CSCI-SHU 220 - Algorithms

Recitation 1 - Solutions

Here are solutions to the exercises which are *not* covered during Recitation 1.

A. Exercises

1. Describe Θ and Ω notations in English words.
2. Suppose $T(n) = a_k n^k + \cdots + a_1 n + a_0$, where k is a non-negative integer and the a_i's are real numbers. Prove that $T(n) = O(n^k)$.
3. Let $k \geq 1$ be a positive integer and let $T(n) = n^k$. Prove that $T(n)$ is not $O(n^{k-1})$.

   Solution. Suppose for the sake of contradiction that $T(n) = O(n^{k-1})$. Then there are two positive constants c_0 and n_0 such that

   $$n^k \leq c_0 n^{k-1}, \quad \text{for all} \quad n \geq n_0 \tag{1}$$
   $$\iff n \leq c_0, \ \text{for all} \quad n \geq n_0. \tag{2}$$

   This is a contradiction, because $n = \lceil \max\{n_0, c_0\} \rceil + 1$ violates (2).
4. For $T(n) = 2^{n+100}$, prove or disprove: $T(n) = O(2^n)$.

   Solution. It suffices to prove $T(n) = O(2^n)$. Take $c_0 = 2^{100} + 1, n_0 = 1$ and apply the definition of the Big-O notation.
5. For $T(n) = 2^{100n}$, prove or disprove: $T(n) = O(2^n)$.

   Solution. We will show that $T(n)$ is not $O(2^n)$. Suppose for the sake of contradiction that $T(n) = O(2^n)$. Then there are two positive constants c_0 and n_0 satisfying

   $$2^{100n} \leq c_0 2^n, \quad \text{for all} \quad n \geq n_0 \tag{3}$$
   $$\iff 2^{99n} \leq c_0, \quad \text{for all} \quad n \geq n_0. \tag{4}$$

   There is some choice of n that voilates (4). Find it.
6. Let f and g denote functions from the positive integers to the non-negative real numbers, and define $T(n) = \max(f(n), g(n))$ for each $n > 1$. Prove that $T(n) = \Theta(f(n) + g(n))$.

   Solution. Let us first prove $T(n) = O(f(n) + g(n))$. This is easy because $\max(f(n), g(n)) \leq f(n) + g(n)$ where we used that both $f(n)$ and $g(n)$ are non-negative (What are c_0 and n_0?). Then we will show that $T(n) = \Omega(f(n) + g(n))$. This is equally easy because $\max(f(n), g(n)) \geq (f(n) + g(n))/2$ (What are c_0 and n_0?).
7. Let f and g be non-decreasing real-valued functions defined on the positive integers, and $f(1) \geq 1$ and $g(1) \geq 1$. Prove or disprove that $f(n) = O(g(n)) \Rightarrow 2^{f(n)} = O(2^{g(n)})$.

   Solution. This solution is motivated by the student who pointed out one of my mistakes in Recitation 1. (My another mistake: we shall have his name here!)

   To disprove it we provide a counter-example. Let $f(n) = 2\log_2 n$ and $g(n) = \log_2 n$. Then we see that $f(n) = O(g(n))$. But $2^{f(n)} = n^2$ and $2^{g(n)} = n$, so $2^{f(n)} \neq O(2^{g(n)})$.

   If given in addition that $f(n) \leq g(n)$ for sufficiently large n, can we prove that $2^{f(n)} = O(2^{g(n)})$?
8. Arrange the following functions in order of increasing growth rate, with $g(n)$ following $f(n)$ in your list if and only if $f(n) = O(g(n))$.
   a) $n^2 \log_2 n$
   b) 2^n
   c) 2^{2^n}
   d) $n^{\log_2 n}$
   e) n^2

Solution. $n^2 = O(n^2\log_2 n) = O(n^{\log_2 n}) = O(2^n) = O(2^{2^n})$. We will only prove $n^2\log_2 n = O(n^{\log_2 n})$. That is, we are going to find some positive c_0 and n_0 such that for every $n \geq n_0$ it holds that

$$n^2 \log_2 n \leq c_0 n^{\log_2 n}. \tag{5}$$

Let $k = \log_2 n$ and so $n = 2^k$ (here k is not necessarily a positive integer). we need to prove for some $c_0 > 0$ and $n_0 > 0$ it holds that

$$(2^k)^2 k \leq c_0 (2^k)^k \iff 2^{2k} k \leq c_0 2^{k^2} \iff k \leq c_0 2^{k^2-2k} \tag{6}$$

for every $2^k \geq n_0$. Which c_0 and n_0 will make the above hold?

9. Prove that

$$\log(n!) = \Theta(n \log n). \tag{7}$$

Hint: To show an upper bound, compare $n!$ with n^n. To show a lower bound, compare it with $(n/2)^{n/2}$.

10. Prove that

$$\sum_{k=1}^{2^n} \frac{1}{k} = \Theta(n). \tag{8}$$

Solution. First of all we have

$$\begin{aligned}
\sum_{k=1}^{2^n} \frac{1}{k} &= 1 + \sum_{i=1}^{n} \sum_{k=2^{i-1}+1}^{2^i} \frac{1}{k} && (9)\\
&\geq 1 + \sum_{i=1}^{n} \sum_{k=2^{i-1}+1}^{2^i} \frac{1}{2^i} && (10)\\
&\geq 1 + \sum_{i=1}^{n} \frac{2^{i-1}}{2^i} && (11)\\
&\geq 1 + \frac{n}{2} && (12)
\end{aligned}$$

This proves $\sum_{k=1}^{2^n} \frac{1}{k} = \Omega(n)$ (*Why?*). On the other hand, we have

$$\begin{aligned}
\sum_{k=1}^{2^n} \frac{1}{k} &= 1 + \sum_{i=1}^{n} \sum_{k=2^{i-1}+1}^{2^i} \frac{1}{k} && (13)\\
&\leq 1 + \sum_{i=1}^{n} \sum_{k=2^{i-1}+1}^{2^i} \frac{1}{2^{i-1}} && (14)\\
&\leq 1 + \sum_{i=1}^{n} \frac{2^{i-1}}{2^{i-1}} && (15)\\
&\leq 1 + n. && (16)
\end{aligned}$$

This proves $\sum_{k=1}^{2^n} \frac{1}{k} = O(n)$ (*Why?*). In sum we finished the proof. To learn more from this problem, check:

https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)

See also the final problem (50 points) and (think about it if you want and) its solution in "Chinese High School Mathematics League, 2012":

https://wenku.baidu.com/view/5e7c0fadd1f34693daef3e9c.html

B. The watermelon problem.

Let us go back to the watermelon problem in mathematical terms.

(a) What is the maximum number L_n of 2D regions defined by n lines of $\mathbb{R}^2$? *Examples*: $L_1 = 2, L_2 = 4, L_3 = 7$. *Hint*: find a relation between L_n and L_{n-1}.

Solution. $L_n = L_{n-1} + n$.

(b) Let there be n planes of $\mathbb{R}^3$; any two of them are not parallel but the intersections of the n planes (which are lines) are parallel. What is the maximum number P_n of 3D regions defined by these n planes of $\mathbb{R}^3$?

Solution. This is the same as (a). So the recurrence relation is $P_n = P_{n-1} + n$.

(c) Next we will ask you for the maximum number P_n of 3D regions defined by n different and non-parallel planes of $\mathbb{R}^3$. Think about the moment when you have $n-1$ such planes and you are going to add a new one to obtain the regions defined by those n planes. At most how many lines does the newly added plane contain as the intersections with other $n-1$ planes? What is the maximum number of 2D regions defined by those lines?

Solution. There are at most $n-1$ lines contained in the newly added plane as the intersections with the other $n-1$ planes. Those $n-1$ lines give rise to at most L_{n-1} regions as per (1).

(d) What is the maximum number P_n of 3D regions defined by n different and non-parallel planes of $\mathbb{R}^3$?

Solution. Combining (a), (b), and (c), we get $P_n = P_{n-1} + L_{n-1}$.