Algorithms in the Time of COVID19

Solving Recurrences[1] *- Recitation 2*

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Recitation 2

- About writing (proofs)
- Approaches of solving recurrences + examples
- Exercises

About writing (proofs)

In general:

- Put reader first.
- Put definition before usage.
- Use English words as much as possible
  - avoid $\forall$ and $\exists$ and their combinations $\forall\exists\forall\exists\exists$.
- Read Knuth.[2]

In homework:

- I am your reader.
- You can use results already known from the class.

[2] "Mathematical writing", `https://jmlr.csail.mit.edu/reviewing-papers/knuth_mathematical_writing.pdf`

Approaches of solving recurrences

Approach 1: The Master Theorem.

Examples

Problem 1. Solve the following recurrence relation:

$$T(n) = 4T(n/2) + n^2\sqrt{n} \tag{1}$$

Approaches of solving recurrences

Approach 2: unfold the recurrence relation directly.

Examples

Problem 2. Solve the following recurrence relation:

$$T(n) = n^{\frac{1}{2}} T(n^{\frac{1}{2}}) + n. \tag{2}$$

Approaches of solving recurrences

Approach 3: Generating functions.[3]

Definition

Given a sequence of numbers $a_0, a_1, \ldots, a_k, \ldots,$, we define its *formal power series* by:

$$F(z) = \sum_{k \geq 0} a_k z^k = a_0 + a_1 z + \cdots + a_k z^k + \cdots \,. \tag{3}$$

We also refer to F as the *generating function* for the sequence $\{a_k\}_{k \geq 0}$. □

Our **primary interest** is in the generating functions for the sequence $\{T(n)\}_{n \geq 0}$ defined by some recurrence relation.

[3] "A generating function is a clothesline on which we hang up a sequence of numbers for display." - Herb Wilf

Examples

Problem 3. Find the generating function for the sequence

- $(1, 1, 1, \dots)$.
- $\{(\binom{n}{k})\}_{k=0}^{n}$.

Problem 4. What sequences are defined by the following generating functions?

- $\frac{1}{1+z}$
- $\frac{1}{1-2z}$
- $\frac{1}{z-3}$
- $\frac{1}{1-z^2}$
- $\frac{1}{(1-z)^2}$

Problem 5. Solve the following recurrence relation using the generating function.

$$T(n) = T(n-1) + 1 \qquad (4)$$

Approaches of solving recurrences

Approach 4: Guess + Induction

Problem 6. Solve the recurrence relation

$$T(n) = T(bn) + T(n - bn) + n, \tag{5}$$

where $b \in (0, 1)$ is a constant number.

Exercises

Problem 7. Consider the recurrence relations

- $T(n) = 2T(n/2) + n$ and
- $f(n) = f(n-1) + \lg n$.

Prove that $T(n) = \Theta(f(n))$.

Exercises

Problem 8. Solve the following recurrence relation.

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n. \qquad (6)$$

Exercises

Problem 9. Solve the recurrence relation

$$T(n) = 2T(\frac{n}{2} - 1) + n \tag{7}$$

Exercises

Problem 10. Let $T(n) = 4T(n/3) + n$. Use Guess + Induction to prove that

$$T(n) = O(n^{\log_3 4}). \tag{8}$$

Exercises

Problem 11. Solve the following recurrence relation:

$$T(n) = T(n/5) + T(7n/10 + 6) + n. \tag{9}$$

Problem 12. Solve the following recurrence relation:

$$T(n) = n^{\frac{3}{4}} T(n^{\frac{1}{4}}) + n. \tag{10}$$

Exercises

Problem 13. Solve the following recurrence relation:

$$T(n) = T(n-2) + n^2. \tag{11}$$

Problem 14. Multiplying a power series by $1/(1-z)$ has nice effect on a power series. If $F(z) = \sum_{k \geq 0} a_j z^j$, what sequence has generating function $F(z)/(1-z)$? That is, if

$$\frac{F(z)}{1-z} = \sum_{k \geq 0} b_k z^k, \tag{12}$$

what are the numbers b_k in terms of the a_j?

Problem 15. Solve the recurrence defined by the Fibnacci sequence F_n using generating functions.

$$F_0 = 0, F_1 = 1,\ F_n = F_{n-1} + F_{n-2}. \tag{13}$$