Stochastic Process

Poisson Processes (1)

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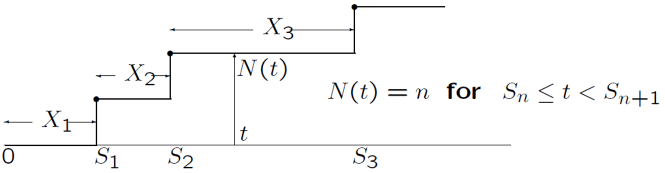
Chapter 2. Poisson process

2.1.1 Arrival Processes

Definition

An arrival process is a sequence of increasing rv.s $0 < S_1 < S_2 < \dots$ *where* $S_i < S_{i+1}$ *means that* $X_1 = S_1$ *and* $X_i = S_i - S_{i-1}$ *for* $i > 1$ *are positive rv.s,* $F_{X_i}(0) = 0$*. The differences* X_i *are call interarrival times and the* S_i *are called arrival epochs.*

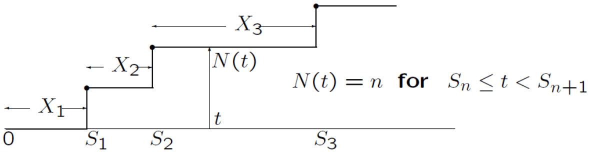


For each $t > 0$, $N(t)$ is the number of arrivals in $(0, t]$. We call $\{N(t): \ t > 0\}$ an arrival counting process.

2.1.1 Arrival Processes

Remark.

The process starts at time 0 and that multiple arrivals cannot occur simultaneously, or we will permit simultaneous arrivals or arrivals at time 0 as events of zero probability, but these can be ignored.



The figure shows how the arrival epochs, interarrival times, and counting variables are interrelated for a generic stair case function.

2.1.1 Arrival Processes

Remark.

- $X_i = S_i - S_{i-1}$ for $i \geqslant 2$ and $X_1 = S_1$; $S_n = \sum_{i=1}^{n} X_i$.
- $\{S_n \leqslant t\} = \{N(t) \geqslant n\}$ for all $n \geqslant 1,\ t > 0$.

If $S_n = \tau$ for some $\tau \leqslant t$, then $N(\tau) = n$ and $N(t) \geqslant n$.

An arrival process can be specified by the joint distributions of the arrival epochs, or of the interarrival times, or of the counting random variables.

§ 2.2 Definition and properties of a Poission process

Definition

A renewal process is an arrival process for which the sequence of interarrival times $X_1, X_2, \ldots$ are i.i.d..

Definition

A Poisson process is a renewal process for which each X_i has an exponential distribution,

$$\Pr\{X > x\} = F_X^c(x) = \exp(-\lambda x) \quad \text{for } x \geqslant 0,$$

where λ is a fixed parameter called the rate, or each X_i has the PDF:

$$f_X(x) = \lambda \exp(-\lambda x) \quad \text{for } x \geqslant 0.$$

2.2.1 Memoryless Property

The remarkable simplicity of Poisson processes is closely related to the memoryless property:

Definition

A rv. X is memoryless if X is positive and, for all real $t > 0$ and $x > 0$,

$$\Pr\{X > t + x\} = \Pr\{X > t\}\Pr\{X > x\}.$$

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Since the interarrival interval for a Poisson process is exponential, i.e., $\Pr\{X > x\} = \exp(-\lambda x)$ for $x \geqslant 0$,

$$\exp(-\lambda(x + t)) = \exp(-\lambda t) \exp(-\lambda x).$$

An arbitrary rv. X is memoryless *iff* it is exponential.

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The reason for the word "memoryless" is more apparent when using conditional probabilities,

$$\Pr\{X > t + x | X > t\} = \Pr\{X > x\}.$$

If people in a checkout line have exponential service, and you have waited 15 minute for the person in front, what is his or her remaining service time?

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If people in a checkout line have exponential service, and you have waited 15 minute for the person in front, what is his or her remaining service time?

Same as when service started. The remaining waiting time has no 'memory' of previous waiting.

Has your time waiting been wasted?

Why do you move to another line if someone takes a long time?

2.2.1 Memoryless Property

Theorem 1

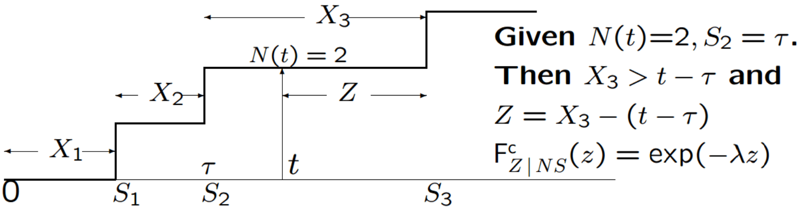
For a Poisson process of rate λ, and any given $t > 0$, the length of the interval from t until the first arrival after t is a positive r.v. Z with the CDF

$$F_Z(z) = 1 - \exp(-\lambda z) \quad \text{for } z \geqslant 0.$$

This r.v. is independent of both $N(t)$ and the $N(t)$ arrival epochs before time t. It is also independent of the set of r.v.s $\{N(\tau) : \tau \leqslant t\}$.

2.2.1 Memoryless Property

Idea of proof: Conditional on $N(t) = n$ and $S_n = \tau$, i.e., the number n of arrivals in $(0, t]$ and the time, τ of the most recent arrival in $(0, t]$.



2.2.1 Memoryless Property

Remark.

- This theorem essentially extends the idea of 'memorylessness' to the entire Poisson process.

  That is, starting at any $t > 0$, the interval Z to the next arrival is also an exponential rv. of rate λ. Z is independent of everything before t.
- Let Z_m be the time from the $(m-1)$th arrival after t to the mth arrival epoch after t. We can see that $Z_1, Z_2, \ldots$ are unconditionally i.i.d and also independent of $\{N(\tau) : \tau \leqslant t\}$ and $S_1, S_2, \ldots, S_{N(t)}$.
- Thus the interarrival process starting at t with first interarrival Z, and continuing with subsequent interarrivals is also a Poisson process.

Stationary increment property

Definition

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Let

$$\widetilde{N}(t, t') = N(t') - N(t).$$

Stationary increments means that $\widetilde{N}(t, t')$ has the same distribution as $N(t' - t)$.

Thus, the distribution of the number of arrivals in an interval depends on the size of the interval but NOT the starting point.

Independent increment property

Definition

A counting process $\{N(t) : t > 0\}$ has the independent increment property if, for every integer $k > 0$ and every k-tuple of times $0 < t_1 < \cdots < t_k$, the k-tuples of rv.s $N(t_1), \widetilde{N}(t_1, t_2), \ldots, \widetilde{N}(t_{k-1}, t_k)$ are statistically independent.

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This implies that the number of arrivals in each of a set of non-overlapping intervals are independent r.v.s.

Theorem 2

Poisson processes have both the stationary increment and independent increment properties.

2.2.2 Probability Density of S_n and joint density of $S_1, \dots, S_n$

Definition

For a Poisson process, $S_n = \sum_{i=1}^{n} X_i$, where X_i are i.i.d. rv.s with the PDF

$$f_X(x) = \lambda \exp(-\lambda x),$$

then the S_n has the *Erlang density* ($S_n \sim \Gamma(n, \lambda)$):

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!}.$$

Idea of proof: $S_1 = X_1$ and $S_n = S_{n-1} + X_n$, where S_{n-1} and X_n are independent. Use induction,

$$f_{S_n}(t) = \int_0^t f_{S_{n-1}}(x) \cdot f_{X_n}(t-x)\mathrm{d}x$$

2.2.2 Probability Density of S_n and joint density of $S_1, \ldots, S_n$

The joint density of $X_1, X_2, \ldots, X_n$ is

$$\begin{aligned} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) &= \lambda^n \exp(-\lambda x_1 - \lambda x_2 - \cdots - \lambda x_n) \\ &= \lambda^n \exp(-\lambda s_n) \quad \text{where } s_n = \sum_{i=1}^{n} x_i. \end{aligned}$$

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Theorem 3

Let $S_1, S_2, \ldots, S_n, \ldots$ be the arrival epoches of a Poisson process with the rate λ. Then the joint density of $S_1, S_2, \ldots, S_n$ is

$$f_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = \lambda^n \exp(-\lambda s_n) \quad \text{for } 0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n.$$

Given that the n-th arrival is at s_n, the other $n-1$ arrivals are uniformly distributed in $(0, s_n)$, subject to the ordering. Integrating, we get the Erlang marginal density.

2.2.3 The probability mass function for $N(t)$

Theorem 4

For a Poisson process of rate λ, and for any $t > 0$, the PMF for $N(t)$, i.e., the number of arrivals in $(0, t]$, is given by the Poisson PMF,

$$P_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}.$$

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proof 1: For some vanishingly small δ,

$$\Pr\{t < S_{n+1} < t+\delta\} = \int_t^{t+\delta} f_{S_{n+1}}(\tau) \mathrm{D}\tau = f_{S_{n+1}}(t)(\delta + o(\delta)).$$

The second way to calculate $\Pr\{t < S_{n+1} < t+\delta\}$ is to first observe that more than one arrival in $(t, t+\delta]$ is $o(\delta)$.

$$\begin{aligned}\Pr\{t < S_{n+1<t+\delta}\} &= \Pr\{t < S_{n+1<t+\delta} | N(t) = n\} P_{N(t)}(n) + o(\delta)\\ &= P_{N(t)}(n)(\lambda\delta + o(\delta)) + o(\delta).\end{aligned}$$

2.2.3 The probability mass function for $N(t)$

Thus, $P_{N(t)}(n)(\lambda\delta + o(\delta)) + o(\delta) = f_{S_{n+1}}(t)(\delta + o(\delta))$. Dividing by δ and taking the limit $\delta \to 0$, we get

$$\lambda P_{N(t)}(n) = f_{S_{n+1}}(t).$$

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proof 2: use $\{N(t) \geqslant n\} = \{S_n \leqslant t\}$, then

$$\sum_{i=n}^{\infty} P_{N(t)}(i) = \int_0^t f_{S_n}(\tau)\mathrm{d}\tau.$$

$$\begin{aligned}
P_{N(t)}(n) &= \int_0^t f_{S_n}(\tau) - f_{S_{n+1}}(\tau)\mathrm{d}\tau \\
&= \int_0^t \left[\frac{\lambda^n \tau^{n-1}\exp(-\lambda\tau)}{(n-1)!} - \frac{\lambda^{n+1}\tau^n \exp(-\lambda\tau)}{n!}\right]\mathrm{d}\tau \\
&= \frac{\lambda^n}{n!}\int_0^t \exp(-\lambda\tau)\mathrm{d}(\tau^n) + \tau^n \mathrm{d}(\exp(-\lambda\tau)) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}.
\end{aligned}$$

2.2.4 Alternate definition of Poisson processes

Theorem 5

If an arrival process has the stationary and independent increment properties and if $N(t)$ has the Poisson PMF for given λ and all $t > 0$, then the process is Poisson.

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Theorem 6

If an arrival process has the stationary and independent increment properties and satisfies

$$\Pr\{\widetilde{N}(t, t+\delta) = n\} = \begin{cases} 1 - \lambda\delta + o(\delta) & \text{for} \quad n = 0, \\ \lambda\delta + o(\delta) & \text{for} \quad n = 1, \\ o(\delta) & \text{for} \quad n \geqslant 2. \end{cases}$$

then it is Poisson.

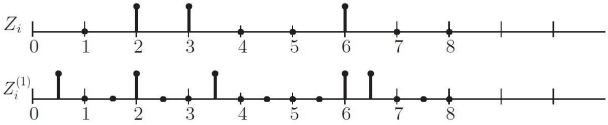
2.2.5 The Poisson process as a limit of shrinking Bernoulli processes

We can view a Bernoulli process as an arrival process, an arrival occurs at discrete time n if and only if $Z_n = 1$. Thus $S_n = Z_1 + \cdots + Z_n$ is the number of arrivals up to and including time n.

$$P_{N(t)}(k) = \binom{\lfloor t \rfloor}{k} p^k (1-p)^{\lfloor t \rfloor - k}, \text{ for } k < \lfloor t \rfloor$$

.

Now we "shrink" the time scale of the process so that for some integer $j > 0$, $Z_i^{(j)}$ is an arrival or no arrival at time $i2^{-j}$.



In order to keep the arrival rate constant, we let $p = \lambda 2^{-j}$ for the jth process.

The expected number of arrivals per unit time is then λ.

Theorem 7 (Poisson's theorem)

Consider the sequence of shrinking Bernoulli processes with arrival probability $\lambda 2^{-j}$ and time-slot size 2^{-j}. Then for every fixed time $t > 0$ and fixed number of arrivals n, the counting PMF $P_{N_j(t)}(n)$ approaches the Poisson PMF (of the parameter λt) with increasing j,

$$\lim_{j \to \infty} P_{N_j(t)}(n) = \frac{(\lambda t)^n \mathrm{e}^{-\lambda t}}{n!}.$$

Proof.

For each j, the jth Bernoulli process has an associated Bernoulli counting process

$$N_j(t) = \sum_{i=1}^{\lfloor t2^j \rfloor} Z_i^{(j)}$$

$$P_{N_j(t)}(n) = \binom{\lfloor t2^j \rfloor}{n} p^n (1-p)^{\lfloor t2^j \rfloor - n}, \text{ for } n < \lfloor t \rfloor$$

where $p = \lambda 2^{-j}$.

Thus,

$$\begin{aligned}
\lim_{j\to\infty} P_{N_j(t)}(n) &= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} (\lambda 2^{-j})^n (1 - \lambda 2^{-j})^{\lfloor t2^j \rfloor - n} \\
&= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1 - \lambda 2^{-j}} \right)^n \exp[\lfloor t2^j \rfloor \ln(1 - \lambda 2^{-j})]
\end{aligned}$$

$$\lim_{j\to\infty} P_{N_j(t)}(n) = \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp[\lfloor t2^j \rfloor \ln(1-\lambda 2^{-j})]$$

$$(\text{use } \ln(1-\lambda 2^{-j}) = -\lambda 2^{-j} + o(2^{-j}).)$$

$$= \lim_{j\to\infty} \binom{\lfloor t2^j \rfloor}{n} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp(-\lambda t)$$

$$= \lim_{j\to\infty} \frac{\lfloor t2^j \rfloor \cdot \lfloor t2^j - 1 \rfloor \cdots \lfloor t2^j - n + 1 \rfloor}{n!} \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right)^n \exp(-\lambda t)$$

$$(\text{for } 0 \leqslant i \leqslant n-1,\ \lim_{j\to\infty} \lfloor t2^j - i \rfloor \left(\frac{\lambda 2^{-j}}{1-\lambda 2^{-j}}\right) = \lambda t.)$$

$$= \frac{(\lambda t)^n}{n!} \exp(-\lambda t).$$

Homework:

请参考教材 Figure 2.1，编写程序仿真。分别画 5 个 $n=10$ 的样本函数，体会到达过程和泊松过程。

1）更新过程，间隔时间通过掷一枚均匀骰子决定，掷到多少点间隔就取多少。

2）更新过程，间隔时间服从[0,1]上的均匀分布。

3）泊松过程 $\lambda=1$。

Thank you for your attention!