4

Interpolation

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In science and engineering problems, if $f(x)$ is difficult to be computed or is determined by the following set of data:

x	x_0	x_1	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_n)$

Such data might be the results of a sequence of laboratory measurements, where x might represent time or temperature and $f(x)$ might represent distance or pressure. Or the data might represent measurements of some natural phenomenon, such as the population of an endangered species or a supernova light curve over time. Or the data might represent stock prices at various times or sales figures over successive periods. Or the data might represent values of some mathematical function for various arguments.

In this chapter we will learn how to represent such discrete data in terms of relatively simple functions that are then easily manipulated.

4.1 Lagrange Interpolating Polynomials

Definition 19. *Suppose $x_0, x_1, \cdots, x_n$ are $n+1$ distinct numbers on $[a,b]$ and f is a function whose values are given at these numbers,*

$$f(x_0), f(x_1), \cdots, f(x_n).$$

If a polynomial $p_n(x)$ of degree at most n exists with

$$p_n(x_i) = f(x_i) \quad (i = 0, 1, 2, \cdots, n) \tag{4.1}$$

then $p_n(x)$ is called nth interpolating polynomial of $f(x)$.

Theorem 25. *The nth interpolating polynomial p_n of f about $n+1$ distinct nodes is unique.*

Proof Let

$$p_n(x) = \sum_{k=0}^{n} c_k x^k,$$

and take it to Eq. (4.1). We have

$$\begin{cases} \sum_{k=0}^{n} x_0^k c_k = f(x_0), \\ \sum_{k=0}^{n} x_1^k c_k = f(x_1), \\ \qquad \vdots \\ \sum_{k=0}^{n} x_n^k c_k = f(x_n) \end{cases}$$

The vector $\boldsymbol{c} = (c_0, c_1, \cdots, c_n)^T$ of coefficients of the polynomial interpolating the data points $(x_k, f(x_k)), k = 0, \ldots, n$, is given by the $(n+1) \times (n+1)$ linear system

$$\boldsymbol{Ac} = \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

A matrix of this form, whose columns are successive powers of some independent variable x, is called a Vandermonde matrix. Since $x_i, i = 1, \ldots n$, are all distinct, the dominant of A is non-zero, i.e. A is non-singular. Hence the linear system has unique solution such that the interpolating polynomial is unique.

□

Next we will discuss how to construct the Lagrange interpolating polynomial $p_n(x)$.

Firstly, $n = 1$ is considered for example. The problem of determining a polynomial of degree one that passes through the distinct points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is the same as approximating a function f for which $p_1(x_0) = f(x_0)$ and $p_1(x_1) = f(x_1)$ by means of a first-degree polynomial interpolating, or agreeing with, the values of f at the given points. Define the functions

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

The linear **Lagrange interpolating polynomial** through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$p_1(x) = l_0(x)f(x_0) + l_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$l_0(x_0) = 1, \quad l_0(x_1) = 0, \quad l_1(x_0) = 0, \quad \text{and} \quad l_1(x_1) = 1,$$

which implies that

$$p_1(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0)$$

and

$$p_1(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1).$$

So p_1 is the unique polynomial of degree at most one that passes through $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $n + 1$ distinct points

$$(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$$

.

In this case, we first construct a polynomial of degree at most n $l_k(x)$ with the property that

$$l_k(x_0) = 0, \cdots, l_k(x_{k-1}) = 0, l_k(x_k) = 1, l_k(x_{k+1}) = 0, \cdots, l_k(x_n) = 0, \tag{4.2}$$

i.e.

$$l_k(x_j) = \begin{cases} 1 & (j = k), \\ 0 & (j \neq k). \end{cases}$$

Since $x_0, x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n$ are the zeros of $l_k(x)$, the numerator of $l_k(x)$ contain the terms:

$$x - x_0, x - x_1, \cdots, x - x_{k-1}, x - x_{k+1}, \cdots, x - x_n.$$

Then $l_k(x)$ can be written as

$$\begin{aligned} l_k(x) &= A_k\left(x-x_0\right)\left(x-x_1\right)\cdots\left(x-x_{k-1}\right)\left(x-x_{k+1}\right)\cdots\left(x-x_n\right) \\ &= A_k \prod_{\substack{i=0 \\ i\neq k}}^{n}\left(x-x_i\right), \end{aligned} \tag{4.3}$$

where A_k are constants to be determined. From $l_k\left(x_k\right)=1$, we obtain

$$A_k \prod_{\substack{i=0 \\ i\neq k}}^{n}\left(x_k-x_i\right)=1$$

So

$$A_k=\frac{1}{\prod\limits_{\substack{i=0 \\ i\neq k}}^{n}\left(x_k-x_i\right)}$$

We take A_k to (4.3), and

$$l_k(x)=\frac{\prod\limits_{\substack{i=0 \\ i\neq k}}^{n}\left(x-x_i\right)}{\prod\limits_{\substack{i=0 \\ i\neq k}}^{n}\left(x_k-x_i\right)}=\prod_{\substack{i=0 \\ i\neq k}}^{n}\frac{x-x_i}{x_k-x_i} \tag{4.4}$$

The interpolating polynomial is easily described once the form of l_k is known. This polynomial, called the **nth Lagrange interpolating polynomial**, is defined in the following theorem.

Theorem 26. *If $x_0, x_1, ..., x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $p_n(x)$ of degree at most n exists with*

$$p_n\left(x_i\right)=f\left(x_i\right) \quad (i=0,1,2,\cdots,n).$$

And the polynomial is given by

$$L_n(x)=\sum_{k=0}^{n} f\left(x_k\right) l_k(x)=\sum_{k=0}^{n} f\left(x_i\right) \prod_{\substack{i=0 \\ i\neq k}}^{n}\frac{x-x_i}{x_k-x_i},$$

which is nth Lagrange interpolating polynomial.

Exercise 4.1. *Suppose $f(x)=\sin x, x\in[0,\pi]$.*

(1) Use the nodes $x_0=0, x_1=\frac{\pi}{2}, x_2=\pi$ to find the second interpolating polynomial $L_2(x)$ for $f(x)$ and plot the figures of $f(x)$ and $L_2(x)$.

(2) Use the nodes $x_0=0, x_1=\frac{\pi}{3}, x_2=\frac{2\pi}{3}, x_3=\pi$ to find the second interpolating polynomial $L_3(x)$ for $f(x)$ and plot the figures of $f(x)$ and $L_3(x)$.

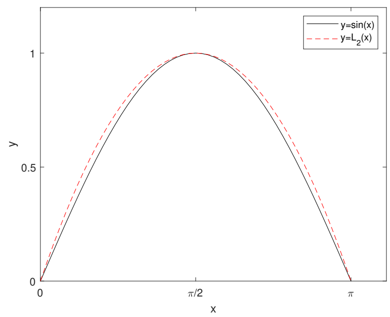

FIGURE 4.1
$L_2(x)$

Solution (1)

$$\begin{aligned}L_2(x) =& f\left(x_0\right) \frac{\left(x-x_1\right)\left(x-x_2\right)}{\left(x_0-x_1\right)\left(x_0-x_2\right)}+f\left(x_1\right) \frac{\left(x-x_0\right)\left(x-x_2\right)}{\left(x_1-x_0\right)\left(x_1-x_2\right)} \\ & +f\left(x_2\right) \frac{\left(x-x_0\right)\left(x-x_1\right)}{\left(x_2-x_0\right)\left(x_2-x_1\right)} \\ =& \frac{x(x-\pi)}{\frac{\pi}{2}\left(\frac{\pi}{2}-\pi\right)}=\frac{4}{\pi^2} x(\pi-x)\end{aligned}$$

The figures of $f(x)$ and $L_2(x)$ can be seen in Fig. 4.1.

(2)

$$\begin{aligned}L_3(x) =& f\left(x_0\right) \frac{\left(x-x_1\right)\left(x-x_2\right)\left(x-x_3\right)}{\left(x_0-x_1\right)\left(x_0-x_2\right)\left(x_0-x_3\right)} \\ & +f\left(x_1\right) \frac{\left(x-x_0\right)\left(x-x_2\right)\left(x-x_3\right)}{\left(x_1-x_0\right)\left(x_1-x_2\right)\left(x_1-x_3\right)} \\ & +f\left(x_2\right) \frac{\left(x-x_0\right)\left(x-x_1\right)\left(x-x_3\right)}{\left(x_2-x_0\right)\left(x_2-x_1\right)\left(x_2-x_3\right)} \\ & +f\left(x_3\right) \frac{\left(x-x_0\right)\left(x-x_1\right)\left(x-x_2\right)}{\left(x_3-x_0\right)\left(x_3-x_1\right)\left(x_3-x_2\right)} \\ =& \frac{\sqrt{3}}{2} \times \frac{(x-0)\left(x-\frac{2\pi}{3}\right)(x-\pi)}{\left(\frac{\pi}{3}-0\right)\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)\left(\frac{\pi}{3}-\pi\right)} \\ & +\frac{\sqrt{3}}{2} \times \frac{(x-0)\left(x-\frac{\pi}{3}\right)(x-\pi)}{\left(\frac{2\pi}{3}-0\right)\left(\frac{2\pi}{3}-\frac{\pi}{3}\right)\left(\frac{2\pi}{3}-\pi\right)} \\ =& \frac{9\sqrt{3}}{4\pi^2} x(\pi-x).\end{aligned}$$

The figures of $f(x)$ and $L_3(x)$ can be seen in the following figure.

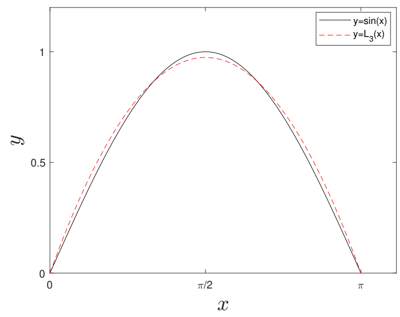

FIGURE 4.2
$L_3(x)$

4.1.1 Interpolation Error

The **interpolation error** at x is

$$R_n(x) = f(x) - p_n(x) \tag{4.5}$$

where $p_n(x)$ is Lagrange interpolating polynomial. $R_n(x)$ is the difference between the original function that provided the data points and the interpolating polynomial, evaluated at x. It is clear that at the distinct nodes $R_n(x_i) = 0, (i = 0, 1, \cdots, n)$. If $x \neq x_i (i = 0, 1, \cdots, n)$, $R_n(x_i) \neq 0, (i = 0, 1, \cdots, n)$ generally. The next theorem gives a formula for the interpolation error that is usually impossible to evaluate exactly, but often can at least lead to an error bound.

Theorem 27. *Suppose $x_0, x_1, \cdots, x_n$ are distinct nodes in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi \in (a, b)$ exists with*

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} W_{n+1}(x), \tag{4.6}$$

where $W_{n+1}(x) = \prod_{i=0}^{n} (x - x_i)$.

Proof Note first that if $x = x_0, x_1, \cdots, x_n$, then $R_n(x_k) = f(x_k) - p_n(x_k) = 0$, and choosing ξ arbitrarily in (a, b) yields Eq. (4.6). If $x \neq x_k$, for all $k = 0, 1, \cdots, n$, define the function φ for t in $[a, b]$ by

$$\varphi(t) = R_n(t) - K(x) W_{n+1}(t).$$

Thus φ is zero at the $n + 2$ distinct nodes $x, x_0, x_1, \cdots, x_n$.

By the Rolle's Theorem, $\varphi'(t)$ has at least $n + 1$ distinct zeros in the interval (a,b). Repeat the above step, $\varphi''(t)$ has at least n distinct zeros in (a,b). Continue this progress, there exists a number ξ in (a,b) for which

$$\varphi^{(n+1)}(\xi) = 0.$$

Since

$$\begin{aligned}\varphi^{(n+1)}(t) &= R_n^{(n+1)}(t) - K(x)\left[W_{n+1}(t)\right]^{(n+1)} \\ &= f^{(n+1)}(t) - (n+1)!K(x),\end{aligned}$$

there is

$$\varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - (n+1)!K(x) = 0.$$

Thus

$$K(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

The theorem is proved. □

Remark

(1) Since

$$\xi \in \left(\min\left\{x, x_0, x_1, \cdots, x_n\right\}, \max\left\{x, x_0, x_1, \cdots, x_n\right\}\right),$$

ξ is related to x, i.e. $\xi = \xi(x)$.

(2) If $f(x)$ is a polynomial of degree at most n, then

$$f(x) - p_n(x) = 0,$$

i.e.

$$p_n(x) = f(x).$$

Specially $f(x) = 1$, there is

$$\sum_{k=0}^{n} l_k(x) \equiv 1,$$

which is the property of l_k.

(3) In fact, ξ exists but is unknown. The estimation of the error is also used to get the error bound

$$|R_n(x)| \leqslant \frac{M_{n+1}}{(n+1)!}\left|W_{n+1}(x)\right|$$

where $\max_{a\leqslant x\leqslant b}\left|f^{(n+1)}(x)\right| = M_{n+1}$.

Exercise 4.2. *The error function defined by*

$$f(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\mathrm{d}t,$$

in the interval $[4,6]$. *If use linear interpolation on the nodes* $x_0 = 4, x_1 = 6$ *to obtain the approximation of* $f(x)$*, try to estimate the interpolation error.*

Solution

Suppose the linear interpolating polynomial is $p_1(x)$, and from Eq. (4.6), there is

$$f(x) - p_1(x) = \frac{1}{2} f''(\xi)\left(x - x_0\right)\left(x - x_1\right).$$

Then

$$\max_{4 \le x \le 6} \mid f(x) - p_1(x) \mid \le \frac{1}{2} \max_{4 \le x \le 6} \mid f''(x) \mid \cdot \max_{4 \le x \le 6} \mid (x - x_0)(x - x_1) \mid .$$

Since

$$f'(x) = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x^2}, \quad f''(x) = -\frac{4x}{\sqrt{x}} \mathrm{e}^{-x^2},$$
$$f'''(x) = \frac{4}{\sqrt{\pi}}\left(2x^2 - 1\right) \mathrm{e}^{-x^2} > 0, \quad x \in (4, 6),$$

we have

$$\max_{4 \in x \in 6} |f''(x)| = |f''(4)| = 1.016 \times 10^{-6}.$$

Hence

$$\max_{4 \leqslant x \leqslant 6} |f(x) - p_1(x)| \leqslant \frac{1}{2} \times 1.016 \times 10^{-6} \times 1 = 0.508 \times 10^{-6}.$$

4.2 Newton's Divided-Difference Formula

In this section we will construct the interpolating polynomial L_k by L_{k-1}.

Let

$$g(x) = L_k(x) - L_{k-1}(x),$$

which is a polynomial of degree at most k and for each $j = 0, 1, \cdots, k-1$,

$$g\left(x_j\right) = L_k\left(x_j\right) - L_{k-1}\left(x_j\right) = f\left(x_j\right) - f\left(x_j\right) = 0.$$

This implies $g(x)$ is zero at $x_0, x_1, \cdots, x_{k-1}$ such that a_k exists with

$$g(x) = a_k\left(x - x_0\right)\left(x - x_1\right) \cdots \left(x - x_{k-1}\right).$$

Thus

$$L_k(x) = L_{k-1}(x) + a_k\left(x - x_0\right)\left(x - x_1\right) \cdots \left(x - x_{k-1}\right). \tag{4.7}$$

If the constant a_k can be determined, $L_k(x)$ can be derived from $L_{k-1}(x)$ by Eq. (4.7) . By recurrence, we have

$$\begin{aligned} L_n(x) = & a_0 + a_1\left(x - x_0\right) + a_2\left(x - x_0\right)\left(x - x_1\right) + \cdots \\ & + a_n\left(x - x_0\right)\left(x - x_1\right) \cdots \left(x - x_{n-1}\right). \end{aligned} \tag{4.8}$$

In Eq. (4.7), let $x = x_k$,

$$
\begin{aligned}
a_k =& \frac{L_k(x_k) - L_{k-1}(x_k)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})} \\
=& \frac{f(x_k) - \sum\limits_{m=0}^{k-1} f(x_m) \prod\limits_{\substack{i=0 \\ i \neq m}}^{k-1} \frac{x_k - x_i}{x_m - x_i}}{\prod\limits_{i=0}^{k-1}(x_k - x_i)} \\
=& \frac{f(x_k)}{\prod\limits_{i=0}^{k-1}(x_k - x_i)} - \sum_{m=0}^{k-1} \frac{f(x_m)}{(x_k - x_m)\prod\limits_{\substack{i=0 \\ i \neq m}}^{k-1}(x_m - x_i)} \\
=& \sum_{m=0}^{k} \frac{f(x_m)}{\prod\limits_{\substack{i=0 \\ i \neq m}}^{k}(x_m - x_i)}.
\end{aligned} \tag{4.9}
$$

4.2.1 Divided difference

It is complexed to compute a_k from (4.9) . Suppose $f(x_0), f(x_1), \cdots, f(x_n)$, come from $f(x)$ with the distinct nodes $x_0, x_1, \cdots, x_n$.

$$\frac{f(x_j) - f(x_i)}{x_j - x_i}$$

is called the **first divided difference** of $f(x)$ with respect to x_i, x_j and denoted $f[x_i, x_j]$, i.e.

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}.$$

The **second divided difference** $f[x_i, x_j, x_k]$ is defined as

$$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i}.$$

Similarly, after the $(k-1)$th divided differences

$$f[x_1, x_2, \cdots, x_{k-1}, x_k], f[x_0, x_1, \cdots, x_{k-2}, x_{k-1}]$$

have been determined, the kth divided difference relative to $x_0, x_1, \cdots, x_k$ is

$$f[x_0, x_1, \cdots, x_{k-1}, x_k] = \frac{f[x_1, x_2, \cdots, x_{k-1}, x_k] - f[x_0, x_1, \cdots, x_{k-2}, x_{k-1}]}{x_k - x_0}.$$

TABLE 4.1

x	x_0	x_1	x_2	x_3
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

TABLE 4.2
The divided differences

k	x_k	$f[x_k]$	$f[x_k,x_{k+1}]$	$f[x_k,x_{k+1},x_{k+2}]$	$f[x_k,x_{k+1},x_{k+2},x_{k+3}]$
0	x_0	$f[x_0]$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
1	x_1	$f[x_1]$	$f[x_1,x_2]$	$f[x_1,x_2,x_3]$	
2	x_2	$f[x_2]$	$f[x_2,x_3]$		
3	x_3	$f[x_3]$			

The zeroth divided difference of the function f with respect to x_i, is denoted $f[x_i]=f(x_i)$.

The generation of the divided differences can be outlined in a table. For example, for the given data in Table 4.1.

The divided differences are list in Table 4.2. It is east to get the divided differences from Table 4.2. And $f[x_0], f[x_0,x_1], f[x_0,\ x_1,x_2], f[x_0,x_1,x_2,x_3]$ is on the first column of the table.

The properties of divided differences is following.

Property 1. *The kth divided difference $f[x_0,x_1,\cdots,x_k]$ can be determined by $f(x_0), f(x_1),\cdots,f(x_k)$ i.e.*

$$f[x_0,x_1,\cdots,x_k]=\sum_{m=0}^{k}\frac{f(x_m)}{\prod\limits_{\substack{i=0\\ i\neq m}}^{k}(x_m-x_i)}. \tag{4.10}$$

Proof

This theorem can be proved by with mathematical induction.

1° The theorem holds for $k=0$ clearly.

2° Suppose it holds for $k=l-1$, i.e.

$$f[x_0,x_2,\cdots,x_{l-1}]=\sum_{m=0}^{l-1}\frac{f(x_m)}{\prod\limits_{\substack{i=0\\ i\neq m}}^{l-1}(x_m-x_i)}$$

and

$$f[x_1,x_2,\cdots,x_l]=\sum_{m=1}^{l}\frac{f(x_m)}{\prod\limits_{\substack{i=1\\ i\neq m}}^{l}(x_m-x_i)}.$$

Thus

$$\begin{aligned}
f[x_1, x_2, \cdots, x_l] &= \frac{1}{x_l - x_0}\left(f[x_1, x_2, \cdots, x_l] - f[x_0, x_1, \cdots, x_{l-1}]\right) \\
&= \frac{1}{x_l - x_0}\left[\sum_{m=1}^{l} \frac{f(x_m)}{\prod\limits_{\substack{i=1\\ i\neq m}}^{l} (x_m - x_i)} - \sum_{m=0}^{l-1} \frac{f(x_m)}{\prod\limits_{\substack{i=0\\ i\neq m}}^{l-1} (x_m - x_i)}\right] \\
&= \frac{1}{x_0 - x_l} \cdot \frac{f(x_0)}{\prod\limits_{i=1}^{l-1} (x_0 - x_i)} \\
&\quad + \frac{1}{x_l - x_0} \sum_{m=1}^{l-1} \left[\frac{1}{\prod\limits_{\substack{i=1\\ i\neq m}}^{l} (x_m - x_i)} - \frac{1}{\prod\limits_{\substack{i=0\\ i\neq m}}^{l-1} (x_m - x_i)}\right] f(x_m) \\
&\quad + \frac{1}{x_l - x_0} \cdot \frac{f(x_l)}{\prod\limits_{i=1}^{l-1} (x_l - x_i)} \\
&= \sum_{m=0}^{l} \frac{f(x_m)}{\prod\limits_{\substack{i=0\\ i\neq m}}^{l} (x_m - x_i)}.
\end{aligned}$$

The Eq. (4.10) holds for $k = l$. By mathematical induction, the property can be proved.

□

Since (4.9) and (4.10), there is

$$a_k = f[x_0, x_1, \cdots, x_k].$$

Taking it to Eq. (4.8), we have

$$\begin{aligned}
L_n(x) =& f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&+ \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}$$

which is called **Newton's divided-difference interpolating polynomial** and is denoted by $N_n(x)$, i.e.

$$\begin{aligned}
N_n(x) =& f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&+ \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1}) \qquad (4.11)
\end{aligned}$$

Exercise 4.3. *Use Newton's divided difference formula to construct interpolating polynomial of degree two for the following data*

x_k	-1	1	2
$f(x_k)$	3	1	-1

TABLE 4.3

k	x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$
0	-1	3	-1	$-\frac{1}{3}$
1	1	1	-2	
2	2	-1		

TABLE 4.4
Divided Difference

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$	$f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$	$f[x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}]$
x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3, \tilde{x}]$
x_1	$f[x_1]$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	$f[x_1, x_2, x_3, \tilde{x}]$	
x_2	$f[x_2]$	$f[x_2, x_3]$	$f[x_2, x_3, \tilde{x}]$		
x_3	$f[x_3]$	$f[x_3, \tilde{x}]$			
$\tilde{x}$	$f[\tilde{x}]$				

Solution The divided difference is listed in the following Table 4.3. Thus,

$$N_2(x) = 3 - (x+1) - \frac{1}{3}(x+1)(x-1).$$

Property 2. *The divided difference is not related with the order of the nodes.*

In fact, the right hand of Eq. (4.11) does not change if we change the order of two numbers x_l and x_m.

Suppose $N_m(x)$ is an interpolating polynomial of degree m on the nodes $x_0, x_1, \cdots, x_m$. If we add a node $\widetilde{x}$, the new interpolating polynomial $N_{m+1}(x)$, can be derived by the table of divided difference in which we only add $\widetilde{x}$ on the last column from the property 2.

Take $m = 3$ for example and the divided difference can be computed in Table 4.4 Thus

$$N_{m+1}(x) = N_m(x) + f[x_0, x_1, \cdots, x_m, \widetilde{x}](x - x_0)(x - x_1) \cdots (x - x_m).$$

Property 3. *Suppose that $f \in C^n[a, b]$ and $x_0, x_1, \cdots, x_n$ are distinct numbers in $[a, b]$. Then a number η exists in (a, b) with*

$$f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)}(\eta)}{k!}$$

Proof The kth Newton's divided difference interpolating polynomial of $f(x)$ is

$$N_k(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1}(x - x_i)$$

about the distinct nodes $x_0, x_1, \cdots, x_k$. The interpolating error is

$$R_k(x) = f(x) - N_k(x).$$

Since $R_k(x_i) = 0 (i = 0, 1, \cdots, k)$, the function $R_k(x)$ has $(k+1)$ distinct zeros. By Rolle's theorem, the function $R_k'(x)$ has at least k distinct zeros in (a, b). Repeat the progress, the function $R_k^{(k)}(x)$ at least one zero η, i.e.

$$R_k^{(k)}(\eta) = f^{(k)}(\eta) - N_k^{(k)}(\eta) = f^{(k)}(\eta) - k! f[x_0, x_1, \cdots, x_k] = 0.$$

Thus

$$f[x_0, x_1, \cdots, x_k] = \frac{f^{(k)}(\eta)}{k!}$$

where $\eta \in (a, b)$.

□

Next, we will derive the interpolating error by divided difference.

$$\begin{aligned} f(x) &= f(x_0) + f[x_0, x](x - x_0) \\ f[x_0, x] &= f[x_0, x_1] + f[x_0, x_1, x](x - x_1) \\ f[x_0, x_1, x] &= f[x_0, x_1, x_2] + f[x_0, x_1, x_2, x](x - x_2) \\ &\vdots \\ f[x_0, \cdots, x_{n-2}, x] &= f[x_0, x_1, \cdots, x_{n-1}] + f[x_0, x_1, \cdots, x_{n-1}, x](x - x_{n-1}) \\ f[x_0, \cdots, x_{n-1}, x] &= f[x_0, x_1, \cdots, x_n] + f[x_0, x_1, \cdots, x_n, x](x - x_n) \end{aligned}$$

Then

$$\begin{aligned} f(x) =& f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}) \\ &+ f[x_0, x_1, \cdots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n) \\ =& N_n(x) + f[x_0, x_1, \cdots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n) \end{aligned}$$

Since $L_n(x) = N_n(x)$, the interpolating error can be written as

$$R_n(x) = f[x_0, x_1, \cdots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n).$$

4.2.2 Forward Difference and Newton forward-difference formula

Suppose the distinct nodes are

$$x_i = a + ih \quad (i = 0, 1, \cdots, n)$$

where h is step.

TABLE 4.5
The forward difference

x_k	f_k	Δf_k	$\Delta^2 f_k$	$\Delta^3 f_k$	$\Delta^4 f_k$
x_0	f_0	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
x_1	f_1	Δf_1	$\Delta^2 f_1$	$\Delta^3 f_1$	
x_2	f_2	Δf_2	$\Delta^2 f_2$		
x_3	f_3	Δf_3			
x_4	f_4				

Definition 20. *Suppose the function $f(x)$ have the values $f(x_i) = f_i (i = 0, 1, \cdots, n)$ on the equidistant nodes x_i. Δf_i is called the* **first forward difference** *and is defined by*

$$\Delta f_i = f_{i+1} - f_i,$$

and in general,

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i.$$

is the kth forward difference.

The difference can be computed in the Table 4.5.

By mathematical induction method, we have

$$f[x_i, x_{i+1}, \cdots, x_{i+k}] = \frac{\Delta^k f_i}{k! h^k}. \tag{4.12}$$

Let

$$x = x_0 + th,$$

then

$$\prod_{j=0}^{k-1} (x - x_j) = \prod_{j=0}^{k-1} [(x_0 + th) - (x_0 + jh)] = h^k \prod_{j=0}^{k-1} (t - j). \tag{4.13}$$

Taking Eqs. (4.12) and (4.13) into Newton's interpolating formula

$$N_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \cdots, x_k] \prod_{j=0}^{k-1} (x - x_j),$$

we obtain the **Newton forward-difference formula**

$$N_n(x_0 + th) = \sum_{k=0}^{n} \frac{\Delta^k f_0}{k!} \prod_{j=0}^{k-1} (t - j), \tag{4.14}$$

where $t = \frac{x - x_0}{h}$.

Let

$$Ef_i = f_{i+1}, \quad If_i = f_i,$$

then

$$\Delta f_i = (E - I)f_i = f_{i+1} - f_i.$$

Thus

$$\begin{aligned}\Delta^k f_i = (E - I)^k f_i &= \sum_{j=0}^{k} C_k^j E^{k-j}(-I)^j f_i \\ &= \sum_{j=0}^{k} (-1)^j C_k^j f_{i+k-j}\end{aligned}$$

i.e. $\Delta^k f_i$ is a linear combination of $f_i, f_{i+1}, \cdots, f_{i+k}$.

4.3 Hermite Interpolation

In addition, since their first derivatives agree with those of f , they have the same "shape" as the function at $(x_i, f(x_i))$ in the sense that the tangent lines to the polynomial and the function agree. We will study the interpolating polynomial of f not only to interpolate a sequence of distinct points $x_0, x_1, \cdots,$ x_n, but also to interpolate multi-order derivative of these points.

Definition 21. *For given* $(n + 1)$ *distinct nodes* $x_i(i = 0, 1, \cdots, n)$ *in* $[a, b]$, *and the values* $f(x_i), f'(x_i), \cdots, f^{(m_i)}(x_i)$, *let* $m = \sum_{i=0}^{n}(m_i + 1) - 1$. *A polynomial* $H_m(x)$ *of at most degree* m *is called* **Hermite interpolating polynomial** *of* f *for*

$$\left\{\begin{array}{c} H_m(x_0) = f(x_0), H_m'(x_0) = f'(x_0), \cdots, H_m^{(m_0)}(x_0) = f^{(m_0)}(x_0) \\ H_m(x_1) = f(x_1), H_m'(x_1) = f'(x_1), \cdots, H_m^{(m_1)}(x_1) = f^{(m_1)}(x_1) \\ \vdots \\ H_m(x_n) = f(x_n), H_m'(x_n) = f'(x_n), \cdots, H_m^{(m_n)}(x_n) = f^{(m_n)}(x_n) \end{array}\right. \tag{4.15}$$

Theorem 28. *The Hermite interpolating polynomial of degree* m *is unique with Eqs.* (4.15).

Proof Let $H_m(x) = \sum\limits_{k=0}^{m} c_k x^k$, and take it to Eqs. (4.15):

$$\begin{cases} \sum\limits_{k=0}^{m} x_i^k c_k = f(x_i) \\ \sum\limits_{k=1}^{m} k x_i^{k-1} c_k = f'(x_i), \qquad (i = 0, 1, \cdots, n) \\ \vdots \\ \sum\limits_{k=m_i}^{m} k(k-1)\cdots(k-m_i+1)\, x_i^{k-m_i} c_k = f^{(m_i)}(x_i) \end{cases}$$

To prove the above linear system has unique solutions, it is only to prove the corresponding homogeneous system of linear equations has only zero solutions. In fact, if the right hand of (4.15) is zero, x_i is a zero of multiplicity $(m_i + 1)$ of $f(x)$. Then $H_m(x)$ has $\sum\limits_{i=0}^{n} (m_i + 1) = m+1$ multiple zeros. Since a polynomial of degree m has at most m zeros, the function $H_m(x)$ must be zero. Thus, Eqs. (4.15) has only unique solutions.

□

Similarly as Theorem 27, we have the following theorem.

Theorem 29. *Suppose $H_m \in C^{m+1}[a, b]$ is the interpolating polynomial of degree at most m agreeing with Eq. (4.15). The $\xi \in (a, b)$, exists with*

$$R_m(x) = f(x) - H_m(x) = \frac{f^{(m+1)}(\xi)}{(m+1)!} \prod_{i=0}^{n} (x - x_i)^{m_i+1}. \tag{4.16}$$

Exercise 4.4. *Suppose $f(x) \in C^{m+1}[a, b], x_0 \in [a, b]$. Construct a polynomial $H(x)$ of degree m such that*

$$H^{(k)}(x_0) = f^{(k)}(x_0) \quad (k = 0, 1, \cdots, m) \tag{4.17}$$

and determine $f(x) - H(x)$.

Solution From Taylor's expansion, we have

$$\begin{aligned} f(x) =& f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m \\ &+ \frac{f^{(m+1)}(\xi)}{(m+1)!}(x - x_0)^{m+1} \end{aligned}$$

where ξ is between x_0 and x. Let

$$H(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m.$$

It is clear that the function $H(x)$ satisfy Eqs. (4.17) and

$$f(x) - H(x) = \frac{f^{(m+1)}(\xi)}{(m+1)!}(x - x_0)^{m+1}.$$

Remark In Eq. (4.16), the exponent of $(m_i + 1)$ of $(x - x_i)$ is the multiplicity of the node x_i. Next, we will discuss the Hermite polynomials by divided differences which uses the connection between the nth divided difference and the nth derivative of f.

Exercise 4.5. *Construct a linear interpolating polynomial with*

$$p(x_0) = f(x_0), \quad p(x_1) = f(x_1) \tag{4.18}$$

Solution From Newton's interpolating formula, $p(x)$ is as follows

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0), \tag{4.19}$$

where

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

It is easy to know

$$\lim_{x_1 \to x_0} f[x_0, x_1] = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0).$$

In the right hand of Eq. (4.19), we can obtain the linear polynomial by $x_1 \to x_0$,

$$H(x) = f[x_0] + f'(x_0)(x - x_0). \tag{4.20}$$

Clearly, the function $H(x)$ satisfy

$$H(x_0) = f(x_0), \quad H'(x_0) = f'(x_0). \tag{4.21}$$

The linear polynomial $H(x)$ in (4.20) is the interpolating polynomial with (4.21). Denote $f[x_0, x_0] = f'(x_0)$, then Eq. (4.20) can be written

$$H(x) = f[x_0] + f[x_0, x_0](x - x_0). \tag{4.22}$$

Next, we will discuss how to construct Hermite interpolating polynomial using divided difference.

Theorem 30. *(Hermite-Gennochi) Given distinct nodes $x_i \in [a, b], i = 0, 1, \cdots, n$, suppose $f \in C^n[a, b]$. Then*

$$f[x_0, \cdots, x_n] = \int \underset{\tau_n}{\cdots} \int f^{(n)}(t_0 x_0 + t_1 x_1 + \cdots + t_n x_n) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n,$$

where $\tau_n = \{(t_1, \cdots, t_n) \mid t_1 \geqslant 0, \cdots, t_n \geqslant 0, \sum_{i=1}^n t_i \leqslant 1\}$ is a simplex in n-dimensional space, and $t_0 = 1 - \sum\limits_{i=1}^{n} t_i$.

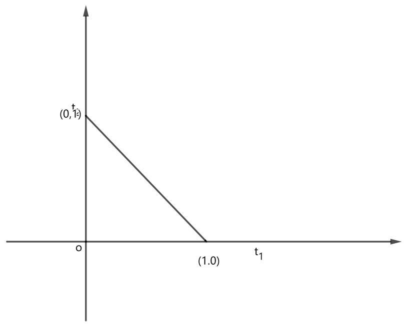

FIGURE 4.3

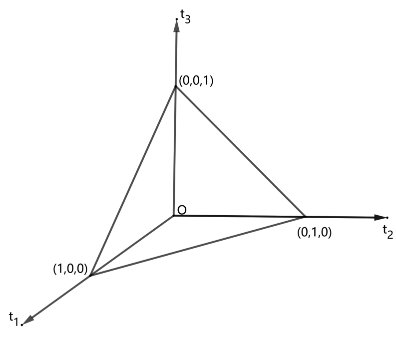

FIGURE 4.4

For example, the simplex on 1-dimensional space is the interval [0,1] and the simplexes on 2 or 3-dimensional space can be seen in Fig. 4.3 and 4.4.

Proof The theorem will be proved by mathematical induction. If $n = 1$, we have

$$\begin{aligned}\int_0^1 f'\left(t_0x_0 + t_1x_1\right) \mathrm{d}t_1 &= \int_0^1 f'\left(x_0 + t_1\left(x_1 - x_0\right)\right) \mathrm{d}t_1 \\ &= \frac{1}{x_1 - x_0} f\left(x_0 + t_1\left(x_1 - x_0\right)\right)\Big|_{t_1=0}^{t_1=1} \\ &= \frac{1}{x_1 - x_0}\left[f\left(x_1\right) - f\left(x_0\right)\right] = f\left[x_0, x_1\right].\end{aligned}$$

Suppose the theorem holds for $n = k$. Let $n = k + 1$, then

$$\begin{aligned}&\int \underset{\tau_{k+1}}{\cdots} \int f^{(k+1)}\left(t_0x_0 + t_1x_1 + \cdots + t_{k+1}x_{k+1}\right) \mathrm{d}t_1 \cdots \mathrm{d}t_{k+1} \\ =&\int \underset{\tau_k}{\cdots} \int \left(\int_0^{1-(t_1+\cdots+t_k)} f^{(k+1)}\left(x_0 + t_1\left(x_1 - x_0\right)\right.\right. \\ &\left.\left.+ \cdots + t_{k+1}\left(x_{k+1} - x_0\right)\right) \mathrm{d}t_{k+1}\right) \mathrm{d}t_1 \cdots \mathrm{d}t_k \\ =&\int \underset{\tau_k}{\cdots} \int \frac{1}{x_{k+1} - x_0} f^{(k)}\left(x_0 + t_1\left(x_1 - x_0\right) + \cdots + t_k\left(x_k - x_0\right)\right. \\ &\left.+t_{k+1}\left(x_{k+1} - x_0\right)\right)\Big|_{t_{k+1}=0}^{t_{k+1}-1-(t_1+t_2+\cdots+t_k)} \mathrm{d}t_1\mathrm{d}t_2 \cdots \mathrm{d}t_k \\ =&\frac{1}{x_{k+1} - x_0} \int \cdots \int \Big(f^{(k)}\left(t_1x_1 + \cdots + t_kx_k + t_0x_{k+1}\right) \\ &-f^{(k)}\left(t_0x_0 + t_1x_1 + \cdots + t_kx_k\right)\Big) \mathrm{d}t_1 \cdots \mathrm{d}t_k \\ =&\frac{1}{x_{k+1} - x_0}\left(f\left[x_1, \cdots, x_k, x_{k+1}\right] - f\left[x_0, x_1, \cdots, x_k\right]\right) \\ =&f\left[x_0, x_1, \cdots, x_k, x_{k+1}\right].\end{aligned}$$

It holds for any positive integer n.

□

It is noted the function integrated is the composition of the function $f^{(n)}(x)$ and linear continuous function in n variables (also continuous for $x_0, x_1, \cdots, x_n$)

$$\begin{aligned}x\left(t_1, \cdots, t_n\right) &= x_0 + t_1\left(x_1 - x_0\right) + \cdots + t_n\left(x_n - x_0\right) \\ &= \sum_{i=0}^{n} t_ix_i.\end{aligned}$$

Thus $f\left[x_0, x_1, \cdots, x_n\right]$ is continuous for $x_0, \cdots, x_n$.

From the property 3, we have

$$f[x_0, x_0] = \lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$f[\underbrace{x_0,\cdots,x_0}_{k+1}] = \lim_{x_1\to x_0} f[x_0,x_1,\cdots,x_k] = \lim_{\substack{x_1\to x_0\\ \vdots \\ x_k\to x_0}} \frac{f^k(\eta)}{k!} = \frac{f^k(x_0)}{k!} \tag{4.23}$$

$$\begin{aligned} f[x_0,x_0,x_1] &= \lim_{x\to x_0} f[x,x_0,x_1] = \lim_{x\to x_0} \frac{f[x_0,x_1]-f[x,x_0]}{x_1-x} \\ &= \frac{f[x_0,x_1]-f[x_0,x_0]}{x_1-x_0} \end{aligned} \tag{4.24}$$

Thus, the interpolating problem (4.15) can be seen the interpolation about the distinct $(m+1)$ nodes and it is

$$\begin{aligned} H_m(x) = {} & f[x_0] + f[x_0,x_0](x-x_0) + \cdots + f[\underbrace{x_0,\cdots,x_0}_{m_0+1}]\,(x-x_0)^{m_0} \\ & + f[\underbrace{x_0,\cdots,x_0}_{m_0+1},x_1]\,(x-x_0)^{m_0+1}\cdots \\ & + f[\underbrace{x_0,\cdots,x_0}_{m_0+1},\underbrace{x_1,\cdots,x_1}_{m_1+1}]\,(x-x_0)^{m_0+1}\,(x-x_1)^{m_1} + \cdots \\ & + f[\underbrace{x_0,\cdots,x_0}_{m_0+1},\cdots,\underbrace{x_{n-1},\cdots,x_{n-1}}_{m_{n-1}+1},x_n]\,(x-x_0)^{m_0+1} \bullet \\ & \cdots \bullet (x-x_{n-1})^{m_{n-1}+1} + \cdot \\ & + f[\underbrace{x_0,\cdots,x_0}_{m_0+1},\cdots,\underbrace{x_{n-1},\cdots,x_{n-1}}_{m_{n-1}+1},\underbrace{x_n,\cdots,x_n}_{m_n+1}] \bullet \\ & (x-x_0)^{m_0+1} \bullet \cdots \bullet (x-x_{n-1})^{m_{n-1}+1}\,(x-x_n)^{m_n} \end{aligned}$$

The interpolating error is

$$\begin{aligned} & f(x) - H_m(x) \\ & \quad = f[\underbrace{x_0,\cdots,x_0}_{m_0+1},\cdots,\underbrace{x_n,\cdots,x_n}_{m_n+1},x]\,(x-x_0)^{m_0+1} \bullet \cdots \bullet (x-x_n)^{m_n+1} \\ & \quad = \frac{f^{(m+1)}(\xi)}{(m+1)!} \prod_{i=0}^{n} (x-x_i)^{m_i+1} \end{aligned}$$

where $\min(x_0,x_1,\cdots,x_n,x) < \xi < \max(x_0,x_1,\cdots,x_n,x)$.

Exercise 4.6. *Construct a Hermite interpolating polynomial $H(x)$ of degree at most four using divide difference such that*

$$H(0)=3, \quad H'(0)=4, \quad H(1)=5, \quad H'(1)=6, \quad H''(1)=7.$$

Solution The table of the divided difference is as follows
Thus the interpolating polynomial is

$$H(x) = 3 + 4(x-0) - 2(x-0)^2 + 6(x-0)^2(x-1) - \frac{13}{2}(x-0)^2(x-1)^2.$$

TABLE 4.6
Table of the divided difference

k	x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	$f[x_k, x_{k+1}, x_{k+2}]$	$f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$	$f[x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}]$
0	0	3	4	-2	6	$-13/2$
1	0	3	2	4	$-1/2$	
2	1	5	6	7/2		
3	1	5	6			
4	1	5				

TABLE 4.7

a	$f[a]$	$f[a,a]$	$f[a,a,b]$	$f[a,a,b,b]$
a	$f[a]$	$f[a,b]$	$f[a,b,b]$	
b	$f[b]$	$f[b,b]$		
b	$f[b]$			

Exercise 4.7. *Suppose $f(x) \in C^4[a,b]$. Construct a interpolating polynomial $H_3(x)$ of degree at most three such that*

$$H_3(a) = f(a), H_3'(a) = f'(a), H_3(b) = f(b), H_3'(b) = f'(b),$$

and give the interpolating error.

Solution The divided difference is list in Table 4.7.
The divided differences in Table 4.7 are

$$\begin{aligned}
f[a,a] &= f'(a), \quad f[b,b] = f'(b) \\
f[a,a,b] &= \frac{f[a,b]-f[a,a]}{b-a} = \frac{1}{b-a}\{f[a,b]-f'(a)\} \\
f[a,b,b] &= \frac{f[b,b]-f[a,b]}{b-a} = \frac{1}{b-a}\{f'(b)-f[a,b]\} \\
f[a,a,b,b] &= \frac{f[a,b,b]-f[a,a,b]}{b-a} \\
&= \frac{1}{(b-a)^2}\{f'(b)-2f[a,b]+f'(a)\}.
\end{aligned}$$

Thus

$$\begin{aligned}
H(x) =& f[a] + f[a,a](x-a) + f[a,a,b](x-a)^2 \\
&+ f[a,a,b,b](x-a)^2(x-b) \\
=& f(a) + f'(a)(x-a) + \frac{1}{b-a}\{f[a,b]-f'(a)\}(x-a)^2 \\
&+ \frac{1}{(b-a)^2}\{f'(b)-2f[a,b]+f'(a)\}(x-a)^2(x-b).
\end{aligned}$$

The interpolating error is

$$f(x) - H_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-a)^2(x-b)^2 \quad (\xi \in (a,b), \xi \text{ relating to } x).$$

4.4 Piecewise-Polynomial Approximation

In this section, will discuss the piecewise-polynomial approximation.

4.4.1 Error analysis of high-degree interpolating polynomials

The previous sections concerned the approximation of arbitrary functions on closed intervals using a single polynomial.

From the formula of interpolating error, the value of the error is related with both the number $(n+1)$ of the distinct nodes and the value of high derivatives of $f(x)$. Suppose $f \in C^\infty[a,b]$ and M exists independent with n such that

$$\max_{a \leqslant x \leqslant b} \left| f^{(n)}(x) \right| \leqslant M.$$

Sine Eq. (4.6), we have

$$\max_{a \leqslant x \leqslant b} |f(x) - L_n(x)| \leqslant \frac{M}{(n+1)!}(b-a)^{n+1} \to 0 \quad (n \to \infty\ ) \tag{4.25}$$

It can be seen that the smaller error, the larger $n+1$ in the condition that the higher order derivatives are uniformly bounded in $[a,b]$.

Let

$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1,1],$$

and the equidistant points are $x_i = -1 + \frac{i}{5} (i = 0, 1, \cdots, 10)$. The interpolating polynomial of degree ten of $f(x)$ is as follows

$$L_{10}(x) = \sum_{i=0}^{10} f(x_i)\, l_i(x),$$

where

$$f(x_i) = \frac{1}{1+25x_i^2}, \quad l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{10} \frac{x - x_j}{x_i - x_j}.$$

The computing results are listed in Table 4.8. The figures of $L_5(x), L_{10}(x), L_{16}(x)$ are plotted in Fig. 4.5.

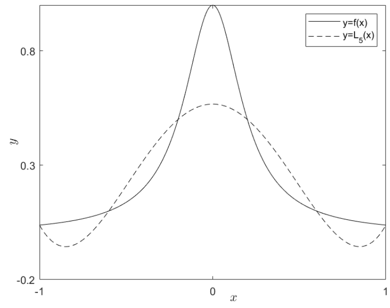

FIGURE 4.5
$f(x)$ and $L_5(x)$

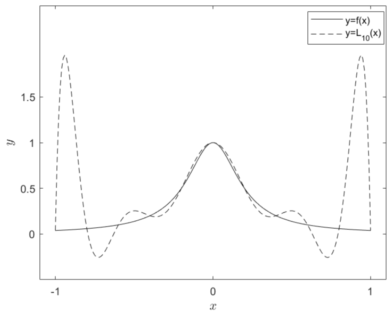

FIGURE 4.6
$f(x)$ and $L_{10}(x)$

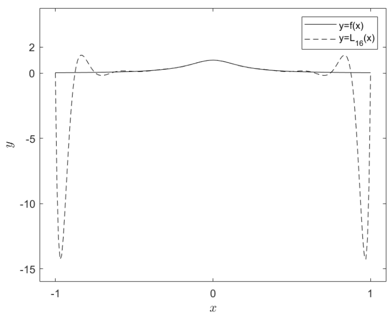

FIGURE 4.7
$f(x)$ and $L_{16}(x)$

TABLE 4.8
The values of functions $f(x)$ and $L_{10}(x)$

x	$f(x)$	$L_{10}(x)$	x	$f(x)$	$L_{10}(x)$
-1.00	0.03846	0.03846	-0.46	0.15898	0.24145
-0.96	0.04160	1.80438	-0.40	0.20000	0.19999
-0.90	0.04706	1.57872	-0.36	0.23585	0.18878
-0.86	0.05131	0.88808	-0.30	0.30769	0.23535
-0.80	0.05882	0.05882	-0.26	0.37175	0.31650
-0.76	0.06477	-0.20130	-0.20	0.50000	0.50000
-0.70	0.07547	-0.22620	-0.16	0.60976	0.64316
-0.66	0.08410	-0.10832	-0.10	0.80000	0.84340
-0.60	0.10000	0.10000	-0.06	0.91743	0.94090
-0.56	0.11312	0.19873	0.00	1.00000	1.00000
-0.50	0.13793	0.25376			

r_i	Δr_i	$\Delta^2 r_i$	$\Delta^3 r_i$	$\Delta^4 r_i$	$\Delta^5 r_i$	$\Delta^6 r_i$
0	0	0	ε	-4ε	10ε	-20ε
0	0	ε	-3ε	6ε	-10ε	
0	ε	-2ε	3ϵ	-4ε		
ε	$-\varepsilon$	ε	$-\varepsilon$			
0	0	0				
0	0					
0						

From Fig. 4.6, the approximation $L_{10}(x)$ to $f(x)$ is better in $[-0.2, 0.2]$, for example $L_{10}(-0.086) = 0.88808; f(-0.96) = 0.04160$. But near the endpoints of the interval, there are $f(-0.86) = 0.05131$, and $L_{10}(-0.96) = 1.80438$ which approximation is not adaptable. This phenomenon is called **Runge phenomenon**.

On the other hand, suppose the values $f_i = f(x_i) = f(x_0 + ih), i = 0, 1, \cdots, n$ are the values of the function $f(x)$ on the equivalence nodes. If there is some oscillation ε on x_j, i.e. $f_0, f_1, \cdots, f_n$ are changed to $f_0, f_1, \cdots, f_{j-1}, f_j + \varepsilon, f_{j+1}, \cdots, f_n$. Denote the new sequence $\widetilde{f}_0, \widetilde{f}_1, \cdots, \widetilde{f}_n$, and

$$r_i = \tilde{f}_i - f_i.$$

Then

$$r_i = \begin{cases} \varepsilon & (i = j) \\ 0 & (i \neq j) \end{cases}$$

$$\Delta^k \widetilde{f}_i - \Delta^k f_i = \Delta^k r_i$$

The forward differences are listed in Table 4.4.1.

From the table, we can see that the oscillation of sixth forward difference is about 20 times the oscillation ε on x_j. An alternative approach is to divide the approximation interval into a collection of subintervals and construct a (generally) different approximating polynomial on each subinterval. This is called piecewise-polynomial approximation.

4.4.2 Piecewise-Linear Interpolation

Given a function f and

x	x_0	x_1	$\cdots$	x_{n-1}	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_{n-1})$	$f(x_n)$

where $a = x_0 < x_1 < \cdots < x_n = b$ are $(n+1)$ distinct nodes. Denote $h_i = x_{i+1} - x_i, h = \max_{0 \le i \le n-1} h_i$. In each subinterval $[x_i, x_{i+1}]$, use the following data

x	x_i	x_{i+1}
$f(x)$	$f(x_i)$	$f(x_{i+1})$

to obtain the linear interpolating polynomial:

$$L_{1,i}(x) = f(x_i) + f[x_i, x_{i+1}](x - x_i), \quad x \in [x_i, x_{i+1}].$$

The interpolating error can be estimated

$$f(x) - L_{1,i}(x) = \frac{1}{2} f''(\xi_i)(x - x_i)(x - x_{i+1}), \quad \xi_i \in (x_i, x_{i+1}).$$

Thus

$$\begin{aligned}\max_{x_i \leqslant x \leq x_{i+1}} |f(x) - L_{1,i}(x)| &= \max_{x_i \leqslant x \leqslant x_{i+1}} \left| \frac{1}{2} f''(\xi_i)(x - x_i)(x - x_{i+1}) \right| \\ &\leqslant \frac{1}{8} h_i^2 \max_{x_i \leqslant x \leqslant x_{i+1}} |f''(x)| \end{aligned} \tag{4.26}$$

Let

$$\tilde{L}_1(x) = \begin{cases} L_{1,0}(x), & x \in [x_0, x_1); \\ L_{1,1}(x), & x \in [x_1, x_2); \\ \vdots & \\ L_{1,n-2}(x), & x \in [x_{n-2}, x_{n-1}); \\ L_{1,n-1}(x), & x \in [x_{n-1}, x_n] \end{cases}$$

and

$$\tilde{L}_1(x_i) = f(x_i) \quad (i = 0, 1, \cdots, n)$$

It is easy to verify $\tilde{L}_1(x_j) = f(x_j)$. The function $\tilde{L}_1(x)$ is called piecewise-linear interpolation of $f(x)$. Besides, we have

$$\begin{aligned}
\max_{a\leqslant x\leqslant b}\left|f(x)-\widetilde{L}_1(x)\right| &= \max_{x_0\leqslant x\leqslant x_n}\left|f(x)-\widetilde{L}_1(x)\right| \\
&= \max_{0\leqslant i\leqslant n-1}\max_{x_i\leqslant x\leqslant x_{i+1}}\left|f(x)-\widetilde{L}_1(x)\right| \\
&= \max_{0\leqslant i\leqslant n-1}\max_{x_i\leqslant x\leqslant x_{i+1}}\left| f(x)-L_{1,i}(x)\right. \\
&\leqslant \max_{0\leqslant i\leqslant n-1}\frac{1}{8}h_i^2\max_{x_i\leqslant x\leqslant x_{i+1}}\left|f''(x)\right| \\
&\leqslant \frac{1}{8}h^2\max_{a\leqslant x\leq b}\left|f''(x)\right|.
\end{aligned}$$

We can see that the piecewise-linear interpolation converge to f with $h\to 0$ if $f\in C^2[a,b]$.

4.4.3 Hermite Piecewise Interpolation

Suppose $f(x)$ has the values on the $(n+1)$ distinct nodes $a=x_0<x_1<\cdots<x_n=b$

x	x_0	x_1	$\cdots$	x_{n-1}	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_{n-1})$	$f(x_n)$
$f'(x)$	$f'(x_0)$	$f'(x_1)$	$\cdots$	$f'(x_{n-1})$	$f(x_n)$

Denote $h_i=x_{i+1}-x_i, h=\max\limits_{0\leqslant i\leqslant n-1}h_i$. Use the following data in $[x_i,x_{i+1}]$

x	x_i	x_{i+1}
$f(x)$	$f(x_i)$	$f(x_{i+1})$
$f'(x)$	$f'(x_i)$	$f'(x_{i+1})$

to obtain the Hermite cubic interpolating polynomial of degree at most three (see Example4.6)

$$\begin{aligned}
H_{3,i}=&f(x_i)+f'(x_i)(x-x_i)+\frac{f[x_i,x_{i+1}]-f'(x_i)}{h_i}(x-x_i)^2 \\
&+\frac{f'(x_{i+1})-2f[x_i,x_{i+1}]+f'(x_i)}{h_i^2}(x-x_i)^2(x-x_{i+1}).
\end{aligned}$$

The error is

$$f(x)-H_{3,i}(x)=\frac{f^{(4)}(\xi_i)}{4!}(x-x_i)^2(x-x_{i+1})^2,\quad \xi_i\in(x_i,x_{i+1}).$$

Thus

$$\max_{x_i\leqslant x\leqslant x_{i+1}}\left|f(x)-H_{3,i}(x)\right|\leqslant\frac{1}{4!}\frac{h_i^4}{16}\max_{x_i\leqslant x<x_{i+1}}\left|f^{(4)}(x)\right| \tag{4.27}$$

Let

$$\widetilde{H}_3(x) = \begin{cases} H_{3,0}(x), & x \in [x_0, x_1)\,; \\ H_{3,1}(x), & x \in [x_1, x_2)\,; \\ \vdots & \\ H_{3,n-2}(x), & x \in [x_{n-2}, x_{n-1})\,; \\ H_{3,n-1}(x), & x \in [x_{n-1}, x_n]\,; \end{cases}$$

then

$$\widetilde{H}_3\left(x_i\right) = f\left(x_i\right), \quad \widetilde{H}_3'\left(x_i\right) = f'\left(x_i\right) \quad (i = 0, 1, \cdots, n)$$

The function $\widetilde{H}_3(x)$ is called **Hermite piecewise interpolation** of $f(x)$. Besides, we have

$$\begin{aligned} \max_{a \leqslant x \leqslant b} \left| f(x) - \widetilde{H}_3(x) \right| &= \max_{x_0 \leqslant x \leqslant x_n} \left| f(x) - \widetilde{H}_3(x) \right| \\ &= \max_{0 \leqslant i \leqslant n-1} \max_{x_i \leqslant x \leqslant x_{i+1}} \left| f(x) - \widetilde{H}_3(x) \right| \\ &= \max_{0 \leqslant i \leqslant n-1} \max_{x_i \leqslant x \leqslant x_{i+1}} \mid f(x) - H_{3,i}(x) \\ &\leqslant \max_{0 \leqslant i \leqslant n-1} \frac{1}{4!} \frac{h_i^4}{16} \max_{x_i \leqslant x \leqslant x_{i+1}} \left| f^{(4)}(x) \right| \\ &\leqslant \frac{1}{384} h^4 \max_{a \leqslant x \leq b} \left| f^{(4)}(x) \right| \end{aligned}$$

We can see that the Hermite piecewise interpolation converge to f with $h \to 0$ if $f \in C^4[a, b]$.

4.5 Cubic Spline Interpolation

The previous sections concerned the approximation of arbitrary functions on closed intervals using a single polynomial. However, high-degree polynomials can oscillate erratically, that is, a minor fluctuation over a small portion of the interval can induce large fluctuations over the entire range.

The simplest piecewise-polynomial approximation is piecewise-linear interpolation. A disadvantage of linear function approximation is that there is likely no differentiability at the endpoints of the subintervals, which, in a geometrical context, means that the interpolating function is not "smooth." Often it is clear from physical conditions that smoothness is required, so the approximating function must be continuously differentiable.

An alternative procedure is to use a piecewise polynomial of Hermite type. To determine the appropriate cubic Hermite polynomial on a given interval is simply a matter of computing $H_3(x)$ for that interval. The Lagrange interpolating polynomials needed to determine H_3 are of first degree, so this can be accomplished without great difficulty. However, to use Hermite piecewise polynomials for general interpolation, we need to know the derivative of the function being approximated, and this is frequently unavailable. The remainder of this section considers approximation using piecewise polynomials that require no specific derivative information, except perhaps at the endpoints of the interval on which the function is being approximated.

4.5.1 Cubic Splines

Definition 22. *Given a function f defined on $[a,b]$ and a set of nodes*

$$a = x_0 < x_1 < \cdots < x_n = b$$

a **cubic spline interpolant** *S for f is a function that satisfies the following conditions:*

$1^\circ S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each each $j = 0, 1, \cdots, n-1$;

$2^\circ S(x) \in C^2[a,b]$;

$3^\circ S(x_j) = y_j, \quad j = 0, 1, 2, \cdots, n$, where $y_j = f(x_j)$.

Let

$$S_j(x) = A_j + B_j x + C_j x^2 + D_j x^3 \quad (j = 0, 1, \cdots, n-1) \tag{4.28}$$

where A_j, B_j, C_j, D_j are to be determined and satisfy the following conditions:

(1)

$$S(x_j) = y_j \quad (j = 0, 1, \cdots, n) \tag{4.29}$$

(2)

$$\begin{cases} S(x_j - 0) = S(x_j + 0) \\ S'(x_j - 0) = S'(x_j + 0); \quad (j = 1, 2, \cdots, n-1) \\ S''(x_j - 0) = S''(x_j + 0) \end{cases} \tag{4.30}$$

The $4n$ parameters are not determined by the $n + 1 + 3(n-1) = 4n - 2$ equations in Eq. (4.29) and Eq. (4.30). Thus, the cubic splines $S(x)$ could be uniquely determined requiring two conditions. The boundary conditions are used generally:

(a)

$$S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n), (\textbf{clamped boundary}) \tag{4.31}$$

(b)

$$S''(x_0) = f''(x_0), \quad S''(x_n) = f''(x_n) \tag{4.32}$$

Specially $S''(x_0) = 0, S''(x_n) = 0$, (**natural boundary**)

(c) If $f(x_0) = f(x_n)$, let

$$S'(x_0) = S'(x_n), \quad S''(x_0) = S''(x_n). \tag{4.33}$$

4.5.2 Construction of a Cubic Spline

Suppose $S''(x_j) = M_j, S''(x_{j+1}) = M_{j+1}$, are known on each subinterval $[x_j, x_{j+1}]$, then $S''(x)$ is decided as follows:

$$S''(x) = M_j + \frac{1}{h_j}(M_{j+1} - M_j)(x - x_j), \quad x \in [x_j, x_{j+1}], \tag{4.34}$$

where $h_j = x_{j+1} - x_j, j = 0, 1, \cdots, n-1$.

Integrate the above equation from x_j to x_{j+1} and we have

$$S'(x) = c_j + M_j(x - x_j) + \frac{1}{2h_j}(M_{j+1} - M_j)(x - x_j)^2, \quad x \in [x_j, x_{j+1}]. \tag{4.35}$$

Integrating the above equation from x_j to x_{j+1} again and by $S(x_j) = y_j$, there is

$$\begin{aligned} S(x) =& y_j + c_j(x - x_j) + \frac{1}{2}M_j(x - x_j)^2 \\ &+ \frac{1}{6h_j}(M_{j+1} - M_j)(x - x_j)^3, \quad x \in [x_j, x_{j+1}]. \end{aligned} \tag{4.36}$$

Applying $S(x_{j+1}) = y_{j+1}$, we obtain

$$c_j = f[x_j, x_{j+1}] - \left(\frac{1}{3}M_j + \frac{1}{6}M_{j+1}\right)h_j. \tag{4.37}$$

Thus

$$\begin{aligned} S(x) = y_j + &\left\{f[x_j, x_{j+1}] - \left(\tfrac{1}{3}M_j + \tfrac{1}{6}M_{j+1}\right)h_j\right\}(x - x_j) \\ &+ \tfrac{1}{2}M_j(x - x_j)^2 + \tfrac{1}{6h_j}(M_{j+1} - M_j)(x - x_j)^3 \\ &(x \in [x_j, x_{j+1}]; j = 0, 1, \cdots, n-1). \end{aligned} \tag{4.38}$$

Since Eq. (4.35) and (4.37), we have

$$S'(x_j + 0) = c_j = f[x_j, x_{j+1}] - \left(\frac{1}{3}M_j + \frac{1}{6}M_{j+1}\right)h_j \quad (j = 0, 1, \cdots, n-1) \tag{4.39}$$

and

$$\begin{aligned} S'(x_{j+1} - 0) &= c_j + M_j h_j + \frac{1}{2}(M_{j+1} - M_j)h_j \\ &= f[x_j, x_{j+1}] + \left(\frac{1}{6}M_j + \frac{1}{3}M_{j+1}\right)h_j \quad (j = 0, 1, 2\cdots, n-1) \end{aligned}$$

i.e.

$$S'(x_j - 0) = f[x_{j-1}, x_j] + \left(\frac{1}{6}M_{j-1} + \frac{1}{3}M_j\right)h_{j-1} \quad (j = 1, 2, \cdots, n). \tag{4.40}$$

Taking Eq. (4.39) and (4.40) into the formula

$$S'(x_j - 0) = S'(x_j + 0) \quad (j = 1, 2, \cdots, n-1),$$

we obtain

$$f[x_{j-1}, x_j] + \left(\tfrac{1}{6}M_{j-1} + \tfrac{1}{3}M_j\right) h_{j-1} = f[x_j, x_{j+1}] - \left(\tfrac{1}{3}M_j + \tfrac{1}{6}M_{j+1}\right) h_j \quad (j = 1, 2, \cdots, n-1)$$

Then

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j \quad (j = 1, 2, \cdots, n-1) \tag{4.41}$$

where

$$\mu_j = \frac{h_{j-1}}{h_{j-1} + h_j}, \quad \lambda_j = \frac{h_j}{h_{j-1} + h_j} = 1 - \mu_j, \quad d_j = 6f[x_{j-1}, x_j, x_{j+1}].$$

Here are $n-1$ equations and we will require another two equations to determine $4n$ constants.

If applying the boundary condition (4.31), $S'(x_0) = f'(x_0), S'(x_n) = f'(x_n)$, we take them to Eqs. (4.39) and (4.40) and obtain

$$f[x_0, x_1] - \left(\frac{1}{3}M_0 + \frac{1}{6}M_1\right) h_0 = f'(x_0)$$
$$f[x_{n-1}, x_n] + \left(\frac{1}{6}M_{n-1} + \frac{1}{3}M_n\right) h_{n-1} = f'(x_n)$$

i.e.

$$2M_0 + M_1 = 6f[x_0, x_0, x_1] \equiv d_0 \tag{4.42}$$
$$M_{n-1} + 2M_n = 6f[x_{n-1}, x_n, x_n] \equiv d_n \tag{4.43}$$

The Eqs. (4.41), (4.42) and (4.43) are written in the matrix form

$$\begin{bmatrix} 2 & 1 & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \tag{4.44}$$

If using the boundary condition (4.32), $M_0 = f''(x_0), M_n = f''(x_n)$, the first and the last equations in (4.41) are

$$2M_1 + \lambda_1 M_2 = d_1 - \mu_1 f''(x_0)$$
$$\mu_{n-1} M_{n-2} + 2M_{n-1} = d_{n-1} - \lambda_{n-1} f''(x_n).$$

The $(n-1)$ equations are rewritten in the matrix form:

$$\begin{bmatrix} 2 & \lambda_1 & & & & \\ \mu_2 & 2 & \lambda_2 & & & \\ & \mu_3 & 2 & \lambda_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-2} & 2 & \lambda_{n-2} \\ & & & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - \mu_1 f''(x_0) \\ d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} f''(x_n) \end{bmatrix} \tag{4.45}$$

If adapting Eq. (4.33), $S'(x_0) = S'(x_n)$, we have

$$f[x_0, x_1] - \left(\frac{1}{3}M_0 + \frac{1}{6}M_1\right) h_0 = f[x_{n-1}, x_n] + \left(\frac{1}{6}M_{n-1} + \frac{1}{3}M_n\right) h_{n-1} \tag{4.46}$$

Since $S''(x_0) = S''(x_n)$, then

$$M_0 = M_n$$

Eq. (4.46) becomes

$$\lambda_n M_1 + \mu_n M_{n-1} + 2M_n = d_n,$$

where

$$\lambda_n = \frac{h_0}{h_0 + h_{n-1}}, \quad \mu_n = \frac{h_{n-1}}{h_0 + h_{n-1}}, \quad d_n = 6 \cdot \frac{f[x_0, x_1] - f[x_{n-1}, x_n]}{h_0 + h_{n-1}}.$$

The first equation in Eq. (4.41) is

$$2M_1 + \lambda_1 M_2 + \mu_1 M_n = d_1$$

The n equations are written in the matrix form:

$$\begin{bmatrix} 2 & \lambda_1 & & & & & \mu_1 \\ \mu_2 & 2 & \lambda_2 & & & & \\ & \mu_3 & 2 & \lambda_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mu_{n-2} & 2 & \lambda_{n-2} & \\ & & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ \lambda_n & & & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} \\ d_n \end{bmatrix} \tag{4.47}$$

The coefficient matrixes of linear systems (4.44),(4.45) and (4.47) are strictly diagonally dominant. The matrixes of the first two equations are tri-angular and the linear systems can be solved by Thomas algorithm.

After obtaining $M_0, M_1, \cdots, M_n$, we can get the cubic spline interpolation.

Exercise 4.8. *Construct the natural cubic spline to approximate $f(x)$ for the given data*

x	1	2	4	5
$f(x)$	1	3	4	2

Use the cubic spline to approximate $f(3)$ and $f(4.5)$.

Solution Let $x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 5$, then

$$f(x_0) = 1, \quad f(x_1) = 3, \quad f(x_2) = 4, \quad f(x_3) = 2$$
$$h_0 = x_1 - x_0 = 1, \quad h_1 = x_2 - x_1 = 2, \quad h_2 = x_3 - x_2 = 1$$
$$\mu_1 = \frac{h_0}{h_0 + h_1} = \frac{1}{3}, \quad \mu_2 = \frac{h_1}{h_1 + h_2} = \frac{2}{3}$$
$$f[x_0, x_1, x_2] = -\frac{1}{2}, \quad f[x_1, x_2, x_3] = -\frac{5}{6}$$

Since it is natural cubic, $M_0 = M_3 = 0$, the linear system is

$$\begin{bmatrix} 2 & \frac{2}{3} \\ \frac{2}{3} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = 6 \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{6} \end{bmatrix}$$

Solving the above equations to obtain $M_1 = -\frac{3}{4}, M_2 = -\frac{9}{4}$.

Taking M_0, M_1, M_2 and M_3 to Eq. (4.38), we have

$$S(x) = \begin{cases} 1 + \frac{17}{8}(x-1) - \frac{1}{8}(x-1)^3 & (1 \leqslant x \leqslant 2); \\ 3 + \frac{7}{4}(x-2) - \frac{3}{8}(x-2)^2 - \frac{1}{8}(x-2)^3 & (2 \leqslant x < 4); \\ 4 - \frac{5}{4}(x-4) - \frac{9}{8}(x-4)^2 + \frac{3}{8}(x-4)^3 & (4 \leqslant x \leqslant 5). \end{cases}$$

Thus $f(3) \approx S(3) = \frac{17}{4}, f(4.5) \approx S(4.5) = \frac{201}{64}$.

4.5.3 Convergence of cubic spline

If $g \in C[a,b]$, denote

$$\|g\|_\infty = \max_{a \leqslant x \leqslant b} |g(x)|$$

Theorem 31. *Suppose $f(x) \in C^4[a,b], S(x)$ is a cubic spline interpolation with boundary condition (4.31) or (4.32). The error can be estimated*

$$\left\| f^{(k)} - S^{(k)} \right\|_\infty \leqslant c_k h^{4-k} \left\| f^{(4)} \right\|_\infty \quad (k = 0, 1, 2) \tag{4.48}$$

where $h = \max\limits_{0 \leqslant j \leqslant n-1} h_j, h_j = x_{j+1} - x_j, c_0 = \frac{1}{16}, c_1 = c_2 = \frac{1}{2}$.

4.6 Exercise

1. Let $f(x)=\sqrt{x}$.

   (1) Use second interpolating polynomial by the nodes $x = 100, 121, 144$ to approximate $\sqrt{115}$, and find an error bound for the approximation;

   (2) Use interpolating polynomial of degree 3 by the nodes $x = 100, 121, 144, 169$ to approximate $\sqrt{115}$, and find an error bound for the approximation;

2. Suppose $f(x)$ is interpolating polynomial of degree at most n by the distinct nodes $x_0, x_1, \cdots, x_n$. Prove

   (1) $f(x)=\sum_{k=0}^{n} f\left(x_k\right) l_k(x)$;

   (2) $\sum_{k=0}^{n} l_k(x)=1$;

3. Suppose $x_j (j=0,1,\cdots,n)$ are distinct. Prove

   (1) $\sum_{j=0}^{n} x_j^k l_j(x)=x^k \quad (k=0,1,\cdots,n)$;

   (2) $\sum_{j=0}^{n}\left(x_j-x\right)^k l_j(x)=0 \quad (k=1,\cdots,n)$

4. Suppose $f \in C^2[a,b]$, and $f(a)=f(b)=0$ and prove

$$\max_{a<x<b}|f(x)| \leqslant \frac{1}{8}(b-a)^2 \cdot \max_{a \leq x<b}\left|f''(x)\right|.$$

5. For the given data,

x	0	1	2	4	5	9
y	9	7	6	4	3	1

   determine all the divided difference and Newton's interpolatory divided difference formula.

6. Suppose $f(x)=x^7+x^4+3x+1$ and compute $f\left[2^0, 2^1, \cdots, 2^7\right]$ and $f\left[2^\circ, 2^1, \cdots, 2^8\right]$.

7. Prove

   (1) $\sum_{j=0}^{n-1} \Delta^2 y_j=\Delta y_n-\Delta y_0$;

   (2)$\Delta\left(f_k g_k\right)=f_k \Delta g_k+g_{k+1} \Delta f_k$.

8. Let $f(x)=\mathrm{e}^x$, $x=0$ is the interpolating node of multiplicity 4 and $x=1$ is the simple interpolating node. Try to determine the Hermite interpolating polynomial and estimate the interpolating error. $(x \in[0,1])$

9. Construct an interpolating polynomial $H_3(x)$ in [a,b] such that

$$H_3(a)=f(a), \quad H_3'(a)=f'(a), \quad H_3''(a)=f''(a), \quad H_3''(b)=f''(b).$$

10. Construct a polynomial of degree at most three with

$$H(a) = 0, \quad H(b) = 0, \quad H''(a) = b, \quad H''(b) = a.$$

11. Derive a Hermite piecewise cubic polynomial to approximate $f(x) = x^4$ in [0,5] and estimate the error. ($h = 1$)

12. Construct a natural cubic spline interpolation with the given data:

i	0	1	2	3
x_i	3	4	6	8
y_i	6	0	2	-1