T\colon P \to P^\perp$ is the map in $\mathrm{Hom}(P, P^\perp)$ corresponding to the subspace V. Now it is a direct verification that h gives a homeomorphism and even diffeomorphism, completing our argument.

Then let $k \to \infty$ and we find the infinite Grassmannian $\mathrm{Gr}(n) = \mathrm{Gr}(n, \infty)$, with the corresponding tautological bundle $\gamma^n_\infty = \gamma^n$. We still need to verify that they are vector bundles. To do this, recall the topology defined at $\mathrm{Gr}(n)$ is given by

2.1.1 Definition. *Assume* $\{A_i\}_{i\in\mathcal{I}}$ *is an indexed sequence of topological spaces with inductive limit* $\operatorname*{colim}_{i\in\mathcal{I}} A_i = A$*, then* A *is a topological space with topology given by the quotient space of the disjoint union* $\coprod_{i\in\mathcal{I}} A_i$ *with equivalence relationship given by the colimit system.*

And in this case, $\mathrm{Gr}(n)$ is given by the limit of an ascending sequence $\mathrm{Gr}(n, 0) \subset \mathrm{Gr}(n, 1) \subset \cdots \subset \mathrm{Gr}(n)$, the topology is given directly by $U \subset \mathrm{Gr}(n)$ open if and only if $U \cap \mathrm{Gr}(n, k)$ is open for all $k \geq 1$. A similar discussion for $\mathbb{R}^\infty$ shows that $\mathbb{R}^\infty$ carries a natural inner product $(\cdot, \cdot)$ which is given by assigning to each pair (x, y) the unique inner product $(x, y)_k$ where $(\cdot, \cdot)_k$ is the inner product in $\mathbb{R}^k$ so that $x, y \in \mathbb{R}^k \subset \mathbb{R}^\infty$. Hence we could define orthogonal projection in $\mathbb{R}^\infty$.(Although this is not a Hilbert space. The completion will be $\ell^2(\mathbb{N})$, a well-known fact in functional analysis.) With this preliminary discussions, we could prove

2.1.2 Proposition. *The topological space* γ^n *constructed above is a vector bundle over* $\mathrm{Gr}(n)$*.*

Proof. For fixed $P \in \mathrm{Gr}(n)$, let U be the set of all n-planes V in $\mathrm{Gr}(n)$ such that the orthogonal projection $p\colon \mathbb{R}^\infty \to P$ sends V isomorphically onto P. By definition, U is open in $\mathrm{Gr}(n)$ and we could define the map $h\colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ as usual. If we write $U_k = U \cap \mathrm{Gr}(n, k)$, then h maps each

$U_k \times \mathbb{R}^n$ homeomorphically onto $\pi^{-1}(U_k)$, and hence if we could see that $\pi^{-1}(U)$ is approximated by $\pi^{-1}(U_k)$, then we have done the proof. This result is given by the following point-set-topology argument. $\square$

2.1.3 Proposition. *Assume that A and B are topological spaces that are limits of two ascending sequences $A_1 \subset A_2 \subset A_3 \subset \cdots$ and $B_1 \subset B_2 \subset B_3 \subset \cdots$ of locally compact spaces, then the product topology and the limit topology on $A \times B$ coincide. Here, the limit topology is the limit of the sequence $A_1 \times B_1 \subset A_2 \times B_2 \subset \cdots$.*

Proof. The product topology of $A \times B$ is generated by $U \times V$, where U is open in A and V is open in B. Now consider the intersection $(U \times V) \cap (A_k \times B_k) = (U \cap A_k) \times (V \times B_k)$. By definition, this is open in $A_k \times B_k$ and hence $U \times V$ is open in $A \times B$ endowed with the limit topology. Conversely, assume $U \subset A \times B$ is open in the limit topology, then $U \cap (A_k \times B_k)$ is open for all k. Pick $x \in U$, then we could find k such that $x \in A_k \times B_k$ and hence we can find $x \in V_k \times W_k \subset U_k$. Since A_k and B_k are locally compact, we could pick V_k and W_k to be precompact and the closure of the product is also contained in U. Then we could pick a precompact neighbourhood V_{k+1} of V_k and W_{k+1} of W_k satisfying the same property that the closure of the product is again contained in U. Proceed by induction, we could find $V \times W$ containing x and is open in U. Therefore U is open in $A \times B$ endowed with the product topology. $\square$

Now comes the main theme of this stop: we will use the universal bundle for classification of vector bundles over any paracompact space X. First of all, we can see what happens to vector bundles over compact Hausdorff spaces. The result is contained in the following theorem

2.2 Theorem. *Assume X is a compact Hausdorff space, then for any vector bundle $\xi^r \to X$ of rank r, there exists a positive integer N and a continuous map $f\colon X \to \mathrm{Gr}(r, N)$ such that $f^*\gamma^r_{r+N} \cong \xi^r$.*

Proof. It suffices to construct for each vector bundle ξ^r a continuous map $f\colon E(\xi^r) \to \mathbb{R}^{N+r}$ linear and injective on each fibre of ξ^r, where $E(\xi^r)$ denotes the total space of ξ^r. Let $\{U_i\}_{i=1}^n$ be an open covering of X such that $\xi^r|_{U_i}$ is trivial and since X is compact and Hausdorff, we could find open subsets $\{V_i\}_{i=1}^n$ such that $\bar{V}_i \subset U_i$ and can further find $\{W_i\}$ so that $\bar{W}_i \subset V_i$. Let h_i be the bump function of W_i supported in $\bar{V}_i$, by trivialization there exists a continuous map $\varphi_i\colon \pi^{-1}(U_i) \to \mathbb{R}^r$ which is linear and injective on each fibre, and we let $\tilde{\varphi}_i(x, v) = h_i(x)\varphi_i(x, v)$ and 0 outside the support of h_i. The global function f is then defined as $f\colon E(\xi^r) \to \mathbb{R}^{nr}$ given by $f = (\tilde{\varphi}_1, \tilde{\varphi}_2, \cdots, \tilde{\varphi}_n)$. The required continuous map $\tilde{f}\colon X \to \mathrm{Gr}(r, rn - r)$ is then given by mapping each x into the vector subspace in $\mathbb{R}^{rn}$ that is the fibre on x. $\square$

Observe that this theorem has inpired us the proof of the case when X is paracompact: here we use a finite covering $\{U_i\}$ and in paracompact case, we have a countable locally finite covering $\{U_i\}_{i=1}^\infty$, and we could use the partition of unity instead of these finite bump functions to construct such a linear map $E(\xi^r) \to \mathbb{R}^\infty$. Here we must pay some attention to the topology since $\mathbb{R}^\infty$ is defined by an inductive limit instead of product, but notice that locally the map is given by a sum of finitely many continuous maps into finite Euclidean spaces, and is hence continuous. The other detailed problem is about the induced base map $\tilde{f}\colon X \to \mathrm{Gr}(r, \infty)$. But since this is again a local argument and locally, all the things are finite, we have proved the following theorem.

2.3 Theorem. *Assume X is paracompact, then any vector bundle ξ^r over X of rank r comes from a continuous map $f\colon X \to \mathrm{Gr}(r, \infty)$.*

Since we have proved that when the base space X is paracompact, X will admit a homotopy argument, therefore we can deduce the following

2.3.1 Theorem. *There is a set-theoretic isomorphism*

$$\mathsf{Vect}_n(X) \cong [X, \mathrm{Gr}(n, \infty)]$$

where $[X, \mathrm{Gr}(n, \infty)]$ *denotes the homotopy class of maps from* X *to* $\mathrm{Gr}(n, \infty)$.

From this classification we can readily obtain all the vector bundles over spheres. Note that $[\mathbb{S}^r, \mathrm{Gr}(n, \infty)] = \pi_r \mathrm{Gr}(n, \infty)$, the vector bundle over spheres corresponding directly to the homotopy group of infinite Grassmannians. Note that we have the fibration $\mathrm{O}(n) \to V_n(\mathbb{R}^\infty) \to \mathrm{Gr}(n, \infty)$, when $n = 1$ we have $\pi_r(\mathrm{Gr}(n, \infty)) = 0$ for $r \geq 2$(Since the Stiefel manifold $V_n(\mathbb{R}^\infty)$ is contractible. See [5]) and $\pi_1(\mathrm{Gr}(n, \infty)) = \mathbb{Z}/2$, hence all real line bundles over $\mathbb{S}^r$ is trivial for $r \geq 2$ and there exists another line bundle ξ over $\mathbb{S}^1$ which is not trivial.

2.3.2 Example. *In fact, this non-trivial line bundle has been well-known to topologist. This is the tautological line bundle over* $\mathbb{RP}^1 \cong \mathbb{S}^1$*, and is not orientable. The total space of* γ_1^2 *is exactly the Mobiüs band.*

Another easy case is $n = 2$, where $\mathrm{SO}(2) \cong \mathbb{S}^1$ and we have $\pi_r \mathrm{Gr}(n, \infty) \cong \pi_{r-1}(\mathrm{O}(2)) = \begin{cases} 0, & r \geq 3; \\ \mathbb{Z}, & r = 2; \\ \mathbb{Z}/2, & r = 1. \end{cases}$ Hence there are again two types of rank 2 real vector bundles over $\mathbb{S}^1$, infinitely many rank 2 real vector bundles over $\mathbb{S}^2$ and only trivial plane bundles over $\mathbb{S}^r$ for r sufficiently large. In general, note that for n greater than 3, the fundamental group of $\mathrm{Gr}(3, \infty)$ is fixed and therefore for all rank there exists two types of real vector bundles over $\mathbb{S}^1$. We will go back to this result later.

2.4 Exercise. *Here's an easy exercise due to Swan's paper [10]. Assume* X *is a* **Tychonoff space**, *that is a Hausdorff space such that for each* $x \in X$ *and a closed subset* $A \subset X$ *with* $x \notin A$*, there exists a continuous function* $f\colon X \to [0, 1]$ *such that* $f(x) \notin f(A)$*. Try to prove:*

1. *Given a vector bundle* $\xi^r \to X$ *of rank* r*, let* $M(\xi)$ *be the module over* $C(X)$ *of global sections of* ξ*, then* $M(\xi \oplus \eta) \cong M(\xi) \oplus M(\eta)$ *and* $M(\xi)$ *is free over* $C(X)$ *if and only if* ξ *is trivial.*

2. *Assume* ξ *satisfies there exists a vector bundle* η *so that* $\xi \oplus \eta$ *is trivial, then* $M(\xi)$ *is a finitely generated projective* $C(X)$*-module. The converse is also true, that is, any finitely generated projective* $C(X)$*-module comes from some vector bundle satisfies the given property.*

3. *Given two vector bundles* ξ *and* η*, show that* $\xi \simeq \eta$ *if and only if* $M(\xi) \cong M(\eta)$.

3 Axiomatic Characteristic Classes

The train then drives to the next stop, where we will introduce characteristic classes to classify vector bundles over topological spaces. By now, we only list the axioms that a characteristic class must satisfy, and the next stop will be how to construct these classes(that is, existence). Firstly we focus on real vector bundles over some base space X.

3.1 Definition. *Assume* ξ^r *is a real vector bundle over* X*, then the* i*-th* **Stiefel-Whitney class** *of* ξ*, written* $w_i(\xi)$*, is a cohomology class in* $H^i(X; \mathbb{Z}/2)$ *satisfying the following axioms:*

1. $w_0(\xi) \equiv 1$*, the unit element in* $H^*(X; \mathbb{Z}/2)$ *and* $w_i(\xi) = 0$ *for* $i > r$.

2. *(Naturality) Assume* $f\colon \xi \to \eta$ *is a bundle map, then* $w_i(\xi) = f^* w_i(\eta)$ *for all* i.

3. *(Whitney Product Theorem) Let* $w(\xi) = \sum_{i=1}^{\infty} w_i(\xi) \in H^*(X; \mathbb{Z}/2)$ *be the* **total Stiefel-Whitney class**, *then for vector bundles* ξ *and* η*, we have* $w(\xi \oplus \eta) = w(\xi) \cup w(\eta)$.

4. $w_1(\gamma_1^1) \neq 0$.

We also have a similar version for complex vector bundles. Firstly let's clarify what complex vector bundle means. From a principal bundle point of view, a complex vector bundle is a real vector bundle of even rank such that the structure group can be reduced to $\mathrm{GL}(n, \mathbb{C})$. From a more intuitive point of view, we firstly introduce:

3.1.1 Definition. *Assume $\xi \to X$ is a real vector bundle of rank $2r$. An* **almost complex structure***(abbr. a.c.s.) on ξ is a global section $J \in \mathrm{Hom}(\xi, \xi)$ such that $J^2 = -\mathbb{1}_\xi$.*

Note that in our familiar linear algebra, the existence of an almost complex structure(in linear algebra, we call it a complex structure) gives a complexification of the real vector space V and hence V becomes a complex vector space of half dimension. Here we simulate the procedure on linear algebra to give the complexification of ξ to make it into a complex vector bundle $\xi_{\mathbb{C}}$. Now we say that the i-th **Chern class** $c_i(\xi) \in H^2(X; \mathbb{Z})$ of a complex vector bundle ξ is a cohomology class satisfying the four axioms listed above, with a modification that $c_1(\gamma(\mathbb{CP}^1))$ must be the generator of $H^2(X; \mathbb{Z})$.

Before going to the long story of uniqueness(and also existence) for these axioms(just as the uniqueness of the homology theory, it takes some great efforts to prove the uniqueness of such axioms, since we need to compute explicitly the cohomology of infinity Grassmannians), we can do some calculation using these axioms. Firstly, note that[5]

3.1.2 Proposition. *$w_i(\underline{\mathbb{R}}^k) = 0$ for all $i > 0$ and $k \geq 1$.*

Proof. This is because trivial bundle comes from the pull-back of vector bundles over a point. □

3.1.3 Proposition. *Assume ξ is any real vector bundle over X, then $w(\xi \oplus \underline{\mathbb{R}}^1) = w(\xi)$.*

Proof. This follows directly from the Whitney Product Theorem. □

3.1.4 Proposition. *For $n \geq 1$, we have $w_1(\gamma_n^1) \neq 0$.*

Proof. There is a natural inclusion $i\colon \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ which pulls γ_n^1 back to γ_1^1, hence $i^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0$ and therefore $w_1(\gamma_n^1) \neq 0$. □

as a corollary, we have

3.1.5 Corollary. *$w_1(\gamma^1) \neq 0$.*

If we view the total Stiefel-Whitney class w as a formal power series $w = 1 + w_1 + w_2 + \cdots$, then since 1 is the unit element in the commutative ring, w is a unit element in the ring $\hat{H}^*(X; \mathbb{Z}/2)$ of all formal series in $H^*(X; \mathbb{Z}/2)$, with the inverse $\bar{w}$ defined as $\bar{w} = w^{-1} = 1 + (w_1 + w_2 + \cdots) + (w_1 + w_2 + \cdots)^2 + \cdots$. Hence the Whitney Product Theorem can help us compute the Stiefel-Whitney class as $w(\xi) = \bar{w}(\eta) w(\xi \oplus \eta)$. If $\xi \oplus \eta$ is trivial, then we have

3.1.6 Theorem (Whitney Duality Theorem)**.** *Assume M is a smooth manifold embedding in the Euclidean space $\mathbb{R}^{n+k}$, τ_M the tangent bundle of M and ν_M^k the normal bundle of M, then $w(\nu_M^k) = \bar{w}(\tau_M)$.*

Proof. This is because $\mathbb{R}^{n+k}$ is contractible and endowed with any Riemannian metric on M, we have the direct sum decomposition $\underline{\mathbb{R}}^{n+k} \simeq \tau_M \oplus \nu_M^k$. □

[5] Here we only compute Stiefel-Whitney classes. The case of Chern classes is exactly the same, and the author feels frustrated to state the same thing twice.

These things can be used directly in the study of vector bundles over real projective spaces. Note that $\mathbb{RP}^n$ is a differential manifold of dimension n, and the tautological bundle γ_n^1 is defined by the subbundle of the trivial bundle $\underline{\mathbb{R}}^{n+1}$, and hence we have the orthogonal complement $\gamma_n^{1,\perp}$ which is a vector bundle of rank n. By Whitney Duality Theorem, we have

3.1.7 Proposition. *$w(\gamma_n^{1,\perp}) = 1 + a + a^2 + \cdots + a^n$ where $a \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ is the generator. Therefore there exists vector bundles of rank n such that w_i does not vanish for all $1 \leq i \leq n$.*

Proof. This follows directly from two facts: one is that $w(\gamma_n^1) = 1 + a$ and the other is that $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[a]/\langle a^{n+1}\rangle$ as graded algebra. $\square$

We can also determine the Stiefel-Whitney class of the tangent bundle $\tau(\mathbb{RP}^n)$. Firstly observe that

3.1.8 Proposition. $\tau(\mathbb{RP}^n) \simeq \mathrm{Hom}(\gamma_n^1, \gamma_n^{1,\perp})$.

Proof. This follows from the differential structure of $\mathbb{RP}^n$, given locally as the space of linear transforms $\mathrm{Hom}(L, L^\perp)$. $\square$

There are no ways to calculate the Stiefel-Whitney class of tensor product of vector bundles by now, but notice that in general, for any vector bundle ξ over some base X, we have

3.1.9 Theorem. *The vector bundle $\mathrm{Hom}(\xi, \xi)$ admits a nowhere zero global section.*

Proof. The global section is defined by $x \mapsto (x, \mathbb{1}_{\xi_x})$. $\square$

Then we could prove

3.1.10 Theorem. *The total Stiefel Whitney class $w(\tau_{\mathbb{RP}^n}) = (1+a)^{n+1} = 1 + \binom{n+1}{1}a + \cdots + \binom{n+1}{n}a^n$.*

3.1.11 Remark. *We often abbreviate $w(X)$ for the total and partial Stiefel-Whitney class of the tangent bundle of the differential manifold X.*

Proof. This follows from the fact that we have bundle isomorphism $\mathrm{Hom}(\gamma_n^1, \gamma_n^{1,\perp}) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \simeq \mathrm{Hom}(\gamma_n^1, \gamma_n^{1,\perp} \oplus \gamma_n^1) \simeq \mathrm{Hom}(\gamma_n^1, \underline{\mathbb{R}}^{n+1})$ and the left-hand side is exactly the direct sum $\tau(\mathbb{RP}^n) \oplus \underline{\mathbb{R}}^1$. The theorem follows from the following exercise. $\square$

3.1.12 Exercise. *Assume that ξ is an Euclidean vector bundle over X, then there is a bundle isomorphism $\xi \simeq \mathrm{Hom}(\xi, \underline{\mathbb{R}}^1)$.*

We then have the following corollary originally proved by Stiefel:

3.1.13 Corollary. *$w(\mathbb{RP}^n) = 1$ if and only if n is a power of 2.*

Proof. This is a number-theoretic result and the proof will be omitted. This depends on the fact that the power of 2 inside a factorial $n!$ is given by the fake power series $\displaystyle\sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor$. $\square$

These results can be applied directly to obtain many fascinating theorems in topology. A classical problem is to find out whether $\mathbb{R}^n$ for $n \geq 1$ admits a division algebra structure, that is to find at least an $\mathbb{R}$-bilinear map $b\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that for fixed $x \in \mathbb{R}^n$, both $b(x, \cdot)$ and $b(\cdot, x)$ are invertible linear transforms. Our conclusion is that

3.1.14 Theorem. *Such a bilinear map exists if n is a power of 2.*

Proof. Assume $\{e_1, e_2, \cdots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$, then $b(e_i, \cdot)$ is invertible and hence we could construct linear transforms $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$ given as the composition $f_i = b(e_i, \cdot) \circ b(e_1, \cdot)^{-1}$. It is obvious that $f_1(x) = x$ and $f_2(x), \cdots, f_n(x)$ forms a system of linearly independent vectors. Passing to projective space $\mathbb{RP}^{n-1}$, we have $f_2, \cdots, f_n$ gives a global frame for $\tau(\mathbb{RP}^{n-1})$, and hence $n-1$ must be a power of 2. □

Another application focuses on immersion problems. Note that if we could immerse $\mathbb{RP}^n$ into $\mathbb{R}^{n+k}$, then the pull-back of the tangent bundle of $\mathbb{R}^{n+k}$ is a trivial bundle $\underline{\mathbb{R}}^{n+k}$ over $\mathbb{RP}^n$, and the tangent bundle $\tau(\mathbb{RP}^n)$ is a subbundle of $\underline{\mathbb{R}}^{n+k}$. Therefore the Stiefel-Whitney class $\bar{w}(\mathbb{RP}^n)$ will tell us informations about k. For example, by direct calculation we find that when $n = 9$, $w(\mathbb{RP}^9) = 1 + a^2 + a^8$ and hence $\bar{w}(\mathbb{RP}^9) = 1 + a^2 + a^4 + a^6$, so we could conclude that $\mathbb{RP}^9 \to \mathbb{R}^{n+k}$ is an immersion only if $k \geq 6$. Another result shows that

3.1.15 Theorem. $\mathbb{RP}^{2^r}$ *can be immersed into* $\mathbb{R}^{2^r+k}$ *only if* $k \geq 2^r - 1$.

In fact, by a theorem of Whitney, the immersion exists if $k = 2^r - 1$, hence the theorem stated gives a sharp bound for the existence of immersion. See [7] for a list of further results on immersion problems[6].

Proof. This depends on the calculation of the dual class $\bar{w}(\mathbb{RP}^n)$. Since when $n = 2^r$ we have $w(\mathbb{RP}^n) = 1 + a + a^n$, the dual class will be $\bar{w}(\mathbb{RP}^n) = 1 + a + a^2 + \cdots + a^{n-1}$ and hence $k \geq n-1$. □

Before going into the odyssey of existence and uniqueness, let's firstly change our point of view toward the problem of classifying vector bundles. Here we assume X is compact and Hausdorff. In this slightly special case, we have

3.2 Proposition. *For any vector bundle $\xi \to X$ there exists a vector bundle $\eta \to X$ such that $\xi \oplus \eta$ is trivial.*

Proof. This follows from the fact that any vector bundle over a compact base is the pull-back of some tautological bundle γ_k^n over the Grassmannian $\mathrm{Gr}(n, k)$. Then the trivial bundle $\underline{\mathbb{R}}^{n+1} := \underline{n+1}$ pulls back to the trivial bundle over X, and we just let η to be the orthogonal complement of ξ provided a Riemannian structure on $\underline{n+1}$. □

Recall that for any given n, we have the set-theoretic isomorphism $\mathsf{Vect}_n(X) \cong [X, \mathrm{Gr}(n)] := [X, \mathrm{BO}(n)]$, where $\mathrm{BO}(n)$ denotes the classifying space for $\mathrm{O}(n)$.[7] We have a natural inclusion from $\mathsf{Vect}_n(X)$ to $\mathsf{Vect}_{n+1}(X)$ given by $\xi \mapsto \xi \oplus \underline{1}$, and on the right-hand side it is just the natural inclusion $\mathrm{BO}(n) \to \mathrm{BO}(n+1)$ and the tautological bundle γ^{n+1} pulls back to $\gamma^n \oplus \underline{1}$(exercise), hence the two maps are the same(or to say the square diagram is commutative). Now we take the direct limit $\mathrm{BO} := \operatorname{colim} \mathrm{BO}(n)$ and we have the bijective correspondence

$$\operatorname*{colim}_n \mathsf{Vect}_n(X) \cong [X, \mathrm{BO}]$$

and the left-hand side, written $\mathsf{StVect}(X)$, is the small category of all vector bundles modulo stable equivalence, *i.e.*two vector bundles ξ and η are isomorphic if there exists positive integers k and l such that $\xi \oplus \underline{k} \simeq \eta \oplus \underline{l}$.

If we write $\mathsf{Vect}_{\mathbb{R}}(X)$ to be the set of isomorphism classes of real vector bundles over X, then there are natural algebraic operators on $\mathsf{Vect}_{\mathbb{R}}(X)$: the Whitney sum served as addition, and the tensor product served as product. By now we only concern with the Whitney sum, and hence the set $\mathsf{Vect}_{\mathbb{R}}(X)$ forms an abelian monoid.

[6] Although this is a sharp bound, there are still things to work on. That is, the sharp bound relies on the dimension of the manifold, and they conjectures a function that gives sharp bounds for each given dimension.

[7] For a definition of classifying space, readers can refer to [8].

3.2.1 Definition. *Assume M is an abelian monoid, the* **Grothendieck group** *associate to M is an abelian group $\tilde{M}$ satisfying the following universal property: for any abelian group G and a monoid homomorphism $f\colon M \to G$, there exists a unique group homomorphism $\tilde{f}\colon \tilde{M} \to G$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & \tilde{M} \arrow[d, "\tilde{f}"] \\ & G \end{tikzcd}$$

commutes.

Then we obtain the **K-group** $KO(X)$ as the Grothendieck group of $\mathsf{Vect}_{\mathbb{R}}(X)$, and by definition two elements ξ and η is identical in $KO(X)$ if and only if there exists a third vector bundle γ such that $\xi \oplus \gamma \simeq \eta \oplus \gamma$. With tensor product, this group forms a commutative ring with identity element $\underline{1}$.

Here we are more focusing on the relation to characteristic classes. Note that the Stiefel-Whitney class w assigns to each element $\xi \in KO(X)$ to a graded cohomology class $w(\xi) \in H^*(X; \mathbb{F}_2)$ where $\mathbb{F}_2$ denotes the field with exactly two elements. From the preceding discussion we know that this class depends not on the rank of the vector bundle but the rank of the "non-trivial" part of the vector bundle, *i.e.*the proposition that $w(\xi \oplus \underline{k}) = w(\xi)$, hence via Stiefel-Whitney class, we can pass the K-group $KO(X)$ to its quotient group $\tilde{KO}(X)$, which is the quotient group of $KO(X)$ identifying all trivial bundles. Notice that this reduced K-group coincides exactly with the Grothendieck group of $\mathsf{StVect}(X)$ and the Stiefel-Whitney class w serves as a group homomorphism $\tilde{\mathsf{StVect}}(X) \to U(H^*(X; \mathbb{F}_2))$.

Now we travel on to the great existence theory of Stiefel-Whitney classes. There are several viewpoint towards understanding, and we will introduce them one by one. The first one is to do via the universal bundle. Recall that any vector bundle over X is determined by some continuous function from X to the classifying space $BO(n)$, hence if we know the Stiefel-Whitney class $w(\gamma^n)$, then we will determine completely all Stiefel-Whitney classes of all vector bundles(without any explicit methods for computation).

References

[1] Section 17.15 (01CA): Tensor product—The Stacks project, Jan 2020. [Online; accessed 12. Jan. 2020].

[2] Section 57.63 (09Y8): Locally constant sheaves—The Stacks project, Jan 2020. [Online; accessed 17. Jan. 2020].

[3] M. F. Atiyah and notes by D. W. Anderson. *K-theory.* W. A. Benjamin, Inc., 1964.

[4] Glen E. Bredon. *Sheaf Theory*, volume 170 of *Graduate Texts in Mathematics.* Springer-Verlag, 1997.

[5] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[6] Max Karoubi. *K-Theory.* Springer, Berlin, Heidelberg, 1978.

[7] Ivan Kolar, Peter W. Michor, and Jan Slovak. *Natural Operations in Differential Geometry | Ivan Kolar | Springer.* Springer-Verlag Berlin Heidelberg, 1993.

[8] J.P. May. *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics. University of Chicago Press, 1999.

[9] John Willard Milnor and James D. Stasheff. *Characteristic Classes.* Princeton University Press, 1974.

[10] R. G. Swan. Vector bundles and projective modules. *Trans. Am. Math. Soc.*, 105:264–277, 1962.