# Finite volume method for shallow water equations

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The system

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ hu \\ hv \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{bmatrix} = 0$$

is known as the shallow water equations in conservative form.

Consider the IV and BV conditions,

$$\begin{aligned} \Omega : (x, y) \in [-1, 1]^2, &\quad t \in [0, 3] \\ U(x, y, 0) = [2, 0, 0]^\top, &\quad \text{for } (x, y) \in [-1/2, 1/2]^2, \\ U(x, y, 0) = [1, 0, 0]^\top, &\quad \text{otherwise.} \end{aligned}$$

Here, $U = [h\ hu\ hv]^\top$.

This Matlab function simulates the above dam breaking problem using a finite volume method with numerical flux function of Lax-Friedrichs type:

$$V_i^{n+1} = V_i^n - \frac{\Delta t}{\Delta x}\left(F^{*n}_{i+\frac{1}{2}} - F^{*n}_{i-\frac{1}{2}}\right),$$

with

$$F^*_{i+\frac{1}{2}} = \frac{1}{2}[F(V_i) + F(V_{i+1})] - \frac{1}{2}|\lambda_{i+\frac{1}{2}}|_{max}(V_{i+1} - V_i),$$

where $|\lambda_{i+\frac{1}{2}}|_{max}$ is the largest eigenvalue in absolute value of the Jacobian matrix of the hyperbolic system at interface $i + \frac{1}{2}$ (in this case, $|\lambda_x|_{max} = |u| + \sqrt{gh}$, or $|\lambda_y|_{max} = |v| + \sqrt{gh}$). For calculating $|\lambda_{i+\frac{1}{2}}|_{max}$, the averages of $u$ (or $v$) and $h$ are used.

At the spatial boundaries, ghost cells are used. Before every time-step, the values of U from the nearest physical cells are copied to the ghost cells. For the velocity component normal to the boundary, however, the value is copied over with the sign reversed. By this trick, the effective normal velocity at the interface between the last physical cell and the ghost cell will be close to zero, as required for a perfect wall.

The time-step is chosen dynamically in every step according to

$$\Delta t = \frac{c}{2} \min \left( \min \left( \frac{\Delta x}{|\lambda_x|_{max}} \right), \min \left( \frac{\Delta y}{|\lambda_y|_{max}} \right) \right).$$

The CFL safety constant $c$ is chosen to be smaller than 1 for this nonlinear system in order to avoid oscillations (for example, $c = 0.8$).