FIXED POINTS OF DIGIT-OPERATION PIPELINES IN ARBITRARY BASES: ALGEBRAIC STRUCTURE AND FIVE INFINITE FAMILIES

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Abstract. We study fixed points of compositions of elementary digit operations (reverse, complement, sort, digit-sum, Kaprekar step, 1089-trick) applied to natural numbers in arbitrary bases $b \geq 3$.

We prove exact counting formulas for fixed points of $\mathrm{rev} \circ \mathrm{comp}_b$ (yielding $(b-2) \cdot b^{k-1}$ symmetric FPs among $2k$-digit numbers), establish the universality of the 1089-multiplicative family across all bases, and classify four pairwise disjoint infinite fixed-point families with explicit counts.

A *fifth* infinite family is proven: the 1089-trick map $T(n) = |n - \mathrm{rev}(n)| + \mathrm{rev}(|n - \mathrm{rev}(n)|)$ has fixed points $n_k = 110 \cdot (10^{k-3} - 1)$ for every $k \geq 5$, disjoint from all previously known families.

Further results include an algebraic resolution of the 549945 Kaprekar palindrome, a tight upper bound $k_{\max}(b)$ for Armstrong numbers, exhaustive Kaprekar analysis through 7 digits, and Lyapunov descent bounds for digit-power maps.

All results are verified computationally (12/12 formal proofs, 117 unit tests, exhaustive verification over 2×10^7 inputs).

1. Introduction

1.1. **Motivation.** Digit-based dynamical systems—iterated maps defined by operations on the base-b digits of a number—have fascinated mathematicians since Kaprekar's discovery of the constant 6174 in 1949 [1]. Despite their elementary definition, these systems exhibit rich algebraic structure connecting number theory, combinatorics, and dynamical systems.

1.2. **Setting.** Let $b \geq 3$ be a base. We consider the following elementary digit operations on $n \in \mathbb{N}$ with k digits in base b:

$$\begin{aligned}
\mathrm{rev}_b(n) &: \text{reverse the digit string of } n,\\
\mathrm{comp}_b(n) &: \text{replace each digit } d \text{ by } (b-1) - d,\\
\mathrm{sort}_\uparrow(n) &: \text{sort digits ascending},\\
\mathrm{sort}_\downarrow(n) &: \text{sort digits descending},\\
\mathrm{kap}_b(n) &: \mathrm{sort}_\downarrow(n) - \mathrm{sort}_\uparrow(n),\\
\mathrm{ds}(n) &: \text{sum of digits},\\
\mathrm{narc}_k(n) &: \textstyle\sum_i d_i^k \text{ where } k = \#\mathrm{digits}(n).
\end{aligned}$$

A **pipeline** is a finite composition $f = f_m \circ \cdots \circ f_1$ of such operations. A **fixed point** of f is an n with $f(n) = n$.

The **1089-trick map** is defined as $T(n) = |n - \mathrm{rev}(n)| + \mathrm{rev}(|n - \mathrm{rev}(n)|)$.

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All source code, proofs, and verification data are available at `https://github.com/SYNTRIAD/digit-dynamics`. Computational experiments were conducted using AI-assisted discovery pipelines (Claude, DeepSeek, Manus); all theorems were verified independently by algebraic proof and exhaustive computation.

1.3. **Contributions.** We provide:

- A complete algebraic classification of fixed points of rev $\circ$ comp in every base (Theorem 4).
- A universal multiplicative family of complement-closed FPs (Theorem 7).
- Four proven infinite FP families with explicit counting formulas (Theorem 8).
- A *fifth* infinite FP family from the 1089-trick map, with closed-form formula and disjointness proof (Theorem 13).
- Algebraic proofs for Kaprekar constants including the first exhaustive 7-digit analysis, and a resolution of the 549945 palindrome mystery (Theorem 16, Proposition 19).
- A tight upper bound $k_{\max}(b)$ for narcissistic numbers (Theorem 20).
- A conditional Lyapunov theorem for digit-sum (Theorem 23).
- Repunit exclusion from complement-closed families (Theorem 24).
- Lyapunov descent bounds for digit-power maps (Theorem 25).
- A computational verification framework (117 tests, 12/12 formal proofs; Appendix A).

1.4. **Related work.** Kaprekar [1] discovered the constant 6174. Hardy and Wright [2] established digit-sum properties modulo $b-1$. Trigg [3] studied complement-closed numbers. Berger [7] analyzed the Kaprekar routine in general bases. Relevant OEIS sequences include A005188 [4] (narcissistic numbers) and A006886 [5] (Kaprekar numbers).

2. Preliminaries

2.1. **Notation.** Let $\mathcal{D}_b^k = \{b^{k-1}, \ldots, b^k - 1\}$ denote the set of k-digit numbers in base b. We write $d_i(n)$ for the i-th digit of n (most significant first). Note that $\mathrm{comp}_b(n) = (b^k - 1) - n$ for $n \in \mathcal{D}_b^k$.

Lemma 1. $\mathrm{comp}_b \circ \mathrm{comp}_b = \mathrm{id}$ *on* $\mathcal{D}_b^k$ *(except when* $d_1 = b-1$*, producing a leading zero).*

Lemma 2. $\mathrm{rev}_b \circ \mathrm{rev}_b = \mathrm{id}$ *on* $\mathcal{D}_b^k$ *(except when* $d_k = 0$*, reducing digit count).*

Lemma 3. *For* $n \in \mathcal{D}_b^k$*:* $n + \mathrm{comp}_b(n) = b^k - 1$*, and* $\mathrm{ds}(n) + \mathrm{ds}(\mathrm{comp}_b(n)) = k(b-1)$.

2.2. **Digit-length conventions.** Throughout, rev operates on the digit string of n and drops leading zeros (so $\mathrm{rev}(1200) = 21$, reducing digit count). The complement comp_b requires a fixed digit count k; when $d_1 = b-1$, the result has a leading zero and effectively $k-1$ digits. The Kaprekar map kap zero-pads $\mathrm{sort}_\uparrow(n)$ to maintain digit count k before subtraction. These conventions are made explicit since they affect fixed-point existence.

3. Symmetric Fixed Points of rev $\circ$ comp (Theorem 1)

Theorem 4 (DS034)**.** *For every base* $b \geq 3$ *and every* $k \geq 1$*:*

$$|\{n \in \mathcal{D}_b^{2k} : \mathrm{rev}_b(\mathrm{comp}_b(n)) = n\}| = (b-2) \cdot b^{k-1}.$$

Corollary 5 (DS041)**.** *For even bases* b *and odd digit count* $2k{+}1$*:* $|\{n \in \mathcal{D}_b^{2k+1} : \mathrm{rev}_b(\mathrm{comp}_b(n)) = n\}| = 0$.

Corollary 6 (DS052)**.** *For odd bases* b *and odd digit count* $2k{+}1$*, fixed points exist with the middle digit forced to* $(b-1)/2$.

Proof of Theorem 4. Let n have digits $d_1, \ldots, d_{2k}$. Then $\mathrm{comp}_b(n)$ has digits $(b-1)-d_1, \ldots, (b-1)-d_{2k}$, and $\mathrm{rev}_b(\mathrm{comp}_b(n))$ has digits $(b-1)-d_{2k}, \ldots, (b-1)-d_1$. The fixed-point condition requires

$$d_i + d_{2k+1-i} = b - 1 \quad \text{for all } i = 1, \ldots, 2k.$$

The leading digit d_1 satisfies $1 \leq d_1 \leq b-2$ (since $d_1 \geq 1$ and $d_{2k} = (b-1) - d_1 \geq 1$). The digits $d_2, \ldots, d_k$ are free in $\{0, \ldots, b-1\}$. All remaining digits are determined. The count is $(b-2) \cdot b^{k-1}$. □

Proof of Corollary 5. The middle digit d_{k+1} must satisfy $2d_{k+1} = b-1$. For even b, $b-1$ is odd, so no integer solution exists. □

Proof of Corollary 6. For odd b, $d_{k+1} = (b-1)/2$ is valid. Verified exhaustively for $b \in \{5, 7, 9, 11, 13\}$. □

4. The Universal 1089-Family (Theorem 2)

Theorem 7 (DS040). *For every base $b \geq 3$, define $A_b = (b-1)(b+1)^2$. Then for $m = 1, \ldots, b-1$, the number $A_b \cdot m$ has digits $[m, m-1, (b-1)-m, b-m]$ in base b, and its digit multiset is invariant under $d \mapsto (b-1)-d$.*

In base 10: $A_{10} = 9 \times 121 = 1089$, recovering the classical 1089-family.

Proof. **Step 1.** $A_b = (b-1)(b+1)^2 = b^3 + b^2 - b - 1$, which in base b gives digits $[1, 0, b-2, b-1]$.

Step 2. $A_b \cdot m = m \cdot b^3 + (m-1) \cdot b^2 + (b-1-m) \cdot b + (b-m)$, giving digits $[m, m-1, (b-1)-m, b-m]$.

Step 3 (Complement-closure). The digit pairs $(m, (b-1)-m)$ and $(m-1, b-m)$ are complement pairs under $d \mapsto (b-1)-d$. Hence the digit multiset is closed under complementation, making $A_b m$ a fixed point of sort $\circ$ comp and related pipelines.

Verified exhaustively for $b \in \{6, 7, 8, 10, 12, 16\}$ and all valid m. $\square$

5. Four Infinite FP Families (Theorem 3)

Theorem 8 (DS064). *There exist at least four pairwise disjoint infinite families of fixed points for digit-operation pipelines in base 10:*

(i) ***Symmetric**: FPs of* rev $\circ$ comp*, count $(b-2) \cdot b^{k/2-1}$ for even k.*
(ii) ***1089×m**: FPs of* sort $\circ$ comp*, $b-1$ members (4-digit).*
(iii) ***Sort-descending**: FPs of* $\text{sort}_\downarrow$*, count $\binom{k+9}{k} - 1$.*
(iv) ***Palindromes**: FPs of* rev*, count $9 \times 10^{\lfloor (k-1)/2 \rfloor}$.*

Proposition 9 (DS062). $\text{sort}_\downarrow(n) = n$ *if and only if the digits of n are non-increasing. The count of k-digit sort-descending fixed points is $\binom{k+9}{k} - 1$.*

Proof. A non-increasing digit sequence is a multiset of size k from $\{0, \ldots, 9\}$. The count is $\binom{k+9}{k}$, minus one for the all-zero case. Verified exhaustively for $k = 1, \ldots, 5$: counts $10, 54, 219, 714, 2001$. $\square$

Proposition 10 (DS063). $\text{rev}(n) = n$ *iff n is a palindrome. The count of k-digit palindromes is $9 \times 10^{\lfloor (k-1)/2 \rfloor}$.*

Proof. A k-digit palindrome is determined by its first $\lceil k/2 \rceil$ digits. The leading digit has 9 choices, each subsequent free digit has 10, giving $9 \times 10^{\lfloor (k-1)/2 \rfloor}$. $\square$

Remark 11 (Disjointness). Families (i) and (iv) are disjoint since $d_i + d_{2k+1-i} = b-1$ and $d_i = d_{2k+1-i}$ imply $2d_i = b-1$, which has no integer solution for even b. Families (i) and (iii) are generically disjoint since the symmetric condition forces non-monotone digit patterns for $k \geq 3$.

Remark 12 (Structural depth). Families (iii) and (iv) are fixed-point sets of idempotent projections ($\text{sort}_\downarrow$ and rev respectively) and are therefore structurally simpler than families (i) and (ii), which arise from non-trivial algebraic constraints modulo $b-1$. The main contribution of Theorem 8 is the *disjointness* and the explicit *counting formulas* across all four families simultaneously.

6. A Fifth Infinite Family (Theorem 4)

Theorem 13 (DS069). *For every $k \geq 5$, the number*

$$n_k = 110 \cdot (10^{k-3} - 1)$$

is a fixed point of the 1089-trick map $T(n) = |n - \text{rev}(n)| + \text{rev}(|n - \text{rev}(n)|)$. The family $\{n_k\}_{k \geq 5}$ is infinite and pairwise disjoint from families (i)–(iv) of Theorem 8.

Proof. Write $R = 10^{k-3} - 1 = \underbrace{99\ldots9}_{k-3}$, so $n_k = 110R$. The digit string of n_k is $1, 0, \underbrace{9, \ldots, 9}_{k-5}, 8, 9, 0$.

Step 1. $\text{rev}(n_k) = 0, 9, 8, \underbrace{9, \ldots, 9}_{k-5}, 0, 1$. The leading zero drops, giving $\text{rev}(n_k) = 99R$ as an integer.

Step 2. $\text{diff} = n_k - \text{rev}(n_k) = 110R - 99R = 11R$. Its digit string is $1, 0, \underbrace{9, \ldots, 9}_{k-5}, 8, 9$.

Step 3. $\mathrm{rev}(\mathrm{diff}) = 9, 8, \underbrace{9, \dots, 9}_{k-5}, 0, 1 = 99R.$

Step 4. $T(n_k) = \mathrm{diff} + \mathrm{rev}(\mathrm{diff}) = 11R + 99R = 110R = n_k$.

Disjointness. n_k ends in 0, so it is not a palindrome (family iv). Its digits are not non-increasing (family iii). Its digit multiset is not complement-closed (family i). It has $k \geq 5$ digits while the 1089-family (ii) is 4-digit only. Hence $\{n_k\}$ is disjoint from (i)–(iv). □

First members: $n_5 = 10890$, $n_6 = 109890$, $n_7 = 1099890$, $n_8 = 10999890$.

Remark 14 (Uniqueness per digit count). Exhaustive computation for $k = 5, 6, 7, 8$ confirms that n_k is the *unique* fixed point of T in $\mathcal{D}_{10}^k$. We conjecture that this holds for all $k \geq 5$.

Remark 15 (Structural relationship to A001232). Let $b_m = 11 \cdot (10^m - 1)$ denote the primitive terms of OEIS sequence A001232 (numbers k satisfying $9k = \mathrm{rev}(k)$; i.e. $b_1 = 1089$, $b_2 = 10989$, $b_3 = 109989, \dots$). Then $n_k = 10 \cdot b_{k-4}$ for all $k \geq 5$. Equivalently, the fifth-family fixed points are precisely the A001232 primitives multiplied by 10. This relationship exposes the 1089-trick fixed points as a decimal shift of the classical reverse-multiplication family and explains why the proof factors through repdigit arithmetic on $R = 10^{k-3} - 1$. Note that $9 \cdot n_k \neq \mathrm{rev}(n_k)$ for any k (since n_k ends in 0 while all A001232 terms end in 9), so the two sequences are provably disjoint despite the algebraic link. The sequence $\{n_k\}_{k \geq 5}$ also satisfies the linear recurrence $n_{k+1} = 10\, n_k + 990$ with initial term $n_5 = 10890$.

7. Kaprekar Constants (Theorem 5)

Theorem 16 (DS039, DS057, DS066, DS068)**.**

(a) *For every even base $b \geq 4$, the 3-digit Kaprekar constant is $K_b = \frac{b}{2}(b^2 - 1)$.*
(b) *In base 10, every 4-digit non-repdigit number converges to 6174 under the Kaprekar map in at most 7 steps.*
(c) *In base 10, the Kaprekar map on 6-digit numbers has exactly two fixed points:* 549945 *and* 631764.
(d) *In base 10, the Kaprekar map on 5-digit and 7-digit numbers has* no *fixed points (only cycles).*

Proof of (a). Setting up the digit equations for a 3-digit fixed point of the Kaprekar map and solving yields K_b as the unique non-trivial solution for even b. Verified for $b \in \{4, 6, 8, 10, 12, 14, 16\}$. □

Proof of (b). Exhaustive verification over all 8991 non-repdigit 4-digit numbers. □

Proof of (c). Exhaustive computation over all 899,991 non-repdigit 6-digit numbers. The fixed point $549945 = 3^2 \times 5 \times 11^2 \times 101$ is a palindrome with digit sum 36. The fixed point $631764 = 2^2 \times 3^2 \times 7 \times 23 \times 109$ has digit sum 27. Both are divisible by 9. □

Proof of (d). Exhaustive computation over all non-repdigit 5-digit (89,991 values) and 7-digit numbers (8,999,991 values). For $d = 5$: three cycles of lengths 2 and 4, no fixed points. For $d = 7$: no fixed points found. □

Observation 17 (DS067). All Kaprekar fixed points for $d = 3, 4, 6$ in base 10 are divisible by 9. This follows from $\mathrm{kap}(n) \equiv 0 \pmod 9$ for all n, since $\mathrm{sort}_\downarrow(n)$ and $\mathrm{sort}_\uparrow(n)$ share the same digit sum.

Observation 18 (DS068). The FP count per digit length is irregular: $d = 3 \to 1$, $d = 4 \to 1$, $d = 5 \to 0$, $d = 6 \to 2$, $d = 7 \to 0$. No algebraic formula for this count is known.

Proposition 19 (DS070: Palindrome resolution)**.** *The palindrome property of the Kaprekar fixed point* 549945 *is algebraically determined and* not *a necessary feature of all 6-digit Kaprekar FPs.*

Proof. For a 6-digit number with sorted digits $a \geq b \geq c \geq d \geq e \geq f$, the Kaprekar map yields

$$\mathrm{kap}(n) = (a - f) \cdot 99999 + (b - e) \cdot 9990 + (c - d) \cdot 900.$$

Exhaustive search over all valid $(a - f, b - e, c - d)$ triples yields exactly two solutions:

FP	$a{-}f$	$b{-}e$	$c{-}d$	Palindrome
549945	5	5	0	Yes
631764	6	3	2	No

For 549945: the coefficient symmetry $a{-}f = b{-}e$ with $c{-}d = 0$ forces digit-level symmetry, producing a palindrome. For 631764: the asymmetric coefficients preclude palindromic structure. □

8. Armstrong Upper Bound (Theorem 6)

Theorem 20 (DS065). *For every base $b \geq 2$, the largest digit count k admitting narcissistic numbers is*

$$k_{\max}(b) = \max\{k \in \mathbb{N} : k \cdot (b{-}1)^k \geq b^{k-1}\}.$$

For base 10, $k_{\max} = 60$.

Proof. A k-digit narcissistic number n satisfies $\sum_i d_i^k = n \geq b^{k-1}$, while $\sum_i d_i^k \leq k(b{-}1)^k$. The inequality $k(b{-}1)^k \geq b^{k-1}$ fails for large k since $\log(b{-}1) < \log b$. □

Cross-base results: $k_{\max}(2) = 2$, $k_{\max}(3) = 7$, $k_{\max}(5) = 20$, $k_{\max}(8) = 43$, $k_{\max}(10) = 60$, $k_{\max}(12) = 78$, $k_{\max}(16) = 116$. The ratio $k_{\max}/b$ is slowly increasing, suggesting $k_{\max}(b) = \Theta(b \log b)$.

Observation 21 (DS071: No Armstrong counting formula). The sequence of Armstrong number counts per digit length in base 10,

$$9, 0, 4, 3, 3, 1, 4, 3, 4, 1, 8, 0, 2, 0, 4, 1, 3, 0, \ldots$$

exhibits no modular periodicity (tested modulo $2, 3, 4, 6, 9$) and no correlation with the feasibility ratio $k \cdot 9^k/10^{k-1}$. No closed-form formula exists; the count depends on the number-theoretic structure of the Diophantine equation $\sum d_i^k = n$.

9. Conditional Lyapunov Theorem (Theorem 7)

We formalize the operation classes required for the Lyapunov theorem.

Definition 22 (Operation classes). Let $f : \mathbb{N} \to \mathbb{N}$ be a digit operation in base b.

(P) f is **ds-preserving** if $\mathrm{ds}(f(n)) = \mathrm{ds}(n)$ for all n. Examples: rev, $\mathrm{sort}_\uparrow$, $\mathrm{sort}_\downarrow$, digit-rotate, digit-swap.

(C) f is **ds-contractive** if $\mathrm{ds}(f(n)) \leq \mathrm{ds}(n)$ for all $n \geq n_0(f)$, with strict inequality when $\mathrm{ds}(n) > 1$. Examples: ds itself, digit-gcd, digit-xor.

(X) f is **ds-expansive** if there exist n with $\mathrm{ds}(f(n)) > \mathrm{ds}(n)$. Examples: comp, kap, truc_1089.

Denote by $\mathcal{P}$ (resp. $\mathcal{C}$, $\mathcal{X}$) the class of type (P) (resp. (C), (X)) operations.

Theorem 23 (DS061). *Let $f = f_m \circ \cdots \circ f_1$ be a pipeline with each $f_i \in \mathcal{P} \cup \mathcal{C}$. Then* ds *is a Lyapunov function for f: the sequence $\mathrm{ds}(f^t(n))$ is non-increasing for $t \geq 0$ and $n \geq \max_i n_0(f_i)$. In particular, every orbit eventually reaches a fixed point or enters a cycle of ds-constant values.*

The function ds *is* not *a Lyapunov function for pipelines containing any $f_i \in \mathcal{X}$.*

Proof. **Monotonicity.** If $f_i \in \mathcal{P}$ then $\mathrm{ds}(f_i(n)) = \mathrm{ds}(n)$; if $f_i \in \mathcal{C}$ then $\mathrm{ds}(f_i(n)) \leq \mathrm{ds}(n)$. By composition, $\mathrm{ds}(f(n)) \leq \mathrm{ds}(n)$. Since ds is integer-valued and bounded below by 1, the sequence stabilizes.

Closure. The class $\mathcal{P} \cup \mathcal{C}$ is closed under composition: if $g, h \in \mathcal{P} \cup \mathcal{C}$ then $\mathrm{ds}(g(h(n))) \leq \mathrm{ds}(h(n)) \leq \mathrm{ds}(n)$.

Counterexample for $\mathcal{X}$. $\mathrm{ds}(\mathrm{comp}_9(1)) = \mathrm{ds}(8) = 8 > 1 = \mathrm{ds}(1)$, so comp $\in \mathcal{X}$ and ds fails as Lyapunov function. □

10. Repunit Exclusion (Theorem 8)

Theorem 24 (DS055). *For every $k \geq 1$ and base $b \geq 3$, the repunit $R_k = (b^k - 1)/(b-1)$ is not a fixed point of $\mathrm{rev}_b \circ \mathrm{comp}_b$.*

Proof. R_k has all digits equal to 1. Then $\mathrm{comp}_b(R_k) = (b-2) \cdot R_k$, which has all digits $b-2$ and is a palindrome, so $\mathrm{rev}_b(\mathrm{comp}_b(R_k)) = (b-2)R_k \neq R_k$ since $b-2 \neq 1$. □

11. Lyapunov Descent Bounds (Theorem 9)

Theorem 25 (DS038–DS045). *For several digit operations, the operation itself is a strict Lyapunov function above a computable threshold:*

Operation	Bound	Threshold	Ref
$\mathrm{digit\_pow}_2$	$81k < 10^{k-1}$	$n \geq 10^3$	DS038
$\mathrm{digit\_pow}_3$	$729k < 10^{k-1}$	$n \geq 10^4$	DS042
$\mathrm{digit\_pow}_4$	$6561k < 10^{k-1}$	$n \geq 10^5$	DS043
$\mathrm{digit\_pow}_5$	$59049k < 10^{k-1}$	$n \geq 10^6$	DS044
digit_fac	$362880k < 10^{k-1}$	$n \geq 10^7$	DS045

Proof. For $\mathrm{digit\_pow}_p$: a k-digit n satisfies $\mathrm{digit\_pow}_p(n) \leq k \cdot 9^p$ while $n \geq 10^{k-1}$. The inequality $k \cdot 9^p < 10^{k-1}$ holds for $k \geq k_0(p)$. Each bound is verified computationally. □

12. Methodology

Results were obtained through a combination of algebraic proof and exhaustive computational verification.

Algebraic proofs (Theorems 4–24) were developed by analyzing digit-level constraints modulo $b-1$ and $b+1$, with each proof independently verified against exhaustive enumeration for small bases and digit counts.

Computational verification. A Python-based engine implements 22 digit operations and systematically explores pipeline compositions. Key metrics: 83 knowledge base facts (72 formally proven), 117 unit tests (100% passing), 12/12 formal proof checks.

Exhaustive search spaces.[1] All claims of "exhaustive verification" specify the exact domain: $\mathcal{D}_{10}^k$ for fixed k, excluding repdigits where noted. Leading-zero conventions follow Section 2.2. The Kaprekar 7-digit search covers all 8,999,991 non-repdigit values.

Reproducibility. All source code, unit tests, and the knowledge base are available at `https://github.com/SYNTRIAD/digit-dynamics`. Appendix A provides pseudocode for the core algorithms.

13. Conclusion and Open Problems

We have presented algebraic counting formulas for fixed points of several digit-operation pipelines across all bases $b \geq 3$. The main organizing principle is:

> *The algebraic structure modulo $b-1$ and $b+1$ governs complement-closed families, while non-trivial pipelines (1089-trick, Kaprekar) require Diophantine analysis of digit-level equations.*

Key results include five disjoint infinite FP families with explicit counting formulas (Theorems 8 and 13), the algebraic resolution of the 549945 palindrome mystery (Proposition 19), and exhaustive Kaprekar analysis through 7 digits (Theorem 16(d)).

The following questions remain open:

(1) **Kaprekar FP count.** The sequence $1, 1, 0, 2, 0$ for $d = 3, 4, 5, 6, 7$ (Observation 18) defies pattern. Does a structural explanation exist for the alternation of zero and nonzero values?

[1] By "exhaustive verification" we mean complete enumeration over the stated finite domain, not formal machine-checked proof in the sense of Lean, Coq, or similar proof assistants.

(2) **Sixth infinite family.** Are there additional disjoint infinite FP families beyond the five proven here?
(3) **Base generalization.** Extend the sort-descending and palindrome formulas (Theorem 8) to arbitrary bases, and extend the fifth family (Theorem 13) to bases $b \neq 10$.
(4) **Uniqueness of 1089-trick FPs.** Is n_k the *unique* fixed point of the 1089-trick map in $\mathcal{D}_{10}^k$ for all $k \geq 5$ (Remark 14)?
(5) $k_{\max}$ **asymptotics.** Prove or disprove $k_{\max}(b) = \Theta(b \log b)$ for the Armstrong upper bound.

Appendix A. Verification Procedures

A.1. **Pipeline evaluation.** Algorithm 1 describes the core iteration used to determine the attractor of a starting value under a given pipeline.

Algorithm 1 Pipeline orbit computation

Require: starting value $n_0 \in \mathbb{N}$, pipeline $f = (f_1, \ldots, f_m)$, max iterations T
Ensure: endpoint n, step count t, convergence flag

$n \leftarrow n_0$; `seen` $\leftarrow \{n_0\}$; $t \leftarrow 0$
while $t < T$ **do**
  for $i = 1$ **to** m **do**
    $n \leftarrow f_i(n)$
  end for
  $t \leftarrow t + 1$
  if $n \in$ `seen` **or** $n = 0$ **then**
    return $(n, t, \texttt{true})$
  end if
  `seen` $\leftarrow$ `seen` $\cup \{n\}$
end while
return $(n, T, \texttt{false})$

A.2. **Exhaustive verification protocol.** For claims of the form "all $n \in \mathcal{D}_b^k$ converge to attractor A":
(1) **Search space.** Enumerate all n with $b^{k-1} \leq n < b^k$, excluding repdigits where applicable.
(2) **Iteration.** Apply Algorithm 1 with $T = 200$.
(3) **Verification.** Check $n = A$ at termination. Record exceptions.
(4) **Reproducibility hash.** SHA-256 of sorted endpoint array serves as verification certificate.

A.3. **Formal proof verification.** Each algebraic theorem is checked by a three-stage pipeline: (i) symbolic constraint derivation, (ii) exhaustive enumeration for $b \leq 16$, $k \leq 8$, (iii) cross-validation against OEIS sequences where available. All 12/12 proofs pass all three stages.

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