Linear Algebra Homework 1

Zhenming Wang NetID: zw1806

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1 exercise1

$$x_1 + 5x_2 = 7$$
$$-2x_1 - 7x_2 = -5$$

solution: writing the linear system in augmented matrix form:

$$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$$

mutlipying row 1 by 2

$$\begin{bmatrix} 2 & 10 & 14 \\ -2 & -7 & -5 \end{bmatrix}$$

Replacing row 2 with $\text{row1} + \text{row2}$

$$\begin{bmatrix} 2 & 10 & 14 \\ 0 & 3 & 9 \end{bmatrix}$$

Go back to equation notation:

$$2x_1 + 10x_2 = 14 \tag{1}$$
$$3x_2 = 9 \tag{2}$$

According to (2), $x_2 = 3$. Then,

$$2x_1 + 10 * (3) = 14$$

So, $2x_1 = -16$, $x_1 = -8$ So, $(x_1, x_2) = (-8, 3)$ is the unique solution of the system.

2 exercise3

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_1 - 2x_2 &= -2 \end{aligned}$$

solution: writing the linear system in augmented matrix form:

$$\begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix}$$

replacing row 2 with row1 − row2

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 7 & 9 \end{bmatrix}$$

Go back to equation notation:

$$x_1 + 5x_2 = 7 \tag{3}$$
$$7x_2 = 9 \tag{4}$$

Thus, $x_2 = \frac{9}{7}$. substitute x_2 in equation(3)

$$x_1 + 5 * \frac{9}{7} = 7$$

$x_1 = 7 - \frac{45}{7}$, so, $x_1 = \frac{4}{7}$ Thus,$(x_1, x_2) = (\frac{4}{7}, \frac{9}{7})$ is the unique solution of the system.

3 exercise4

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ 3x_1 - 7x_2 &= 5 \end{aligned}$$

solution:
Start by writing the augmented matrix of the system.

$$\begin{bmatrix} 1 & -5 & 1 \\ 3 & -7 & 5 \end{bmatrix}$$

Replacing row1 with row2 - 3 $*$ row1

$$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

Return to equation notation:

$$x_1 - 5x_2 = 1 \tag{5}$$
$$8x_2 = 2 \tag{6}$$

Thus, $x_2 = \frac{1}{4}$, substituting x_2 in equation 1

$$x_1 - 5 * \frac{1}{4} = 1$$

Thus, $x_1 = 1 + \frac{5}{4} \Rightarrow x_1 = \frac{9}{4}$.
Therefore, $(x_1, x_2) = (\frac{9}{4}, \frac{1}{4})$ is the unique solution of the system

4 exercise 11

$$x_2 + 4x_3 = -5 \tag{7}$$
$$x_1 + 3x_2 + 5x_3 = -2 \tag{8}$$
$$3x_1 + 7x_2 + 7x_3 = 6 \tag{9}$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix}$$

Interchange row 1 and row 3:

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

Replace row 2 with $3 * row2 - row1$:

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & 2 & 8 & -12 \\ 0 & 1 & 4 & -5 \end{bmatrix}$$

Replace row 3 with $row2 - 2 * row3$:

$$\begin{bmatrix} 3 & 7 & 7 & 6 \\ 0 & 2 & 8 & -12 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

From row 3, we can see that we have reached a contradiction, since

$$0 \neq -2$$

. Thus, the system is inconsistent.

5 exercise 12

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 - x_3 &= 7 \end{aligned}$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix}$$

Replace row3 with $row3 + 4 * row1$:

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ 0 & -6 & 15 & -9 \end{bmatrix}$$

Replace row 2 with $row2 - 3 * row1$:

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix}$$

replace row3 with $row3 + 3 * row2$:

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

From row 3, we are able to see there is a contradiction, since

$$0 = 3$$

Therefore, the linear system is inconsistent and has no solution.

6 exercise 13

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

Replace row 2 with $row2 - 2 * row1$:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

Replace row 3 with $row2 - 2 * row3$:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

Equation Notation:

$$x_1 - 3x_3 = 8 \quad (10)$$
$$2x_2 + 15x_3 = -9 \quad (11)$$
$$5x_3 = -5 \quad (12)$$

Thus, $x_3 = -1$. Substituting x_3 in equation 13 and 14 :

$$x_1 - 3 * (-1) = 8$$
$$2x_2 + 15 * (-1) = -9$$

Thus, $x_1 = 5$,$x_2 = 3$. Therefore, $(x_1, x_2, x_3) = (5, 3, -1)$ is the unique solution of the linear system.

7 exercise 14

$$x_1 - 3x_2 = 5$$
$$-x_1 + x_2 + 5x_3 = 2$$
$$x_2 + x_3 = 0$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Replace row 2 with $row1 + row2$:

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Replace row 3 with $2 * row3 + row2$:

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 0 & 7 & 7 \end{bmatrix}$$

Equation Notation:

$$
\begin{align}
x_1 - 3x_2 &= 5 \tag{13} \\
-2x_2 + 5x_3 &= 7 \tag{14} \\
7x_3 &= 7 \tag{15}
\end{align}
$$

Thus, $x_3 = 1$, substituting x_3 in equation 17:

$$-2x_2 + 5 * 1 = 7$$

Thus,$x_2 = -1$, substituting x_2 in equation 16:

$$x_1 - 3 * (-1) = 5$$

Thus, $x_1 = 2$.Therefore $(x_1, x_2, x_3) = (2, -1, 1)$ is the unique solution of the system.

8 exercise 15

$$
\begin{aligned}
x_1 + 3x_3 &= 2 \\
x_2 - 3x_4 &= 3 \\
-2x_2 + 3x_3 + 2x_4 &= 1 \\
3x_1 + 7x_4 &= -5
\end{aligned}
$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Replace row 3 with $row3 + 2 * row2$:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

replace row 1 with $row1 - row3$:

$$\begin{bmatrix} 1 & 0 & 0 & 4 & -5 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

replace row 4 with $row4 - 3 * row1$:

$$\begin{bmatrix} 1 & 0 & 0 & 4 & -5 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

We are able to tell that the system is consistent by looking at the 4th row. Since we know x_4, we can substitute x_4 to get x_3, x_2, x_1.

9 exercise 16

$$\begin{aligned} x_1 - 2x_4 &= -3 \\ 2x_2 + 2x_3 &= 0 \\ x_3 + 3x_4 &= 1 \\ -2x_1 + 3x_2 + 2x_3 + x_4 &= 5 \end{aligned}$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

Interchange row 3 and row 4:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ -2 & 3 & 2 & 1 & 5 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

Replace row 3 with $row3 + 2 * row1$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 3 & 2 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

Replace row 3 with $row3 - \frac{3}{2} * row2$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

Equation notation:

$$x_1 - 2x_4 = -3 \tag{16}$$
$$2x_2 + 2x_2 = 0 \tag{17}$$
$$-x_3 - 3x_4 = -1 \tag{18}$$
$$x_3 + 3x_4 = 1 \tag{19}$$

equation 21 and 22 are identical. Therefore, we have three equations for four parameters.
The system is consistent, with infinite solutions.

10 exercise 17

$$x_1 - 4x_2 = 1$$
$$2x_1 - x_2 = -3$$
$$-x_1 - 3x_2 = 4$$

solution:
Augmented Matrix:

$$\begin{bmatrix} 1 & -4 & 0 & 1 \\ 2 & -1 & 0 & -3 \\ -1 & -3 & 0 & 4 \end{bmatrix}$$

Replace row 2 with $row2 - 2 * row1$:

$$\begin{bmatrix} 1 & -4 & 0 & 1 \\ 0 & 7 & 0 & -5 \\ -1 & -3 & 0 & 4 \end{bmatrix}$$

Interchange row 2 with row3:

$$\begin{bmatrix} 1 & -4 & 0 & 1 \\ -1 & -3 & 0 & 4 \\ 0 & 7 & 0 & -5 \end{bmatrix}$$

Replace row 2 with $row1 + row2$:

$$\begin{bmatrix} 1 & -4 & 0 & 1 \\ 0 & -7 & 0 & 5 \\ 0 & 7 & 0 & -5 \end{bmatrix}$$

Therefore, row 2 and row 3 are equivalent.
Equation Notation:

$$x_1 - 4x_2 = 1 \tag{20}$$
$$-7x_2 = 5 \tag{21}$$

Therefore, the system has an unique solution. Namely, $(x_1, x_2) = (-\frac{13}{7}, -\frac{5}{7})$. Thus, the three line have a common point of intersection.