Matrices Summary

To **transpose** a matrix write the rows as columns.

For example:

$$A = \begin{bmatrix} 2 & 1 & 5 & 0 \\ 1 & 5 & 0 & 9 \\ 2 & 1 & 4 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 1 \\ 5 & 0 & 4 \\ 0 & 9 & 2 \end{bmatrix}$$

To **add or subtract** matrices they must be the same dimensions. Just add or subtract the corresponding numbers.

For example:

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & 1 & 2 \\ -3 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 3 \\ 7 & 4 & 2 \\ 10 & 5 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 2 \\ 11 & 5 & 4 \\ 7 & 8 & -1 \end{bmatrix}$$

To **multiply** matrices the number of columns in the 1^{st} must equal the number of rows in the 2^{nd} because you multiply **across the row and down the column**.

For example:

$$\begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 1 & -5 \end{bmatrix} \times \begin{bmatrix} 2 & -3 & 7 \\ 5 & 7 & 8 \end{bmatrix} = \begin{bmatrix} (6+10) & (-9+14) & (21+16) \\ (8+30) & (-12+42) & (28+48) \\ (2-25) & (-3-35) & (7-40) \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 5 & 37 \\ 38 & 30 & 76 \\ -23 & -38 & -33 \end{bmatrix}$$

Transformations

When setting up the multiplication for a transformation the **transformation matrix** always goes **first**.

Rotations:

Clockwise: $\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$ Anticlockwise: $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

Reflections:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: swaps x and y.

Scaling / magnification:

$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$: makes x 3x bigger and y 2x bigger

Composite:

If a matrix A is reflected by T_1 then magnified by T_2 to give matrix B then the single transformation that would transform A into B is T_2T_1 (in that order).

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The Inverse (A^{-1}):

$\boxed{AA^{-1} = \mathbf{I}}$ (the identity matrix) e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Finding the Inverse:

$$\boxed{A^{-1} = \frac{1}{|A|} \text{ x adj}}$$

(Where: $|A|$ = determinant
adj = adjoint)

If: A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = (ad) - (bc)$

adj = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

n.b. If det=0 matrix is singular and there is no inverse. If det≠0, matrix is non-singular.

Solving Simultaneous Equations:

If : $Ax = B$
then: $x = A^{-1}B$

where : (A is the matrix
x is the x, y, z (etc) values
B is the answers)

All you need to remember to find the values of x, y, z etc is to do: **inverse x answers**

Example: Solve $5x - 6y = -8$
and $-3x + 4y = 6$ simultaneously

We have: $\begin{bmatrix} 5 & -6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$

$(A \quad x = B)$

So: $\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} -8 \\ 6 \end{bmatrix}}$

(Use the method shown above to find the inverse then multiply by the answers.)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -8 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

3 X 3 Matrices

Most operations are done exactly the same for a 3x3 matrix as for a 2x2. The formula for finding the inverse is the same but finding the determinant and the adjoint are a bit trickier.

The Minor:

The **minor** of each element is the determinant of everything that is **not** in the same row or column as the element. It is used when finding the **determinant** and the **adjoint**.

e.g. Find the minor of each element in the top row of the matrix: $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix}$

To find the minor of element 'a' cover up everything in the same row or column and the minor is the determinant of what is left:

$$\begin{matrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{matrix}$$

the minor of a is: $\begin{vmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{h} & \boldsymbol{i} \end{vmatrix} = \boldsymbol{ei - fh}$

To find the minor of element 'b' cover up everything in the same row or column and the minor is the determinant of what is left:

$$\begin{matrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{matrix}$$

the minor of b is: $\begin{vmatrix} \boldsymbol{d} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{i} \end{vmatrix} = \boldsymbol{di - fg}$

To find the minor of element 'c' cover up everything in the same row or column and the minor is the determinant of what is left:

$$\begin{matrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{matrix}$$

the minor of c is: $\begin{vmatrix} \boldsymbol{d} & \boldsymbol{e} \\ \boldsymbol{g} & \boldsymbol{h} \end{vmatrix} = \boldsymbol{dh - eg}$

The signs:

When finding the **determinant** or the **adjoint** we use alternate signs as follows:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Finding the Inverse

Remember:

$$\boxed{A^{-1} = \frac{1}{|A|} \times \text{adj}}$$

(Where: $|A|$ = determinant
adj = adjoint)

Determinant:

For the top row only, multiply each number by its minor and apply the alternate signs:

The minor of each element in the top row

If: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$|A| = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Adjoint:

The adjoint is the transpose of the cofactors (c^T)

To find the cofactors you find the '**minor'** of each element (and use the alternate signs as shown above).

If: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

The minors of each element

Matrix of cofactors =

$$\begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Nb. To find the adjoint we then need to <u>transpose</u> this matrix.

e.g. Find the inverse of the following matrix: **A =** $\begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{5} \\ \mathbf{3} & \mathbf{3} & \mathbf{2} \\ \mathbf{4} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Determinant: $|A| \quad = +2\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 3 \\ 4 & 1 \end{vmatrix}$

$= 2(1) - 1(-5) + 5(-9) \qquad = -38$

Cofactors = $\begin{bmatrix} +\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 3 \\ 4 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 5 & -9 \\ 4 & -18 & 2 \\ -13 & 11 & 3 \end{bmatrix}$

We have transposed the matrix of cofactors to get the adjoint.

Adjoint $(\boldsymbol{C^T})$: = $\begin{bmatrix} 1 & 4 & -13 \\ 5 & -18 & 11 \\ -9 & 2 & 3 \end{bmatrix}$

Inverse: $A^{-1} = \frac{1}{|A|}$ x adj $\qquad = -\frac{\mathbf{1}}{\mathbf{38}}\begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{-13} \\ \mathbf{5} & \mathbf{-18} & \mathbf{11} \\ \mathbf{-9} & \mathbf{2} & \mathbf{3} \end{bmatrix}$

Solving Simultaneous Equations Using Cramer's Rule:

Cramers method is a way to solve simultaneous equations using matrices.

e.g. Solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ to find $x \ and \ y$

Replace 1st column with answer column

To find x:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \frac{\text{Determinant of matrix with } \textbf{1st} \text{ column replaced with the answer column.}}{\text{Determinant of original matrix}}$$

To find y:

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \frac{\text{Determinant of matrix with } \textbf{2nd} \text{ column replaced with the answer column.}}{\text{Determinant of original matrix}}$$

Notice that the bottom of the fraction is always the determinant of the original matrix.

e.g. Solve the simultaneous equations: $\quad \mathbf{3x - 4y = 10}$
$\mathbf{x + 7y = -5}$

Think of these as the answers.

Set up a matrix: $\begin{bmatrix} 3 & -4 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$

Determinant of matrix: $\begin{vmatrix} 3 & -4 \\ 1 & 7 \end{vmatrix} = \mathbf{25}$

For help with finding the determinant, see Determinant Summary sheet.

To find x:

answers

$$\frac{\begin{vmatrix} 10 & -4 \\ -5 & 7 \end{vmatrix}}{25} = \frac{50}{25} = \underline{\underline{\mathbf{2}}}$$

To find y:

answers

$$\frac{\begin{vmatrix} 3 & 10 \\ 1 & -5 \end{vmatrix}}{25} = \frac{-25}{25} = \underline{\underline{\mathbf{-1}}}$$

e.g. Solve the simultaneous equations:

$$\begin{aligned} \mathbf{2x + y - 3z} &= \mathbf{7} \\ \mathbf{x + 3y + z} &= \mathbf{6} \\ \mathbf{x - 4y - 5z} &= \mathbf{-2} \end{aligned}$$

Think of these as the answers.

Set up the matrix: $\begin{bmatrix} 2 & 1 & -3 \\ 1 & 3 & 1 \\ 1 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -2 \end{bmatrix}$

Determinant of matrix: $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 3 & 1 \\ 1 & -4 & -5 \end{vmatrix} = \underline{\underline{\mathbf{5}}}$

To find x

$$\begin{array}{c} \begin{array}{ccc} \text{answers} & & \\ \downarrow & & \end{array} \\ \dfrac{\begin{vmatrix} 7 & 1 & -3 \\ 6 & 3 & 1 \\ -2 & -4 & -5 \end{vmatrix}}{5} = \dfrac{5}{5} = \underline{\underline{\mathbf{1}}} \end{array}$$

To find y

$$\begin{array}{c} \begin{array}{ccc} & \text{answers} & \\ & \downarrow & \end{array} \\ \dfrac{\begin{vmatrix} 2 & 7 & -3 \\ 1 & 6 & 1 \\ 1 & -2 & -5 \end{vmatrix}}{5} = \dfrac{10}{5} = \underline{\underline{\mathbf{2}}} \end{array}$$

To find z

$$\begin{array}{c} \begin{array}{ccc} & & \text{answers} \\ & & \downarrow \end{array} \\ \dfrac{\begin{vmatrix} 2 & 1 & 7 \\ 1 & 3 & 6 \\ 1 & -4 & -2 \end{vmatrix}}{5} = \dfrac{-5}{5} = \underline{\underline{\mathbf{-1}}} \end{array}$$