8

Numerical Methods for Partial Differential Equations

CONTENTS

A **partial differential equation**, or PDE, is an equation involving partial derivatives of an unknown function with respect to more than one independent variable. PDEs are of fundamental importance in modeling all types of continuous phenomena in nature.

8.1 Parabolic Partial Differential Equations

The **parabolic partial differential equation** describe time-dependent, dissipative physical processes, such as diffusion, that are evolving toward a steady state. A simple one is in the form

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + d(x,t)u = f(x,t) \tag{8.1}$$

where $(x,t) \in D$. The function $a(x,t) > 0$ in D and $d \geqslant 0$. To specify a unique solution, there are three conditions in general:

(1) If $D = \{(x,t) \mid -\infty < x < \infty, 0 \leqslant t \leqslant T\}$, we prescribe initial data for

all x at some initial time t_0, say $t_0 = 0$. A suitable initial condition therefore has the form

$$u\big|_{t=0} = \varphi(x) \quad (-\infty < x < \infty) \tag{8.2}$$

where $\varphi(x)$ is a given function defined on the real line. Such a pure initial value problem i.e. Eqs. (8.1) and (8.2) is called a **Cauchy problem**.

(2) If $D = \{(x,t) \mid 0 \leqslant x < \infty, 0 \leqslant t \leqslant T\}$, the initial condition is given by

$$u\big|_{t=0} = \varphi(x) \quad (0 < x < \infty) \tag{8.3}$$

and the boundary condition is given by

$$\left[\alpha_0(t)u - \alpha_1(t)\frac{\partial u}{\partial x}\right]\bigg|_{x=0} = \alpha_2(t) \quad (0 \leqslant t \leqslant T) \tag{8.4}$$

where $\alpha_0(t) \geqslant 0, \alpha_1(t) \geqslant 0$ and $\alpha_0(t) + \alpha_1(t) > 0$.

(3) If $D = \{(x,t) \mid 0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T\}$, we give the initial condition and boundary condition as follows:

$$u\big|_{t=0} = \varphi(x) \quad (0 \leqslant x \leqslant l) \tag{8.5}$$

and

$$\begin{cases} \left[\alpha_0(t)u - \alpha_1(t)\frac{\partial u}{\partial x}\right]\big|_{x=0} = \alpha_2(t) & (0 \leqslant t \leqslant T) \\ \left[\beta_0(t)u + \beta_1(t)\frac{\partial u}{\partial x}\right]\big|_{x=l} = \beta_2(t) & (0 \leqslant t \leqslant T) \end{cases} \tag{8.6}$$

where $\alpha_0(t), \beta_0(t), \alpha_1(t), \beta_1(t) \geqslant 0, \alpha_0(t) + \alpha_1(t) > 0, \beta_0(t) + \beta_1(t) > 0$. There are numerous possibilities for the boundary conditions that must be specified on the boundary of the domain or portions thereof:

- **Dirichlet** boundary conditions, sometimes called essential boundary conditions, in which the solution u is specified;
- **Neumann** boundary conditions, sometimes called natural boundary conditions, in which one of the derivatives u_x is specified;
- **Robin** boundary conditions, or mixed boundary conditions, in which a combination of solution values and derivative values is specified.

In this section, we consider the heat, or diffusion, equation in one dimension

$$\begin{cases} \frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = f(x,t) & (0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T) \\ u\big|_{t=0} = \varphi(x) & (0 < x < l) \\ u\big|_{x=0} = \alpha(t),\, u\big|_{x=l} = \beta(t) & (0 \leqslant t \leqslant T) \end{cases} \tag{8.7}$$

where a is a positive constant, and the given functions $f, \varphi, \alpha, \beta$ satisfy $\varphi(0) = \alpha(0), \varphi(l) = \beta(0)$. Here we suppose (8.7) has the solution $u(x,t)$, and $u(x,t)$ is enough continuous we need.

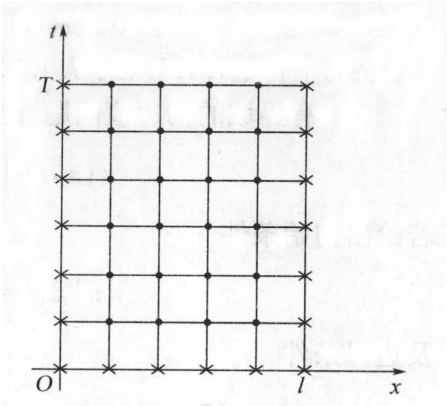

FIGURE 8.1
Grid

8.1.1 Selecting a Grid

Choose integers M and N to define space step size $h = l/M$, and time step size $\tau = T/N$. Partition the interval $[0, l]$ into M equal parts of width h and the interval [0, T] into N equal parts of width τ (see Figure 8.1).

Place a grid on the rectangle by drawing vertical and horizontal lines through the points with coordinates (x_i, t_k), where $x_i = ih$, for each $i = 0, 1, \cdots, M$, and $t_k = k\tau$, for each $k = 0, 1, \cdots, N$. The lines $x = x_i$ and $t = t_k$ are **grid lines**, and their intersections (x_i, t_k) are the **mesh points** of the grid. The set of mesh nodes on the boundary, i.e, on $t = 0, x = 0$ and $x = l$ is denoted Γ_h and D_h denotes the set of the others in the interior $\{(x, t) \mid 0 < x < l, 0 < t \leqslant T\}$. Let $\bar{D}_h = D_h \cup \Gamma_h$.

Next we will give some useful formulas and suppose $g(x)$ is enough continuous on the interval $[x_0 - h, x_0 + h]$. By Taylor series, there are

$$g'(x_0) = \frac{1}{h}[g(x_0 + h) - g(x_0)] - \frac{h}{2}g''(\xi_1) \quad (x_0 < \xi_1 < x_0 + h) \tag{8.8}$$

$$g'(x_0) = \frac{1}{h}[g(x_0) - g(x_0 - h)] + \frac{h}{2}g''(\xi_2) \quad (x_0 - h < \xi_2 < x_0) \tag{8.9}$$

$$g'(x_0) = \frac{1}{h}\left[g\left(x_0 + \frac{h}{2}\right) - g\left(x_0 - \frac{h}{2}\right)\right] - \frac{h^2}{24}g'''(\xi_3) \quad \left(x_0 - \frac{h}{2} < \xi_3 < x_0 + \frac{h}{2}\right) \tag{8.10}$$

$$g''(x_0) = \frac{1}{h^2}[g(x_0 + h) - 2g(x_0) + g(x_0 - h)] - \frac{h^2}{12}g^{(4)}(\xi_4)$$
$$(x_0 - h < \xi_4 < x_0 + h) \tag{8.11}$$

$$g(x_0) = \frac{1}{2}[g(x_0 + h) + g(x_0 - h)] - \frac{h^2}{2}g''(\xi_5) \quad (x_0 - h < \xi_5 < x_0 + h) \tag{8.12}$$

8.1.2 Forward Difference Method

Consider the differential equation at the interior gridpoint (x_i, t_k)

$$\frac{\partial u}{\partial t}(x_i, t_k) - a\frac{\partial^2 u}{\partial x^2}(x_i, t_k) = f(x_i, t_k) \tag{8.13}$$

From Eqs. (8.8) and (8.11), we have

$$\frac{\partial u}{\partial t}(x_i, t_k) = \frac{1}{\tau}\left[u(x_i, t_{k+1}) - u(x_i, t_k)\right] - \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i, \eta_i^k\right) \quad \left(t_k < \eta_i^k < t_{k+1}\right) \tag{8.14}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x_i, t_k) =& \frac{1}{h^2}\left[u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k)\right] \\ & - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right) \quad \left(x_{i-1} < \xi_i^k < x_{i+1}\right)\end{aligned} \tag{8.15}$$

Taking (8.14) and (8.15) to (8.13), there is

$$\begin{aligned}&\frac{1}{\tau}\left[u(x_i, t_{k+1}) - u(x_i, t_k)\right] - \frac{a}{h^2}\left[u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k)\right] \\ &\quad = f(x_i, t_k) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i, \eta_i^k\right) - \frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right) \\ &\qquad\qquad (1 \leqslant i \leqslant M-1, 0 \leqslant k \leqslant N-1)\end{aligned} \tag{8.16}$$

Noticing the initial and boundary conditions, we obtain

$$\begin{cases} u(x_i, t_0) = \varphi(x_i) & (1 \leqslant i \leqslant M-1) \\ u(x_0, t_k) = \alpha(t_k), \quad u(x_M, t_k) = \beta(t_k) & (0 \leqslant k \leqslant N) \end{cases} \tag{8.17}$$

Neglecting the **truncation error**

$$R_{ik}^{(1)} = \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i, \eta_i^k\right) - \frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right) \tag{8.18}$$

and u_i^k approximating $u(x_i, t_k)$, the difference equations are written in the form

$$\begin{aligned}&\frac{1}{\tau}\left(u_i^{k+1} - u_i^k\right) - \frac{a}{h^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) = f(x_i, t_k) \\ &\qquad (1 \leqslant i \leqslant M-1, 0 \leqslant k \leqslant N-1)\end{aligned} \tag{8.19}$$

$$u_i^0 = \varphi(x_i)\,(1 \leqslant i \leqslant M-1) \tag{8.20}$$

$$u_0^k = \alpha(t_k), \quad u_M^k = \beta(t_k)\,(0 \leqslant k \leqslant N) \tag{8.21}$$

which are called **Forward Difference Method** and the grid points used can be seen in Figure 8.2.

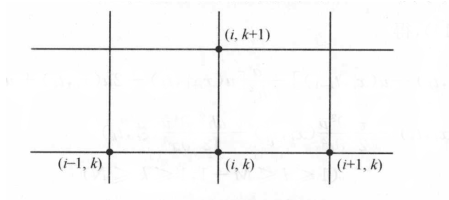

FIGURE 8.2
Grid points

Let $r = a\frac{\tau}{h^2}$. The difference equation (8.19) is rewritten in the form

$$u_i^{k+1} = (1-2r)u_i^k + r\left(u_{i-1}^k + u_{i+1}^k\right) + \tau f\left(x_i, t_k\right) \quad (1 \leqslant i \leqslant M-1, 0 \leqslant k \leqslant N-1).$$

or

$$\begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ \vdots \\ u_{M-2}^{k+1} \\ u_{M-1}^{k+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ \ddots & & \ddots & \ddots & \\ & r & & 1-2r & r \\ & & & r & 1-2r \end{bmatrix} \begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{M-2}^k \\ u_{M-1}^k \end{bmatrix} + \begin{bmatrix} \tau f\left(x_1, t_k\right) + r\alpha\left(t_k\right) \\ \tau f\left(x_2, t_k\right) \\ \vdots \\ \tau f\left(x_{M-2}, t_k\right) \\ \tau f\left(x_{M-1}, t_k\right) + r\beta\left(t_k\right) \end{bmatrix} \quad (k = 0, 1, \cdots, N-1) \tag{8.22}$$

So $\left\{u_i^{k+1} \mid 0 \leqslant i \leqslant M\right\}$ can be calculate by $\left\{u_i^k \mid 0 \leqslant i \leqslant M\right\}$. The method is explicit method.

8.1.3 Backward Difference Method

By (8.9) and (8.11), there is

$$\frac{\partial u}{\partial t}\left(x_i, t_k\right) = \frac{1}{\tau}\left[u\left(x_i, t_k\right) - u\left(x_i, t_{k-1}\right)\right] + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i, \eta_i^k\right) \quad \left(t_{k-1} < \eta_i^k < t_k\right)$$

$$\frac{\partial^2 u}{\partial x^2}\left(x_i, t_k\right) = \frac{1}{h^2}\left[u\left(x_{i+1}, t_k\right) - 2u\left(x_i, t_k\right) + u\left(x_{i-1}, t_k\right)\right] - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right) \quad \left(x_{i-i} < \xi_i^k < x_{i+1}\right)$$

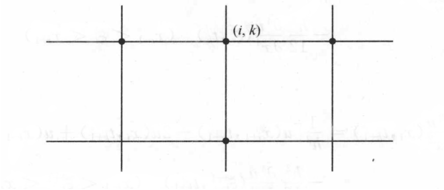

FIGURE 8.3
Grid points

Take the above equations to (8.13) and we have

$$\frac{1}{\tau}\left[u\left(x_i,t_k\right)-u\left(x_i,t_{k-1}\right)\right]-\frac{a}{h^2}\left[u\left(x_{i+1},t_k\right)-2u\left(x_i,t_k\right)+u\left(x_{i-1},t_k\right)\right]$$
$$=f\left(x_i,t_k\right)-\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,\eta_i^k\right)-\frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right)$$
$$\left(1\leqslant i\leqslant M-1,1\leqslant k\leqslant N\right) \tag{8.23}$$

And the initial and boundary conditions are as follows

$$\begin{cases} u\left(x_i,t_0\right)=\varphi\left(x_i\right) & \left(1\leqslant i\leqslant M-1\right) \\ u\left(x_0,t_k\right)=\alpha\left(t_k\right), \quad u\left(x_M,t_k\right)=\beta\left(t_k\right) & \left(0\leqslant k\leqslant N\right) \end{cases} \tag{8.24}$$

Neglecting the truncation error

$$R_\lambda^{(2)}=-\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,\eta_i^k\right)-\frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right) \tag{8.25}$$

in (8.23) and u_i^k approximating $u\left(x_i,t_k\right)$, we obtain

$$\frac{1}{\tau}\left(u_i^k-u_i^{k-1}\right)-\frac{a}{h^2}\left(u_{i+1}^k-2u_i^k+u_{i-1}^k\right)=f\left(x_i,t_k\right)$$
$$\left(1\leqslant i\leqslant M-1,1\leqslant k\leqslant N\right) \tag{8.26}$$
$$u_i^0=\varphi\left(x_i\right)\left(1\leqslant i\leqslant M-1\right) \tag{8.27}$$
$$u_0^k=\alpha\left(t_k\right), \quad u_M^k=\beta\left(t_k\right)\left(0\leqslant k\leqslant N\right) \tag{8.28}$$

which is called **Backward Difference Method** and the Backward-Difference method involves the mesh points (x_i,t_{k-1}), (x_{i-1},t_k), and (x_{i+1},t_k) to approximate the value at (x_i,t_k), as illustrated in Figure 8.3.

The difference equation can be written in the matrix form

$$\begin{bmatrix} 1+2r & -r & & & \\ -r & 1+2r & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & 1+2r & -r \\ & & & -r & 1+2r \end{bmatrix}\begin{bmatrix} u_1^k \\ u_2^k \\ \vdots \\ u_{M-2}^k \\ u_{M-1}^k \end{bmatrix}=\begin{bmatrix} u_1^{k-1} \\ u_2^{k-1} \\ \vdots \\ u_{M-2}^{k-1} \\ u_{M-1}^{k} \end{bmatrix}+\begin{bmatrix} \tau f\left(x_1,t_k\right)+r_\alpha\left(t_k\right) \\ \tau f\left(x_2,t_k\right) \\ \vdots \\ \tau f\left(x_{M-2},t_k\right) \\ \tau f\left(x_{M-1},t_k\right)+\eta\beta\left(t_k\right) \end{bmatrix} \tag{8.29}$$

where the coefficient matrix is tridiagonal matrix. And $\left\{u_i^k \mid 0 \leqslant i \leqslant M\right\}$ can be obtained by solving linear system by Thomas algorithm.

8.1.4 Richardson's Method

Consider the differential equation on mesh node (x_i, t_k)

$$\frac{\partial u}{\partial t}\left(x_i, t_k\right) - a\frac{\partial^2 u}{\partial x^2}\left(x_i, t_k\right) = f\left(x_i, t_k\right).$$

Since

$$\frac{\partial u}{\partial t}\left(x_i, t_k\right) = \frac{1}{2\tau}\left[u\left(x_i, t_{k+1}\right) - u\left(x_i, t_{k-1}\right)\right] - \frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}\left(x_i, \eta_i^k\right)$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}\left(x_i, t_k\right) =& \frac{1}{h^2}\left[u\left(x_{i+1}, t_k\right) - 2u\left(x_i, t_k\right) + u\left(x_{i-1}, t_k\right)\right] \\ &- \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right)\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2\tau}\left[u\left(x_i, t_{k+1}\right) - u\left(x_i, t_k\right)\right] - \frac{a}{h^2}\left[u\left(x_{i+1}, t_k\right) - 2u\left(x_i, t_k\right) + u\left(x_{i-1}, t_k\right)\right] \\ &\quad = f\left(x_i, t_k\right) + \frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}\left(x_i, \eta_i^k\right) - \frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right) \\ &\qquad\qquad (1 \leqslant i \leqslant M-1, 0 \leqslant k \leqslant N-1)\end{aligned} \tag{8.30}$$

The initial and boundary conditions are

$$\begin{cases} u\left(x_i, t_0\right) = \varphi\left(x_i\right) & (1 \leqslant i \leqslant M-1) \\ u\left(x_0, t_k\right) = \alpha\left(t_k\right), \quad u\left(x_M, t_k\right) = \beta\left(t_k\right) & (0 \leqslant k \leqslant N) \end{cases} \tag{8.31}$$

Neglecting the truncation error

$$R_{ik}^{(4)} = \frac{\tau^2}{6}\frac{\partial^3 u}{\partial t^3}\left(x_i, \eta_i^k\right) - \frac{ah^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right)$$

in (8.31) and u_i^k approximating $u\left(x_i, t_k\right)$, we have

$$\begin{aligned}&\frac{1}{2\tau}\left(u_i^{k+1} - u_i^{k-1}\right) - \frac{a}{h^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) = f\left(x_i, t_k\right) \\ &\qquad\qquad (1 \leqslant i \leqslant M-1, 1 \leqslant k \leqslant N-1);\end{aligned} \tag{8.32}$$

$$u_i^0 = \varphi\left(x_i\right) \qquad (1 \leqslant i \leqslant M-1) \tag{8.33}$$

$$u_0^k = \alpha\left(t_k\right), \quad u_M^k = \beta\left(t_k\right) \qquad (0 \leqslant k \leqslant N) \tag{8.34}$$

which is called **Richardson's method** and the grid points used are illustrated in Figure 8.5.

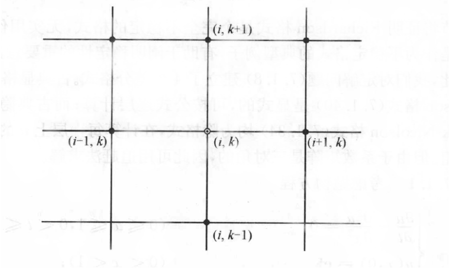

FIGURE 8.4
Grid points

The difference equation (8.32) is written in the matrix form as follows

$$\begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ \vdots \\ u_{M-2}^{k+1} \\ u_{M-1}^{k+1} \end{bmatrix} = \begin{bmatrix} u_1^{k-1} \\ u_2^{k-1} \\ \vdots \\ u_{M-2}^{k-1} \\ u_{M-1}^{k-1} \end{bmatrix} + \begin{bmatrix} -4r & 2r & & & \\ 2r & -4r & 2r & & \\ & \ddots & \ddots & \ddots & \\ & & 2r & -4r & 2r \\ & & & 2r & -4r \end{bmatrix} \begin{bmatrix} u_1^{k} \\ u_2^{k} \\ \vdots \\ u_{M-2}^{k} \\ u_{M-1}^{k} \end{bmatrix}$$
$$+ \begin{bmatrix} 2\tau f(x_1, t_k) + 2r_\alpha(t_k) \\ 2\tau f(x_2, t_k) \\ \vdots \\ 2\tau f(x_{M-2}, t_k) \\ 2\tau f(x_{M-1}, t_k) + 2r\beta(t_k) \end{bmatrix} \tag{8.35}$$

In the Forward difference method and Backward Difference method, the u^{k+1} can be calculated by u^k. These methods are called two-level difference method. While in Richardson's method, it is an explicit method and the u^{k+1} are obtained by u^k and u^{k-1}. So it is three-level difference method. Since

$$\begin{aligned} u(x_i, \tau) &= u(x_i, 0) + \tau \frac{\partial u}{\partial t}(x_i, 0) + \frac{1}{2}\tau^2 \frac{\partial^2 u}{\partial t^2}(x_i, \bar{\eta}_i) \\ &= \varphi(x_i) + \tau \left[a \frac{\mathrm{d}^2 \varphi(x_i)}{\mathrm{d}x^2} + f(x_i, 0)\right] + \frac{1}{2}\tau^2 \frac{\partial^2 u}{\partial t^2}(x_i, \bar{\eta}_i), \end{aligned}$$

the u^1 can be approximated by $u_i^1 = \varphi(x_i) + \tau \left[a \frac{\mathrm{d}^2\varphi(x_i)}{\mathrm{d}x^2} + f(x_i, 0)\right]$ $(1 \leqslant i \leqslant M-1)$.

8.1.5 Crank-Nicolson Method

Consider the differential equation on the grid nodes $\left(x_i, t_k + \frac{\tau}{2}\right)$

$$\frac{\partial u}{\partial t}\left(x_i, t_k + \frac{\tau}{2}\right) - a \frac{\partial^2 u}{\partial x^2}\left(x_i, t_k + \frac{\tau}{2}\right) = f\left(x_i, t_k + \frac{\tau}{2}\right)$$

By Eq. (8.12), we have

$$\begin{aligned}&\frac{\partial u}{\partial t}\left(x_i,t_k+\frac{\tau}{2}\right)-a\cdot\frac{1}{2}\cdot\left[\frac{\partial^2 u}{\partial x^2}\left(x_i,t_k\right)+\frac{\partial^2 u}{\partial x^2}\left(x_i,t_{k+1}\right)\right]\\&=f\left(x_i,t_k+\frac{\tau}{2}\right)-\frac{a\tau^2}{8}\frac{\partial^4 u}{\partial x^2\partial t^2}\left(x_i,\tilde{\eta}_i^t\right)\quad\left(t_k<\tilde{\eta}_i^k<t_{k+1}\right)\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial u}{\partial t}\left(x_i,t_k+\frac{\tau}{2}\right)&=\frac{1}{\tau}\left[u\left(x_i,t_{k+1}\right)-u\left(x_i,t_k\right)\right]-\frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}\left(x_i,\eta_i^k\right)\quad\left(t_k<\eta_i^k<t_{k+1}\right)\\\frac{\partial^2 u}{\partial x^2}\left(x_i,t_k\right)&=\frac{1}{h^2}\left[u\left(x_{i+1},t_k\right)-2u\left(x_i,t_k\right)+u\left(x_{i-1},t_k\right)\right]\\&\quad-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right)\quad\left(x_{i-1}<\xi_i<x_{i+1}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}\left(x_i,t_{k+1}\right)&=\frac{1}{h^2}\left[u\left(x_{i+1},t_{k+1}\right)-2u\left(x_i,t_{k+1}\right)+u\left(x_{i-1},t_{k+1}\right)\right]\\&\quad-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k+1},t_{k+1}\right)\quad\left(x_{i-1}<\xi_i^{k+1}<x_{i+1}\right)\end{aligned}$$

there is

$$\begin{aligned}&\frac{1}{\tau}\left[u\left(x_i,t_{k+1}\right)-u\left(x_i,t_k\right)\right]-\frac{a}{2}\left\{\frac{1}{h^2}\left[u\left(x_{i+1},t_k\right)-2u\left(x_i,t_k\right)+u\left(x_{i-1},t_k\right)\right]\right.\\&\quad\left.+\frac{1}{h^2}\left[u\left(x_{i+1},t_{k+1}\right)-2u\left(x_i,t_{k+1}\right)+u\left(x_{i-1},t_{k+1}\right)\right]\right\}\\&=f\left(x_i,t_{k+\frac{1}{2}}\right)+\frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}\left(x_i,\eta_i^k\right)-\frac{ah^2}{24}\left[\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right)+\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k+1},t_{k+1}\right)\right]\\&\quad-\frac{a\tau^2}{8}\frac{\partial^4 u}{\partial x^2\partial t^2}\left(x_i,\tilde{\eta}_i^k\right)\quad\left(1\leqslant i\leqslant M-1,0\leqslant k\leqslant N-1\right)\end{aligned}\tag{8.36}$$

The initial and boundary condition is

$$u\left(x_i,t_0\right)=\varphi\left(x_i\right)\qquad\left(1\leqslant i\leqslant M-1\right)\tag{8.37}$$

$$u\left(x_0,t_k\right)=\alpha\left(t_k\right),\quad u\left(x_M,t_k\right)=\beta\left(t_k\right)\qquad\left(0\leqslant k\leqslant N\right)\tag{8.38}$$

where $t_{k+\frac{1}{2}}=t_k+\frac{1}{2}\tau$.

Neglecting the truncation error

$$\begin{aligned}R_{ik}^{(3)}&=\frac{\tau^2}{24}\left[\frac{\partial^3 u}{\partial t^3}\left(x_i,\eta_i^k\right)-3a\frac{\partial^4 u}{\partial x^2\partial t^2}\left(x_i,\tilde{\eta}_i^k\right)\right]\\&\quad-\frac{ah^2}{24}\left[\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right)+\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k+1},t_{k+1}\right)\right]\end{aligned}$$

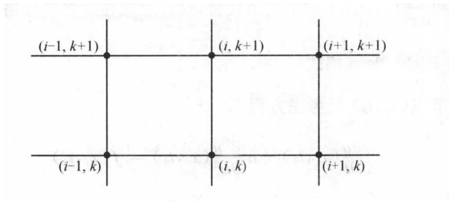

FIGURE 8.5
Grid points

in (8.36) and u_i^k approximating $u\left(x_i, t_k\right)$, the difference equations are

$$\begin{aligned}
&\frac{1}{\tau}\left(u_i^{k+1}-u_i^k\right)-\frac{a}{2}\left[\frac{1}{h^2}\left(u_{i+1}^k-2 u_i^k+u_{i-1}^k\right)+\frac{1}{h^2}\left(u_{i+1}^{k+1}-2 u_i^{k+1}+u_{i-1}^{k+1}\right)\right] \\
&\quad=f\left(x_i, t_{k+\frac{1}{2}}\right) \qquad (1 \leqslant i \leqslant M-1,0 \leqslant k \leqslant N-1) \qquad (8.39)\\
&u_i^0=\varphi\left(x_i\right), \qquad (1 \leqslant i \leqslant M-1) \qquad (8.40)\\
&u_0^k=\alpha\left(t_k\right), \quad u_M^k=\beta\left(t_k\right) \qquad (0 \leqslant k \leqslant N) \qquad (8.41)
\end{aligned}$$

which is called **Crank-Nicolson method** and the mesh nodes used in the method are illustrated in Figure (8.4).

The Crank-Nicolson method (8.39) can be written in the form

$$\begin{aligned}
&\left[\begin{array}{ccccc}
1+r & -\frac{r}{2} & & & \\
-\frac{r}{2} & 1+r & -\frac{r}{2} & & \\
& \ddots & \ddots & \ddots & \\
& & -\frac{r}{2} & 1+r & -\frac{r}{2} \\
& & & -\frac{r}{2} & 1+r
\end{array}\right]\left[\begin{array}{c}
u_1^{k+1} \\
u_2^{k+1} \\
\vdots \\
u_{M-2}^{k+1} \\
u_{M-1}^{k+1}
\end{array}\right] \\
&=\left[\begin{array}{ccccc}
1-r & \frac{r}{2} & & & \\
\frac{r}{2} & 1-r & \frac{r}{2} & & \\
& \ddots & \ddots & \ddots & \\
& & \frac{r}{2} & 1-r & \frac{r}{2} \\
& & & \frac{r}{2} & 1-r
\end{array}\right]\left[\begin{array}{c}
u_1^k \\
u_2^k \\
\vdots \\
u_{M-2}^k \\
u_{M-1}^k
\end{array}\right] \\
&+\left[\begin{array}{c}
\tau f\left(x_1, t_{k+\frac{1}{2}}\right)+\frac{r}{2}\left(\alpha\left(t_k\right)+\alpha\left(t_{k+1}\right)\right) \\
\tau f\left(x_2, t_{k+\frac{1}{2}}\right) \\
\vdots \\
\tau f\left(x_{M-2}, t_{k+\frac{1}{2}}\right) \\
\tau f\left(x_{M-1}, t_{k+\frac{1}{2}}\right)+\frac{r}{2}\left(\beta\left(t_k\right)+\beta\left(t_{k+1}\right)\right)
\end{array}\right] \qquad (8.42)
\end{aligned}$$

The coefficient matrix is tridiagonal matrix and the linear system can be solved by Thomas algorithm.

TABLE 8.1
By Forward Difference method ($h = 1/10, \tau = 1/200$)

k	(x,t)	u_5^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_5^k\rvert$
0	(0.5,0.00)	1.648721	1.648721	0.000000
10	(0.5,0.05)	1.733119	1.733253	0.000134
20	(0.5,0.10)	1.821888	1.822119	0.000231
30	(0.5,0.15)	1.915244	1.915541	0.000297
40	(0.5,0.20)	2.013408	2.013753	0.000345
50	(0.5,0.25)	2.116618	2.117000	0.000382
60	(0.5,0.30)	2.225127	2.225541	0.000414
70	(0.5,0.35)	2.339205	2.339647	0.000442
80	(0.5,0.40)	2.459134	2.459603	0.000469
90	(0.5,0.45)	2.585214	2.585710	0.000496
100	(0.5,0.50)	2.717759	2.718282	0.000523
110	(0.5,0.55)	2.857100	2.857651	0.000551
120	(0.5,0.60)	3.003587	3.004166	0.000579
130	(0.5,0.65)	3.157583	3.158193	0.000610
140	(0.5,0.70)	3.319476	3.320117	0.000641
150	(0.5,0.75)	3.489669	3.490343	0.000674
160	(0.5,0.80)	3.668588	3.669297	0.000709
170	(0.5,0.85)	3.856681	3.857425	0.000744
180	(0.5,0.90)	4.054417	4.055200	0.000783
190	(0.5,0.95)	4.262291	4.263114	0.000823
200	(0.5,1.00)	4.480824	4.481689	0.000865

The Crank-Nicolson method has the truncation error of order $O\left(\tau^2 + h^2\right)$, which is higher than that of Forward Difference method and Backward Difference method.

Exercise 8.1. *Use Forward Difference method, Backward Difference method, Richard's method and Crank-Nicolson method to approximate the solution of the parabolic equation*

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & (0 \leqslant x \leqslant 1, 0 < t \leqslant 1) \\ u(x,0) = \mathrm{e}^x & (0 < x < 1) \\ u(0,t) = \mathrm{e}^t, \quad u(1,t) = \mathrm{e}^{1+t} & (0 \leqslant t \leqslant 1) \end{cases}$$

Solution

The numerical solutions by Forward Difference method (8.19)-(8.21) are listed in Table 8.1 with $h = 1/10$ and $\tau = 1/200$ and we can see that the numerical solutions are good approximations to the exact solutions.

The numerical solutions also by Forward Difference method (8.19)-(8.21) are listed in Table 8.2 with $h = 1/10$ and $\tau = 1/100$ and the error increase with larger k.

The numerical solutions by Backward Difference method (8.26)-(8.28) are listed in Table 8.3 with $h = 1/10$ and $\tau = 1/200$ and the numerical solutions are good agreement with the exact solution.

TABLE 8.2
By Forward Difference method ($h = 1/10, \tau = 1/100$)

k	(x,t)	u_5^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_5^k\rvert$
0	(0.5,0.00)	1.648721	1.648721	0.000000
1	(0.5,0.01)	1.665222	1.665291	0.000069
2	(0.5,0.02)	1.681888	1.682028	0.000140
3	(0.5,0.03)	1.698722	1.698932	0.000210
4	(0.5,0.04)	1.715721	1.716007	0.000286
5	(0.5,0.05)	1.732898	1.733253	0.000355
6	(0.5,0.06)	1.750386	1.750672	0.000286
7	(0.5,0.07)	1.767305	1.768267	0.000962
8	(0.5,0.08)	1.787434	1.786038	0.001396
9	(0.5,0.09)	1.797024	1.803988	0.006964
10	(0.5,0.10)	1.842189	1.822119	0.020070
11	(0.5,0.11)	1.775457	1.840431	0.064974
12	(0.5,0.12)	2.054643	1.858928	0.195715
13	(0.5,0.13)	1.286095	1.877611	0.591516
14	(0.5,0.14)	3.655966	1.896481	1.759485
15	(0.5,0.15)	3.292654	1.915541	1.377113
16	(0.5,0.16)	17.26231	1.934792	15.32752
17	(0.5,0.17)	43.00026	1.954237	41.04602
18	(0.5,0.18)	133.4547	1.973878	131.4808

The Table 8.4 list the numerical results by Richard's method (8.32)-(8.34) with $h = 1/10, \tau = 1/10$ and the error increase with larger k.

The Table 8.5 shows the numerical results by Crank-Nicolson method (8.39)-(8.41) with $h = 1/10, \tau = 1/100$ and the error increases with larger k.

□

8.1.6 Stability and Convergence

We have now seen a variety of finite difference schemes for the parabolic equations, but how do we know that the approximate solutions they produce bear any relation to the true solutions of the corresponding PDEs? We do not expect an approximate solution to be exactly correct, of course, but we would hope to obtain arbitrarily good accuracy by taking sufficiently small step sizes in time and space. The finite difference scheme must have the following two important properties:

• Stability: the error remains bounded;

• Convergence: the approximate solution is close to the exact solution with smaller step sizes.

For the grid function

$$\left\{v_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N\right\}$$

TABLE 8.3
By Backward Difference method with $(h = 1/10, \tau = 1/200)$

k	(x,t)	u_5^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_5^k\rvert$
0	(0.5,0.00)	1.648721	1.648721	0.000000
10	(0.5,0.05)	1.733511	1.733253	0.000258
20	(0.5,0.10)	1.822564	1.822119	0.000445
30	(0.5,0.15)	1.916117	1.915541	0.000576
40	(0.5,0.20)	2.014426	2.013753	0.000673
50	(0.5,0.25)	2.117749	2.117000	0.000749
60	(0.5,0.30)	2.226355	2.225541	0.000814
70	(0.5,0.35)	2.340518	2.339647	0.000871
80	(0.5,0.40)	2.460529	2.459603	0.000926
90	(0.5,0.45)	2.586689	2.585710	0.000979
100	(0.5,0.50)	2.719316	2.718282	0.001034
110	(0.5,0.55)	2.858741	2.857651	0.001090
120	(0.5,0.60)	3.005313	3.004166	0.001147
130	(0.5,0.65)	3.159399	3.158193	0.001206
140	(0.5,0.70)	3.321386	3.320117	0.001269
150	(0.5,0.75)	3.491677	3.490343	0.001334
160	(0.5,0.80)	3.670699	3.669297	0.001402
170	(0.5,0.85)	3.858900	3.857425	0.001475
180	(0.5,0.90)	4.056750	4.055200	0.001550
190	(0.5,0.95)	4.264744	4.263114	0.001630
200	(0.5,1.00)	4.483403	4.481689	0.001714

TABLE 8.4
By Richardson's method with $(h = 1/10, \tau = 1/100)$

k	(x,t)	u_5^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_5^k\rvert$
0	(0.5,0.00)	1.648721	1.648721	0.000000
1	(0.5,0.01)	0.116501	1.665291	1.548790
2	(0.5,0.02)	8.837941	1.682028	7.155913
3	(0.5,0.03)	-22.13696	1.698932	23.83589
4	(0.5,0.04)	90.79588	1.716007	89.07988
5	(0.5,0.05)	-329.1597	1.733253	330.8929
6	(0.5,0.06)	1267.675	1.750672	1265.924
7	(0.5,0.07)	-4908.295	1.768267	4910.063
8	(0.5,0.08)	19285.74	1.786083	19283.96

TABLE 8.5
By Crank-Nicolson method with ($h = 1/10, \tau = 1/10$)

k	(x,t)	u_5^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_5^k \rvert$
0	(0.5,0.0)	1.648721	1.648721	0.000000
1	(0.5,0.1)	1.822349	1.822119	0.000230
2	(0.5,0.2)	2.014105	2.013753	0.000352
3	(0.5,0.3)	2.225953	2.225541	0.000412
4	(0.5,0.4)	2.460072	0.459603	0.000469
5	(0.5,0.5)	2.718802	2.718282	0.000520
6	(0.5,0.6)	3.004743	3.004166	0.000577
7	(0.5,0.7)	3.320755	3.320117	0.000638
8	(0.5,0.8)	3.670002	3.669297	0.000705
9	(0.5,0.9)	4.055979	4.005200	0.000779
10	(0.5,1.0)	4.482550	4.481681	0.000861

on $\bar{D}_h$, let

$$v^k = \left\{ v_i^k \mid 0 \leqslant i \leqslant M \right\}.$$

The norms of v^k are defined as follows

(1) L_2 norm

$$\left\| v^k \right\|_2 = \sqrt{h \left[\frac{1}{2} \left(v_0^k \right)^2 + \sum_{i=1}^{M-1} \left(v_i^k \right)^2 + \frac{1}{2} \left(v_M^k \right)^2 \right]}.$$

(2) L_∞ norm

$$\left\| v^k \right\|_\infty = \max_{0 \leqslant i \leqslant M} \left| v_i^k \right|.$$

(3) L_1 norm

$$\left\| v^k \right\|_1 = h \left(\frac{1}{2} \left| v_0^k \right| + \sum_{i=1}^{M-1} \left| v_i^k \right| + \frac{1}{2} \left| v_M^k \right| \right).$$

For simplicity, we suppose the error is only related with the initial conditions.

Definition 46. *The finite difference method is called* **stable** *in norm* $\| \cdot \|$ *if* C *independent with* h, τ *exists with*

$$\left\| \varepsilon^k \right\| \leqslant C \left\| \varepsilon^0 \right\| \quad (1 \leqslant k \leqslant N) \quad (\textit{two-level method}),$$

or

$$\left\| \varepsilon^k \right\| \leqslant C \left(\left\| \varepsilon^0 \right\| + \left\| \varepsilon^1 \right\| \right) \quad (2 \leqslant k \leqslant N) \quad (\textit{ three-level method});$$

otherwise, the method is unstable.

Theorem 48. *The Forward Difference method* (8.19)-(8.21) *is stable in* L_∞ *norm with* $r \leqslant \frac{1}{2}$*; it is unstable in* L_∞ *norm when* $r > \frac{1}{2}$.

Proof Suppose $\left\{\tilde{u}_i^k\right\}$ are solutions of

$$\begin{cases} \frac{1}{\tau}\left(\tilde{u}_i^{k+1}-\tilde{u}_i^k\right)-\frac{a}{h^2}\left(\tilde{u}_{i+1}^k-2\bar{u}_i^k+\bar{u}_{i-1}^k\right)=f\left(x_i,t_k\right) & \\ & (1\leqslant i\leqslant M-1, 0\leqslant k\leqslant N-1); \\ \tilde{u}_i^0=\varphi\left(x_i\right)+\varepsilon_i^0 & (1\leqslant i\leqslant M-1) \\ \bar{u}_0^k=\alpha\left(t_k\right), \quad \bar{u}_M^k=\beta\left(t_k\right) & (0\leqslant k\leqslant N) \end{cases} \tag{8.43}$$

Let $\varepsilon_i^k=\tilde{u}_i^k-u_i^k$ and

$$\frac{1}{\tau}\left(\varepsilon_i^{k+1}-\varepsilon_i^k\right)-\frac{a}{h^2}\left(\varepsilon_{i+1}^k-2\varepsilon_i^k+\varepsilon_{i-1}^k\right)=0 \qquad (1\leqslant i\leqslant M-1, 0\leqslant k\leqslant N-1) \tag{8.44}$$

$$\varepsilon_0^k=0, \quad \varepsilon_M^k=0 \quad (0\leqslant k\leqslant N) \tag{8.45}$$

Eq. (8.44) can be rewritten in the form

$$\varepsilon_i^{k+1}=(1-2r)\varepsilon_i^k+r\left(\varepsilon_{i-1}^k+\varepsilon_{i+1}^k\right) \quad (1\leqslant i\leqslant M-1, 0\leqslant k\leqslant N-1). \tag{8.46}$$

If $r\leqslant\frac{1}{2}$, there is

$$\begin{aligned} \left|\varepsilon_i^{k+1}\right| &\leqslant(1-2r)\left\|\varepsilon^k\right\|_\infty+r\left(\left\|\varepsilon^k\right\|_\infty+\left\|\varepsilon^k\right\|_\infty\right) \\ &=\left\|\varepsilon^k\right\|_\infty \quad (1\leqslant i\leqslant M-1). \end{aligned}$$

Since $\varepsilon_0^{k+1}=0$ and $\varepsilon_M^{k+1}=0$, there is

$$\left\|\varepsilon^{k+1}\right\|_\infty\leqslant\left\|\varepsilon^k\right\|_\infty \quad (0\leqslant k\leqslant N-1).$$

Then

$$\left\|\varepsilon^k\right\|_\infty\leqslant\left\|\varepsilon^0\right\|_\infty \quad (1\leqslant k\leqslant N).$$

Thus the Forward Difference method is stable in L_∞ norm with $r\leqslant\frac{1}{2}$.

Theorem 49. *The Backward Difference method* (8.26)-(8.28) *is stable in L_∞ norm for any r.*

Theorem 50. *Richardson's method* (8.32)-(8.34) *is unstable in L_∞ norm and L_2 norm for any r.*

Theorem 51. *Crank-Nicolson method* (8.39)-(8.41) *is stable in L_2 norm for any r.*

Suppose $\left\{u_i^k\right\}$ is the approximation to the exact solutions $\left\{u\left(x_i,t_k\right)\right\}$. Let

$$e_i^k=u\left(x_i,t_k\right)-u_i^k \quad (0\leqslant i\leqslant M, 0\leqslant k\leqslant N).$$

Definition 47. *If*

$$\lim_{\substack{h\to 0\\ \tau\to 0}} \max_{1\leqslant k<N} \left\|e^k\right\| = 0,$$

the finite difference method is called **convergent** *in norm* $\|\cdot\|$. *If*

$$\max_{1\leq k<N} \left\|e^k\right\| = O\left(h^p+\tau^q\right)$$

the method has convergence order p *in space and order* q *in time in* $\|\cdot\|$ *norm.*

Theorem 52. *The Forward Difference method* (8.19)-(8.21) *has second order convergence in space and first order convergence in time in* L_∞ *norm if* $r \leqslant \frac{1}{2}$.

Proof

Subtracting (8.16)-(8.17) from (8.19)-(8.21), we can get

$$\frac{1}{\tau}\left(e_i^{k+1}-e_i^k\right)-\frac{a}{h^2}\left(e_{i+1}^k-2e_i^k+e_{i-1}^k\right)=R_{ik}^{(1)}(1\leqslant i\leqslant M-1, 0\leqslant k\leqslant N-1) \tag{8.47}$$

$$e_i^0=0 \quad (1\leqslant i\leqslant M-1, 0\leqslant k\leqslant N-1) \tag{8.48}$$

$$e_0^k=0, \quad e_M^k=0 \quad (0\leqslant k\leqslant N) \tag{8.49}$$

Since Eq. (8.18), there is

$$\left|R_{ik}^{(1)}\right| \leqslant C_1\left(\tau+h^2\right) \tag{8.50}$$

where

$$C_1=\max\left\{\frac{1}{2}\max_{(x,t)\in D}\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right|, \frac{\alpha}{12}\max_{(x,t)\in D}\left|\frac{\partial^4 u(x,t)}{\partial x^4}\right|\right\}.$$

Eq. (8.47) can be rewritten in the form

$$e_i^{k+1}=(1-2r)e_i^k+r\left(e_{i+1}^k+e_{i-1}^k\right)+\tau R_{ik}^{(1)} \quad (1\leqslant i\leqslant M-1, 0\leqslant k\leqslant N-1).$$

Then

$$\left\|e^{k+1}\right\|_\infty \leqslant \left\|e^k\right\|_\infty + C_1\tau\left(\tau+h^2\right) \quad (0\leqslant k\leqslant N-1).$$

So

$$\left\|e^k\right\|_\infty \leqslant \left\|e^0\right\|_\infty + C_1 k\tau\left(\tau+h^2\right) \leqslant C_1 T\left(\tau+h^2\right) \quad (1\leqslant k\leqslant N).$$

□

Theorem 53. *The Backward Difference method* (8.26)-(8.28) *has second order convergence in space and first order convergence in time in* L_∞ *norm for any* r.

Theorem 54. *Crank-Nicolson method* (8.39)-(8.41) *has second order convergence in space and time in* L_2 *norm for any* r.

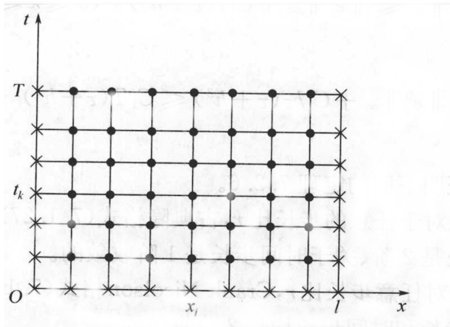

FIGURE 8.6
Grid points

8.2 Hyperbolic Partial Differential Equations

Hyperbolic PDEs describe time-dependent, conservative physical processes, such as wave propagation, that are not evolving toward a steady state. In this section, we consider wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) & (0 < x < l, 0 < t \leqslant T) \quad (8.51a) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) & (0 < x < l) \quad (8.51b) \\ u(0,t) = \alpha(t), \quad u(l,t) = \beta(t) & (0 \leqslant t \leqslant T) \quad (8.51c) \end{cases}$$

where $a > 0$. First select an integer $M > 0$ and define the x-axis step size $h = l/M$. Then select a time step size $\tau = \frac{T}{N}$. The grid points for this situation are (x_i, t_k), where $x_i = i_h$, for $i = 0, 1, \cdots, M$, and $t_k = kh, fork = 0, 1, \cdots N$ and are illustrated in Figure 8.6.

8.2.1 Explicit Method

Consider the differential equation on grid point (x_i, t_k)

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_k) - a^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_k) = f(x_i, t_k). \tag{8.52}$$

Taking

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(x_i,t_k) =& \frac{1}{\tau^2}\left[u(x_i,t_{k+1})-2u(x_i,t_k)+u(x_i,t_{k-1})\right]\\ &-\frac{\tau^2}{12}\frac{\partial^4 u}{\partial t^4}\left(x_i,y_i^k\right)\quad \left(t_{k-1}<\eta_i^k<t_{k+1}\right)\\ \frac{\partial^2 u}{\partial x^2}(x_i,t_k) =& \frac{1}{h^2}\left[u(x_{i+1},t_k)-2u(x_i,t_k)+u(x_{i-1},t_k)\right]\\ &-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right)\quad \left(x_{i-1}<\xi_i^k<x_{i+1}\right)\end{aligned}$$

to Eq. (8.52), we have

$$\begin{aligned}&\frac{1}{\tau^2}\left[u(x_i,t_{k+1})-2u(x_i,t_k)+u(x_i,t_{k-1})\right]-\frac{a^2}{h^2}\left[u(x_{i+1},t_k)-2u(x_i,t_k)+u(x_{i-1},t_k)\right]\\ &=f(x_i,t_k)+\frac{\tau^2}{12}\frac{\partial^4 u}{\partial t^4}\left(x_i,\eta_i^k\right)-\frac{a^2h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k,t_k\right)\\ &\quad(1\leqslant i\leqslant M-1,1\leqslant k\leqslant N-1)\end{aligned}\tag{8.53}$$

From Eq. (8.51b), the initial condition and u^1 is approximated by Taylor series as follows

$$u(x_i,t_0)=\varphi(x_i)$$

$$\begin{aligned}u(x_i,t_1)&=u(x_i,t_0)+\tau\frac{\partial u}{\partial t}(x_i,t_0)+\frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,t_0)+\frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i,\eta_i)\\ &=\varphi(x_i)+\tau\psi(x_i)+\frac{\tau^2}{2}\left[a^2\frac{\mathrm{d}^2\varphi(x_i)}{\mathrm{d}x^2}+f(x_i,t_0)\right]+\frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i,\eta_i)\end{aligned}\tag{8.54}$$

From boundary conditions (8.51c), we have

$$u(x_0,t_k)=\alpha(t_k),\quad u(x_M,t_k)=\beta(t_k)\quad(0\leqslant k\leqslant N)\tag{8.55}$$

Neglecting the high order term in (8.53)-(8.55) and using u_i^k to approximate $u(x_i,t_k)$, we obtain the finite difference scheme

$$\begin{cases}\dfrac{1}{\tau^2}\left(u_i^{k+1}-2u_i^k+u_i^{k-1}\right)-\dfrac{a^2}{h^2}\left(u_{i+1}^k-2u_i^k+u_{i-1}^k\right) & =f(x_i,t_k)\\ & (1\leqslant i\leqslant M-1,1\leqslant k\leqslant N-1)\ ; \quad (8.56a)\\ u_i^0=\varphi(x_i),\quad u_i^1=\Psi(x_i) & (1\leqslant i\leqslant M-1)\ ; \quad (8.56b)\\ u_0^k=\alpha(t_k),\quad u_M^k=\beta(t_k) & (0\leqslant k\leqslant N) \quad (8.56c)\end{cases}$$

where

$$\Psi(x_i)=\varphi(x_i)+\tau\psi(x_i)+\frac{\tau^2}{2}\left[a^2\frac{\mathrm{d}^2\varphi(x_i)}{\mathrm{d}x^2}+f(x_i,t_0)\right].$$

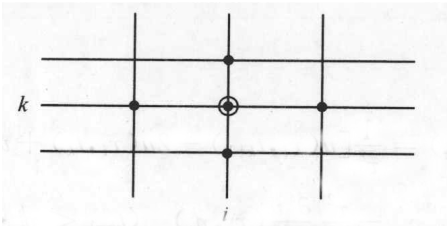

FIGURE 8.7
Grid nodes

The grid nodes used in the difference scheme (8.56a) are illustrated in Figure 8.7.

Let $s = \frac{a\tau}{h}$. The difference scheme (8.56a) can be written in the form

$$u_i^{k+1} = s^2\left(u_{i+1}^k + u_{i-1}^k\right) + 2\left(1 - s^2\right)u_i^k - u_i^{k-1} + \tau^2 f\left(x_i, t_k\right)$$

The $\left\{u_i^{k+1} \mid 0 \leqslant i \leqslant M\right\}$ can be calculated from $\left\{u_i^k \mid 0 \leqslant i \leqslant M\right\}$ and $\left\{u_i^{k-1} \mid 0 \leqslant i \leqslant M\right\}$. So (8.56a) is an explicit method.

Exercise 8.2. *Use the explicit method* (8.56a) *to approximate the solution of*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & & (0 < x < 1, 0 < t \leqslant); \\ u(x,0) = \exp(x), & \frac{\partial u(x,0)}{\partial t} = \exp(x) & (0 < x < 1); \\ u(0,t) = \exp(t), & u(1,t) = \exp(1+t) & (0 < t \leqslant 1) \end{cases}$$

The exact solution is $u(x,t) = \exp(x+t)$.

Solution

Table 8.6 give the numerical results with $h = 1/100, \tau = 1/100$. From the table, the numerical solution is agreement with the exact solution. Table 8.7 shows the numerical results with $h = 1/100$, $\tau = 1/80$ and the error increases with larger k.

Theorem 55. 1° *The explicit method* (8.56a)-(8.56c) *is stable in* L_2 *norm for* $s \leqslant 1$ *and it is unstable in* L_2 *norm when* $s > 1$;

2° *The explicit method* (8.56a)-(8.56c) *has second order convergence in time and space in* L_2 *norm with* $s \leqslant 1$.

8.2.2 Implicit Method

Since Eq. (8.52), there is

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}\left(x_i, t_k\right) - \frac{1}{2}a^2\left[\frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+1}\right) + \frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k-1}\right)\right] \\ &= f\left(x_i, t_k\right) - \frac{1}{2}a^2\tau^2\frac{\partial^4 u}{\partial x^2 \partial t^2}\left(x_i, \bar{\eta}_i^k\right) \quad \left(t_{k-1} < \bar{\eta}_i^k < t_{k+1}\right) \end{aligned} \tag{8.57}$$

TABLE 8.6
Numerical results with $(h = 1/100, \tau = 1/100)$

k	(x,t)	u_{50}^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_{50}^k \rvert$
0	(0.5,0.0)	1.648721	1.648721	0.000000
10	(0.5,0.1)	1.822116	1.822119	0.000003
20	(0.5,0.2)	2.013747	2.013753	0.000006
30	(0.5,0.3)	2.225532	2.225541	0.000009
40	(0.5,0.4)	2.459592	2.459603	0.000011
50	(0.5,0.5)	2.718268	2.718282	0.000014
60	(0.5,0.6)	3.004154	3.004166	0.000012
70	(0.5,0.7)	3.320108	3.320117	0.000009
80	(0.5,0.8)	3.669291	3.669297	0.000006
90	(0.5,0.9)	4.055199	4.055200	0.000001
100	(0.5,1.0)	4.481688	4.481689	0.000001

TABLE 8.7
Numerical results with $(h = 1/100, \tau = 1/80)$

k	(x,t)	u_{50}^k	$u(0.5,t_k)$	$\lvert u(0.5,t_k) - u_{50}^k \rvert$
0	(0.5000,0.0000)	1.648721	1.648721	0.000000
1	(0.5000,0.0125)	1.669459	1.669460	0.000001
2	(0.5000,0.0250)	1.690458	1.690459	0.000001
3	(0.5000,0.0375)	1.711720	1.711722	0.000002
4	(0.5000,0.0500)	1.733254	1.733253	0.000001
5	(0.5000,0.0625)	1.755042	1.755055	0.000013
6	(0.5000,0.0750)	1.777168	1.777130	0.000038
7	(0.5000,0.0875)	1.799323	1.799484	0.000161
8	(0.5000,0.1000)	1.822730	1.822119	0.000611
9	(0.5000,0.1125)	1.842627	1.845038	0.002411
10	(0.5000,0.1250)	1.877649	1.868246	0.009403
11	(0.5000,0.1375)	1.854973	1.891746	0.036773
12	(0.5000,0.1500)	2.059241	1.915541	0.143700
13	(0.5000,0.1625)	1.377665	1.939635	0.561970
14	(0.5000,0.1750)	4.163094	1.964033	2.199061
15	(0.5000,0.1875)	-6.623787	1.988737	8.612524
16	(0.5000,0.2000)	35.774890	2.013753	33.761137

Taking

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x_i,t_k) =&\frac{1}{\tau^2}\left[u(x_i,t_{k+1})-2u(x_i,t_k)+u(x_i,t_{k-1})\right]\\
&-\frac{\tau^2}{12}\frac{\partial^4 u}{\partial t^4}\left(x_i,\eta_i^k\right)\quad\left(t_{k-1}<\eta_i^k<t_{k+1}\right)\\
\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+1}) =&\frac{1}{h^2}\left[u(x_{i+1},t_{k+1})-2u(x_i,t_{k+1})+u(x_{i-1},t_{k+1})\right]\\
&-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k+1},t_{k+1}\right)\quad\left(x_{i-1}<\xi_i^{k+1}<x_{i+1}\right)\\
\frac{\partial^2 u}{\partial x^2}(x_i,t_{k-1}) =&\frac{1}{h^2}\left[u(x_{i+1},t_{k-1})-2u(x_i,t_{k-1})+u(x_{i-1},t_{k-1})\right]\\
&-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k-1},t_{k-1}\right)\quad\left(x_{i-1}<\xi_i^{k-1}<x_{i+1}\right)
\end{aligned}$$

into Eq. (8.57), we have

$$\begin{aligned}
&\frac{1}{\tau^2}\left[u(x_i,t_{k+1})-2u(x_i,t_k)+u(x_i,t_{k-1})\right]\\
&\quad-\frac{a^2}{2}\left\{\frac{1}{h^2}\left[u(x_{i+1},t_{k+1})-2u(x_i,t_{k+1})+u(x_{i-1},t_{k+1})\right]\right.\\
&\quad\left.+\frac{1}{h^2}\left[u(x_{i+1},t_{k-1})-2u(x_i,t_{k-1})+u(x_i-1,t_{k-1}\right]\right\}\\
&=f(x_i,t_k)-\frac{1}{2}a^2\tau^2\frac{\partial^4 u}{\partial x^2\partial t^2}\left(x_i,\bar{\eta}_i^k\right)+\frac{\tau^2}{12}\frac{\partial^4 u}{\partial t^4}\left(x_i,\eta_i^k\right)\\
&\quad-\frac{a^2h^2}{24}\left[\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k+1},t_{k+1}\right)+\frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k-1},t_{k-1}\right)\right]\\
&\quad(1\leqslant i\leqslant M-1,1\leqslant k\leqslant N-1)
\end{aligned}\tag{8.58}$$

The initial and boundary conditions are

$$u(x_i,t_0)=\varphi(x_i),u(x_i,t_1)=\Psi(x_i)+\frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i,\eta_i)\quad(1\leqslant i\leqslant M-1)\tag{8.59}$$

$$u(x_0,t_k)=\alpha(t_k),\quad u(x_M,t_k)=\beta(t_k)\quad(0\leqslant k\leqslant N)\tag{8.60}$$

Neglecting high order term in Eq. (8.58)-(8.60) and using u_i^k to approximate $u(x_i,t_k)$, we have

$$\begin{cases}
\frac{1}{\tau^2}\left(u_i^{k+1}-2u_i^k+u_i^{k-1}\right)-\quad\frac{a^2}{2}\left[\frac{1}{h^2}\left(u_{i+1}^{k+1}-2u_i^{k+1}+u_{i-1}^{k+1}\right)+\frac{1}{h^2}\left(u_{i+1}^{k-1}-2u_i^{k-1}+u_i^{k-1}\right)\right]\\
=f(x_i,t_k)\qquad(1\leqslant i\leqslant M-1,1\leqslant k\leqslant N-1) & (8.61\text{a})\\
u_i^0=\varphi(x_i),u_i^1=\Psi(x_i)\qquad(1\leqslant i\leqslant M-1) & (8.61\text{b})\\
u_0^k=\alpha(t_k),\quad u_M^k=\beta(t_k)\quad(0\leqslant k\leqslant N) & (8.61\text{c})
\end{cases}$$

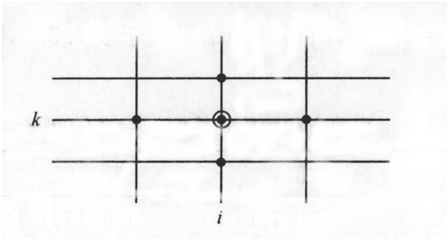

FIGURE 8.8
Grid points

which can be written in the form

$$\begin{bmatrix} 1+s^2 & -\frac{1}{2}s^2 & & & \\ -\frac{1}{2}s^2 & 1+s^2 & -\frac{1}{2}s^2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2}s^2 & 1+s^2 & -\frac{1}{2}s^2 \\ & & & -\frac{1}{2}s^2 & 1+s^2 \end{bmatrix} \begin{bmatrix} u_1^{k+1} \\ u_2^{k+1} \\ \vdots \\ u_{M-2}^{k+1} \\ u_{M-1}^{k+1} \end{bmatrix}$$

$$= 2\begin{bmatrix} u_1^{k} \\ u_2^{k} \\ \vdots \\ u_{M-2}^{k} \\ u_{M-1}^{k} \end{bmatrix} + \begin{bmatrix} -\left(1+s^2\right) & \frac{1}{2}s^2 & & & \\ \frac{1}{2}s^2 & -\left(1+s^2\right) & \frac{1}{2}s^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2}s^2 & -\left(1+s^2\right) & \frac{1}{2}s^2 \\ & & & \frac{1}{2}s^2 & -\left(1+s^2\right) \end{bmatrix}$$

$$\begin{bmatrix} u_1^{k-1} \\ u_2^{k-1} \\ \vdots \\ u_{M-2}^{k-1} \\ u_{M-1}^{k-1} \end{bmatrix} + \begin{bmatrix} \tau^2 f\left(x_1, t_k\right) + \frac{1}{2}s^2\left(\alpha\left(t_{k-1}\right) + \alpha\left(t_{k+1}\right)\right) \\ \tau^2 f\left(x_2, t_k\right) \\ \vdots \\ \tau^2 f\left(x_{M-1}, t_k\right) + \frac{1}{2}s^2\left(\beta\left(t_{k-1}\right) + \beta\left(t_{k+1}\right)\right) \end{bmatrix} \tag{8.62}$$

The coefficient matrix is tridiagonal matrix and the linear system can be solved by Thomas algorithm. So the method is called implicit method. The grid nodes used in the method (8.61a) are illustrated in Figure 8.8.

Exercise 8.3. *Use implicit method* (8.61a)-(8.61c) *to approximate the solution of the problem in 8.2.*

Solution Table 8.8 shows the numerical solutions with $h = 1/100, \tau = 1/100$. From the table, the numerical solutions are better approximations to the exact solutions.

TABLE 8.8
Numerical Results with $(h = 1/100, \tau = 1/100)$

k	(x, t)	u_{50}^k	$u(0.5, t_k)$	$\lvert u(0.5, t_k) - u_{50}^k \rvert$
0	(0.5,0.0)	1.648721	1.648721	0.000000
10	(0.5,0.1)	1.822114	1.822119	0.000005
20	(0.5,0.2)	2.013740	2.013753	0,000013
30	(0.5,0.3)	2.225511	2.225541	0.000030
40	(0.5,0.4)	2.459551	2.459603	0.000052
50	(0.5,0.5)	2.718201	2.718282	0.000081
60	(0.5,0.6)	3.004062	3.004166	0.000104
70	(0.5,0.7)	3.319985	3.320117	0.000132
80	(0.5,0.8)	3.669143	3.669297	0.000154
90	(0.5,0.9)	4.055023	4.055200	0.000177
100	(0.5,1.0)	4.481491	4.481689	0.000198

8.3 Elliptic Partial Differential Equations

Elliptic PDEs describe systems that have already reached a steady state, or equilibrium, and hence are time-independent. For an elliptic boundary value problem, the solution at every point in the problem domain depends on all of the boundary data (in contrast to the limited domain of dependence for time-dependent problems), and consequently an approximate solution must be computed everywhere simultaneously, rather than being generated step by step using a recurrence. Consequently, discretization of an elliptic boundary value problem results in a single system of algebraic equations to be solved for some finite-dimensional approximation to the solution.

In this section, we discuss the finite difference method for Poisson equation in two dimensions:

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x, y) \quad ((x, y) \in \Omega) \\ u|_{\partial\Omega} = \varphi(x, y) \end{cases} \tag{8.63}$$

where $\Omega = \{(x, y) \mid a < x < b, c < y < d\}$.

Choose integers M and N to define step sizes $h_1 = (b - a)/M$ and $h_2 = (d - c)/N$. Partition the interval $[a, b]$ into M equal parts of width h_1 and the interval $[c, d]$ into N equal parts of width h_2 (see Figure 8.9).

Place a grid on the rectangle Ω by drawing vertical and horizontal lines through the points with (x_i, y_j), where $x_i = a + ih_1$, for each $0 \leqslant i \leqslant M$ and $y_j = c + jh_2$, for each $0 \leqslant j \leqslant N$. Let $\Omega_h = \{(x_i, y_j) \mid 0 \leqslant i \leqslant M, 0 \leqslant j \leqslant N\}$, $\overset{\circ}{\Omega}_h = \{(x_i, y_i) \mid 1 \leqslant i \leqslant M - 1, 1 \leqslant j \leqslant N - 1\}$, $\omega_h = \Omega_h \backslash \overset{\circ}{\Omega}_h$. We call h_1 x-axis step size and h_2 y-axis step size.

(x_i, y_j) is called interior mesh point if $(x_i, y_j) \in \Omega_h$, i.e. $(i, j) \in \Omega_h$; (x_i, y_j) is called boundary mesh point $(x_i, y_j) \in \omega_h$, i.e. $(i, j) \in \omega_h$. It easy to know

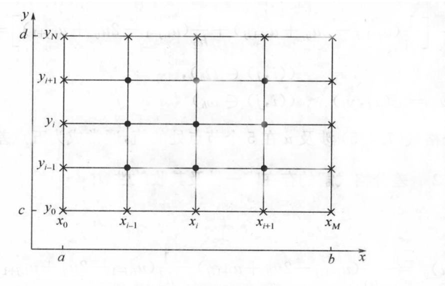

FIGURE 8.9
Grid

$(i,j) \in \Omega_h$ if and only if $1 \leqslant i \leqslant M-1, 1 \leqslant j \leqslant N-1$;$(i,j) \in \omega_h$ if and only if $i=0$ or $i=M, j=0$ or $j=N$.

Consider the differential equation on the interior node (x_i, y_j)

$$-\left[\frac{\partial^2 u\left(x_i, y_j\right)}{\partial x^2}+\frac{\partial^2 u\left(x_i, y_j\right)}{\partial y^2}\right]=f\left(x_i, y_j\right). \tag{8.64}$$

Taking

$$\frac{\partial^2 u\left(x_i, y_j\right)}{\partial x^2}=\frac{1}{h_1^2}\left[u\left(x_{i-1}, y_j\right)-2 u\left(x_i, y_j\right)+u\left(x_{i+1}, y_j\right)\right]-\frac{h_1^2}{12} \frac{\partial^4 u\left(\xi_{ij}, y_i\right)}{\partial x^4}$$
$$\left(\xi_{ij} \in\left(x_{i-1}, x_{i+1}\right)\right)$$

and

$$\frac{\partial^2 u\left(x_i, y_j\right)}{\partial y^2}=\frac{1}{h_2^2}\left[u\left(x_i, y_{j-1}\right)-2 u\left(x_i, y_j\right)+u\left(x_i, y_{j+1}\right)\right]-\frac{h_2^2}{12} \frac{\partial^4 u\left(x_i, \eta_{ij}\right)}{\partial y^4}$$
$$\left(\eta_{ij} \in\left(y_{j-1}, y_{j+1}\right)\right)$$

into Eq. (8.64), we have

$$\begin{aligned} -\frac{1}{h_1^2}\left[u\left(x_i-1, y_j\right)-2 u\left(x_i, y_j\right)+u\left(x_{i+1}, y_j\right)\right]-\frac{1}{h_2^2}\left[u\left(x_i, y_{j-1}\right)-2 u\left(x_i, y_j\right)+u\left(x_i, y_{j+1}\right)\right] \\ =f\left(x_i, y_j\right)-\frac{h_1^2}{12} \frac{\partial^4 u\left(\xi_{ij}, y_j\right)}{\partial x^4}-\frac{h_2^2}{12} \frac{\partial^4 u\left(x_i, \eta_{ij}\right)}{\partial y^4} \quad\left((i, j) \in \Omega_h\right) \end{aligned} \tag{8.65}$$

The boundary condition is

$$u\left(x_i, y_j\right)=\varphi\left(x_i, y_j\right) \quad\left((i, j) \in \omega_h\right). \tag{8.66}$$

Neglecting

$$R_{ij}=-\frac{h_1^2}{12} \frac{\partial^4 u\left(\xi_{ij}, y_j\right)}{\partial x^4}-\frac{h_2^2}{12} \frac{\partial^4 u\left(x_i, \eta_{ij}\right)}{\partial y^4}$$

in Eq. (8.65) and using u_{ij} to approximate $u(x_i, y_j)$, the difference scheme is obtained as follows

$$\begin{cases} -\left[\dfrac{1}{h_1^2}(u_{i-1,j} - 2u_{ij} + u_{i+1,j}) + \dfrac{1}{h_2^2}(u_{i,j-1} - 2u_{ij} + u_{i,j+1})\right] = f(x_i, y_j) & ((i,j) \in \Omega_h), \quad (8.67\text{a}) \\ u_{ij} = \varphi(x_i, y_j) & ((i,j) \in \omega_h) \quad (8.67\text{b}) \end{cases}$$

which is called five-point difference scheme. Let

$$(L_h u)_{ij} = -\frac{1}{h_1^2}(u_{i-1,j} - 2u_{ij} + u_{i+1,j}) - \frac{1}{h_2^2}(u_{i,j-1} - 2u_{ij} + u_{i,j+1}) \quad \left((i,j) \in \mathring{\Omega}_h\right)$$

Corollary 4. *Suppose* $u = \{u_{ij} \mid 0 \leqslant i \leqslant M, 0 \leqslant j \leqslant N\}$ *is grid function on* Ω_h.

1° *If* u *satisfies*

$$(L_h u)_{ij} \leqslant 0 \quad \left((i,j) \in \mathring{\Omega}_h\right),$$

then

$$\max_{(i,j)\in\mathring{\Omega}_h} u_{ij} \leqslant \max_{(i,j)\in\omega_h} u_{ij};$$

2° *If* u *satisfies*

$$(L_h u)_{ij} \geqslant 0 \quad \left((i,j) \in \mathring{\Omega}_h\right),$$

then

$$\min_{(i,j)\in\mathring{\Omega}_h} u_{ij} \geqslant \min_{(i,j)\in\omega_h} u_{ij}.$$

Theorem 56. *The five-point difference scheme has unique solution.*

Theorem 57. *Suppose* $\Omega = \{(x,y) \mid a < x < b, c < y < d\}, u(x,y)$ *is the solution of* (8.63) *and the fourth order partial derivative of* $u(x,y)$ *to* x *and* y *is continuous on* $\bar{\Omega}$. *Let* $u = \{u_{ij} \mid 0 \leqslant i \leqslant M, 0 \leqslant j \leqslant N\}$ *is the solution of finite difference scheme* (8.67a)-(8.67b). *Then*

$$\max_{(i,j)\in\mathring{\Omega}_h} |u(x_i, y_j) - u_{ij}| \leqslant \frac{M_4}{48}\left[\left(\frac{b-a}{2}\right)^2 + \left(\frac{d-c}{2}\right)^2\right](h_1^2 + h_2^2),$$

where

$$M_4 = \max\left\{\max_{(x,y)\in\bar{\Omega}}\left|\frac{\partial^4 u(x,y)}{\partial x^4}\right|, \max_{(x,y)\in\bar{\Omega}}\left|\frac{\partial^4 u(x,y)}{\partial y^4}\right|\right\}.$$

We use the information from the boundary conditions whenever appropriate in the system given by the five-point difference scheme; that is, at all points (x_i, y_j) adjacent to a boundary mesh point. This produces a linear system with the $(M-1) \times (N-1)$ unknowns being the approximations u_{ij} to $u(x_i, y_j)$ at the interior mesh points.

The difference scheme (8.67a)-(8.67b) can be written in the form

$$\begin{cases} u_{ij} = \left[\frac{1}{h_1^2}\left(u_{i-1,j} + u_{i+1,j}\right) + \frac{1}{h_2^2}\left(u_{i,j-1} + u_{i,j+1}\right) + f\left(x_i, y_j\right)\right] / \left(\frac{2}{h_1^2} + \frac{2}{h_2^2}\right) \\ \qquad (i = 1, 2, \cdots, M-1; j = 1, 2, \cdots, N-1) \\ u_{ij} \equiv \varphi\left(x_i, y_i\right) \quad ((i, j) \in \omega_h) \end{cases}$$

The values of u_{ij} can be obtained by applying Jacobi method to this linear system

$$\begin{cases} u_{ij}^{(k+1)} = \left[\frac{1}{h_1^2}\left(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)}\right) + \frac{1}{h_2^2}\left(u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\right) + f\left(x_i, y_j\right)\right] / \left(\frac{2}{h_1^2} + \frac{2}{h_2^2}\right) \\ \qquad (i = 1, 2, \cdots, M-1; j = 1, 2, \cdots, N-1) \\ u_{ij}^{(k+1)} \equiv \varphi\left(x_i, y_j\right) \quad ((i, j) \in \omega_h) \end{cases} \tag{8.68}$$

or by applying Gauss-Seidel method

$$\begin{cases} u_{ij}^{(k+1)} = \left[\frac{1}{h_1^2}\left(u_{i-1}^{(k+1)} + u_{i+1,j}^{(k)}\right) + \frac{1}{h_2^2}\left(u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)}\right) + f\left(x_i, y_j\right)\right] / \left(\frac{2}{h_1^2} + \frac{2}{h_2^2}\right) \\ \qquad (i = 1, 2, \cdots, M-1; j = 1, 2, \cdots, N-1) \\ u_{ij}^{(k+1)} \equiv \varphi\left(x_i, y_j\right) ((i, j) \in \omega_h) \end{cases} \tag{8.69}$$

Exercise 8.4. *Use five-point difference scheme to approximate the solution to the elliptic partial differential equation*

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -2\exp(x+y) \\ u|_{\partial\Omega} = \exp(x+y) \end{cases}$$

where $\Omega = (0,1) \times (0,1)$, *and compare the results to the actual solution* $u(x, y) = \exp(x+y)$.

Soluiton Take $M = 10, N = 10$, and apply the difference scheme (8.67a)-(8.67b) to compute the values of u_{ij} by Gauss-Seidel method. The initial vector is set to be $\boldsymbol{u}^{(0)} = \mathbf{0}$. If

$$\left\|\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}\right\|_\infty \leqslant \frac{1}{2} \times 10^{-6}$$

we need 139 iterations. The numerical results are listed in Table 8.9.

Take $M = N = 100$ and if

$$\left\|\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}\right\|_\infty \leqslant \frac{1}{2} \times 10^{-8},$$

13,918 iterations are required. The numerical results are listed in Table 8.10.

□

TABLE 8.9
Numerical Results with($M = 10, N = 10$)

(x, y)	u_{ij}	$u(x_i, y_j)$	$\mid u(x_i, y_j) - u_{ij} \mid$
(0.2,0.2)	1.491940	1.491825	0.000115
(0.2,0.5)	2.013950	2.013753	0.000197
(0.2,0.8)	2.718440	2.718282	0.000158
(0.5,0.2)	2.013950	2.013753	0.000197
(0.5,0.5)	2.718621	2.718282	0.000339
(0.5,0.8)	3.669563	3.669297	0.000266
(0.8,0.2)	2.718440	2.718282	0.000158
(0.8,0.5)	3.669563	3.669297	0.000266
(0.8,0.8)	4.953257	4.953032	0.000225

TABLE 8.10
Num ($M = 100, N = 100$)

(x, y)	u_{ij}	$u(x_i, y_j)$	$\mid u(x_i, y_j) - u_{ij} \mid$
(0.2,0.2)	1.4918241	1.4918247	0.0000006
(0.2,0.5)	2.0137517	2.0137527	0.0000010
(0.2,0.8)	2.7182817	2.7182818	0.0000001
(0.5,0.2)	2.0137517	2.0137527	0.0000010
(0.5,0.5)	2.7182802	2.7182818	0.0000016
(0.5,0.8)	3.6692964	3.6692967	0.0000003
(0.8,0.2)	2.7182817	2.7182818	0.0000001
(0.8,0.5)	3.6692964	3.6692967	0.0000003
(0.8,0.8)	4.9530330	4.9530324	0.0000006

8.4 Exercise

1. Approximate the solution to the following partial differential equation using the Forward Difference method.

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & (0 \leqslant x \leqslant 1, t > 0) \\ u(x,0) = \sin \pi x & (0 \leqslant x \leqslant 1) \\ u(0,t) = u(1,t) = 0 & (t \geqslant 0) \end{cases}$$

Use $h = 0.1, r = 0.5$ and $t = 0.015$.

2. Use Backward Difference method to approximate the solution of

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & (0 < x < 1, t > 0) \\ u(x,0) = x(1-x) & (0 \leqslant x \leqslant 1) \\ u(0,t) = t, \quad u(1,t) = 0 & (t > 0) \end{cases}$$

with $h = 0.2, r = 0.5$ and $t = 0.04$.

3. Construct an explicit difference method to approximate the solution for

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu$$

and give the truncation error where a, b, c are constants and $a > 0$.

4. Show the Theorem 49.

5. Show the Theorem 53.

6. Use the explicit and implicit methods to approximate the solution for

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & (0 < x < 1, t > 0) \\ u(x,0) = \sin \pi x, \quad \frac{\partial u(x,0)}{\partial t} = x(1-x) & (0 \leqslant x \leqslant 1) \\ u(0,t) = u(1,t) = 0 & (t \geqslant 0) \end{cases}$$

with $h = 0.2, s = 1$ and $t = 0.6$

7. Given the partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 & (0 < x \leqslant 1, 0 < t \leqslant T) \\ u(x,0) = \varphi(x) & (0 < x \leqslant 1) \\ u(0,t) = \psi(t) & (0 < t \leqslant T) \end{cases}$$

and

$$h = \tfrac{1}{M}, \quad \tau = \tfrac{T}{N}, \quad x_i = ih \quad (0 \leqslant i \leqslant M), t_k = k\tau \quad (0 \leqslant k \leqslant N)$$

the finite difference scheme is constructed as follows

$$\begin{cases} \frac{1}{\tau}\left(u_i^{k+1}-u_i^k\right)+\frac{1}{h}\left(u_i^k-u_{i-1}^k\right)=0 & (1 \leqslant i \leqslant M, 0 \leqslant k \leqslant N-1) \\ u_i^0=\varphi\left(x_i\right) & (1 \leqslant i \leqslant M) \\ u_0^k=\psi\left(t_k\right) & (0 \leqslant k \leqslant N) \end{cases}$$

(1) Derive the truncation error;

(2) Show the scheme is stable when $\frac{\tau}{h} \leqslant 1$;

(3) Show the convergence of the scheme.

8. Use five-point difference scheme to approximate the solution of the elliptic partial differential equation

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=6x-2 \quad ((x, y) \in \Omega) \\ u|_{a\Omega}=x^3-y^3 \end{cases}$$

where $\Omega=\{(x, y) \mid 0<x, y<3\}$. Use step size $h_1=1$ and $h_2=1$.

9. (1) Program Crank-Nicolson method to approximate the solution for the parabolic equation with $a=1, f(x,t)=0, \varphi(x)=\exp x, \alpha(t)=\exp t, \beta(t)=\exp(1+t), M=40, N=40$. Output the numerical solutions on the mesh points (0.2,1.0),(0.4,1.0),(0.6,1.0),(0.8,1.0).

(2) Compare the numerical results at (0.2,1.0),(0.4,1.0),(0.6,1.0),(0.8,1.0) to the actual solution $u(x,t)=\exp(x+t)$ with $M=N=40, 80, 160$.