

Final Exam Review, Spring 2024

This document contains several practice problems to help prepare for the final exam. These are taken from various texts and are modeled after the types of problems you have seen before, albeit with some extra direction. The goal is to understand the “how,” not just the “what.” Ask yourself the following questions when attempting an exercise:

- What theorem/result/formula should I use? Why can I use it?
- How can I move logically from one step to the next?
- Are there possibly multiple ways to solve the problem?
- Does my work flow? Can a partner (or my instructor) follow my work with minimal stress?
This is probably the most important!

Attempt each exercise. Show all work as you would on an exam.

1 Techniques of Integration

1.1. Evaluate $\int x^3 \sin(2x) dx$.

1.2. Evaluate $\int \arctan x \, dx$.

- 1.3. Evaluate $\int x^4 \ln x \, dx$. (As an extension, evaluate $\int x^p \ln x \, dx$ for $p > 0$.) **Suggestion:**
The integral can be attacked in two ways.

1.4. Evaluate $\int \cos(\sqrt{x}) \, dx$.

1.5. Evaluate $\int \cos^3(4x) \, dx$.

1.6. Evaluate $\int \sin^2(3x) \, dx$. (Hint: Use the power-reducing/half-angle/golden identity.)

1.7. Evaluate $\int \sin^3 x \cos^2 x \, dx$.

1.8. Evaluate $\int \sec^4 x \tan^2 x \, dx$.

1.9. Evaluate $\int \sec x \tan^3 x \, dx$.

- 1.10. By using an appropriate trigonometric substitution, evaluate $\int x\sqrt{x^2 + 16} \, dx$. The final answer should not include expressions such as $\sin(\arctan x)$ or $\arccos(\sin x)$.

1.11. Evaluate the integral $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

1.12. Evaluate the integral $\int \frac{dx}{(x^2 - 9)^{3/2}}$.

1.13. Evaluate the integral $\int \frac{dx}{x(x^2 + 1)}$ by decomposing into partial fractions.

1.14. Evaluate the integral $\int \frac{2x + 1}{x^2 - 7x + 12} dx$.

1.15. Evaluate the integral $\int \frac{1}{x^2(x-1)} dx$.

1.16. Evaluate the improper integral $\int_0^{\infty} x e^{-3x} dx$.

1.17. Does the integral $\int_e^\infty \frac{\ln x}{x^2} dx$ converge? If so, find the value.

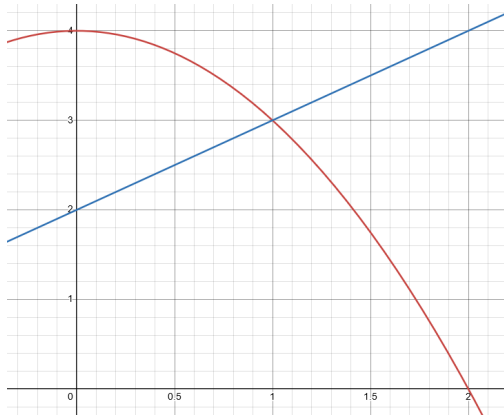
1.18. Does the integral $\int_0^1 x \ln x \, dx$ converge? If so, find its value. (Hint: Remember that $\ln x$ is only defined for $x > 0$.)

2 Applications of Definite Integrals

For problems 2.1 – 2.4, **write, but do not evaluate** the integrals. As an added exercise, you may finish evaluating the integrals. Use the included plots to aid your setups.

- 2.1. Consider the region in the plane bounded by the curves $y = 4 - x^2$, $y = x + 2$, $y = 0$, $x = 1$, and $x = 2$.

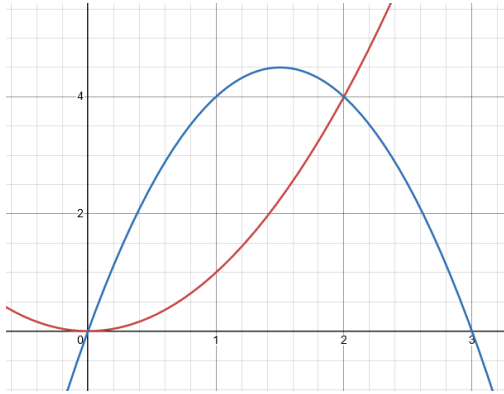
a. Write an integral that represents the area of this region.



b. Rotate the region about the line $y = 0$. What integral represents this volume?

2.2. Consider the region in the plane bounded by the curves $y = x^2$ and $y = 6x - 2x^2$.

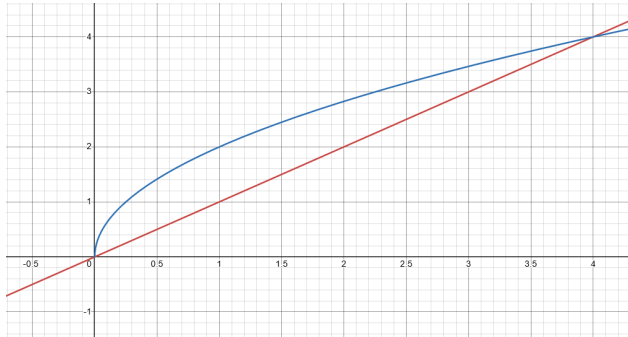
a. Write an integral that represents the area of this region.



b. Rotate this region about the line $x = 0$. What integral represents this volume?

2.3. Consider the region in the plane bounded by the curves $y = x$ and $y = 2\sqrt{x}$.

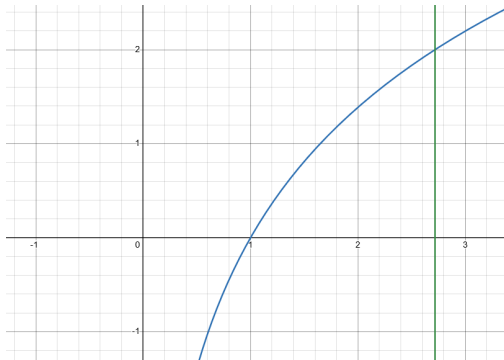
a. Rotate this region about the line $x = 4$. What integral represents this volume?



b. Rotate this region about the line $y = -1$. What integral represents this volume?

2.4. Consider the region in the plane bounded by the curves $y = 2 \ln x$, $y = 0$, $x = 1$, and $x = e$.

a. Rotate this region about the line $x = -1$. What integral represents this volume?



b. Rotate this region about the line $y = 2$. What integral represents this volume?

- 2.5. A spring requires 20 N of force to displace it 0.4 m beyond its natural length. How much work is done in stretching it 0.25 m beyond its natural length?

- 2.6. A spring requires 20 J of energy to stretch it from 0.2 m to 0.25 m beyond its natural length. How much work is required to stretch it 0.15 m beyond its natural length? (Hint: The total energy in the system is the integral of the force function – but what is the general law to determine the force needed to displace a spring?)

- 2.7. A cable that weighs 2 lb/ft is used to lift 800 pounds of coal from a mine shaft 500 feet deep. How much work is done in lifting all of the cable and the coal?

- 2.8. A rope that is 10 meters long has a mass of 80 kilograms. The rope hangs off the edge of a building. How much work is needed to lift half of the rope to the top?

- 2.9. From the side of a building 30 meters tall, a cable with weight 120 Newtons per meter has a box with mass 150 kilograms attached to it. How much work is needed to lift the cable and box 25 meters?

3 Sequences and Series

- 3.1. Expand the function $f(x) = \frac{1}{1 + 2x^2}$ as power series near $a = 0$. What is its radius and interval of convergence?

- 3.2. Expand the function $g(x) = 3 \ln(x + 1)$ as a power series near $a = 2$. What is its radius and interval of convergence? (First expand the derivative as a power series, then integrate that expansion term by term.)

3.3. On what interval does the power series $\sum_{n=0}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$ converge?

3.4. On what interval does the power series $\sum_{n=0}^{\infty} \frac{2^n (x-3)^2}{n!}$ converge?

3.5. On what interval does the power series $\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{\sqrt{n+3}}$ converge?

3.6. Write the Taylor series for $f(x) = \sin x$ near $a = \frac{\pi}{2}$.

- 3.7. Write the Taylor expansion for $g(x) = e^{-x^2}$ near $a = 0$. (Hint: First expand e^x near $a = 0$, then replace x with $-x^2$.)

3.8. Without using the “geometric series trick,” expand $h(x) = \frac{2}{x}$ near $a = 2$.

Math 1224 Formula Sheet

- **Area Between Curves:** $A = \int_a^b [f(x) - g(x)] dx$
- **Volume by Slicing:** $V = \int_a^b A(x) dx$
- **Disk Method:** $V = \pi \int_a^b (\text{Radius})^2 dx$
- **Washer Method:** $V = \pi \int_a^b [(\text{Outer Radius})^2 - (\text{Inner Radius})^2] dx$
- **Shell Method:** $V = 2\pi \int_a^b (\text{Radius})(\text{Height}) dx$
- **Arc Length:** $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$
- **Surface Area:** $SA = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx$
- **Work:** $W = \int_a^b F(x) dx$
- **Hooke's Law:** $F(x) = kx$
- **Integration by Parts:** $\int u dv = uv - \int v du$
- **Trig. Identities:** $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin 2x = 2 \sin x \cos x$
- **Trig. Substitution:** If $\sqrt{a^2 - x^2}$, use $x = a \sin \theta$. If $\sqrt{x^2 - a^2}$, use $x = a \sec \theta$. If $\sqrt{x^2 + a^2}$, use $x = a \tan \theta$.
- **Partial Fraction Examples:**

$$\begin{aligned} \frac{1}{x(x+1)(x+2)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \\ \frac{1}{x^3(x+2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2} \\ \frac{1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\ \frac{1}{(x^2+1)^2(x+2)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+2} \end{aligned}$$

- **Sequences:** A sequence a_n converges to a limit L if $\lim_{n \rightarrow \infty} a_n = L$ where L is a real number. Otherwise, the sequence diverges.
- **Series:** A series $\sum a_n$ converges to a sum S if $\lim_{n \rightarrow \infty} S_n = S$ where S is a real number and S_n denotes the n th partial sum.
- **Geometric Series Test:** A geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$. Otherwise, the series diverges.
- **Divergence Test:** A series $\sum a_n$ diverges if $\lim_{n \rightarrow \infty} a_n \neq 0$.
- **Integral Test:** Consider a series $\sum a_n$, where $f(n) = a_n$ is positive, continuous, and eventually decreasing. Then $\sum a_n$ and $\int_a^{\infty} f(x) dx$ both converge or both diverge.
- **p -Series Test:** A series of the form $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

- **Direct Comparison Test:** Consider two series $\sum a_n$ and $\sum b_n$ with positive terms.

1. If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
2. If $a_n \geq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

- **Limit Comparison Test:** Consider two series $\sum a_n$ and $\sum b_n$ with positive terms and let

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

1. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

- **Alternating Series Test:** Consider an alternating series $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$. The series converges if

1. $b_n \geq 0$
2. $b_{n+1} \leq b_n$
3. $\lim_{n \rightarrow \infty} b_n = 0$

- **Ratio Test:** Consider a series $\sum a_n$ with nonzero terms and let

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

1. If $L < 1$, the series converges absolutely.
2. If $L > 1$, the series diverges.
3. If $L = 1$, the ratio test is inconclusive.

- **Root Test:** Consider a series $\sum a_n$ with nonzero terms and let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$$

1. If $L < 1$, the series converges absolutely.
2. If $L > 1$, the series diverges.
3. If $L = 1$, the root test is inconclusive.

- **Term-By-Term Differentiation and Integration:** Consider the function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Then

1. $f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}$
2. $\int f(x) dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

- **Taylor/Maclaurin Series:** The Taylor series for $f(x)$ centered at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

The Maclaurin series for $f(x)$ is the Taylor series for $f(x)$ centered at $x = 0$.