5

Approximation Theory

CONTENTS

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5.1 Best Uniform Approximation

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5.1.1 Normed Linear Space

Definition 23. *A sequence of $\varphi_1, \varphi_2, \cdots, \varphi_m$ from a linear space X is said to be* **linearly dependent***, if there exist $a_1, a_2, \cdots, a_m \in \mathbf{R}$, not all zero, such that*

$$a_1\varphi_1 + a_2\varphi_2 + \cdots + a_m\varphi_m = 0.$$

Otherwise, they are **linearly independent** *if the above equation can only be satisfied by $a_1 = a_2 = \cdots = a_m = 0$.*

Suppose $\varphi_1, \varphi_2, \cdots, \varphi_m \in X$ are linearly independent, and for any $\varphi \in X$, it can be expressed

$$\varphi = a_1\varphi_1 + a_2\varphi_2 + \cdots + a_m\varphi_m,$$

where $\varphi_1, \varphi_2, \cdots, \varphi_m$ are basis of X, i.e. $X = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_m\}$, and the dimension of X is m.

Suppose M_n is a set of polynomial of degree at most n, and $M_n = \text{span}\{1, x, \cdots, x^n\}$ or $M_n = \text{span}\{l_0(x), l_1(x), \cdots, l_n(x)\}$. The dimension of M_n is $m+1$.

$C[a,b]$ is called a linear space such that for any $f \in C[a,b], g \in C[a,b], \lambda \in \mathbf{R}$, and

$$(\lambda f)(x) = \lambda \cdot f(x)$$
$$(f+g)(x) = f(x) + g(x).$$

The finite dimensional linear space $M_n \subset C[a,b]$, for any n, and $C[a,b]$ is an infinite dimensional linear space.

Definition 24. *A* **norm** *on linear space X is a function, $\|\cdot\|$, from X into $\mathbf{R}$ with the following properties:*

1° $\|x\| \geqslant 0$ *for any* $x \in X$, $\|x\| = 0$ *if and only if* $x = 0$*;*

2° $\|\lambda x\| = |\lambda| \cdot \|x\|$ *for any* $\lambda \in \mathbf{R}, x \in X$*;*

3° $\|x + y\| \leqslant \|x\| + \|y\|$ *for any* $x \in X, y \in X$.

Then X is called **normed linear space**.

Definition 25. *Suppose X is a normed linear space. $\|x - y\|$ is called a* **distance** *between x and y for any $x \in X, y \in X$.*

Exercise 5.1. *(Normed linear space $\mathbf{R}^n$)* *Suppose $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbf{R}^n, \boldsymbol{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \in \mathbf{R}^n, \lambda \in \mathbf{R}$, and the linear operations are defined as follows*

$$\lambda x = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)^{\mathrm{T}},$$
$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)^{\mathrm{T}}.$$

Then $\mathbf{R}^n$ is a linear space. Let

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|\boldsymbol{x}\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|, \quad \|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \tag{5.1}$$

It is easy to verify $\|\cdot\|_1, \|\cdot\|_\infty$ and $\|\cdot\|_2$ are norms on $\mathbf{R}^n$.

Exercise 5.2. *(Normed linear space $C[a,b]$)* *Suppose $f \in C[a,b]$. Let*

$$\|f\|_1 = \int_a^b |f(x)| \mathrm{d}x, \quad \|f\|_\infty = \max_{a \leqslant x \leqslant b} |f(x)|, \|f\|_2 = \sqrt{\int_a^b f^2(x) \mathrm{d}x}. \tag{5.2}$$

It is easy to verify $\|\cdot\|_1, \|\cdot\|_\infty$ and $\|\cdot\|_2$ are norms on $C[a,b]$.

It is natural to use infinity norm to describe the approximation between two functions, i.e.

$$\|f - g\|_\infty = \max_{a \leqslant x \leqslant b} |\ f(x) - g(x)\ |$$

Definition 26. *For given $f \in X$, suppose X is a normed linear space and $M \subseteq X$. A $\varphi \in M$ is said* **best approximation** *to f such that*

$$\|f-\varphi\| \leqslant \|f-\psi\|$$

for any $\psi \in M$.

5.1.2 Polynomial of Best Uniform Approximation

Definition 27. *For given $f \in C[a,b]$, a polynomial $p_n \in M_n$, said* **a polynomial of best uniform approximation** *, exists with*

$$\|f-p_n\|_\infty \leqslant \|f-q_n\|_\infty\,,$$

i.e.

$$\max_{a \leqslant x \leqslant b} |f(x)-p_n(x)| \leqslant \max_{a \leqslant x \leqslant b} |f(x)-q_n(x)|$$

for any $q_n \in M_n$.

Theorem 32. *For given $f \in C[a,b]$, the polynomial $p_n(x)$ of best uniform approximation of degree at most n to f is unique.*

Definition 28. *Suppose $g \in C[a,b]$. The node $x_k \in [a,b]$ is said a* **point of maximum deviation** *such that*

$$|g(x_k)| = \|g\|_\infty.$$

If

$$g(x_k) = \|g\|_\infty,$$

the node x_k is said **a point of positive deviation** *or* **a** $(+)-$*point.*

If

$$g(x_k) = -\|g\|_\infty,$$

the node x_k is said **a point of negative deviation** *or* **a** $(-)-$*point.*

Lemma 1. *For given function $f \in C[a,b], p_n(x)$ is a polynomial of best uniform approximation of degree $\leq n$ to $f(x)$. Then $f-p_n$ must have points of positive and negative deviation.*

Proof Let $E_n = \|f-p_n\|_\infty$. By contradiciton, assume $f-p_n$ has only points of positive deviation, i.e.

$$-E_n < f(x)-p_n(x) \leqslant E_n.$$

Since the functions f and p_n are continuous, $f-p_n$ can achieve the minimum value on $[a,b]$. Then there exist $\varepsilon \in (0, E_n)$, such that

$$-E_n + 2\varepsilon \leqslant f(x)-p_n(x) \leqslant E_n.$$

That is

$$-\left(E_n-\varepsilon\right) \leqslant f(x)-\left(p_n(x)+\varepsilon\right) \leqslant E_n-\varepsilon.$$

Thus

$$\left\|f-\left(p_n+\varepsilon\right)\right\|_{\infty}<E_n,$$

which is contradict with that p_n is a polynomial of best uniform approximation of degree at most n to f.

□

We now proceed to the main theorem which establishes the necessary and sufficient conditions for a polynomial p_n to be a polynomial of best approximation of degree n to a continuous real function f defined on $[a, b]$. This theorem was proved by Chebyshev in 1854 and marked the onset of development of the theory of approximation of functions.

Theorem 33. *Assume that a continuous real function f is defined on a segment $[a, b]$. In order that a polynomial p_n of degree $\leq n$ be a polynomial of the best approximation to f, it is necessary and sufficient that there exist at least one system of $n+2$ points $a \leqslant x_0<x_1<\cdots<x_n<x_{n+1} \leqslant b$, such that the difference $f(x)$–$p_n(x)$ satisfying*

$$f\left(x_i\right)-p_n\left(x_i\right)=(-1)^i \sigma\left\|f-p_n\right\|_{\infty} \quad(i=0,1, \cdots, n+1), \tag{5.3}$$

where $\sigma=1$ or -1. A system of points $\{x_j\}_{j=0}^{n+1}$ satisfying Eq. (5.3) *is called* **alternation** *or* **Chebyshev alternation**.

Proof (**Sufficiency**) By contradiction, assume that there exists a polynomial q_n approximating the function f better than p_n,

$$\left\|f-q_n\right\|_{\infty}<\left\|f-p_n\right\|_{\infty}.$$

Then

$$\begin{aligned}
q_n\left(x_i\right)-p_n\left(x_i\right) & =\left[f\left(x_i\right)-p_n\left(x_i\right)\right]-\left[f\left(x_i\right)-q_n\left(x_i\right)\right] \\
& =(-1)^i \sigma\left\|f-p_n\right\|_{\infty}-\left[f\left(x_i\right)-q_n\left(x_i\right)\right] \\
& =(-1)^i \sigma\left\{\left\|f-p_n\right\|_{\infty}-(-1)^i \sigma\left[f\left(x_i\right)-q_n\left(x_i\right)\right]\right\} \\
& =(-1)^i \sigma \varepsilon_i \quad(i=0,1, \cdots, n+1),
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_i & =\left\|f-p_n\right\|_{\infty}-(-1)^i \sigma\left[f\left(x_i\right)-q_n\left(x_i\right)\right] \\
& \geqslant\left\|f-p_n\right\|_{\infty}-\left|f\left(x_i\right)-q_n\left(x_i\right)\right| \\
& \geqslant\left\|f-p_n\right\|_{\infty}-\left\|f-q_n\right\|_{\infty}>0 .
\end{aligned}$$

Thus the difference q_n-p_n changes its sign on $[a, b]$ at least $n+1$ times. This means that the polynomial q_n-p_n has at least $n+1$ zeros on the interval $[a, b]$ but this is impossible because q_n-p_n is a polynomial of degree n by virtue of the assumption that $q_n \neq p_n$.

(**Necessity**) The proof can be seen in detail in the book: *Theory of Uniform Approximation of Functions by Polynomials.*

□

Corollary 2. *For given function $f \in C[a,b]$, suppose p_n is the polynomial of best uniform approximation of degree at most n to f. If $f^{(n+1)}$ exists and keep sign in (a,b), the alternation has only $(n+2)$ points in $[a,b]$ including the endpoints a,b.*

Proof

By contradiction, assume that the number of points of maximum deviation more than $(n+2)$ or at least one of the endpoints is not a point of maximum deviation. Then the sequence of points of maximum deviation in (a,b) would have at least $n+1$ points, i.e. $x_0 < x_1, \cdots < x_n$. From Fermat Theorem, we have

$$f'(x_i) - p_n'(x_i) = 0 \quad (i = 0, 1, \cdots, n)$$

By Rolle Theorem, the function $f'' - p_n''$ has at least n distinct zeros. Hence the function $f^{(n+1)} - p_n^{(n+1)}$ has at least one zero, $\xi \in (a,b)$, i.e.

$$f^{(n+1)}(\xi) - p_n^{(n+1)}(\xi) = 0.$$

Since $p_n^{(n+1)}(x) = 0$, there is

$$f^{(n+1)}(\xi) = 0,$$

which is contract with $f^{(n+1)}(x)$ keeps sign in (a,b).

□

From Corollary 2, the polynomial of best uniform approximation to a given function $f \in C[a,b]$ and $f^{(n+1)}$ keeping sign in (a,b) is

$$p_n(x) = c_0 + c_1 x + \cdots + c_n x^n,$$

then there exists $(n+2)$ points $a < x_1 < x_2 < \cdots < x_n < b$ satisfying

$$\begin{aligned} f(a) - p_n(a) &= -\left[f(x_1) - p_n(x_1)\right] = f(x_2) - p_n(x_2) \\ &= \cdots \\ &= (-1)^n \left[f(x_n) - p_n(x_n)\right] = (-1)^{n+1}\left[f(b) - p_n(b)\right] \end{aligned}$$

$$f'(x_i) - p_n'(x_i) = 0 \quad (i = 1, 2, \cdots, n) \tag{5.4}$$

which is nonlinear equations about the $(2n+1)$ unknowns $(c_0, c_1, \cdots, c_n, x_1, x_2, \cdots, x_n)$.

Let $\mu = \sigma \|f - p_n\|$, then the above equations can be rewritten as follows

$$\begin{cases} f(a) - p_n(a) - \mu = 0 \\ f(x_i) - p_n(x_i) - (-1)^i \mu = 0, \quad (i = 1, 2, \cdots, n) \\ f(b) - p_n(b) - (-1)^{n+1}\mu = 0, \\ f'(x_i) - p_n'(x_i) = 0 \end{cases} \tag{5.5}$$

The above nonlinear equations can be solved by iterative method.

Exercise 5.3. *For given $f \in C^2[a,b], f''(x) > 0$ for each $x \in (a,b)$. Construct a linear polynomial $p_1(x)$ of best uniform approximation to $f(x)$.*

Solution Suppose $p_1(x) = c_0 + c_1 x$, then there exist three points $a < x_1 < b$ satisfying

$$\begin{cases} f(a) - p_1(a) = -\left[f\left(x_1\right) - p_1\left(x_1\right)\right] = f(b) - p_1(b) \\ f'\left(x_1\right) - p_1'\left(x_1\right) = 0 \end{cases}$$

Taking $p_1(x)$ to the following equation

$$f(a) - p_1(a) = f(b) - p_1(b),$$

we have

$$f(a) - \left(c_0 + c_1 a\right) = f(b) - \left(c_0 + c_1 b\right).$$

Thus

$$c_1 = \frac{f(b) - f(a)}{b - a}.$$

Since

$$f'\left(x_1\right) - p_1'\left(x_1\right) = 0,$$

there is

$$f'\left(x_1\right) - c_1 = 0.$$

So

$$x_1 = \left(f'\right)^{-1}\left(c_1\right) = \left(f'\right)^{-1}\left(\frac{f(b) - f(a)}{b - a}\right).$$

The constant

$$c_0 = \frac{f(a) + f\left(x_1\right)}{2} - c_1 \frac{a + x_1}{2}$$

can be obtained by

$$f(a) - p_1(a) = -\left[f\left(x_1\right) - p_1\left(x_1\right)\right]$$

The linear polynomial $p_1(x)$ of best uniform approximation to $f(x)$ is plotted in Fig. 5.1.

Corollary 3. *For given function $f(x) \in C[a,b]$, the polynomial $p_n(x)$ of best uniform approximation to $f(x)$ of degree n can be seen as an interpolating polynomial about some nodes.*

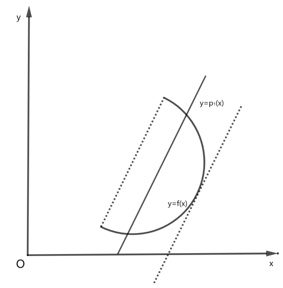

FIGURE 5.1
Linear polynomial of best uniform approximation

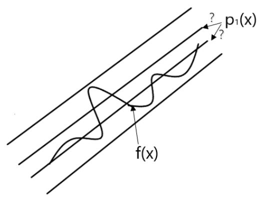

FIGURE 5.2

5.2 Least Square Approximation

In the above section, the maximum of the absolute error in the interval is considered in the best uniform approximation. But it is difficult to obtain the polynomial of best uniform approximation by solving a nonlinear system and it can not be adaptable for the function which changing larger on some small interval. For example, we think the lower line is better than the upper line to approximate $f(x)$ in Fig. 5.2, because the lower line is closed to original curve as a whole.

5.2.1 Inner Product Space

Definition 29. *An* **inner product space** *is a linear space X over* **R** *together with a map*

$$(\cdot, \cdot) : X \times X \to R$$

called an **inner product** *that satisfies the following conditions for any $x, y, z \in X$*

1° $(x, y) = (y, x)$*;*

2° $(\lambda x, y) = \lambda(x, y)$ *for* $\lambda \in \mathbf{R}$*;*
3° $(x + y, z) = (x, z) + (y, z)$;
4° $(x, x) \geqslant 0$, *and* $(x, x) = 0$ *if and only if* $x = 0$.

Definition 30. *Suppose* X *is an inner product space, and* $x, y \in X$. x *and* y *is called* **orthogonal** *if their inner product is zero, i.e.* $(x, y) = 0$.

Exercise 5.4. *For any* $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}, y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$ *of* $\mathbf{R}^n$, *let*

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

It is easy to verify (x, y) *is an inner product on* $\mathbf{R}^n$.

Exercise 5.5. *For any* $f, g \in C[a, b]$, *let*

$$(f, g) = \int_a^b f(x)g(x)\mathrm{d}x$$

It is also easy to verify (f, g) *is an inner product on* $C[a, b]$.

Lemma 2. *(Cauchy-Schwarz Inequality)* *For any* x, y *of an inner product space* X, *it holds*

$$(x, y)^2 \leqslant (x, x)(y, y) \tag{5.6}$$

Proof If $x = 0$, (5.6) holds naturally. If $x \neq 0$, then

$$(\lambda x + y, \lambda x + y) = (x, x)\lambda^2 + 2(x, y)\lambda + (y, y) \geqslant 0$$

where λ is a real number. The above equation holds if and only if

$$\Delta = 4(x, y)^2 - 4(x, x)(y, y) \leqslant 0$$

i.e.

$$(x, y)^2 \leqslant (x, x)(y, y).$$

$\square$

For an inner product space X, and any $x \in X$. Denote

$$\|x\| = \sqrt{(x, x)}. \tag{5.7}$$

Next we will show it is a norm in X.

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) \\ &= (x, x) + 2(x, y) + (y, y) \\ &\leqslant (x, x) + 2\sqrt{(x, x)} \cdot \sqrt{(y, y)} + (y, y) \\ &= \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Then

$$\|x+y\| \leqslant \|x\| + \|y\|.$$

It easy to verify $\|x\| = \sqrt{(x,x)}$ satisfies the definition of norm. Hence, the inner product space with the norm (5.7) is also a normed linear space.

Exercise 5.6. $\mathbf{R}^n$ *:*

$$(x,y) = \sum_{i=1}^{n} x_i y_i.$$

The norm is defined

$$\|x\| = \sqrt{(x,x)}.$$

It is just the $\|x\|_2$ *on* $\mathbf{R}^n$*.*

Exercise 5.7. $C[a,b]$*:*

$$(f,g) = \int_a^b f(x)g(x)\mathrm{d}x$$

The norm is

$$\|f\| = \sqrt{(f,f)}$$

It is $\|f\|_2$ *on* $C[a,b]$*.*

5.2.2 Least Squares Approximation

Suppose X is an inner product space and $M \subseteq X$ is a finite dimensional linear space. $M = \mathrm{span}\{\varphi_0, \varphi_1, \cdots, \varphi_m\}$. For given $f \in X$, $\varphi \in M$ is called **least squares approximation** to f if

$$\|f-\varphi\| \leqslant \|f-\psi\|, \quad \forall \psi \in M \tag{5.8}$$

or

$$\|f-\varphi\| = \min_{\varphi \in M} \|f-\psi\|$$

Let $\varphi = \sum\limits_{i=0}^{m} c_i \varphi_i, \psi = \sum\limits_{i=0}^{m} a_i \varphi_i$, then the problem of (5.8) is to find the constants $c_0, c_1, \cdots, c_m$ such that

$$\left\| f - \sum_{i=0}^{m} c_i \varphi_i \right\|^2 = \min_{a_0, a_1, \cdots, a_m \in \mathbf{R}} \left\| f - \sum_{i=0}^{m} a_i \varphi_i \right\|^2.$$

Denote

$$\Phi\left(a_0, a_1, \cdots, a_m\right) = \left\| f - \sum_{i=0}^{m} a_i \varphi_i \right\|^2,$$

then the constants $(c_0, c_1, \cdots, c_m)$ satisfy

$$\Phi(c_0, c_1, \cdots, c_m) = \min_{a_0, a_1, \cdots, a_m \in \mathbf{R}} \Phi(a_0, a_1, \cdots, a_m).$$

It is noticed that

$$\begin{aligned}\Phi(a_0, a_1, \cdots, a_m) &= \left(f - \sum_{i=0}^{m} a_i \varphi_i, f - \sum_{j=0}^{m} a_j \varphi_j\right) \\ &= (f, f) - 2\sum_{i=0}^{m} a_i (f, \varphi_i) + \sum_{i,j=0}^{m} a_i a_j (\varphi_i, \varphi_j).\end{aligned}$$

Then

$$\frac{\partial \Phi}{\partial a_k} = -2(f, \varphi_k) + 2\sum_{i=0}^{m} a_i (\varphi_i, \varphi_k) \quad (k = 0, 1, \cdots, m).$$

Hence we get the linear **normal equations**

$$\begin{bmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) & \cdots & (\varphi_0, \varphi_m) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) & \cdots & (\varphi_1, \varphi_m) \\ \vdots & \vdots & & \vdots \\ (\varphi_m, \varphi_0) & (\varphi_m, \varphi_1) & \cdots & (\varphi_m, \varphi_m) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} (f, \varphi_0) \\ (f, \varphi_1) \\ \vdots \\ (f, \varphi_m) \end{bmatrix} \tag{5.9}$$

Lemma 3. *The coefficient matrix A of linear normal equations* (5.9) *is symmetric and positive definite where A is*

$$\boldsymbol{A} = \begin{bmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) & \cdots & (\varphi_0, \varphi_m) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) & \cdots & (\varphi_1, \varphi_m) \\ \vdots & \vdots & & \vdots \\ (\varphi_m, \varphi_0) & (\varphi_m, \varphi_1) & \cdots & (\varphi_m, \varphi_m) \end{bmatrix}$$

Proof It is easy to prove A is symmetric. Let $\boldsymbol{x} = (x_0, x_1, \cdots, x_m)^{\mathrm{T}} \neq \mathbf{0}$ and

$$\varphi = \sum_{i=0}^{m} x_i \varphi_i.$$

Since $\varphi_0, \varphi_1, \cdots, \varphi_m$ are linearly independent, $\varphi \neq 0$. Then

$$\boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x} = \|\varphi\|^2 > 0$$

So A is positive.

□

Theorem 34. *The linear normal equations* (5.9) *has only solutions* $(c_0, c_1, \cdots, c_m)^{\mathrm{T}}$.

Theorem 35. *The unique solutions $(c_0, c_1, \cdots, c_m)$ of normal equations* (5.9) *minimize $\Phi(a_0, a_1, \cdots, a_m)$.*

5.2.3 Least Squares Approximation to continuous function

Suppose $f(x) \in C[a,b], M = \operatorname{span}\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_m(x)\} \subseteq C[a,b]$. For each $q(x) \in M$, it can be written as follows

$$q(x) = \sum_{i=0}^{m} a_i \varphi_i(x).$$

Let

$$\Phi\left(a_0, a_1, \cdots, a_m\right) = \|f - q\|^2 = \int_a^b \left[f(x) - \sum_{i=0}^{m} a_i \varphi_i(x)\right]^2 \mathrm{d}x$$

The function $p(x)$ is said least square approximation with

$$\|f - p\|_2 \leqslant \|f - q\|_2, \quad \text{for all } q \in M. \tag{5.10}$$

Let $p(x) = \sum\limits_{i=0}^{m} c_i \varphi_i(x)$. The problem (5.10) is to find $(c_0, c_1, \cdots, c_m)$, such that

$$\Phi\left(c_0, c_1, \cdots, c_m\right) = \min_{a_0, a_1, \cdots, a_m \in \mathbf{R}} \Phi\left(a_0, a_1, \cdots, a_m\right).$$

From the above section, $\left(c_0, c_1, \cdots, c_m\right)^{\mathrm{T}}$ are solutions of the following normal equations

$$\begin{bmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) & \cdots & (\varphi_0, \varphi_m) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) & \cdots & (\varphi_1, \varphi_m) \\ \vdots & \vdots & & \vdots \\ (\varphi_m, \varphi_0) & (\varphi_m, \varphi_1) & \cdots & (\varphi_m, \varphi_m) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} (f, \varphi_0) \\ (f, \varphi_1) \\ \vdots \\ (f, \varphi_m) \end{bmatrix}$$

where

$$(\varphi_i, \varphi_j) = \int_a^b \varphi_i(x) \varphi_j(x) \mathrm{d}x, \quad (f, \varphi_i) = \int_a^b f(x) \varphi_i(x) \mathrm{d}x.$$

Definition 31. *The function $p(x)$ is said a least squares approximating polynomial of degree m for $f(x)$ on $[a,b]$ if $\varphi_i(x) = x^i (i = 0, 1, \cdots, m)$.*

Exercise 5.8. *Find the least squares approximating polynomial p_2 of degree 2 for $f(x) = \mathrm{e}^x$ on $[0,1]$*

$$p_2(x) = c_0 + c_1 x + c_2 x^2$$

Solution Let $\varphi_0(x) = 1, \varphi_1(x) = x, \varphi_2(x) = x^2$, we have

$$(\varphi_0, \varphi_0) = \int_0^1 1\mathrm{d}x = 1, \quad (\varphi_0, \varphi_1) = \int_0^1 x\mathrm{d}x = \frac{1}{2}$$
$$(\varphi_0, \varphi_2) = \int_0^1 x^2\mathrm{d}x = \frac{1}{3}, \quad (\varphi_1, \varphi_1) = \int_0^1 x^2 = \frac{1}{3}$$
$$(\varphi_1, \varphi_2) = \int_0^1 x^3\mathrm{d}x = \frac{1}{4}, \quad (\varphi_2, \varphi_2) = \int_0^1 x^4\mathrm{d}x = \frac{1}{5}$$
$$(f, \varphi_0) = \int_0^1 \mathrm{e}^x\mathrm{d}x = \mathrm{e} - 1, \quad (f, \varphi_1) = \int_0^1 x\mathrm{e}^x\mathrm{d}x = 1$$
$$(f, \varphi_2) = \int_0^1 x^2\mathrm{e}^x\mathrm{d}x = \mathrm{e} - 2$$

The normal equations are

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}-1 \\ 1 \\ \mathrm{e}-2 \end{bmatrix}$$

These equations in three unknowns can be solved to obtain $c_0 = 39\mathrm{e} - 105, c_1 = 588 - 216\mathrm{e}, c_2 = 210\mathrm{e} - 570$. Consequently, the least squares polynomial approximation of degree 2 for f on $[0, 1]$ is

$$\begin{aligned} p_2(x) &= 39\mathrm{e} - 105 + (588 - 216\mathrm{e})x + (210\mathrm{e} - 570)x^2 \\ &= 1.0130 + 0.8515x + 0.8392x^2. \end{aligned}$$

Exercise 5.9. *Find c, d such that minimizing*

$$\int_0^1 \left(x^3 - c - dx^2\right) \mathrm{d}x.$$

Solution This problem is to find the least squares polynomial approximation $p(x) = c + dx^2$ for $f(x) = x^3$ on $[0, 1]$. Let $\varphi_0(x) = 1, \varphi_1(x) = x^2$, then

$$(\varphi_0, \varphi_0) = \int_0^1 1\mathrm{d}x = 1, \quad (\varphi_0, \varphi_1) = \int_0^1 x^2 = \frac{1}{3}$$
$$(\varphi_1, \varphi_1) = \int_0^1 x^4\mathrm{d}x = \frac{1}{5}, \quad (f, \varphi_0) = \int_0^1 x^3\mathrm{d}x = \frac{1}{4}$$
$$(f, \varphi_1) = \int_0^1 x^5\mathrm{d}x = \frac{1}{6}$$

The normal equations are

$$\begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{6} \end{bmatrix}$$

By solving the above equations, we get $c = -\frac{1}{16}, d = \frac{15}{16}$.

5.2.4 Least Squares Solution of Overdetermined Linear Equations

For the given linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

where $m > n$, suppose the column vectors of the coefficient matrix A are linearly independent. The equations is said **overdetermined linear equations** and there are no exact solutions in general. Let

$$\boldsymbol{A}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad (j = 1, 2, \cdots, n), \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots & \boldsymbol{A}_n. \end{pmatrix}$$

The linear equations can be written as

$$x_1 \boldsymbol{A}_1 + x_2 \boldsymbol{A}_2 + \cdots + x_n \boldsymbol{A}_n = \boldsymbol{b}.$$

Denote $M = \text{span}\{\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n\} \subseteq \mathbf{R}^m$. Let

$$\Phi(x_1, x_2, \cdots, x_n) = \left\| \boldsymbol{b} - \sum_{i=1}^{n} x_i \boldsymbol{A}_i \right\|_2^2$$

Find $(x_1^*, x_2^*, \cdots, x_n^*)$, such that

$$\Phi(x_1^*, x_2^*, \cdots, x_n^*) = \min_{x_1, x_2, \cdots, x_n \in \mathbb{R}} \Phi(x_1, x_2, \cdots, x_n)$$

From Sec. 5.2.2, $(x_1^*, x_2^*, \cdots, x_n^*)^{\mathrm{T}}$ are the solutions of the following linear equations

$$\begin{bmatrix} (\boldsymbol{A}_1, \boldsymbol{A}_1) & (\boldsymbol{A}_1, \boldsymbol{A}_2) & \cdots & (\boldsymbol{A}_1, \boldsymbol{A}_n) \\ (\boldsymbol{A}_2, \boldsymbol{A}_1) & (\boldsymbol{A}_2, \boldsymbol{A}_2) & \cdots & (\boldsymbol{A}_2, \boldsymbol{A}_n) \\ \vdots & \vdots & & \vdots \\ (\boldsymbol{A}_n, \boldsymbol{A}_1) & (\boldsymbol{A}_n, \boldsymbol{A}_2) & \cdots & (\boldsymbol{A}_n, \boldsymbol{A}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (\boldsymbol{b}, \boldsymbol{A}_1) \\ (\boldsymbol{b}, \boldsymbol{A}_2) \\ \vdots \\ (\boldsymbol{b}, \boldsymbol{A}_n) \end{bmatrix}$$

The above equations can be rewritten as

$$\boldsymbol{A}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A}^{\mathrm{T}} \boldsymbol{b}$$

Thus $(x_1^*, x_2^*, \cdots, x_n^*)$ are said **least squares solution of overdetermined linear equations**.

Exercise 5.10. *Find least squares solutions of the following overdetermined linear equations*

$$\begin{cases} 3x + 4y = 5 \\ -4x + 8y = 1 \\ 6x + 3y = 3 \end{cases}$$

Solution The coefficient matrix and righthand vector are

$$\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ -4 & 8 \\ 6 & 3 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{bmatrix} 61 & -2 \\ -2 & 89 \end{bmatrix}, \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 29 \\ 37 \end{bmatrix}$$

Then we have

$$\begin{bmatrix} 61 & -2 \\ -2 & 89 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 29 \\ 37 \end{bmatrix}$$

By solving the above equations, we get $x = 0.4894, y = 0.4267$.

5.2.5 Discrete Least Square Approximation

Definition 32. *A set of* $\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_m(x)\}$ *is said* **linearly independent** *about nodes* $x_1, x_2, \cdots, x_n$ *if, whenever*

$$c_0\varphi_0(x_k) + c_1\varphi_1(x_k) + \cdots + c_m\varphi_m(x_k) = 0 \quad (k = 1, 2, \cdots, n)$$

we have $c_0 = c_1 = \cdots = c_m = 0$. *Otherwise the set of functions is said to be* **linearly dependent** *about the nodes.*

Given the data

x	x_1	x_2	x_3	$\cdots$	x_n
y	y_1	y_2	y_3	$\cdots$	y_n

suppose $\varphi_0(x), \varphi_1(x), \cdots, \varphi_m(x)$ are linearly independently about $x_1, x_2, \cdots, x_n$. Let

$$q(x) = \sum_{i=0}^{m} a_i\varphi_i(x), \quad \Phi(a_0, a_1, \cdots, a_m) = \sum_{k=1}^{n} (q(x_k) - y_k)^2$$

Find the values of $(c_0, c_1, \cdots, c_m)$, to minimize

$$\Phi(c_0, c_1, \cdots, c_m) = \min_{a_0, a_1, \cdots, a_m \in \mathbb{R}} \Phi(a_0, a_1, \cdots, a_m)$$

And $p(x) = \sum\limits_{i=0}^{m} c_i\varphi_i(x)$ is said **discrete least squares function**.

If $\varphi_k(x) = x^k, 0 \leqslant k \leqslant m$, $p(x)$ is said **discrete least squares polynomial** of degree at most m. Let

$$\boldsymbol{\varphi}_k = \begin{bmatrix} \varphi_k(x_1) \\ \varphi_k(x_2) \\ \vdots \\ \varphi_k(x_n) \end{bmatrix} \quad (k = 0, 1, \cdots, m), \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

then $(c_0, c_1, \cdots, c_m)$ are solutions of the following normal equations

$$\begin{bmatrix} (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_m) \\ (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_m) \\ \vdots & \vdots & & \vdots \\ (\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_m, \boldsymbol{\varphi}_m) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} (\boldsymbol{y}, \boldsymbol{\varphi}_0) \\ (\boldsymbol{y}, \boldsymbol{\varphi}_1) \\ \vdots \\ (\boldsymbol{y}, \boldsymbol{\varphi}_m) \end{bmatrix} \tag{5.11}$$

Exercise 5.11. *Fit the following data with the discrete least squares polynomial $f(t) = c_0 + c_1 t + c_2 t^2$.*

t	*0*	*0.9*	*1.9*	*3.0*	*3.9*	*5.0*
s	*0*	*10*	*30*	*51*	*80*	*111*

Solution Let $\varphi_0(t) = 1, \varphi_1(t) = t, \varphi_2(t) = t^2$, then

$$\boldsymbol{\varphi}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{\varphi}_1 = \begin{bmatrix} 0 \\ 0.9 \\ 1.9 \\ 3.0 \\ 3.9 \\ 5.0 \end{bmatrix}, \quad \boldsymbol{\varphi}_2 = \begin{bmatrix} 0 \\ 0.81 \\ 3.61 \\ 9 \\ 15.21 \\ 25 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ 10 \\ 30 \\ 51 \\ 80 \\ 111 \end{bmatrix}$$

Taking them to normal equations (5.11), there is

$$\begin{bmatrix} 6 & 14.7 & 53.63 \\ 14.7 & 53.63 & 218.907 \\ 53.63 & 218.907 & 951.0323 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 282 \\ 1086 \\ 4567.2 \end{bmatrix}$$

Solve the system to get $c_0 = -0.6170, c_1 = 11.1586, c_2 = 2.2687$. Thus the least squares polynomial of degree 2 fitting the data is

$$f(t) = -0.6170 + 11.1586t + 2.2687t^2,$$

whose graph is shown in Fig. 5.3.

For some nonlinear least squares approximation, for example, the approximating function to be in the form $y = a\mathrm{e}^{bx}$ for some constants a and b. Consider the logarithm of the approximating equation:

$$\ln y = \ln a + bx.$$

In this case, a linear problem now appears, and solutions for $\ln a$ and b can be obtained by appropriately modifying the normal equations.

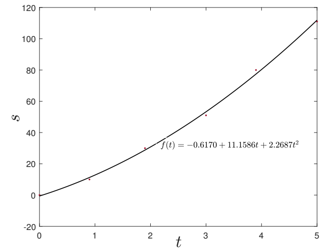

FIGURE 5.3

5.3 Exercise

1. Construct a a polynomial of best uniform approximation of degree zero to this function $f(x) \in C[a, b]$.
2. Construct a polynomial of best uniform approximation of degree six to $f(x) = \sin 4x$ on $[0, 2\pi]$.
3. Confirm a constant a such that $\max_{0 \le x \le 1} \left| x^3 - ax \right|$ achieving a minimum.
4. Construct a linear polynomial of best uniform approximation to $f(x) = x^3$ on [0,1] and estimate the maximum error.
5. Suppose $M_3 = \text{span}\left\{1, x^2, x^4\right\}$. Find a least squares polynomial $p \in M_3$for $f(x) = |x|$ on $[-1, 1]$.
6. Find the least squares solution of overdetermined linear equations
$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 + x_2 = 5 \\ 2x_1 + 2x_2 = 6 \\ -x_1 + 2x_2 = 2 \\ 3x_1 - x_2 = 4 \end{cases}$$
7. Fit the data with the discrete least squares polynomial $y = a\mathrm{e}^{-bx}$.

x	0	1	2	4
y	2.010	1.210	0.740	0.450