The Euler-Maruyama Method

An Extension of ODE Methods to SDEs

Sean Zachary Roberson

The University of Texas at San Antonio

AIM 5113: Introduction to Industrial Mathematics

November 23, 2022

Background

- There are many methods to numerically solve differential equations.
- The classical methods many students see and professionals implement are based on deterministic functions.
- Certain phenomena are random in nature and require a different class of methods.

Objectives

In this talk, we will...

- introduce the basic notion of stochastic integration,
- develop a scheme to numerically solve stochastic differential equations, and
- solve a simple example with a known solution.

Crash Course in Stochastic Integration

To develop the stochastic integral, we begin with the Riemann-Stieltjes integral. For a function of bounded variation $g(x)$, the Riemann-Stieltjes integral of $f(x)$ with respect to $g(x)$ is defined as a limit:

$$\int_a^b f(x)\ dg(x) = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\left(g(x_j) - g(x_{j-1})\right)$$

where the limit is interpreted to allow the step size between the tags x_j to go to zero.

The aim is to allow the integrator $g(x)$ to be a random variable.

Crash Course in Stochastic Integration

To achieve our goal, we choose to integrate with respect to a Brownian motion B_t. By using an appropriate limiting process, replace $g(t)$ in the previous limit by B_t. Thus, we create the Ito integral:

$$\int_a^b f(t)\, dB_t = \lim_{n\to\infty} \sum_{j=1}^{n} f(t_j) \left(B_{t_j} - B_{t_{j-1}} \right)$$

An extended treatment can be found in [1], with a slightly general explanation in [2].

The Ito Lemma

The primary tool used in the stochastic calculus is the Ito lemma. An abridged version is stated here.

Ito Lemma

Let $f(t,x)$ be $\mathcal{C}^2([0,\infty) \times \mathbb{R})$. Then the process $X_t = f(t, B_t)$ has the following "derivative:"

$$dX_t = \left(f_t + \frac{1}{2} f_{xx} \right) \, dt + f_x \, dB_t$$

Note that the derivative here is interpreted in the loose sense.

The Ito Lemma

For our purposes, we present the integral form of the Ito lemma.

Ito Lemma, Integral Form

Suppose $f(t,x)$ satisfies the previous hypothesis. Then

$$X_t = X_0 + \int_0^t \left(f_s(s, B_s) + \frac{1}{2} f_{xx}(s, B_s) \right) \, ds + \int_0^t f_x(s, B_s) \, dB_s$$

One interpretation of this formula is the stochastic variant of the Fundamental Theorem of Calculus.

The Euler-Maruyama Method

Given the stochastic differential equation

$$dX_t = u(t, X_t)\ dt + v(t, X_t)\ dB_t$$

can we find a solution X_t or numerically approximate it? Here, the functions u and v are given. The aim is to mimic the deterministic Euler method.

The Euler-Maruyama Method

The primary tool is to approximate the integrals previously seen by a one-point quadrature:

$$\int_t^{t+h} u(s, X_s)\ ds + \int_t^{t+h} v(s, X_s)\ dB_s \approx hu(t, X_t) + v(t, X_t)(B_{t+h} - B_t)$$

where the increment $B_{t+h} - B_t$ is a normal random variable with mean 0 and standard deviation $\sqrt{t}$. This develops the iterative method by creating a discretization Y_t of the process X_t over a desired time interval.

Euler-Maruyama Method

The stochastic differential equation

$$dX_t = u(t, X_t)\ dt + v(t, X_t)\ dB_t$$

can be numerically approximated by the Markov chain partitioned on the time interval $[0, T]$ with N equally spaced points

$$Y_n = Y_{n-1} + hu(t_{n-1}, Y_{t_{n-1}}) + v(t_{n-1}, Y_{t_{n-1}})(B_{t_{n-1}} - B_{t_n})$$

where $Y_0 = X_0, h = \frac{T}{N}$, and $n = 0, 1, \ldots, N$.

Example

An example will involve the Ornstein-Uhlenbeck process:

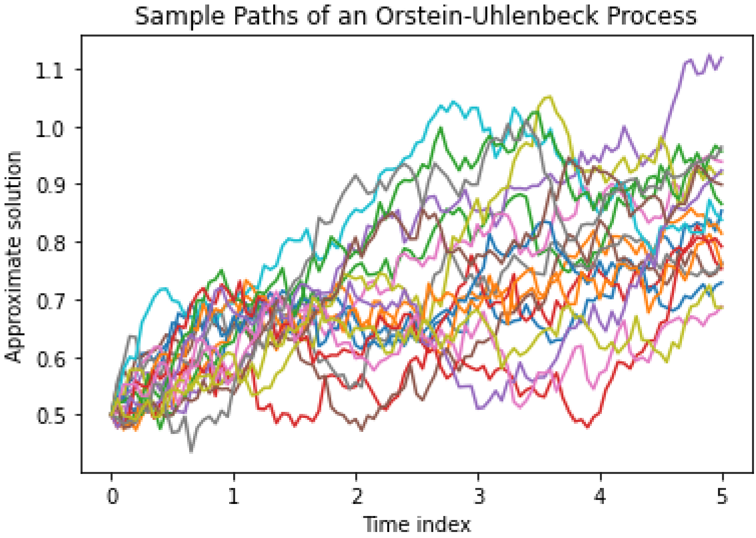
$$dX_t = \theta(\mu - X_t)\ dt + \sigma\ dB_t$$

where $\theta, \sigma > 0$ are the drift and volatility constants, respectively, and μ is the mean of X_t, in limit. Specifically, we consider the equation

$$dX_t = 1.3(1 - X_t)\ dt + 0.225\ dB_t$$

with initial condition $X_0 = 0.5$.

Below is a plot of 20 sample paths of this process.



References

Peter E. Kloeden and Eckhard Platen.
Numerical Solution of Stochastic Differential Equations.
Springer, 1999.

Sean Zachary Roberson.
A quick drift into stochastic calculus.
2019.