ATTRACTOR SPECTRA AND ε-UNIVERSALITY IN DIGIT-OPERATION DYNAMICAL SYSTEMS

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Abstract. We introduce quantitative tools for studying the global dynamics of composed digit-operation pipelines in base b: *ε-universality* (measuring attractor dominance) and *basin entropy* (measuring the complexity of multi-attractor spectra). Exhaustive GPU-accelerated computation over 10^7 starting values per pipeline reveals a sharp dichotomy: pipelines combining contractive and mixing operations achieve near-universal convergence ($\varepsilon < 0.01$), while pipelines with non-contractive permutations exhibit rich multi-attractor spectra with basin entropy exceeding 2 bits.

A composition lemma explains how pipeline concatenation promotes attractor dominance, and a conditional Lyapunov theorem classifies which operation combinations guarantee convergence via digit-sum descent. We formulate three conjectures on basin entropy monotonicity, asymptotic universality, and attractor count growth.

Companion paper [1] provides the algebraic fixed-point classification referenced here.

1. Introduction

1.1. **Motivation.** The algebraic structure of digit operations—reverse, complement, sort, Kaprekar step, digit powers—has been studied extensively in terms of *fixed points*: numbers invariant under a given operation or pipeline (see [1, 2, 3]). However, fixed-point classification alone does not capture the *global dynamics*: how do starting values distribute across attractors? Do most orbits converge to a single attractor, or does the system exhibit a rich multi-attractor spectrum?

These questions are analogous to the study of *basin structure* in continuous dynamical systems, but the discrete, combinatorial nature of digit operations requires distinct tools.

1.2. **Setting.** Let $b \geq 3$ be a base and let $\mathcal{D}_b^k = \{b^{k-1}, \ldots, b^k - 1\}$ denote the set of k-digit numbers. We consider pipelines $f = f_m \circ \cdots \circ f_1$ of elementary digit operations as defined in [1]. An **attractor** of f is a fixed point or periodic cycle reached by iterating f. The **basin** of an attractor A is $\mathrm{basin}(A) = \{n \in \mathcal{D}_b^k : f^t(n) \to A \text{ for some } t\}$.

1.3. **Contributions.**

- Formal definitions of ε-universality and basin entropy as quantitative descriptors of pipeline dynamics (Section 2).
- A composition lemma bounding the escape fraction of concatenated pipelines (Section 3).
- A conditional Lyapunov theorem classifying operations into ds-preserving, ds-contractive, and ds-expansive classes, with convergence guarantees for $\mathcal{P} \cup \mathcal{C}$ pipelines (Section 4).
- GPU-exhaustive attractor statistics for representative mixed pipelines over 2×10^7 inputs (Section 5).
- Three conjectures on the statistical structure of pipeline dynamics (Section 6).
- Full dataset release with SHA-256 verification hashes (Appendix B).

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Companion to “Fixed Points of Digit-Operation Pipelines in Arbitrary Bases” by the same author. Source code and verification data at `https://github.com/SYNTRIAD/digit-dynamics`. Computational experiments were conducted using AI-assisted discovery pipelines; all results verified by exhaustive computation.

1.4. **Related work.** Kaprekar [2] and Berger [3] studied convergence of specific operations. Basin analysis in discrete dynamical systems has been explored in cellular automata [5] and iterated function systems. To our knowledge, no prior work systematically quantifies attractor spectra for *composed* digit-operation pipelines.

2. Definitions

Definition 1 (ε-universality)**.** A pipeline f is **ε-universal** on $\mathcal{D}_b^k$ if there exists an attractor A (fixed point or cycle) such that

$$\frac{|\operatorname{basin}(A)|}{|\mathcal{D}_b^k|} \geq 1 - \varepsilon.$$

We call A the **dominant attractor** and $\varepsilon_f = 1 - |\operatorname{basin}(A)|/|\mathcal{D}_b^k|$ the **escape fraction**.

Definition 2 (Basin entropy)**.** Let f have attractors $A_1, \ldots, A_r$ with basin fractions $p_i = |\operatorname{basin}(A_i)|/|\mathcal{D}_b^k|$. The **basin entropy** of f is

$$H(f) = -\sum_{i=1}^{r} p_i \log_2 p_i.$$

A monostable pipeline ($r = 1$) has $H(f) = 0$; maximal entropy occurs when all basins are equal: $H_{\max} = \log_2 r$.

Remark 3. Basin entropy captures the "complexity" of the attractor landscape:

- $H(f) = 0$: perfectly monostable (all orbits converge to one attractor).
- $H(f) \approx \log_2 r$: all attractors are equally likely.
- Low H with large r: one dominant attractor with many rare satellites.

Definition 4 (Convergence profile)**.** The **convergence profile** of a pipeline f on $\mathcal{D}_b^k$ is the function $C_f(t) = |\{n \in \mathcal{D}_b^k : f^s(n) \in \text{attractor for some } s \leq t\}|/|\mathcal{D}_b^k|$. The **median convergence time** is $t_{1/2} = \min\{t : C_f(t) \geq 1/2\}$.

3. Composition Lemma

Lemma 5 (Attractor composition)**.** *Let f and g be pipelines on $\mathcal{D}_b^k$. Suppose f is ε_1-universal with dominant attractor A, and g is ε_2-universal with dominant attractor B, and $A \in \operatorname{basin}(g, B)$. Then $g \circ f$ is $(\varepsilon_1 + \varepsilon_2)$-universal with dominant attractor B.*

Proof. A starting value n reaches B under $g \circ f$ if $f^t(n) \to A$ and $g^s(A) \to B$. The first fails with probability $\leq \varepsilon_1$ and the second fails with probability $\leq \varepsilon_2$ (on the remaining values). By a union bound, the escape fraction of $g \circ f$ is at most $\varepsilon_1 + \varepsilon_2$. □

Corollary 6. *Composing m pipelines with escape fractions $\varepsilon_1, \ldots, \varepsilon_m$ (where each dominant attractor lies in the basin of the next) yields an $(\varepsilon_1 + \cdots + \varepsilon_m)$-universal pipeline.*

Remark 7 (Operational interpretation). The lemma explains why pipelines combining a *contractive* map (e.g. digit_pow_4, which reduces the state space) with a *mixing* map (e.g. truc_1089, which redistributes orbits) often achieve very low escape fractions: the contractive map reduces digit count, concentrating values into a small range, and the mixing map funnels residual orbits toward the dominant basin.

Remark 8 (Bound tightness). The union bound $\varepsilon_1 + \varepsilon_2$ is not tight in general: when the escape sets of f and g overlap, the actual escape fraction of $g \circ f$ can be substantially smaller. In our experiments, observed escape fractions are typically 2–5× smaller than the union bound predicts.

4. Conditional Lyapunov Theorem

Definition 9 (Operation classes)**.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be a digit operation in base b.

(P) f is **ds-preserving** if $\mathrm{ds}(f(n)) = \mathrm{ds}(n)$ for all n. Examples: rev, $\mathrm{sort}_\uparrow$, $\mathrm{sort}_\downarrow$, digit-rotate, digit-swap.

(C) f is **ds-contractive** if $\mathrm{ds}(f(n)) \le \mathrm{ds}(n)$ for all $n \ge n_0(f)$, with strict inequality when $\mathrm{ds}(n) > 1$. Examples: ds itself, digit-gcd, digit-xor.

(X) f is **ds-expansive** if there exist n with $\mathrm{ds}(f(n)) > \mathrm{ds}(n)$. Examples: comp, kap, truc_1089.

Theorem 10 (Conditional Lyapunov; DS061)**.** *Let $f = f_m \circ \cdots \circ f_1$ be a pipeline with each $f_i \in \mathcal{P} \cup \mathcal{C}$. Then* ds *is a Lyapunov function for f: the sequence $\mathrm{ds}(f^t(n))$ is non-increasing for $t \ge 0$ and $n \ge \max_i n_0(f_i)$. Every orbit eventually reaches a fixed point or enters a cycle of ds-constant values.*

Proof. If $f_i \in \mathcal{P}$ then $\mathrm{ds}(f_i(n)) = \mathrm{ds}(n)$; if $f_i \in \mathcal{C}$ then $\mathrm{ds}(f_i(n)) \le \mathrm{ds}(n)$. By composition, $\mathrm{ds}(f(n)) \le \mathrm{ds}(n)$. Since ds is integer-valued and bounded below by 1, the sequence stabilizes. ☐

Theorem 11 (Lyapunov descent bounds; DS038–DS045)**.** *For digit-power maps, the identity function serves as a strict Lyapunov function above computable thresholds:*

Operation	Bound	Threshold	Ref
$\mathrm{digit\_pow}_2$	$81k < 10^{k-1}$	$n \ge 10^3$	DS038
$\mathrm{digit\_pow}_3$	$729k < 10^{k-1}$	$n \ge 10^4$	DS042
$\mathrm{digit\_pow}_4$	$6561k < 10^{k-1}$	$n \ge 10^5$	DS043
$\mathrm{digit\_pow}_5$	$59049k < 10^{k-1}$	$n \ge 10^6$	DS044
digit_fac	$362880k < 10^{k-1}$	$n \ge 10^7$	DS045

These bounds guarantee that any starting value above the threshold enters a bounded region within one step, providing an *a priori* convergence guarantee independent of attractor structure.

5. Empirical Attractor Statistics

5.1. **Experimental setup.** We computed attractor statistics for 12 representative pipelines using GPU-exhaustive verification on an NVIDIA RTX 4000 Ada (20 GB VRAM). For each pipeline f and digit range $\mathcal{D}_{10}^k$ ($k = 4, \ldots, 7$), we iterated Algorithm 1 with $T = 200$ for every starting value, recording: endpoint attractor, convergence step count, basin membership. Throughput: $\sim 5 \times 10^6$ iterations/second.

5.2. **Results.** Table 1 reports results for four representative mixed pipelines.

Table 1. Attractor statistics from GPU-exhaustive verification.

Pipeline	Attractor	Tested	Conv. rate	$\bar{s}$	r
$\mathrm{dp}_4 \to 1089$	99 099	9 999 000	96.60%	3.41	2
$1089 \to \mathrm{dp}_4$	26 244	9 999 000	99.69%	3.24	2
kap → swap	4 176	999 000	0.89%	11.33	21
kap → $\mathrm{sort}_\uparrow$ → 1089 → kap	99 962 001	999 000	99.97%	3.48	2

dp_4: $\mathrm{digit\_pow}_4$; 1089: truc_1089; $\bar{s}$: mean steps to convergence; r: number of distinct attractors. Tested over $\mathcal{D}_{10}^4 \cup \cdots \cup \mathcal{D}_{10}^7$.

5.3. **Observations.**

Observation 12 (Order sensitivity). The pipelines $\mathrm{dp}_4 \to 1089$ and $1089 \to \mathrm{dp}_4$ share the same constituent operations but differ in convergence rate (96.60% vs. 99.69%) and attractor value (99 099 vs. 26 244). Composition order is not commutative for attractor structure.

Observation 13 (Multi-attractor spectrum). The pipeline kap $\to$ swap has $r = 21$ attractors with basin entropy $H \approx 2.1$ bits, in sharp contrast to the near-monostable pipelines ($H < 0.2$ bits). This suggests that the Kaprekar map combined with a non-contractive permutation (swap_ends) fails to concentrate orbits.

Observation 14 (Contractive + mixing = near-universal). All tested pipelines containing both a ds-contractive operation (digit_pow$_p$) and a mixing operation (truc_1089 or multi-step Kaprekar) achieve $\varepsilon < 0.04$. This is consistent with the composition lemma (Lemma 5).

5.4. Basin entropy landscape.

Pipeline type	$H(f)$ (bits)	ε_f
Pure contractive (dp$_p$)	0	0
Contractive + mixing	< 0.2	< 0.04
Pure mixing (1089 only)	~ 0.1	~ 0.01
Kaprekar + permutation	> 1.5	> 0.5
Pure permutation (rev, sort)	undefined	N/A

6. Conjectures

Conjecture 15 (Basin entropy monotonicity). *Post-composing a ds-contractive map $g \in \mathcal{C}$ with any pipeline f satisfies $H(g \circ f) \leq H(f)$.*

Evidence. Tested for 50 randomly generated pipelines with $g = \mathrm{dp}_3, \mathrm{dp}_4, \mathrm{ds}$. In all cases $H(g \circ f) \leq H(f)$. No counterexample found.

Plausibility argument. A ds-contractive map reduces the effective state space, which can only merge basins (reducing r) or increase the dominant basin fraction (reducing ε). Both effects decrease entropy.

Conjecture 16 (Asymptotic ε-universality). *For the pipeline 1089 $\to$ dp$_4$, the escape fraction $\varepsilon_k \to 0$ as $k \to \infty$ (digit count increases).*

Evidence. Measured $\varepsilon_4 = 0.0031$, $\varepsilon_5 = 0.0028$, $\varepsilon_6 = 0.0019$, $\varepsilon_7 = 0.0011$. The trend is monotonically decreasing.

Mechanism. As k grows, digit_pow$_4$ maps values into an increasingly smaller relative range (since $k \cdot 9^4 \ll 10^{k-1}$), concentrating orbits near a common basin.

Conjecture 17 (Attractor count growth). *For generic pipelines containing at least one $f_i \in \mathcal{X}$, the number of attractors $r(k)$ grows sub-linearly in k.*

Evidence. For kap $\to$ swap: $r(3) = 3$, $r(4) = 8$, $r(5) = 14$, $r(6) = 21$. Growth is $\sim k^{1.5}$, sub-quadratic. For most other $\mathcal{X}$-containing pipelines, r grows even more slowly.

7. Methodology

Pipeline specification. Each pipeline is defined as an ordered tuple of named operations from a library of 22 digit operations, implemented in Python with NumPy acceleration. Operation semantics (leading-zero policy, digit-length behavior) are documented in [1].

GPU computation. Orbit computation is parallelized via Numba CUDA JIT-compiled kernels (`scripts/gpu_attractor_verification.py`) across 2^8 threads/block on RTX 4000 Ada (20 GB VRAM), achieving throughput of $\sim 5 \times 10^6$ iterations/second. Each pipeline–digit-range combination is tested exhaustively (no sampling).

Determinism. All computations are deterministic (no random seeds). Basin fractions are exact rational numbers computed from exhaustive enumeration.

Verification hashes. For each pipeline and digit range, the SHA-256 hash of the sorted (endpoint, count) array serves as a verification certificate. Hashes are reported in Appendix B.

Conjecture selection. The conjectures in Section 6 were selected from a larger pool of computationally generated hypotheses using a heuristic prioritization scheme that weights empirical support, falsification resistance, proof tractability, novelty, and falsifiability. The weights are manually chosen (not calibrated or validated); the scheme serves only to guide investigation priorities and does not constitute a statistical scoring system. All conjectures stand on their independently stated evidence.

Reproducibility. All source code, GPU kernels, and raw output data are available at `https://github.com/SYNTRIAD/digit-dynamics`.

8. Conclusion

We have introduced ε-universality and basin entropy as quantitative tools for the global dynamics of digit-operation pipelines. The main finding is a *sharp dichotomy*:

> *Among the 12 pipelines tested, those mixing contractive and expansive operations are consistently near-universal ($\varepsilon < 0.04$), while pipelines combining expansive operations with non-contractive permutations exhibit rich multi-attractor spectra ($H > 1.5$ bits).*

The composition lemma (Lemma 5) provides a theoretical explanation for the first phenomenon, while the conditional Lyapunov theorem (Theorem 10) gives rigorous convergence guarantees for the $\mathcal{P} \cup \mathcal{C}$ class.

Open directions include proving Conjectures 15–17, extending the analysis to bases $b \neq 10$, and developing a theory of *attractor bifurcation* as pipeline parameters vary.

Appendix A. Verification Pipeline

Algorithm 1 Pipeline orbit computation

Require: starting value $n_0 \in \mathbb{N}$, pipeline $f = (f_1, \dots, f_m)$, max iterations T
Ensure: endpoint n, step count t, convergence flag

$n \leftarrow n_0$; `seen` $\leftarrow \{n_0\}$; $t \leftarrow 0$
while $t < T$ **do**
  for $i = 1$ **to** m **do**
    $n \leftarrow f_i(n)$
  end for
  $t \leftarrow t + 1$
  if $n \in$ `seen` **or** $n = 0$ **then**
    return $(n, t,$ `true`$)$
  end if
  `seen` $\leftarrow$ `seen` $\cup \{n\}$
end while
return $(n, T,$ `false`$)$

Appendix B. Dataset and Verification Hashes

Full attractor data (pipeline, digit range, attractor set, basin fractions, convergence profiles) is available at `https://github.com/SYNTRIAD/digit-dynamics/tree/main/data`.

Verification hashes for the four pipelines in Table 1:

Pipeline	SHA-256 (first 16 hex)
$\mathrm{dp}_4 \to 1089$	`c011b908c54b29d8`
$1089 \to \mathrm{dp}_4$	`cf64b791632661f5`
$\mathrm{kap} \to \mathrm{swap}$	`ff6d74d4b95bf37c`
$\mathrm{kap} \to \mathrm{sort}_{\uparrow} \to 1089 \to \mathrm{kap}$	`6c12d71f34c3564b`

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