Final Exam Review, Spring 2024

This document contains several practice problems to help prepare for the final exam. These are taken from various texts and are modeled after the types of problems you have seen before, albeit with some extra direction. The goal is to understand the "how," not just the "what." Ask yourself the following questions when attempting an exercise:

- What theorem/result/formula should I use? Why can I use it?
- How can I move logically from one step to the next?
- Are there possibly multiple ways to solve the problem?
- Does my work flow? Can a partner (or my instructor) follow my work with minimal stress? **This is probably the most important!**

Attempt each exercise. Show all work as you would on an exam.

1 Techniques of Integration

1.1. Evaluate $\int x^3 \sin(2x)\, dx$.

Use IBP - better with tabular. Set $u = x^3$, $dv = \sin 2x \, dx$.

	D	I
+	x^3	$\sin 2x$
−	$3x^2$	$-\frac{1}{2}\cos 2x$
+	$6x$	$-\frac{1}{4}\sin 2x$
−	6	$\frac{1}{8}\cos 2x$
+	0	$\frac{1}{16}\sin 2x$

$$\int x^3 \sin 2x \, dx = \boxed{-\frac{1}{2}x^3\cos 2x + \frac{3}{4}x^2 \sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + C.}$$

1.2. Evaluate $\int \arctan x \, dx$. Use parts: $u = \arctan x$, $dv = dx$.

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx^{*}$$

$$= \boxed{x \arctan x - \frac{1}{2} \ln\left|1+x^2\right|^{*} + C.}$$

* This integral can be done in two ways. Most natural is using, e.g., $t = 1+x^2$.

* The absolute value isn't needed as $1+x^2 \geq 0$ for all x, but it's good to keep it in.

1.3. Evaluate $\int x^4 \ln x \, dx$. (As an extension, evaluate $\int x^p \ln x \, dx$ for $p > 0$.) **Suggestion:** The integral can be attacked in two ways.

Use parts - $u = \ln x$, $dv = x^4 \, dx$. *

$$\int x^4 \ln x \, dx = \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4 \, dx$$

$$= \boxed{\frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C.}$$

* Another solution uses $x = e^y$ with $dx = e^y \, dy$. (So $y = \ln x$.)

$$\int x^4 \ln x \, dx = \int (e^y)^4 \ln(e^y) \cdot e^y \, dy$$

$$= \int y e^{4y} \cdot e^y \, dy \qquad u = y \quad dv = e^{5y} \, dy$$

$$= \int y e^{5y} \, dy$$

	D	I
+	y	e^{5y}
−	1	$\frac{1}{5}e^{5y}$
+	0	$\frac{1}{25}e^{5y}$

$$= \frac{1}{5}ye^{5y} - \frac{1}{25}e^{5y} + C$$

$$= \boxed{\frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C.} \; *$$

* Note:

$$e^{5y} = e^{5\ln x} = e^{\ln x^5} = x^5.$$

1.4. Evaluate $\int \cos(\sqrt{x})\, dx$. First, $x = y^2$ so $dx = 2y\, dy$. *

$$\int \cos\sqrt{x}\, dx = \int 2y \cos y\, dy$$

Now parts: $u = 2y$, $dv = \cos y\, dy$.

	D	I
$+$	$2y$	$\cos y$
$-$	2	$\sin y$
$+$	0	$-\cos y$

$$= 2y \sin y + 2\cos y + C$$

$$= \boxed{2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.}$$

1.5. Evaluate $\int \cos^3(4x)\,dx$. Save a cosine.

$$\int \cos^3 4x \, dx = \int \cos 4x \underline{\cos^2 4x} \, dx$$

Use the basic Pythagorean identity.

$$= \int \cos 4x \left(\underline{1 - \sin^2 4x}\right) dx$$

Now $z = \sin 4x$, $dz = 4\cos 4x\, dx$.

$$= \frac{1}{4}\int z(1 - z^2)\, dz$$

$$= \frac{1}{4}\int z - z^3 \, dz$$

$$= \frac{1}{4}\left(\frac{1}{2}z^2 - \frac{1}{3}z^3\right) + C$$

$$= \boxed{\frac{1}{8}\sin^2(4x) - \frac{1}{12}\sin^3(4x) + C.}$$

1.6. Evaluate $\int \sin^2(3x)\, dx$. (Hint: Use the power-reducing/half-angle/golden identity.)

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$$

$$\int \sin^2 3x \, dx = \int \frac{1}{2}(1 - \cos 6x)\, dx \qquad \text{Here } \alpha = 3x.$$

$$= \int \frac{1}{2} - \frac{1}{2}\cos 6x \, dx$$

$$= \boxed{\frac{1}{2}x - \frac{1}{12}\sin 6x + C.}$$

1.7. Evaluate $\int \sin^3 x \cos^2 x \, dx$. Save a sine.

$$\int \sin^3 x \cos^2 x \, dx = \int \underline{\sin x} \sin^2 x \cos^2 x \, dx$$

$$= \int \sin x \left(\underline{1-\cos^2 x}\right) \cos^2 x \, dx$$

Use the Pythagorean identity.
Now $u = \cos x$, $du = -\sin x \, dx$.

$$= -\int u^2 (1-u^2) \, du$$

$$= -\int u^2 - u^4 \, du$$

$$= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.}$$

1.8. Evaluate $\int \sec^4 x \tan^2 x \, dx$.

Save two secants.

Use the Pythagorean identity, and then $y = \tan x$.

$$\int \sec^4 x \tan^2 x \, dx = \int \underline{\sec^2 x} \sec^2 x \tan^2 x \, dx.$$

$$= \int \sec^2 x \left(\underline{1 + \tan^2 x} \right) \tan^2 x \, dx$$

$$= \int y^2 (1 + y^2) \, dy$$

$$= \int y^2 + y^4 \, dy$$

$$= \frac{1}{3} y^3 + \frac{1}{5} y^5 + C$$

$$= \boxed{\frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C.}$$

1.9. Evaluate $\int \sec x \tan^3 x \, dx$. Save a tangent.

$$\int \sec x \tan^3 x \, dx = \int \sec x \underline{\tan x} \tan^2 x \, dx$$

$$= \int \sec x \tan x \left(\underline{\sec^2 x - 1} \right) dx$$

Use the Pythagorean identity, then $w = \sec x$.

$$= \int w^2 - 1 \, dw$$

$$= \frac{1}{3} w^3 - w + C$$

$$= \boxed{\frac{1}{3} \sec^3 x - \sec x + C.}$$

1.10. By using an appropriate trigonometric substitution, evaluate $\int x\sqrt{x^2+16}\,dx$. The final answer should not include expressions such as $\sin(\arctan x)$ or $\arccos(\sin x)$.

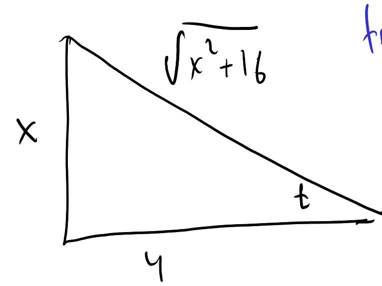
Use $x = 4\tan t$ so $dx = 4\sec^2 t\,dt$. Now,

$$\begin{aligned}
\sqrt{x^2+16} &= \sqrt{(4\tan t)^2+16} \\
&= \sqrt{16\tan^2 t+16} \\
&= \sqrt{16(\tan^2 t+1)} \\
&= 4\sec t.
\end{aligned}$$

This gives

$$\begin{aligned}
\int x\sqrt{x^2+16}\,dx &= \int 4\tan t\cdot 4\sec t\cdot 4\sec^2 t\,dt \\
&= 64\int \tan t\,\sec t\cdot\sec^2 t\,dt \qquad y = \sec t. \\
&= 64\int y^2\,dy \\
&= \frac{64}{3}y^3 + C \\
&= \frac{64}{3}\sec^3 y + C \\
&= \frac{64}{3}\left(\frac{\sqrt{x^2+16}}{4}\right)^3 + C \\
&= \boxed{\frac{1}{3}(x^2+16)^{3/2} + C.}
\end{aligned}$$

Use the triangle.



$\cos t = \dfrac{4}{\sqrt{x^2+16}}$

$\sec t = \dfrac{\sqrt{x^2+16}}{4}$

1.11. Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Use $x = 2\sin t$, so $dx = 2\cos t\, dt$.

Then $4 - x^2 = 4 - 4\sin^2 t = 4\cos^2 t$.

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos t \, dt}{(2\sin t)^2\sqrt{4\cos^2 t}}$$

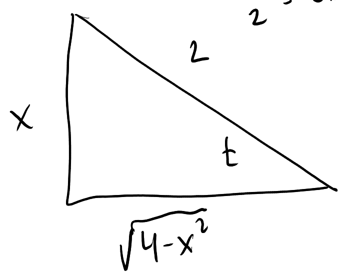
$$= \frac{1}{4}\int \frac{dt}{\sin^2 t}$$

$$= \frac{1}{4}\int \csc^2 t \, dt$$

$$= \frac{1}{4}\cot t + C.$$

$$= \boxed{\frac{\sqrt{4-x^2}}{4x} + C}$$

$\frac{x}{2} = \sin t$



$\tan t = \frac{x}{\sqrt{4-x^2}}$

$\cot t = \frac{\sqrt{4-x^2}}{x}$

1.12. Evaluate the integral $\int \frac{dx}{(x^2-9)^{3/2}}$.

Use $x = 3\sec t$, so $dx = 3\sec t \tan t \, dt$.

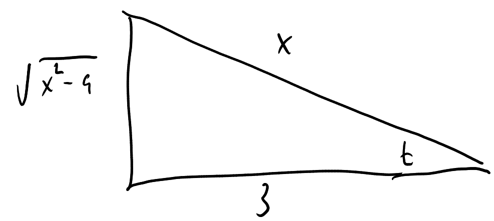
Then
$$x^2 - 9 = (3\sec t)^2 - 9 = 9\sec^2 t - 9 = 9(\sec^2 t - 1) = 9\tan^2 t.$$

$$\int \frac{dx}{(x^2-9)^{3/2}} = \int \frac{3\sec t \tan t \, dt}{(9\tan^2 t)^{3/2}}$$
$$= 3\int \frac{\sec t \tan t \, dt}{9^{3/2}\tan^3 t}$$
$$= \frac{3}{27}\int \frac{\sec t}{\tan^2 t}\, dt$$
$$= \frac{1}{9}\int \frac{1}{\cos t}\cdot\frac{\cos^2 t}{\sin^2 t}\, dt$$

After cancellation, $u = \sin t$.

$$= \frac{1}{9}\int \frac{du}{u^2} = -\frac{1}{9}\cdot\frac{1}{u} + C$$
$$= -\frac{1}{9}\cdot\frac{1}{\sin t} + C.$$
$$= \boxed{-\frac{1}{9}\cdot\frac{x}{\sqrt{x^2-9}} + C}$$

$\frac{x}{3} = \sec t \rightarrow \frac{3}{x} = \cos t$



$\sin t = \frac{\sqrt{x^2-9}}{x}$

1.13. Evaluate the integral $\int \frac{dx}{x(x^2+1)}$ by decomposing into partial fractions.

Write $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$. Clearing fractions gives $1 = A(x^2+1) + (Bx+C)x$.

Set $x=0$:

$1 = A$

Set $x=1$:

$1 = 1(1^2+1) + (B+C)$

$1 = 2 + B + C$

$B + C = -1$.

Set $x=-1$:

$1 = 1((-1)^2+1) + (B(-1)+C)(-1)$

$1 = 2 + (-B+C)\cdot(-1)$

$-1 = B - C$.

Solve the system $\begin{cases} B+C = -1 \\ B-C = -1 \end{cases} \longrightarrow \begin{matrix} B = -1, \\ C = 0. \end{matrix}$

So $\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$. Now,

$$\int \frac{1}{x} - \frac{x}{x^2+1}\,dx = \boxed{\ln|x| - \frac{1}{2}\ln|x^2+1| + C.}$$

1.14. Evaluate the integral $\int \frac{2x+1}{x^2-7x+12}\,dx$.

Write $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3}$. Clearing gives $2x+1 = A(x-3) + B(x-4)$.

Set $x=3$:

$2(3)+1 = -B$.

$B = -7$.

Set $x=4$.

$2(4)+1 = A$

$A = 9$.

So $\frac{2x+1}{x^2-7x+12} = \frac{9}{x-4} - \frac{7}{x-3}$.

Now $\int \frac{9}{x-4} - \frac{7}{x-3}\,dx = \boxed{9\ln|x-4| - 7\ln|x-3| + C.}$

1.15. Evaluate the integral $\displaystyle\int \frac{1}{x^2(x-1)}\,dx$.

Write $\dfrac{1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$. Clearing gives $1 = Ax(x-1) + B(x-1) + Cx^2$.

Set $x=0$:

$1 = -B$

$B = 1$.

Set $x=1$:

$1 = C$.

Set $x=-1$.

$1 = A(-1)(-1-1) + (-1-1) + (-1)^2$

$1 = 2A - 2 + 1$

$2A = 2$

$A = 1$.

So, $\displaystyle\int \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-1}\,dx = \boxed{\ln|x| - \frac{1}{x} + \ln|x-1| + C}$

1.16. Evaluate the improper integral $\int_0^\infty xe^{-3x}\,dx$.

Use parts: $u = x$, $dv = e^{-3x}\,dx$.

	D	I
+	x	e^{-3x}
−	1	$-\frac{1}{3}e^{-3x}$
+	0	$\frac{1}{9}e^{-3x}$

$$\int_0^\infty xe^{-3x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-3x}\,dx$$

$$= \lim_{t\to\infty}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right)\Bigg|_0^t$$

$$= \lim_{t\to\infty}\left(\left[-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}\right] - \left[-\frac{1}{9}\right]\right)$$

$$= \boxed{\frac{1}{9}.}$$

$$\lim_{t\to\infty} -\frac{1}{3}te^{-3t} = \lim_{t\to\infty} -\frac{1}{3}\cdot\frac{t}{e^{3t}} = 0.$$

$$\lim_{t\to\infty} -\frac{1}{9}e^{-3t} = 0.$$

1.17. Does the integral $\int_e^\infty \frac{\ln x}{x^2}\, dx$ converge? If so, find the value.

Use parts with $u = \ln x$, $dv = \frac{1}{x^2}\,dx$:

$$\int_e^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{\ln x}{x}\Big|_e^t + \int_e^t \frac{1}{x^2}\,dx \right)$$

$$= \lim_{t\to\infty}\left(-\frac{\ln t}{t} + \frac{1}{e} + \left(-\frac{1}{x}\right)\Big|_e^t \right)$$

$$= \lim_{t\to\infty}\left(\underbrace{\frac{-\ln t}{t}} + \frac{1}{e} - \underbrace{\frac{1}{t}} + \frac{1}{e} \right)$$

$$= \boxed{\frac{2}{e}}$$

The integral converges.

$$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0. \quad \text{(by l'Hôpital.)}$$

$$\lim_{t\to\infty} \frac{1}{t} = 0.$$

1.18. Does the integral $\int_0^1 x \ln x \, dx$ converge? If so, find its value. (Hint: Remember that $\ln x$ is only defined for $x > 0$.)

Use parts: see 1.3.

$$\int_0^1 x \ln x \, dx = \lim_{t \to 0^+} \int_t^1 x \ln x \, dx$$

$$= \lim_{t \to 0^+} \left(\frac{1}{2}x^2 \ln x \Big|_t^1 - \int_t^1 \frac{1}{2} x \, dx \right)$$

$$= \lim_{t \to 0^+} \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \right) \Big|_t^1$$

$$= \lim_{t \to 0^+} \left(-\frac{1}{2} \underbrace{t^2 \ln t} - \frac{1}{4} + \frac{1}{4}t^2 \right) = \boxed{-\frac{1}{4}}$$

The integral converges.

$$\lim_{t \to 0^+} t^2 \ln t = \lim_{t \to 0^+} \frac{\ln t}{1/t^2} = \lim_{t \to 0^+} \frac{1/t}{-2/t^3} = \lim_{t \to 0^+} \frac{1}{t} \cdot \frac{-t^3}{2} = 0.$$

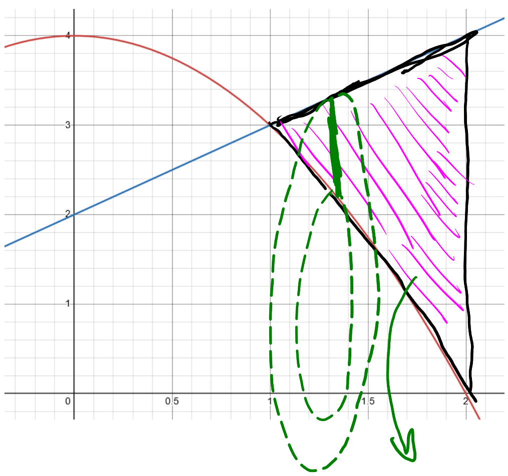
2 Applications of Definite Integrals

For problems 2.1 – 2.4, **write, but do not evaluate** the integrals. A an added exercise, you may finish evaluating the integrals. Use the included plots to aid your setups.

This was a typo.

2.1. Consider the region in the plane bounded by the curves $y = 4 - x^2$, $y = x + 2$ [redacted] $x = 1$, and $x = 2$.

a. Write an integral that represents the area of this region.



$$A = \int_1^2 (x+2) - (4-x^2) \, dx$$

$$= \int_1^2 x^2 + x - 2 \, dx.$$

b. Rotate the region about the line $y = 0$. What integral represents this volume?

Use disks.

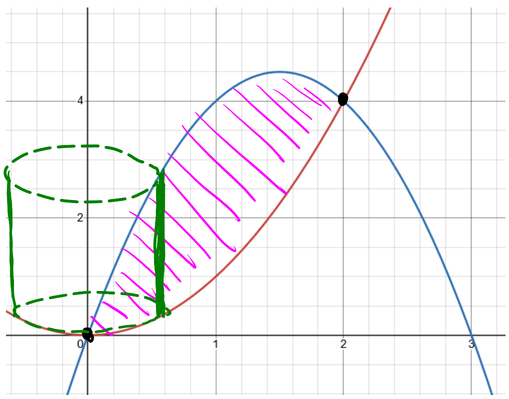
Outer radius $= x+2$.

Inner radius $= 4-x^2$

$$V = \pi \int_1^2 (x+2)^2 - (4-x^2)^2 \, dx.$$

2.2. Consider the region in the plane bounded by the curves $y = x^2$ and $y = 6x - 2x^2$.

a. Write an integral that represents the area of this region.



$x^2 = 6x - x^2$

$2x^2 - 6x = 0$

$2x(x-3) = 0$

$x = 0 \qquad x = 3$

$$A = \int_0^3 (6x - x^2) - x^2 \, dx$$

b. Rotate this region about the line $x = 0$. What integral represents this volume?

Shells.

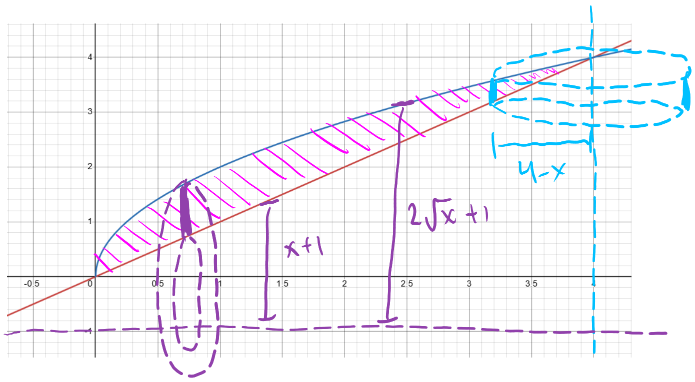
radius $= x$

height $= 6x - x^2 - x^2$.

$$V = 2\pi \int_0^3 x(6x - x^2) \, dx.$$

2.3. Consider the region in the plane bounded by the curves $y = x$ and $y = 2\sqrt{x}$.

a. Rotate this region about the line $x = 4$. What integral represents this volume?



Shells are natural.

radius: $4-x$

height: $2\sqrt{x} - x$

$$V = 2\pi \int_0^4 (4-x)(2\sqrt{x} - x)\, dx$$

$2\sqrt{x} = x$

$4x = x^2$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x=0 \quad x=4$

b. Rotate this region about the line $y = -1$. What integral represents this volume?

Disks.

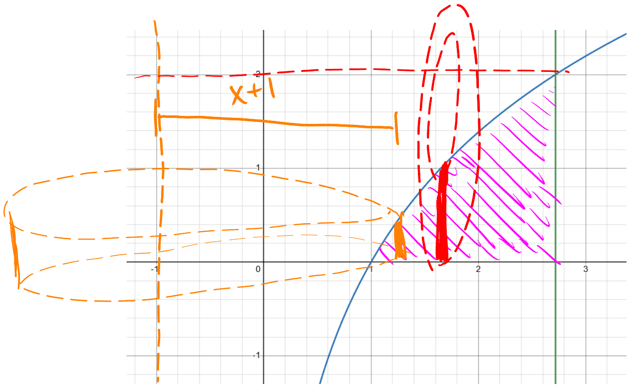
Outer radius $= 2\sqrt{x} + 1$

Inner radius $= x+1$

$$V = \pi \int_0^4 (2\sqrt{x}+1)^2 + (x+1)^2\, dx$$

2.4. Consider the region in the plane bounded by the curves $y = 2\ln x,\ y = 0,\ x = 1$, and $x = e$.

a. Rotate this region about the line $x = -1$. What integral represents this volume?



Use shells.

radius: $x+1$

height: $\ln x$

$$V = 2\pi \int_1^e (x+1)\ln x \, dx$$

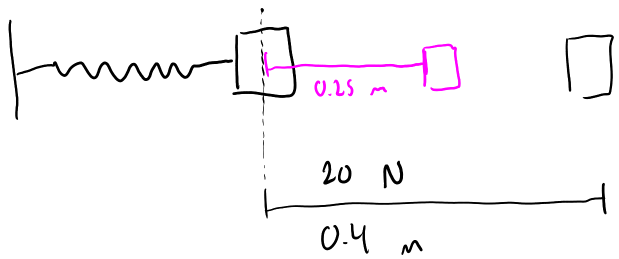
b. Rotate this region about the line $y = 2$. What integral represents this volume?

outer radius $= 2$

$$V = \pi \int_1^e 2^2 - (2-\ln x)^2 \, dx$$

inner radius $= 2 - \ln x$

2.5. Aspring requires $20\ N$ of force to displace it $0.4\ m$ beyond its natural length. How much work is done in stretching it $0.25\ m$ beyond its natural length?



Use Hooke's Law:

$$F = kx$$

$$20 = 0.4k$$

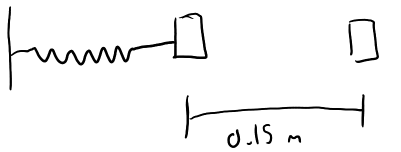
$$k = 50\ N/m$$

$$F(x) = 50x.$$

$$\text{Total Work} = \int_0^{0.25} 50x\, dx = 25x^2 \Big|_0^{0.25}$$

$$= \boxed{78.125\ J}$$

2.6. A spring requires $20\ J$ of energy to stretch it from $0.2\ m$ to $0.25\ m$ beyond its natural length. How much work is required to stretch it $0.15\ m$ beyond its natural length? (Hint: The total energy in the system is the integral of the force function – but what is the general law to determine the force needed to displace a spring?)



$$F(x) = kx$$

This is the energy.

$$W(x) = \int_0^x ks\, ds = \frac{1}{2}kx^2$$

Energy:

$$20 = \int_{0.2}^{0.25} kx\, dx = \frac{1}{2}kx^2\Big|_{0.2}^{0.25}$$

$$= \frac{1}{2}k\left(0.25^2 - 0.2^2\right)$$

$$= \frac{1}{2}\cdot 0.0225\, k.$$

$$k = \frac{2\cdot 20}{0.0225}$$

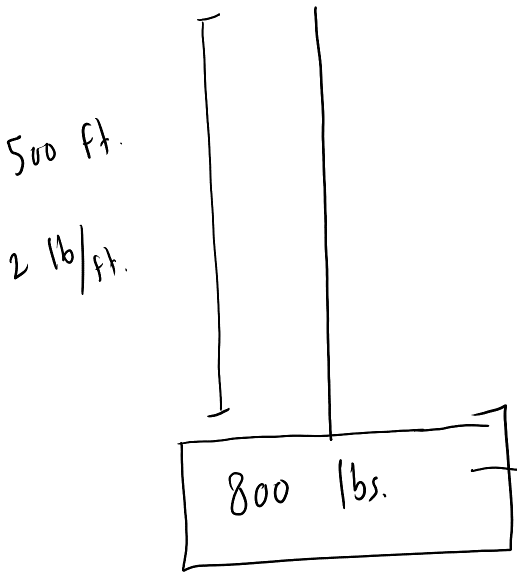
So,

Work to stretch 0.15 m

$$= \int_0^{0.15} kx\, dx = \frac{1}{2}kx^2\Big|_0^{0.15}$$

$$= \frac{1}{2}k\cdot 0.15^2 = \boxed{20\ J.}$$

2.7. A cable that weighs 2 lb/ft is used to lift 800 pounds of coal from a mine shaft 500 feet deep. How much work is done in lifting all of the cable and the coal?



This weight is constant.

Cable weight $= 2 \cdot 500 = 1000.$

$F_{\text{cable}}(x) = 1000 - 2x.$

$$\text{Total work} = \int_0^{500} 1000 - 2x + 800 \, dx$$

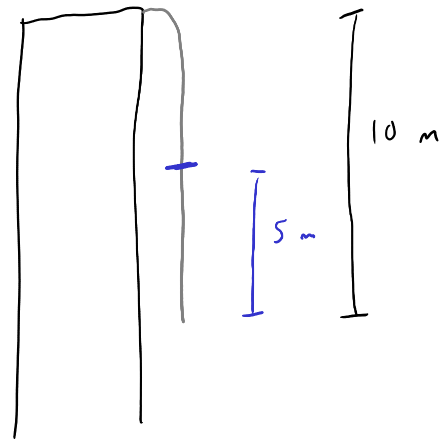
$$= \int_0^{500} 1800 - 2x \, dx = 1800x - x^2 \Big|_0^{500}$$

$$= \boxed{650{,}000 \text{ ft} \cdot \text{lbs.}}$$

2.8. A rope that is 10 meters long has a ~~mass of 80 kilograms.~~ weight of 784 N. The rope hangs off the edge of a building. How much work is needed to lift half of the rope to the top?

$$\text{average weight per meter} = \frac{784}{10} = 78.4 \ N/m.$$

$$F(x) = 784 - 78.4x.$$

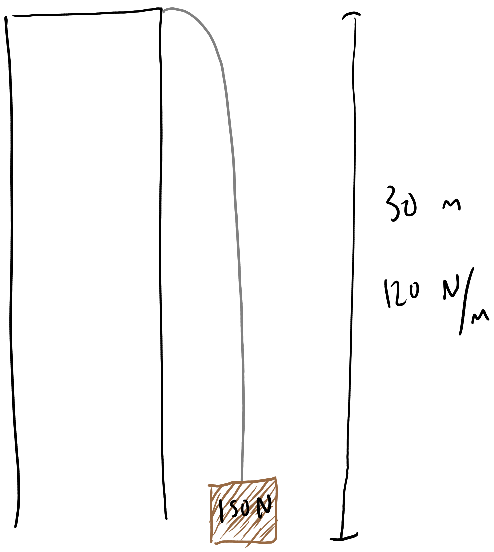


$$\text{Total Work} = \int_0^5 784 - 78.4x \, dx$$

$$= 784x - \frac{1}{2}(78.4)x^2 \Big|_0^5$$

$$= \boxed{2940 \text{ J.}}$$

2.9. From the side of a building 30 meters tall, a cable with weight 120 Newtons per meter has a box with ~~mass~~ weight 150 ~~kilograms~~ Newtons attached to it. How much work is needed to lift the cable and box 25 meters?



Total weight of cable $= 120 \cdot 30 = 3600$ N

Total Force $= 3600 - 120x + 150$

$= 3750 - 120x$

Total Work $= \displaystyle\int_0^{25} 3750 - 120x \, dx$

$= 3750x - 60x^2 \Big|_0^{25}$

$= \boxed{56250 \text{ J}}$

3 Sequences and Series

3.1. Expand the function $f(x) = \dfrac{1}{1+2x^2}$ as power series near $a = 0$. What is its radius and interval of convergence?

We use $\dfrac{1}{1-q} = \sum_{n=0}^{\infty} q^n$. Here, $q = -2x^2$, as $1+2x^2 = 1-(-2x^2)$.

Now,

$$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty} (-2x^2)^n = \sum_{n=0}^{\infty} (-2)^n x^{2n}.$$

Because this is a geometric series, we need $|-2x^2| < 1$. Now, as $|-2x^2| = 2x^2$ ($x^2 \geq 0$, and $|-2| = 2$), we solve $2x^2 < 1$. The solution is $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. The radius of convergence is $\frac{1}{\sqrt{2}}$.

Note the power series diverges when $x = \frac{1}{\sqrt{2}}$ or $x = -\frac{1}{\sqrt{2}}$, as $2x^2 = 1$, and $\sum_{n=0}^{\infty} (-1)^n$ is certainly divergent. The interval of convergence is $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

3.2. Expand the function $g(x) = 3\ln(x+1)$ as a power series near $a = 2$. What is its radius and interval of convergence? (First expand the derivative as a power series, then integrate that expansion term by term.)

* See 3.1, $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$

First, $g'(x) = \frac{3}{x+1}$. Now,

$$\frac{3}{x+1} = \frac{3}{x-2+1+2} = \frac{3}{(x-2)+3} = \frac{3}{3\left(\frac{x-2}{3}+1\right)} = \frac{1}{1+\frac{x-2}{3}} \overset{*}{=} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-2}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x-2)^n$$

Now integrate to recover g:

$$\int \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x-2)^n \, dx = \sum_{n=0}^{\infty} \int \left(-\frac{1}{3}\right)^n (x-2)^n \, dx = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{(x-2)^{n+1}}{n+1} + C.$$

To find C, set $x=2$ - all terms in the sum are zero, and

$g(2) = 3\ln 3 \implies C = 3\ln 3$. So, $g(x) = 3\ln 3 + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{(x-2)^{n+1}}{n+1}$.

For the interval,

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-2)^{n+2}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(-1)^n(x-2)^n} \right| = \frac{|x-2|}{3} \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = \frac{|x-2|}{3}.$$

We need $\frac{1}{3}|x-2| < 1$, which is satisfied for $x \in (-1, 5)$. The radius of convergence is 3. Notice:

$$g(-1) = 3\ln 3 + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{(-3)^{n+1}}{n+1}$$

divergent by comparison against $\frac{1}{n}$.

$$g(5) = 3\ln 3 + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \frac{3^{n+1}}{n+1}$$

convergent by AST.

The interval of convergence is $(-1, 5]$.

3.3. On what interval does the power series $\sum_{n=0}^{\infty}(-1)^n \frac{n^2 x^n}{2^n}$ converge?

Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2 x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n^2 x^n}\right| = \frac{|x|}{2}\lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\right| = \frac{|x|}{2} < 1.$$

So, $x \in (-2, 2)$.

At $x = 2$,

$$\sum_{n=0}^{\infty}(-1)^n \frac{n^2 2^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n n^2,$$ which certainly diverges by the n^{th} term test.

Likewise, at $x = -2$,

$$\sum_{n=0}^{\infty}(-1)^n \frac{n^2(-2)^n}{2^n} = \sum_{n=0}^{\infty} n^2$$ is also divergent.

The interval of convergence is $(-2, 2)$.

3.4. On what interval does the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{n!}$ converge?

Use the ratio test.

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n(x-3)^n} \right| = 2|x-3| \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right|$$

$$= 2|x-3| \lim_{n\to\infty} \left| \frac{n!}{(n+1)n!} \right|$$

$$= 2|x-3| \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = 0.$$

The series converges everywhere, i.e.,

on $(-\infty, \infty]$.

3.5. On what interval does the power series $\sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{\sqrt{n+3}}$ converge?

Use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x+1)^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{(-1)^n(x+1)^n}\right| = |x+1| \lim_{n\to\infty}\left|\frac{\sqrt{n+3}}{\sqrt{n+4}}\right| = |x+1|.$$

We need $|x+1|<1$. Any x in $(-2,0)$ satisfies this inequality.

For $x=0$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$ is convergent by AST, for $\frac{1}{\sqrt{n+3}}$ is positive, decreasing, with limit zero. However, for $x=-2$, $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+3}}$ diverges by LCT against $\frac{1}{\sqrt{n}}$.

The interval of convergence is $[-2,0]$.

3.6. Write the Taylor series for $f(x) = \sin x$ near $a = \frac{\pi}{2}$.

$f(x) = \sin x$ $\quad f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x$ $\quad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x$ $\quad f''\left(\frac{\pi}{2}\right) = -1$

$f^{(3)}(x) = -\cos x$ $\quad f^{(3)}\left(\frac{\pi}{2}\right) = 0$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}\left(\frac{\pi}{2}\right) = 1$

$f^{(5)}(x) = \cos x$ $\quad f^{(5)}\left(\frac{\pi}{2}\right) = 0$

$f^{(6)}(x) = -\sin x$ $\quad f^{(6)}\left(\frac{\pi}{2}\right) = -1$

$\vdots$

Even-numbered derivatives contribute. Odd-numbered derivatives are zero. So,

$$f^{(2n)}\left(\frac{\pi}{2}\right) = (-1)^n.$$

$$\frac{f^{(2n)}\left(\frac{\pi}{2}\right)}{(2n)!} = \frac{(-1)^n}{(2n)!}$$

Now,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x - \pi/2\right)^{2n}}{(2n)!}.$$

3.7. Write the Taylor expansion for $g(x) = e^{-x^2}$ near $a = 0$. (Hint: First expand e^x near $a = 0$, then replace x with $-x^2$.)

Expand e^x.

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x \longrightarrow \frac{f^{(n)}(x)}{n!} = \frac{e^x}{n!} \longrightarrow \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

So, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Now replace x with $-x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

3.8. Without using the "geometric series trick," expand $h(x) = \dfrac{2}{x}$ near $a = 2$.

$$h(x) = \frac{2}{x}$$

$$h'(x) = \frac{-2}{x^2}$$

$$h''(x) = \frac{2 \cdot 2}{x^3}$$

$$h^{(3)}(x) = -\frac{2 \cdot 2 \cdot 3}{x^4}$$

$$h^{(4)}(x) = \frac{2 \cdot 2 \cdot 3 \cdot 4}{x^5}$$

$$\vdots$$

$$h^{(n)}(x) = (-1)^n \frac{2 \cdot n!}{x^{n+1}} \longrightarrow \frac{h^{(n)}(x)}{n!} = \frac{(-1)^n \cdot 2}{x^{n+1}} \longrightarrow \frac{h^{(n)}(2)}{n!} = \frac{(-1)^n 2}{2^{n+1}}$$

$$= \frac{(-1)^n}{2^n}.$$

So, $$\frac{2}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-2)^n.$$

Math 1224 Formula Sheet

- **Area Between Curves:** $A = \int_a^b [f(x) - g(x)]\,dx$
- **Volume by Slicing:** $V = \int_a^b A(x)\,dx$
- **Disk Method:** $V = \pi \int_a^b (\text{Radius})^2\,dx$
- **Washer Method:** $V = \pi \int_a^b \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] dx$
- **Shell Method:** $V = 2\pi \int_a^b (\text{Radius})(\text{Height})\,dx$
- **Arc Length:** $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$
- **Surface Area:** $SA = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx$
- **Work:** $W = \int_a^b F(x)\,dx$
- **Hooke's Law:** $F(x) = kx$
- **Integration by Parts:** $\int u\,dv = uv - \int v\,du$
- **Trig. Identities:** $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin 2x = 2 \sin x \cos x$
- **Trig. Substitution:** If $\sqrt{a^2 - x^2}$, use $x = a \sin\theta$. If $\sqrt{x^2 - a^2}$, use $x = a \sec\theta$. If $\sqrt{x^2 + a^2}$, use $x = a \tan\theta$.
- **Partial Fraction Examples:**

$$\begin{aligned}
\frac{1}{x(x+1)(x+2)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \\
\frac{1}{x^3(x+2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2} \\
\frac{1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \\
\frac{1}{(x^2+1)^2(x+2)} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+2}
\end{aligned}$$

- **Sequences:** A sequence a_n converges to a limit L if $\lim_{n\to\infty} a_n = L$ where L is a real number. Otherwise, the sequence diverges.
- **Series:** A series $\sum a_n$ converges to a sum S if $\lim_{n\to\infty} S_n = S$ where S is a real number and S_n denotes the nth partial sum.
- **Geometric Series Test:** A geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$. Otherwise, the series diverges.
- **Divergence Test:** A series $\sum a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$.
- **Integral Test:** Consider a series $\sum a_n$, where $f(n) = a_n$ is positive, continuous, and eventually decreasing. Then $\sum a_n$ and $\int_a^{\infty} f(x)\,dx$ both converge or both diverge.
- **p-Series Test:** A series of the form $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

- **Direct Comparison Test:** Consider two series $\sum a_n$ and $\sum b_n$ with positive terms.
  1. If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
  2. If $a_n \geq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
- **Limit Comparison Test:** Consider two series $\sum a_n$ and $\sum b_n$ with positive terms and let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

  1. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
  2. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
  3. If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
- **Alternating Series Test:** Consider an alternating series $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$. The series converges if
  1. $b_n \geq 0$
  2. $b_{n+1} \leq b_n$
  3. $\lim_{n\to\infty} b_n = 0$
- **Ratio Test:** Consider a series $\sum a_n$ with nonzero terms and let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

  1. If $L < 1$, the series converges absolutely.
  2. If $L > 1$, the series diverges.
  3. If $L = 1$, the ratio test is inconclusive.
- **Root Test:** Consider a series $\sum a_n$ with nonzero terms and let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

  1. If $L < 1$, the series converges absolutely.
  2. If $L > 1$, the series diverges.
  3. If $L = 1$, the root test is inconclusive.
- **Term-By-Term Differentiation and Integration:** Consider the function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

  Then
  1. $f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}$
  2. $\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$
- **Taylor/Maclaurin Series:** The Taylor series for $f(x)$ centered at $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

  The Maclaurin series for $f(x)$ is the Taylor series for $f(x)$ centered at $x = 0$.