3. Encuentre la regla de Simpson simple.

Se tiene el conjunto soporte $\Omega = \{(a, f(a)), (x_m, f(x_m)), (b, f(b))\}$ dónde

$x_m = \frac{a+b}{2}$

El polinomio interpolador de orden 2 es:

$$P_2(x) = f(a)\frac{(x-x_m)(x-b)}{(a-x_m)(a-b)} + f(x_m)\frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} + f(b)\frac{(x-a)(x-x_m)}{(b-x_m)(b-a)}$$

Dada la equipartición del intervalo.

$$P_2(x) = \frac{f(a)}{2h^2}(x^2-(x_m+b)x+x_m b) - \frac{f(x_m)}{h^2}(x^2-(a+b)x+ab) + \frac{f(b)}{2h^2}(x^2-(x_m+a)x+x_m a)$$

La aproximación de la integral es:

$$\int_a^b f(x)dx \approx \int_a^b P_2(x)dx = \frac{f(a)}{2h^2}\int_a^b x^2-(x_m+b)x+x_m b\,dx - \frac{f(x_m)}{h^2}\int_a^b x^2-(a+b)x+ab\,dx$$

$$+\frac{f(b)}{2h^2}\int_a^b x^2-(x_m+a)x+x_m a\,dx$$

$$= \frac{f(a)}{2h^2}\left[\frac{b^3-a^3}{3} - \frac{3b+a}{2}\cdot\frac{b^2-a^2}{2} + \frac{b^2+ab}{2}\cdot(b-a)\right]$$

$$-\frac{f(x_m)}{h^2}\left[\frac{b^3-a^3}{3} - (a+b)\cdot\frac{b^2-a^2}{2} + ab(b-a)\right]$$

$$+\frac{f(b)}{2h^2}\left[\frac{b^3-a^3}{3} - \frac{b+3a}{2}\cdot\frac{b^2-a^2}{2} + \frac{ba+a^2}{2}(b-a)\right]$$

$$= \frac{f(a)}{2h^2}\left[\frac{b^3-a^3}{3} - \frac{3b^3+b^2a-3ba^2-a^3}{4} + \frac{b^3+ab^2-ab^2-a^2b}{2}\right]$$

$$-\frac{f(x_m)}{h^2}\left[\frac{b^3-a^3}{3} - \frac{b^2a+b^3-a^3-ba^2}{2} + ab^2-a^2b\right]$$

$$+\frac{f(b)}{2h^2}\left[\frac{b^3-a^3}{3} - \frac{b^3+3b^2a-ba^2-3a^3}{4} + \frac{b^2a+ba^2-ba^2-a^3}{2}\right]$$

$$= \frac{f(a)}{2h^2}\left(\frac{b^3}{12} - \frac{1}{4}b^2a + \frac{1}{4}ba^2 - \frac{a^3}{12}\right) + \frac{f(x_m)}{h^2}\left(\frac{b^3}{6} - \frac{1}{2}ab^2 + \frac{1}{2}a^2b - \frac{a^3}{6}\right)$$

$$+\frac{f(b)}{2h^2}\left(\frac{b^3}{12} - \frac{1}{4}b^2a + \frac{1}{4}ba^2 - \frac{a^3}{12}\right)$$

$$= \frac{f(a)}{3h^2}\left(\frac{b^3-3b^2a+3ba^2-a^3}{8}\right)+\frac{4f(x_m)}{3h^2}\left(\frac{b^3-3b^2a+3ba^2-a^3}{8}\right)$$

$$+\frac{f(b)}{3h^2}\left(\frac{b^3-3b^2a+3ba^2-a^3}{8}\right)$$

$$= \frac{f(a)}{3h^2}\left(\frac{b-a}{2}\right)^3+\frac{4f(x_m)}{3h^2}\left(\frac{b-a}{2}\right)^3+\frac{f(b)}{3h^2}\left(\frac{b-a}{2}\right)^3$$

$$= \frac{f(a)}{3h^2}\cdot h^3+\frac{4f(x_m)}{3h^2}\cdot h^3+\frac{f(b)}{3h^2}\cdot h^3 = \frac{f(a)h}{3}+\frac{4f(x_m)h}{3}+\frac{f(b)h}{3}$$

$$= \frac{h}{3}\left(f(a)+4f(x_m)+f(b)\right)$$

4. Verifique el resultado presentado en la ecuación 1.89.

El error del polinomio interpolador es:

$$E(x) = \frac{f'''(\xi)}{3!}(x-a)(x-b)\left(x-\frac{a+b}{2}\right) = \frac{f'''(\xi)}{3!}\left(x^3-x^2\left(\frac{a+b}{2}\right)-(a+b)x^2\right.$$

$$\left.-\frac{(a+b)^2}{2}x+abx-ab\left(\frac{a+b}{2}\right)\right)$$

$$= \frac{f'''(\xi)}{3!}\left(x^3-\frac{3}{2}(a+b)x^2+\frac{(a^2+4ab+b^2)}{2}x-\frac{ab(a+b)}{2}\right)$$

Entonces:

$$\int_a^b E(x)dx = \frac{f'''(\xi)}{3!}\left[\frac{b^4-a^4}{4}-\frac{3}{2}(a+b)\frac{b^3-a^3}{3}+\frac{a^2+4ab+b^2}{2}\cdot\frac{b^2-a^2}{2}-\frac{ab(a+b)(b-a)}{2}\right]$$

$$= \frac{f'''(\xi)}{3!}(b^2-a^2)\left[\frac{b^2+a^2}{4}-\frac{b^2+ab+a^2}{2}+\frac{a^2+4ab+b^2}{4}-\frac{ab}{2}\right]$$

$$= \frac{f'''(\xi)}{3!}(b^2-a^2)\left[\frac{b^2}{4}-\frac{b^2}{2}+\frac{b^2}{4}-\frac{ab}{2}+\frac{4ab}{4}-\frac{ab}{2}+\frac{a^2}{4}-\frac{a^2}{2}+\frac{a^2}{4}\right]$$

$$= \frac{f'''(\xi)}{3!}(b^2-a^2)(0) = \emptyset$$