Problems 2.1.

$a, b \in \mathbb{Z}$ $\quad b > 0$ $\quad \exists! q, r$ $\quad a = bq + r$ $\quad 2b \leq r < 3b$

we already have $a = b\bar{q} + \bar{r}$ where $0 \leq \bar{r} < b$

$$\bar{r} = a - b\bar{q} \Rightarrow \bar{r} + 2b = a - b\bar{q} + 2b \Rightarrow a = (\bar{q} - 2) \cdot b + (\bar{r} + 2b)$$

Define $r = \bar{r} + 2b$, $q = \bar{q} - 2$. $r$ and $q$ are unique since $\bar{r}, \bar{q}$ are unique.

2.1/2 Show that any integer of the form $6k+5$ is also of the form $3j+2$ but not conversely.

Let $x=6k+5$.
$x=6k+3+2 = 3(2k+1)+2 = 3j+2$ for $j=2k+1$

Converse is not true. for $j=0$ $3\cdot(j)+2= 3\cdot(0)+2=2$

but $2= 6k+5 \Rightarrow k=-\frac{1}{2} \notin \mathbb{Z}$.

2.1/3 Use division algorithm to establish following

a) The square of any integer is either of the form $3k$ or $3k+1$

By division algorithm we get that any integer is in the form of one of the following $3k-1$, $3k$, $3k+1$

(case 1). $(3k-1)^2 = 9k^2-6k+1 = 3(3k^2-2k)+1 = 3p+1$, $p=3k^2-2k \in \mathbb{Z}$

(case 2) $3k^2 = 9k^2 = 3(3k^2) = 3p$, $p=3k^2$

(case 3) $(3k+1)^2 = 9k^2+6k+1 = 3(3k^2+2k)+1 = 3p+1$, $p=3k^2+2k$

So The square of an integer is either in $3p+1$ form or $3p$ form.

b) The cube of any integer has one of the forms: $9k$, $9k+1$ or $9k+8$

By division algorithm any integer must have one of the forms: $3k$, $3k+1$ or $3k+2$.

For $3k$; $(3k)^3 = 27k^3 = 9(3k^3) = 9.p$
For $3k+1$; $(3k+1)^3 = 27k^3+27k^2+9k+1 = 9(3k^3+8k^2+k)+1 = 9p+1$
For $3k+2$; $(3k+2)^3 = 27k^3+54k^2+18k+8 = 9(3k^3+6k^2+3k)+8 = 9p+8$

c) The forth power of any integer is either of the form $5k$ or $5k+1$.

By division algorithm any intege has one of the forms: $5k-2$, $5k-1$, $5k$, $5k+1$ or $5k+2$.

For $5k-2$; $(5k-2)^4 = (5k)^4+4(5k)^3(-2)^1+6(5k)^2(-2)^2+4(5k)(-2)^3+(-2)^4$
$= 5(5^3k^4-8.5^2.k^3+32.k^2 5 + -32k+3)+1 = 5\bar{k}+1$
For $5k+2$; $(5k+2)^4 = 5(5^3k^4+8.5^2k^3+32k^2.5+32k+3)+1 = 5\bar{k}+1$
For $5k-1$; $(5k-1)^4 = (5k)^4+(4)(5k)^3(-1)+(6)(5k)^2(-1)^2+(4)(5k)(-1)^3+(-1)^4$
$= 5(5^3k^4-4.5^2.k^3+6.5k^2-4k)+1 = 5\bar{k}+1$
For $5k+1$; $(5k+1)^4 = 5(5^3k^4+4 5^2k^3+30k^2+4k)+1 = 5\bar{k}+1$
For $5k$; $5k^4 = 5\bar{k}+0$

2.1/4. Prove that $3a^2-1$ is never a perfect square.

Any perfect square has one of the forms $3k$ or $3k+1$

Assume $3a^2-1=3k$ for $k\in\mathbb{Z}$ $\Rightarrow 3(a^2-k)=1$

$\Rightarrow (a^2-k)=\frac{1}{3}$ for $a,k\in\mathbb{Z}$ not possible. so $3a^2-1$ can not have $3k$ form.

Assume $3a^2-1=3k+1 \Rightarrow 3(a^2-k)=2 \Rightarrow (a^2-k)=2/3$ which is not possible

Hence $3a^2-1$ can not have forms $3k$ and $3k+1$ so it may not be a perfect square

2.1/5 for $n\geq 1$ prove that $n(n+1)(2n+1)/6$ is an integer.

By division algorithm $n$ has one of the forms: $3k$, $3k+1$ or $3k-1$

for $n=3k$. $(3k)(3k+1)(6k+1) = 3k(18k^2+9k+1)$ where $k$ is either even or odd. if $k$ is even $3k$ is divisible by 6 and we are done. if $k$ is odd then $(18k^2+9k+1)$ is even and so $(3k)(18k^2+9k+1)$ can be written as $(3k)(2\bar{k}) = 6(k)(\bar{k})$ which is divisible by 6

for $n=3k+1$ $(3k+1)(3k+2)(6k+3) = (3k+1)(3k+2)\,3(2k+1)$ either $(3k+1)$ or $(3k+2)$ is an even number so $(3k+1)(3k+2)$ is even and hence $(3k+1)(3k+2)(6k+3) = 2\bar{k}\cdot 3(2k+1)$ which is divisible by 6

for $n=3k-1$ $(3k-1)(3k)(6k-1) = (3k)(18k^2-9k+1)$ where $k$ is even or odd if $k$ is even $3k$ is divisible by 6 we are done if $k$ is odd then $18k^2-9k+1$ is even and so $(3k)(18k^2-9k+1)$ is divisible by 6

2.1/6 Show that the cube of any integer is of the form $7k$ or $7k\pm1$.

Any integer is one of the forms $7k\pm1$, $7k\pm2$ or $7k$

for $7k\pm1$; $(7k\pm1)^3 = (7k)^3+(3)(7k)^2(\pm1)+(3)(7k)(\pm1)^2+(\pm1)^3 = 7(\bar{k})\pm1$

for $7k\pm2$; $(7k\pm2)^3 = (7k)^3+(3)(7k)^2(\pm2)+(3)(7k)(\pm2)^2+(\pm2)^3 = 7(\bar{k})\pm8$

$= 7(\bar{k}\pm1)\pm1$

$= 7\bar{\bar{k}}\pm1$

for $7k$; $(7k)^3 = 7(\bar{k})$

2.1/7. For integers $a,b$ $b\neq0$ there exist unique integers $q,r$ such that $a=bq+r$ $-\frac{|b|}{2}<r\leq\frac{|b|}{2}$

By division algorithm there exist unique $q,r$ satisfying $a=bq+r$ $0\leq r<|b|$. For $r$ there are two possibilities (I) $0\leq r<\frac{|b|}{2}$ or II $r\geq\frac{|b|}{2}$

If $0\leq r<\frac{|b|}{2}$ then we are done since $0\leq r<\frac{|b|}{2} \Rightarrow$

$-\frac{|b|}{2}<r<\frac{|b|}{2} \Rightarrow -\frac{|b|}{2}\leq r\leq\frac{|b|}{2}$.

else if $r \geq \frac{|b|}{2}$ then $\bar{r} = (r - |b|)$, $\bar{q} = (q \pm 1)$ satisfies

$a = b \cdot \bar{q} + \bar{r}$ and $\bar{r}_1 = (r - |b|)$, $\bar{r}_2 = (r - |b|)$ $\Rightarrow$ $\bar{r}_1 = \bar{r}_2$

$\bar{q}_1 = (q \pm 1)$, $\bar{q}_2 = (q \pm 1)$ $\Leftrightarrow$ $\bar{q}_1 = \bar{q}_2$

$\bar{q}$ and $\bar{r}$ are both unique

by omission

$(\bar{r} = r - |b|)$ $\quad \frac{|b|}{2} \leq r < |b| \Rightarrow \frac{|b|}{2} - |b| \leq r - |b| < |b| - |b|$

$\Rightarrow -\frac{|b|}{2} \leq r - |b| < 0 < \frac{|b|}{2} \Rightarrow -\frac{|b|}{2} < \bar{r} \leq \frac{|b|}{2}$ $\square$

2.1./8. Prove that no integer in the following sequence is perfect square 11, 111, 1111, 11111, ....

Any number in this sequence can be written as

$\underbrace{111 \cdots 11}_{\text{0 or more many 1's}}08 + 3$ that is equal $4k+3$ for some $k \in \mathbb{N}$

followed by 08

Since sequence contains only odd numbers if a number from this sequence is squared an integer that integer must be odd.

Assume there is an integer in this sequence that is squared integer than must hold for some k and p

$4k+3 = (2p+1)^2$

$4k+3 = 4p^2 + 4p + 1 \Rightarrow (4p^2 + 4p - 4k) = 2 \Rightarrow (p^2 + p - k) = \frac{1}{2}$

which is not possible

Hence there is no perfect square in sequence

2.1/9 Verify that if an integer is simultaneously a square and a cube then it must be either $7k$ or $7k+1$.

by 2.1/6 any cube of an integer is in form: $7k$ or $7k \pm 1$

Any integer must be in one of the forms: $7k$; $7k \pm 1$, $7k \pm 2$

For $7k$: $(7k)^2 = 7(7k^2) = 7(\bar{k})$

For $7k \pm 1$: $(7k \pm 1)^2 = 49k^2 \pm 14k + 1 = 7\bar{k} + 1$

For $7k \pm 2$: $(7k \pm 2)^2 = 49k^2 \pm 28k + 4 = 7\bar{k} + 4$

By 2.1/6 only $7(\bar{k})$ and $7\bar{k}+1$ can be cubes hence if a number is both cube and square it has one of the form $7\bar{k}, 7\bar{k}+1$

2.1/10 For $n \geq 1$ establish that the integer $n(7n^2+5)$ is of the form $6k$.

$n \in \{3k \pm 1, 3k\}$ for some $k \in \mathbb{Z}$

$$n \in \left\{ (3k\pm1)\left[7(3k\pm1)^2+5\right], (3k)\left[7(3k)^2+5\right] \right\}$$

$$n \in \left\{ (3k\pm1)(63k^2 \pm 42k + 6), (3k)(63k^2+5) \right\}$$

for $(3k\pm1)(63k^2 \pm 42k + 6)$ if $k$ is even $63k^2 \pm 42k + 6$ is $6\bar{k}$ for some $\bar{k}$ hence $(3k\pm1)(63k^2\pm42k+6) = 6.\bar{\bar{k}}$ for some $\bar{\bar{k}}$.

if $k$ is odd then $(3k\pm1)$ is even hence $(3k\pm1) = 2\bar{k}$ for some $\bar{k}$ and $(3k\pm1)(63k^2 \pm 42k + 6) = 2\bar{k} \cdot 3(21k^2 \pm 14k + 2)$ $= 6.\bar{k}(21k^2 \pm 14k + 2) = 6.\bar{\bar{k}}$ for some $\bar{\bar{k}}$.

for $(3k)(63k^2+5)$ if $k$ is even we are done
If $k$ is odd then $63k^2+5$ is even and we are done

so $n \in \{6\bar{\bar{k}}, 6.\bar{\bar{k}}\}$ $k \in \mathbb{Z}$ ☒

2.1/11 If $n$ is odd integer show that $n^4 + 4n^2 + 11$ is of the form $16k$

$$(2k+1)^4 + 4(2k+1)^2 + 11 = 16k^4 + (4)(2k)^3 + 6 \cdot (2k)^2 + 4(2k) + 1 + 4\left((2k)^2 + 4k + 1\right) + 11$$

$$= 16k^4 + 32k^3 + 40k^2 + 24k + 16$$

$$= 16(k^4 + 2k^3) + \underbrace{8(5k^2 + 3k + 2)}_{\text{always even}} = 16(k^4 + 2k^3 + 8(2\bar{k}))$$

$$= 16(k^4 + 2k^3 + \bar{k}) = 16.\bar{\bar{k}} \text{ for some } \bar{\bar{k}} \in \mathbb{Z}$$ ☒