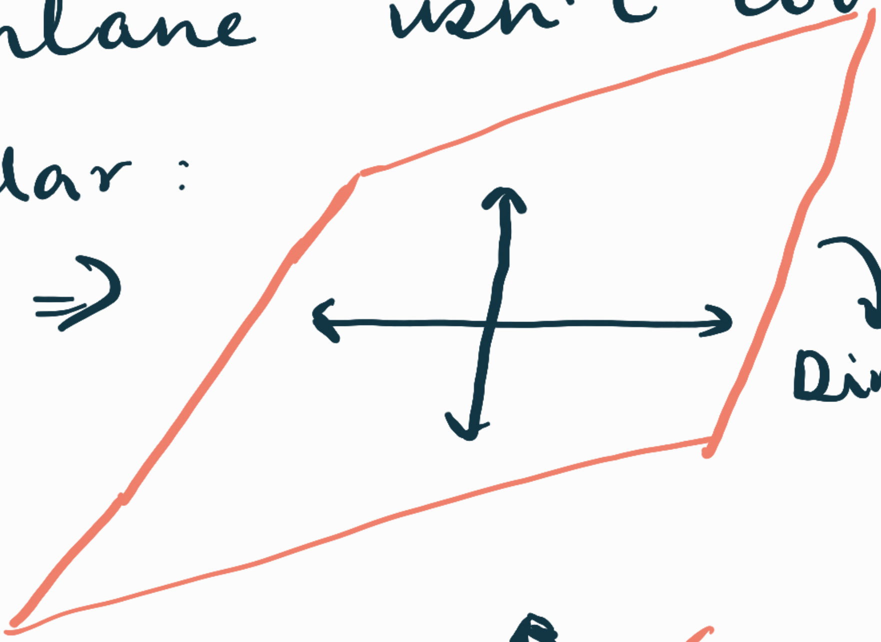
Singularity & rank of linear transformations

When a matrix is singular entire plane isn't covered

Non-singular:

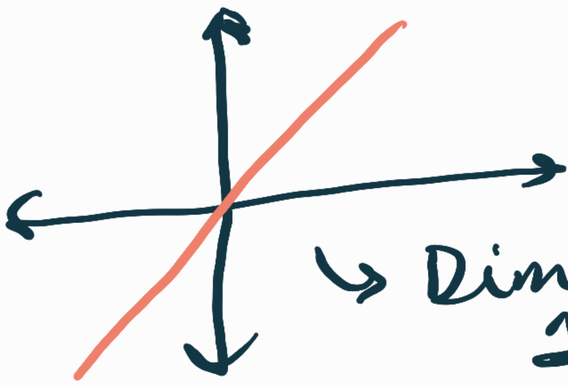
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow$$



Dimension 2

Singular

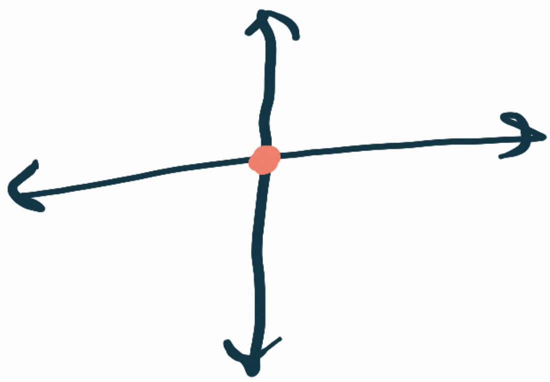
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow$$



Dimension 1

Singular

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



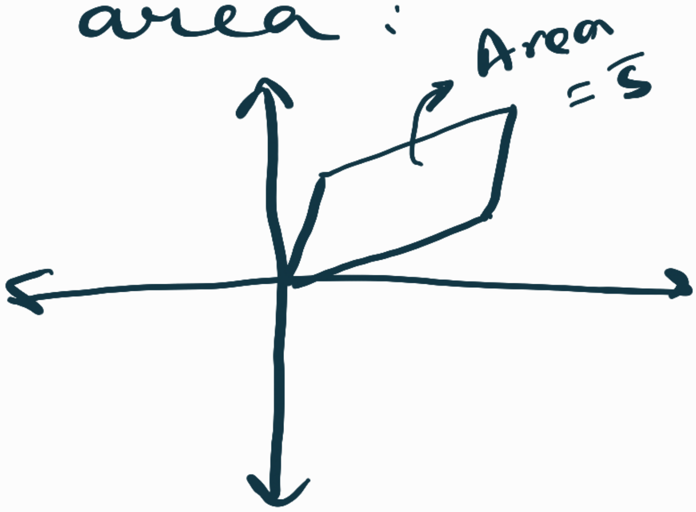
Dimension 0

Rank = Dimension of linear transformation

Determinant as area:

Consider:

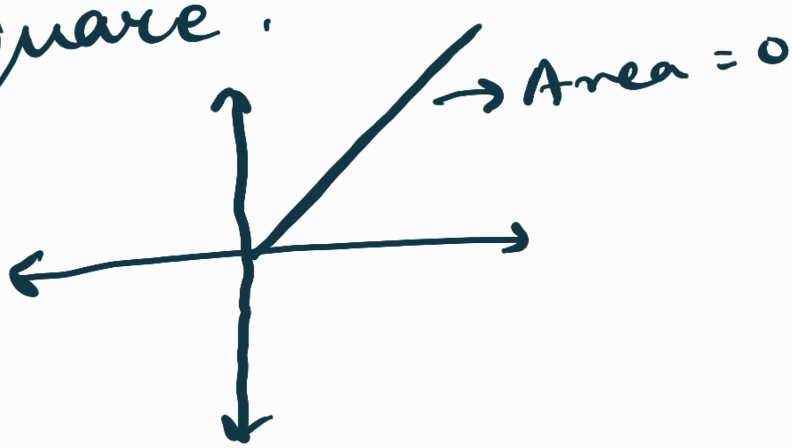
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow$$



$Det = 3 \times 2 - 1 \times 1$
$= 5$

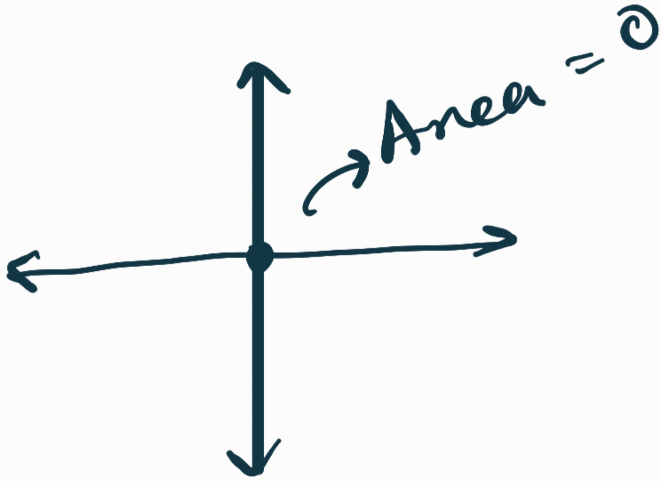
A determinant of a matrix is the fundamental basis formed by unit square.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow$$



$Det = 2 - 2$
$= 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$



$Det = 0$

If vectors in clockwise = positive,
else negative.

Determinant of product of matrices

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

Det = 5 Det = 3 Det = 15

det (AB) = det (A) det (B)

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix}$$

$\Rightarrow$ det = 0

Also because of above rule if any det = 0, it will result in a singular matrix.

Determinants of inverse :

i) Find det of $\begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$

$= 0.4 \times 0.6 - (-0.2 \times -0.2)$

$= 0.2$

$\begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$

$= 0.25 \times 0.625 - (-0.25 \times -0.125)$

$= 0.125$

=) When matrix is invertible,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof :

$\det(AB) = \det(A) \times \det(B)$

Let $B = A^{-1}$

$\Rightarrow \det(AA^{-1}) = \det(A) \times \det(A^{-1})$

But, $AA^{-1} = I$ (Identity matrix)

$\Rightarrow \det(I) = \det(A) \times \det(A^{-1})$

Practice Quiz:

i) what is det of $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= 1.0.0 + 2.1.0 + (-1).1.1$
$- ((-1).0.0) - 2.1.0 - 1.1.1$
$= 0 + 0 - 1 + 0 + 0 - 1 = -2$

ii) Calculate inverse of above matrix $\to A$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad = \frac{1}{|A|} \text{Adj } A$

$$= \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & -2 & -2 \end{bmatrix}$$ Now, transpose the matrix i.e. switch rows to columns

$$= 1 \begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & -2 \\ 1 & -1 & -2 \end{bmatrix} \times \frac{1}{-2}$$

Devide by determinant

$$= \begin{bmatrix} 1/2 & 1/2 & -1 \\ 0 & 0 & 1 \\ -1/2 & 1/2 & 1 \end{bmatrix}$$

iii) Any matrix multiplied with an identity matrix results in itself

iv) Rank of any identity matrix is non-singular

v) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$

$3 \times \cancel{3}$ $\quad$ $\cancel{3} \times 1$

$= \begin{bmatrix} 5 \; -4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$

vii) Dot product :

$\begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 3 \times 2 + 1 \times 2 + 7 \times 0$

$= 8$

viii) Multiply matrix A

w/ matrix B

$$\begin{bmatrix} 5 & 2 & 3 \\ -1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -4 \\ 2 & 1 & 0 \\ 8 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5+4+24 & 0+2-3 & -20 \\ -1-6+16 & 0-3-2 & 4 \\ 0+2-8 & 0+1+1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 33 & -1 & -20 \\ 9 & -5 & 4 \\ -6 & 2 & 0 \end{bmatrix}$$

ix) Calculate determinant of inverse of above matrix

→ Determinant of an inverse is the inverse of determin

Every point in space can be expressed as a linear combination of elements in basis.

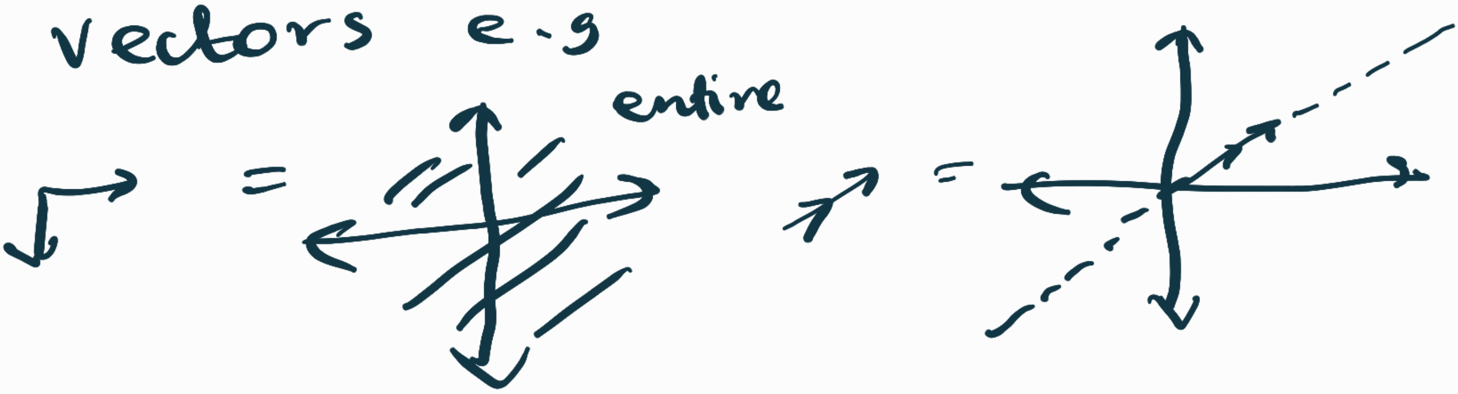
Almost all vectors are basis, so what isn't a basis?

e.g.

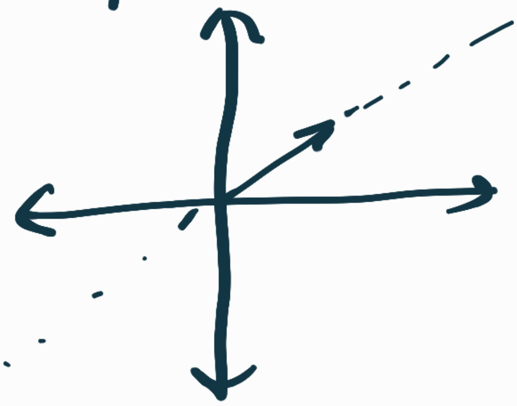
i.e. any two vectors that form a line are not basis.

Span in linear algebra:

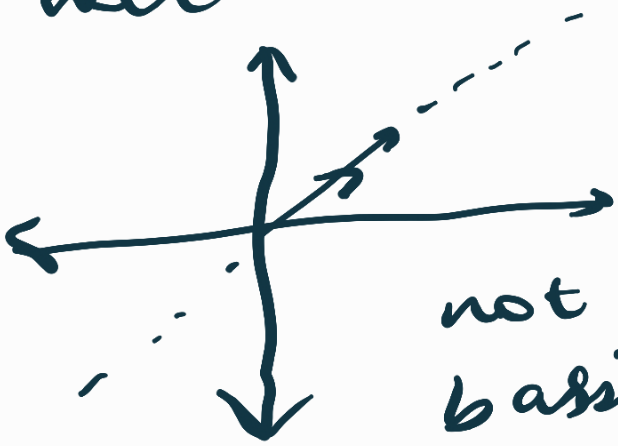
Any point that can be reached by a pair of vectors e.g



A basis is a minimal spanning set

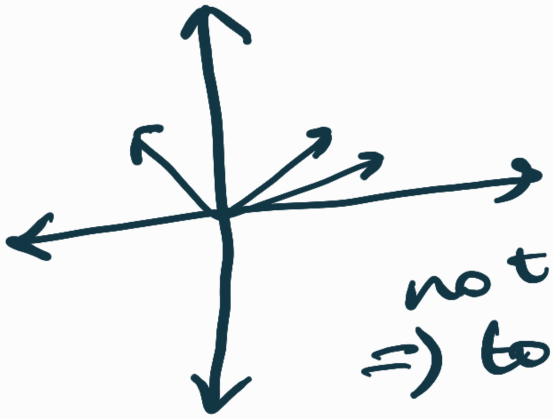


basis



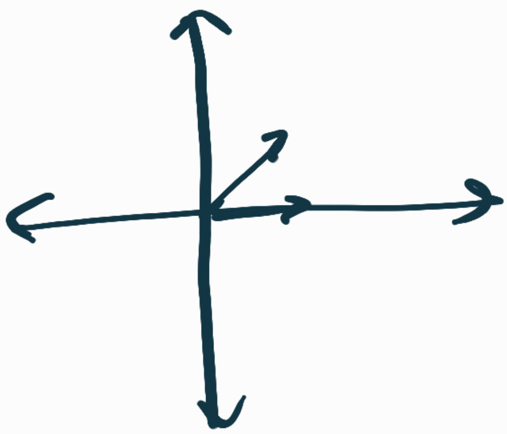
not a basis

Number of elements in basis is the dimension.

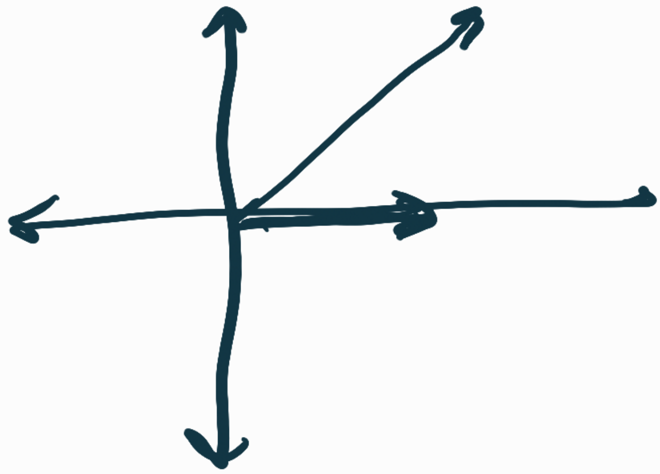


not a basis $\Rightarrow$ too many

<u>**Eigenbases**</u>: (Useful for PCA)



$\Rightarrow$



It sends a parallelogram to another parallelogram of size parallel to the original one.

Two vectors in this basis are called eigenvectors and stretching factors are called eigenvalues.

Finding eigenvalues:

If λ is an eigenvalue,

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has ∞ sol^n

$$\Rightarrow \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \underbrace{(2-\lambda)(3-\lambda) - 1.0}_{\text{Characteristic polynomial}} = 0$$

Eigen values are those that make above 0

$\Rightarrow \quad \lambda = 2 \;, \; \lambda = 3$

Solve the equations,

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow 2x + y = 2x \quad \Rightarrow x = 1 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$0x + 3y = 2y \qquad y = 0$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow 2x + y = 3x \quad \Rightarrow x = 1 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$0x + 3y = 3y \qquad y = 1$

eigen vector

Quiz: Find eigen values & eigen vectors:

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (9-\lambda)(3-\lambda) - (4\times 4) = 0$$

$$27 - 9\lambda - 3\lambda + \lambda^2 - 16 = 0$$

$$\Rightarrow \lambda^2 - 12\lambda + 11 = 0$$

$$\Rightarrow (\lambda - 11)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 11, 1$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 11 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Quiz:

1. Select characteristic:

$$\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ -3 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

determinant =

$$(2-\lambda)(6-\lambda) - (1 \cdot -3) = 0$$

$$\Rightarrow 12 - 2\lambda - 6\lambda + \lambda^2 + 3 = 0$$

$$\Rightarrow \lambda^2 - 8\lambda + 15$$

$$a^2 + (a+b)a + ab = 0$$

$$\Rightarrow ab = 15$$

$$a+b = -8$$

Possible combinations $= (-1, -15), (-3, -5)$

But $(-3, -5)$ sums to -8

$\Rightarrow$ Eigen value is $3, 5$

$$(\lambda - 3)(\lambda - 5) = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad 2x + y = 5x$$

$$3x + 6y = 5y$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$=$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det = (1-\lambda)^2 - (0) + (0)$$

$$= (1-\lambda)^2$$

$$\Rightarrow \quad 1$$

$$\rightarrow \quad A = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}$$

$$= (1-\lambda)(4-\lambda) - 0 = 0$$

$$\Rightarrow \quad (1, 4)$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x + 2y = x \qquad = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$0x + 4y = y$$

$$x + 2y = 4x \qquad = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$0x + 4y = 4y$$

$$\begin{bmatrix} 3-\lambda & 1 & -2 \\ 4 & -\lambda & 1 \\ 2 & 1 & -1 \end{bmatrix}$$