Homework 1

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Measure Theory

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Problem 1.3.
(1) is not an algebra
(2) is not an algebra
(3) is a σ algebra

Problem 1.7.
Suppose power set is not the largest possible σ algebra, then there exists a larger σ algebra that contains sets that are not subsets of X $\rightarrow$ contradiction by defination
Suppose $\{\phi$,x$\}$ is not the smallest possible σ algebra, then there exists a smaller σ algebra A $\subset \{\phi$,x$\}$. However, such set is not closed under complement $\rightarrow$ contradiction by defination

Problem 1.10.
Let $\{S_a\}$ be a family of σ algebras.
A $\in \cap\, S_a \Rightarrow$ A $\in S_i\ \forall$ i $\Rightarrow A^{\complement} \in S_i\ \forall$ i $\Rightarrow A^{\complement} \cap S_a$
Let $A_j \in \cap\, S_i$ for j $\in$ J. Then $A_j \in S_i\ \forall$ i $\forall$ j $\Rightarrow \cup\, A_j \subset \cap\, S_i$

Problem 1.17.
(1) A $\subset$ B $\Rightarrow \exists\ B_i$ such that $B_i \in$ B and $B_i \notin$ A $\Rightarrow \mu$(B) = $\mu(B_i)$+μ(A) $\geq \mu$(A)
(2) If $\{A_i\}$ forms a partition of A, then the equality holds, otherwise $\sum_{i=1}^{\infty} \mu(A_i) \geq \mu(\cup A_i)$

Problem 1.18.
(1) If A is empty, then A $\cap$ B is also empty $\Rightarrow \lambda$(A) = μ(A $\cap$ B) = 0
(2) Let $\{A_i\}$ be a partition of A $\Rightarrow \lambda(\cup\, A_i)$= μ(A$\cap$B)= $\sum_{i=1}^{\infty} \mu(A_i\cap$B)=$\sum_{i=1}^{\infty} \lambda(A_i)$

Problem 2.14.
Let A be an algebra. By lemma, $\forall$ sets in A can be written as countable unions of open intervals $\Rightarrow A_i \in \sigma(O)$ $\forall$i $\Rightarrow \sigma$(A)$\subset\sigma(O)$. Also, $\forall$ sets in O can be written as countable union of sets of A$\Rightarrow$ $O\subset$A$\Rightarrow\sigma(O)\subset\sigma$(A)$\Rightarrow\sigma(O)$=$\sigma$(A)

Problem 3.1.
Let a$\in$R. Then $\{$a$\}\subset\{$a-ϵ, a+$\epsilon\}$ $\forall\epsilon \Rightarrow 2\epsilon$=$\lambda^*$($\{$a-$\epsilon$,a+$\epsilon\}$)$\geq\lambda^*$($\{a\}$) $\Rightarrow \lambda^*$($\{$a$\}$)=0 $\forall$a. If A=$\cup\{a_i\}$ is a countable set, then λ^*(A)=0

Problem 3.4.
Each of 4 sets can be written as either countable unions of other sets or complements of other set, hence, if one of them is measurable, others are also measurable.

Problem 3.7.
{f+g<b}=$\bigcup_{q+r}$ {f<q}cap{g<r}$\Rightarrow$ f+g is measurable
Observe: {x$\in$A:f(x)^2>a}={x$\in$A:f(x)>$\sqrt{a}$}$\cup${x$\in$A:f(x)<-$\sqrt{a}$}, hence f^2 is measurable $\Rightarrow$fg=1/2((f+g)^2-f^2-g^2) is measurable