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Problem 2.

1).   denote $P(y=1) = \phi$. $P(y=0) = 1-\phi$

$$P(y=1 \mid x; \mu_0, \mu_1, \Sigma) = \frac{P(x \mid y=1) * P(y=1)}{P(x \mid y=1) * P(y=1) + P(x \mid y=0) P(y=0)}$$

$$= \frac{\phi\, P(x \mid y=1)}{\phi\, P(x \mid y=1) + (1-\phi) \cdot P(x \mid y=0)}$$

$$= \frac{\phi \exp[-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1} (x-\mu_1)]}{\phi \exp[-\frac{1}{2}(x-\mu_1)^T \Sigma^{-1} (x-\mu_1)] + (1-\phi) \cdot \exp[-\frac{1}{2}(x-\mu_0)^T \Sigma^{-1} (x-\mu_0)]}$$

$$= \frac{\phi e^{a_1}}{\phi e^{a_1} + (1-\phi) \cdot e^{a_0}} = \frac{1}{1 + \frac{1-\phi}{\phi} e^{a_0 - a_1}}$$

$$a_c \overset{\text{def}}{=} -\frac{1}{2} (x-\mu_c)^T \Sigma (x-\mu_c)$$

$$\frac{1-\phi}{\phi} = \left(\frac{\phi}{1-\phi}\right)^{-1} = e^{-\log \frac{\phi}{1-\phi}}$$

$$\rightarrow P(y=1 \mid x) = \frac{1}{1 + \exp[-\log \frac{\phi}{1-\phi} + a_0 - a_1]}$$

$$a_0 - a_1 = -\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0) + \frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)$$

$$= -\frac{1}{2}(x^T-\mu_0^T)\Sigma^{-1}(x-\mu_0) + \frac{1}{2}(x^T-\mu_1^T)\Sigma^{-1}(x-\mu_1)$$

$$= -\frac{1}{2}(x^T-\mu_0^T)\Sigma^{-1}x + \frac{1}{2}(x^T-\mu_0^T)\Sigma^{-1}\mu_0 + \frac{1}{2}(x^T-\mu_1^T)\Sigma^{-1}x - \frac{1}{2}(x^T-\mu_1^T)\Sigma^{-1}\mu_1$$

$$\boxed{= -\frac{1}{2}\cancel{x^T\Sigma^{-1}x} + \frac{1}{2}\mu_0^T\Sigma^{-1}x + \frac{1}{2}x^T\Sigma^{-1}\mu_0 - \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0 + \frac{1}{2}\cancel{x^T\Sigma^{-1}x} - \frac{1}{2}\mu_1^T\Sigma^{-1}x - \frac{1}{2}x^T\Sigma^{-1}\mu_1 + \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1}$$

Note that $x^T\Sigma^{-1}\mu_k \in \mathbb{R}^{1\times 1}$, since $x^T \in 1\times d$, $\Sigma^{-1} \in d\times d$, $\mu_k \in d\times 1$

$$\rightarrow x^T\Sigma^{-1}\mu_k = (x^T\Sigma^{-1}\mu_k)^T = (\Sigma^{-1}\mu_k)^T x = \mu_k^T(\Sigma^{-1})^T x$$

since Σ is symmetric, and hence Σ^{-1}.

$$\rightarrow \mu_k^T(\Sigma^{-1})^T x = \mu_k^T\Sigma^{-1}x.$$

Continue the red box equation.

$$x^T\Sigma^{-1}\mu_0 = \mu_0^T\Sigma^{-1}x, \quad x^T\Sigma^{-1}\mu_1 = \mu_1^T\Sigma^{-1}x$$

this gives us,

$$\mu_0^T\Sigma^{-1}x - \mu_1^T\Sigma^{-1}x + \frac{1}{2}\mu_1^T\Sigma^{-1}\mu_1 - \frac{1}{2}\mu_0^T\Sigma^{-1}\mu_0$$

$$= \boxed{(\mu_0-\mu_1)^T\Sigma^{-1}x + \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_0^T\Sigma^{-1}\mu_0)} \rightsquigarrow a_0 - a_1$$

so, $$p(Y=1|x) = \frac{1}{1+\exp(-\theta_0+\theta^Tx)}$$

$$\theta^T = (\mu_0-\mu_1)^T\Sigma^{-1} \quad \in \mathbb{R}^{1\times d}$$

$$\rightarrow \theta = \Sigma^{-1}(\mu_0-\mu_1) \in \mathbb{R}^{d\times 1}$$

$$-\theta_0 = -\log\frac{\phi}{1-\phi} + \frac{1}{2}(\mu_1^T\Sigma^{-1}\mu_1 - \mu_0^T\Sigma^{-1}\mu_0)$$

$$\rightarrow \theta_0 = \log \frac{\phi}{1-\phi} + \frac{1}{2}(\mu_0^T \Sigma^{-1} \mu_0 - \mu_1^T \Sigma^{-1} \mu_1) \in \mathbb{R}.$$