Problem 10-1 (Party Propaganda)   18 Points

You have an undirected graph $G = (V, E)$ and two special nodes $r, d \in V$. At time 0, node r is republican, node d is democratic, while all the other nodes $v \notin \{r, d\}$ are initially "undecided". For every $i = 1, 2, 3, \ldots$, the following 2-stage "conversion" process is performed at time time i. At the first stage, all republicans at time $(i - 1)$ look at all their neighboring nodes v which are still undecided, and convert those undecided nodes to become republican. Similarly, at the second stage, all democratic nodes at time $(i - 1)$ look at all their neighboring nodes v which are still undecided by the end of the first stage above, and convert those undecided nodes to become democratic. The process is repeated until no new conversions can be made. For example, if G is a 5-cycle $1, 2, 3, 4, 5$ where $r = 1$, $d = 5$, after time 1 node 2 becomes republican and node 4 becomes democratic, and after time 2 the last remaining node 3 becomes republican (as republicans move first).

<div align="center">at the start   after time 1   after time 2</div>
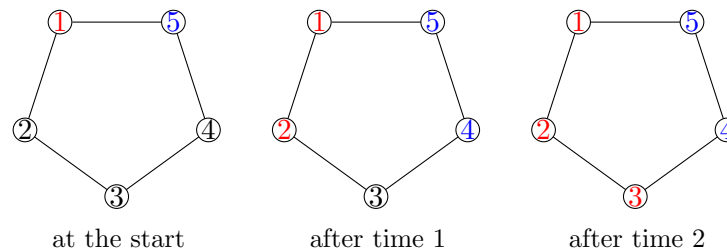
On the other hand, if the initial democratic node was $d = 3$ instead, then already after step 1 nodes 2 and 5 become republican, and node 4 becomes democratic, and no step 2 is needed.

<div align="center">at the start   after time 1</div>
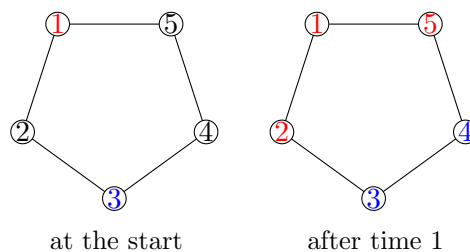
Assume each node v have a field $v.color$, where *red* means republican, *blue* means democratic, and *white* means undecided, so that, at time 0, $r.color = red$, $d.color = blue$, and all other nodes v have $v.color = white$.

(a) (5 points) Using two BFS calls, show how to properly fill the final color of each node.

(b) (8 points) Show how to speed up your procedure in part (a) by a factor of 2 (or more, depending on your implementation) by directly modifying the BFS procedure given in the

book. Namely, instead of computing distances from the root node, you are computing the final colors of each node, by essentially performing a *single*, appropriately modified BFS traversal of G. Please write pseudocode, as it is *very* similar to the standard BFS pseudocode, and is much easier to grade. But briefly explain your code.

(c) (5 points) Now assume that at time 0 more than one node could be republican or democratic. Namely, you are given as inputs some disjoint subsets R and D of V, where nodes in R are initially republican and nodes in D are initially democratic, but otherwise the conversion process is the same. For concreteness, assume $|R| = |D| = t$ for some $t \geq 1$ (so that parts (a) and (b) correspond to $t = 1$). Show how to generalize your solutions in parts (a) and (b) to this more general setting. Given parts (a) and (b) took time $O(|V| + |E|)$ (with different constants), how long would their modifications take as a function of t, $|V|$, $|E|$? Which procedure gives a faster solution?

Problem 10-2 (Counting Shortest Paths)   10 Points

(a) (4 points) Describe how to modify the algortihm for Breadth First Search so that for each vertex v, the number of shortest paths from s to v is stored in $v.n$

(b) (4 points) For the next two parts, assume that all graphs are undirected. Recall that a simple cycle is a cycle without repeated vertices. Prove that if there is more than one path from v to s, the graph must contain a simple cycle.

(c) (2 points) After solving part b, Gary claims that if the number of shortest paths from s to v is 1 for every vertex v, then the graph must have no cycles. Either prove that Gary is correct, or prove that he is wrong by giving a counterexample.

Problem 10-3 (Rescue Mission)   8 (+4) points

A big earthquake in the island of Coco destroyed many (but not all) two-way roads connecting some of the n houses present in the island. The Red Cross would like to check all the n houses to ensure that all the inhabitants are safe. The Red Cross can fly in an inspector to any particular house u by a helicopter. However, landing the helicopter is very dangerous, so the officials would like to minimize the number of such landings. Luckily, once the inspector is delivered to any node u, he can drive to any neighboring node v, if the road (u, v) is not destroyed (but not otherwise). Unfortunately, the inspector can only tell if the road (u, v) is safe after arriving at u, so one cannot plan the whole rescue operation in advance. In particular, the Red Cross does not even know in advance the number m of safe roads. Nevertheless, help Red Cross to design a rescue operation with the smallest number of helicopter landings.

(a) (8 points) Design a strategy that minimizes the number of landings. Prove that your strategy always gives an optimal, i.e., no other plan has fewer landings.

(b) (4 points) [**Extra credit**] Design a plan that has minimal number of landings, and less than $2n$ trips along safe roads.

Problem 10-4 (Lonely Vertex)   8 points

(a) (4 points) Explain how a vertex u of a directed graph can end up in a depth-first tree containing only u, although u has both incoming and outgoing edges.

(b) (4 points) Assume u is part of some directed cycle in G. Can u still end up all by itself in the depth-first forest of G? Justify your answer.
(**Hint**: Recall the White Path Theorem.)