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→ only true when A is diagonalizable

*) Given $A = (\quad)_{n\times m}$, Then $A^n = QD^nQ^{-1}$

eg:

$$D = Q^{-1}AQ$$
$$QD = AQ$$
$$QDQ^{-1} = A$$

$$A^2 = (QDQ^{-1})(QDQ^{-1})$$
$$= QDDQ^{-1}$$
$$A^2 = QD^2Q^{-1}$$

System of Differential Equation:-

① Solve following sys of differential eqn.

$$\left.\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + 4x_2 + 2x_3 \\ x_3' &= -x_1 - x_2 + x_3 \end{aligned}\right\} \longrightarrow ①$$

Soln:

$x_i = x_i(t)$, $i = 1, 2, 3$

Define $x : \mathbb{R} \to \mathbb{R}^3$ by ~~$x'(t$~~ $x' : \mathbb{R} \to \mathbb{R}^3$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}$$

① $\Rightarrow$ $x' = Ax$

where $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$ shld be diagonalizable

The char eqn is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 3+4+1 = 8$

$S_2 = (4+2)+(3+1)+(12-2) = \cancel{8}\,6 + 4 + 10 = 20$

$S_3 = 3(4+2) - 1(2+2) + 1(-2+4) = 18 - 4 + 2 = 16$

$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$

$(\lambda-2)(\lambda^2 - 6\lambda + 8) = 0$

$(\lambda-2)(\lambda-4)(\lambda-2) = 0$

$\lambda = 2, 2, 4$

2	1	-8	20	-16
	0	2	-12	16
	1	-6	8	0

8, -6: -4, -2

To find eigen vectors.

$(A - \lambda I)x = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (3-\lambda)x_1 + 1x_2 + 1x_3 &= 0 \\ 2x_1 + (4-\lambda)x_2 + 2x_3 &= 0 \\ -1x_1 + (-1)x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow ②$$

$\lambda = 2$

② $\Rightarrow$ $1x_1 + 1x_2 + 1x_3 = 0$

$2x_1 + 2x_2 + 2x_3 = 0$

$-1x_1 + (-1)x_2 + (-1)x_3 = 0$

$\Rightarrow$ $x_1 = -x_2 - x_3$

Let $x_2 = k$, $x_3 = t$ $\Rightarrow$ $x_1 = -k - t$

Ex

$$x = \left\{ \begin{pmatrix} -k-t \\ k \\ t \end{pmatrix} / k, t \in R \right\} \qquad x = \begin{pmatrix} -k-t \\ k \\ t \end{pmatrix}$$

$k=0, t=1 \qquad k=1, t=0$

$$x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

<u>when $\lambda = 4$</u>

② ⇒ $-1x_1 + x_2 + x_3 = 0$

$2x_1 + 0x_2 + 2x_3 = 0 \longrightarrow x_1 + x_3 = 0$

$-1x_1 + (-1)x_2 + (-3)x_3 = 0 \qquad \boxed{x_1 = -x_3}$

$2x_3 + x_2 = 0$

$-2x_3 - x_2 = 0$

$x_2 = -2x_3$

Let $x_3 = s$, $x_2 = -2s$, $x_1 = -s$

$$x = \begin{pmatrix} -s \\ -2s \\ s \end{pmatrix}, s \in R$$

Let $s = 1$,

$$x_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\boxed{x' = Ax}$$

$D = Q^{-1}AQ$

$$\boxed{A = QDQ^{-1}}$$

$Q^{-1} : 3 \times 3$

$x : 3 \times 1$

$Q^{-1}x : 3 \times 1$

↓

y defined as new

$x' = QDQ^{-1}x$

$Q^{-1}x' = DQ^{-1}x$

Define $y : R \to R^3$ by $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$

Let $Q^{-1}x = y \Rightarrow Q^{-1}x' = y'$

$Q^{-1}x^{-1} = Dy$

$y' = Dy$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2y_1(t) \\ 2y_2(t) \\ 4y_3(t) \end{pmatrix}$$

$y_1'(t) = 2y_1(t)$

$y_2'(t) = 2y_2(t) \Rightarrow y_2(t) = c_2 e^{2t}$

$y_3'(t) = 4y_3(t) \Rightarrow y_3(t) = c_3 e^{4t}$

$$\frac{y_1'(t)}{y_1(t)} = 2$$

$$\int \frac{y_1'(t)}{y_1(t)} dt = \int 2 dt$$

$\log(y_1(t)) = 2t + c$

$y_1(t) = e^{2t+c}$

$y_1(t) = e^{2t} e^c$

$\boxed{y_1(t) = c_1 e^{2t}}$

wkt, $Q^{-1}x = y$

$x = Qy$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{2t} \\ c_3 e^{4t} \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -c_1e^{2t} - c_2e^{2t} - c_3e^{4t} \\ 0 + c_2e^{2t} - 2c_3e^{4t} \\ c_1e^{2t} + 0 + c_3e^{4t} \end{pmatrix}$$

$$x(t) = e^{2t}\left[c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right] + e^{4t}c_3\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$x(t) = e^{2t}z_1 + e^{4t}z_2$$

where $z_1 \in E_{\lambda_1}$, $z_2 \in E_{\lambda_2}$

Prob

① Find general soln for each eqn of differential eqns.

i) $x' = x + y$, $y' = 3x - y$

ii) $x_1' = 8x_1 + 10x_2$, $x_2' = -5x_1 - 7x_2$

iii) $x_1' = x_1 + x_3$, $x_2' = x_2 + x_3$, $x_3' = 2x_3$

MODULE - 5

INNER PRODUCT SPACE :-

*) Let V be a vs over F, $V(F)$ an inner product on V is a function defined from $V \times V \to F$ (field) from a vector space to some scalars. It is represented by $\langle x, y \rangle$ or (x, y)

$\langle x, y \rangle : V \times V \to F$ if satisfies following condns,

i) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

ii) $\langle cx, y \rangle = c\langle x, y \rangle$

iii) Conjugate of $\langle x, y \rangle = \langle y, x \rangle \Rightarrow \overline{\langle x, y \rangle} = \langle y, x \rangle$

iv) always $\langle x, x \rangle > 0$ if $x \neq 0$ (positive)

$\langle x_1 + x_2 + \dots + x_n, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle + \dots + \langle x_n, y \rangle$

$\langle \sum_{i=1}^{n} x_i, y \rangle = \sum_{i=1}^{n} \langle x_i, y \rangle$

Example

$V = R^n$, suppose $x, y \in V$

$\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b_i}$ is defined, Check it is inner product space in V.

Soln:

$x = (a_1, a_2, \dots, a_n)$, $y = (b_1, b_2, \dots, b_n)$

$\langle x, y\rangle = a_1\bar{b_1} + a_2\bar{b_2} + \dots + a_n\bar{b_n}$

$\langle x, y\rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$ ($\because F = R$)

let $z = (c_1, c_2, \dots, c_n)$

i) $\langle x+z, y\rangle = \langle((a_1+c_1), (a_2+c_2), \dots (a_n+c_n)), (b_1, b_2, \dots b_n)\rangle$

$= (a_1+c_1)\bar{b_1} + (a_2+c_2)\bar{b_2} + \dots + (a_n+c_n)\bar{b_n}$

$\cancel{\langle x, y\rangle + \langle z, y\rangle}$

$\langle x, y\rangle = a_1\bar{b_1} + \dots + a_n\bar{b_n}$

$\langle z, y\rangle = c_1\bar{b_1} + \dots + c_n\bar{b_n}$

$\langle x+y\rangle + \langle z+y\rangle = (a_1+c_1)\bar{b_1} + \dots + (a_n+c_n)\bar{b_n}$

(ii) prop is true

(iii) $\overline{\langle x, y\rangle} = \overline{a_1\bar{b_1} + a_2\bar{b_2} + \dots + a_n\bar{b_n}}$

$= \bar{a_1}b_1 + \bar{a_2}b_2 + \dots + \bar{a_n}b_n$

$\langle y, x\rangle = b_1\bar{a_1} + \dots + b_n\bar{a_n}$

(iv) $\langle x, x\rangle = a_1\bar{a_1} + \dots + a_n\bar{a_n}$ $\to F = C$ $\downarrow$ $|a_1|^2 + \dots + |a_n|^2$

$= a_1^2 + \dots + a_n^2 = \sum_{i=1}^{n} a_i^2 = \sum |a_i|^2$

Prob

① V be V.S , $\langle x,y\rangle' = r\langle x,y\rangle$

$\langle x,y\rangle$ is a inner pdt. Is $\langle x,y\rangle'$ a inner pdt ?

Soln :

as $\langle x,y\rangle$ is inner pdt,

wkt $\langle x,x\rangle > 0$, $x \neq 0$.

$\langle x,y\rangle' = r\langle x,y\rangle$

It does not satisfy (iv) prop for $r<0$.

$\therefore$ Not a inner pdt.

② $V = C[0,1]$ → real valued conti function on $[0,1]$. $f,g \in V$, $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$,

Check whether $\langle f,g\rangle$ is inner pdt ?

Soln :

i) $f,g,h \in V$

$\langle f+h, g\rangle = \langle f,g\rangle + \langle h,g\rangle$

$\langle f+h,g\rangle = \int_0^1 (f+h)(t)\,g(t)\,dt$

$= \int_0^1 [f(t)+h(t)]\,g(t)\,dt$

$= \int_0^1 [f(t)g(t) + h(t)g(t)]\,dt$

$= \langle f,g\rangle + \langle h,g\rangle$ (i) prop is true

iii) $\overline{\langle f,g\rangle} = \int_0^1 f(t)g(t)dt$.

$\langle g,f\rangle = \int_0^1 g(t)f(t)\,dt$

(iv) $\langle f,f\rangle = \int_0^1 f(t)f(t)dt = \int_0^1 (f(t))^2 dt > 0$

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Result:

*) Let V be inner pdt space, then $x, y, z \in V$, $c \in F$, following are true

(i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(ii) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(iii) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

(iv) $\langle x, x \rangle = 0$ iff $x = 0$

v) If $\langle x, y \rangle = \langle x, z \rangle$ $\forall x \in V$, (x·) then $y = z$

Definition:

*) Let V be inner pdt space for $x \in V$, we define norm (or) length of x. It ~~is def~~ by square root of $\langle x, x \rangle$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$\|x\|^2 = \langle x, x \rangle$$

eg:

$F = \mathbb{R}$, $V = \mathbb{R}^n$

Standard i.p.s
$\langle x, y \rangle = \sum a_i \overline{b_i}$

Let $x = (a_1, a_2, \dots, a_n)$

$$\|x\|^2 = \langle x, x \rangle$$

$$= \langle (a_1, a_2, \dots, a_n), (a_1, a_2, \dots, a_n) \rangle$$

$$= |a_1|^2 + |a_2|^2 + \dots + |a_n|^2$$

$$\|x\|^2 = \sum_{i=1}^{n} |a_i|^2$$

$$\|x\| = \sqrt{\sum_{i=1}^{n} |a_i|^2}$$

$$\|x\| = \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2}$$

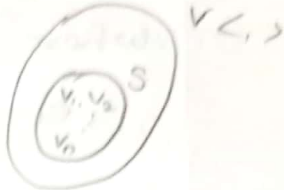
Definition :

V is a i.p.s , S is a subset of V

Orthogonal $\Rightarrow \langle v_i, v_j \rangle = 0 \ \forall i \neq j$

Orthonormal $\Rightarrow \|v_i\| = 1.$

$\hookrightarrow \langle v_i, v_j \rangle = 0 \ \forall i \neq j$



→ If these two condns are true, then set S is called as orthonormal set.

→ Subset of i.p.s is said to be orthonormal basis if it is a basis & also satisfies orthonormal condns.

Gram - Schmidt Orthogonalisation Process :-

Theorem (Stmt)

⇒ Let V be an i.p.s and let $S = \{w_1, w_2, w_3, \ldots, w_n\}$ be an linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots, v_n\}$ where

$v_1 = w_1$

For $2 \leq k \leq n$,

$$\boxed{v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j}$$

eg: $V = \mathbb{R}^4$ $\quad S = \{w_1, w_2, w_3, w_4\}$

To find S'

$v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$v_4 = w_4 - \frac{\langle w_4, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_4, v_2 \rangle}{\|v_2\|^2} v_2 - \frac{\langle w_4, v_3 \rangle}{\|v_3\|^2} v_3$$

$S' = \{v_1, v_2, v_3, v_4\}$

$\{e_1, e_2, e_3, e_4\}$

where e_i is unit vector of v_i

$e_1 = \frac{v_1}{\|v_1\|}$ | $e_2 = \frac{v_2}{\|v_2\|}$ | $e_3 = \frac{v_3}{\|v_3\|}$ | $e_4 = \frac{v_4}{\|v_4\|}$

Prob

① Let $V = R^4$ be an i.p.s with std i.p and let $S = \{(1,0,1,0), (1,1,1,1), (0,1,2,1)\}$ be linearly inde sub set of V. Using Gram Schmidt process, find orthonormal set.

Soln.

Let $w_1 = (1,0,1,0)$

$w_2 = (1,1,1,1)$

$w_3 = (0,1,2,1)$

$v_1 = w_1 = (1,0,1,0)$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$$

$\langle w_2, v_1\rangle = \langle (1,1,1,1), (1,0,1,0)\rangle = 1+0+1+0 = 2$

$\cancel{= (1,0,1,0)}$

$\|v_1\|^2 = \langle v_1, v_1\rangle = \langle (1,0,1,0), (1,0,1,0)\rangle = 1+0+1+0 = 2$

$v_2 = (1,1,1,1) - \frac{2}{2}(1,0,1,0)$

$= (1,1,1,1) - (1,0,1,0)$

$v_2 = (0,1,0,1)$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$\langle w_3, v_1\rangle = \langle (0,1,2,1), (1,0,1,0)\rangle = 0+0+2+0 = 2$

$\langle w_3, v_2\rangle = \langle (0,1,2,1), (0,1,0,1)\rangle = 0+1+0+1 = 2$

Scanned with CamScanner

$\|v_1\|^2 = 2$ $\quad \|v_2\|^2 = \langle(0,1,0,1),(0,1,0,1)\rangle$

$= 0+1+0+1 = 2$

$v_3 = (0,1,2,1) - \frac{2}{2}(1,0,1,0) - \frac{2}{2}(0,1,0,1)$

$= (0,1,2,1) - (1,0,1,0) - (0,1,0,1)$

$v_3 = (-1,0,1,0)$

$e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1,0,1,0) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$

$e_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(0,1,0,1) = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$e_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}(-1,0,1,0) = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$

$\|v_3\|^2 = \langle(-1,0,1,0),(-1,0,1,0)\rangle = 1+0+1+0 = 2$

$\{v_1, v_2, v_3\} \rightarrow$ orthogonal set

$\{e_1, e_2, e_3\} \rightarrow$ ortho~~gonal~~normal set

) $V = P_2(\mathbb{R})$, $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$.

Consider a subspace β is std ordered basis of V. Use Gram Schmidt process to find orthonormal process.

Soln:

$\beta = \{1, x, x^2\}$

Let $w_1 = 1$

$w_2 = x$

$w_3 = x^3$

$v_1 = w_1 = 1$

$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$

$\langle w_2, v_1\rangle = \langle x, 1\rangle = \int_{-1}^{1} t \cdot 1\,dt = \left[\frac{t^2}{2}\right]_{-1}^{1} = \frac{1}{2}(1-1)$

$= 0$

Scanned with CamScanner

$$v_2 = w_2 - \frac{0}{\|v_1\|^2} v_1 = x$$

$$v_2 = x$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$\langle w_3, v_1\rangle = \langle x^2, 1\rangle = \int_{-1}^{1} t^2 \cdot 1\, dt = \left[\frac{t^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{3}(1+1) = \frac{2}{3}$$

$$\|v_1\|^2 = \langle v_1, v_1\rangle = \langle 1, 1\rangle = \int_{-1}^{1} 1\, dt = [t]_{-1}^{1} = 2$$

$$\langle w_3, v_2\rangle = \langle x^2, x\rangle = \int_{-1}^{1} t^2 \cdot t\, dt = 0 \quad \left[\begin{matrix}\text{odd func}^n \\ -a \text{ to } +a\end{matrix} = 0\right]$$

$$v_3 = x^2 - \frac{2/3}{2} 1 - 0 = x^2 - \frac{1}{3}$$

Orthogonal set or basis $= \left\{1, x, x^2 - \frac{1}{3}\right\}$

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$$

$$e_2 = \frac{v_2}{\|v_2\|} = \frac{x}{\sqrt{2}/\sqrt{3}}$$

$$e_2 = \frac{\sqrt{3}}{\sqrt{2}} x$$

$$e_3 = \frac{v_3}{\|v_3\|} = \frac{x^2 - 1/3}{\sqrt{8/45}}$$

$$= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

Orthonormal basis

$$= \left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\right\}$$

$$\|v_2\|^2 = \langle v_2, v_2\rangle$$

$$= \langle x, x\rangle \int_{-1}^{1} t^2\, dt$$

$$= \frac{1}{3}(1+1) = \frac{2}{3}$$

$$\|v_3\|^2 = \langle x^2 - \tfrac{1}{3}, x^2 - \tfrac{1}{3}\rangle$$

$$= \int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)\left(t^2 - \frac{1}{3}\right) dt$$

$$= \int_{-1}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right) dt$$

$$= \left[\frac{t^5}{5} - \frac{2}{3}\left(\frac{t^3}{3}\right) + \frac{1}{9}t\right]_{-1}^{1}$$

$$= \left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) - \left(\frac{-1}{5} + \frac{2}{9} - \frac{1}{9}\right)$$

$$= \frac{9 - 10 + 5 + 9 - 10 + 5}{45}$$

$$\|v_3\|^2 = \frac{8}{45}$$

Theorem:

→ Let V be a non-zero finite dimensional i.p.s then V has a orthonormal basis β. If $x \in V$, then x can be written as

$$x = \sum_{i=1}^{n} \underbrace{\boxed{\langle x, v_i \rangle}}_{\text{fourier coefficients.}} v_i$$

eg:

$V = P_2(\mathbb{R})$ $\quad \beta = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$

$x+1 \in V$,

$$x+1 = \left\langle x+1, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \left\langle x+1, \sqrt{\frac{3}{2}}\, x \right\rangle \sqrt{\frac{3}{2}}\, x + \left\langle x+1, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\rangle \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

Prob

① $V = \mathbb{R}^2$, $\beta = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\}$ → orthonormal basis. Let $x \in V$, $x = (3, 4)$. Find the fourier coeff. of x.

Soln:

$$(3,4) = \left\langle (3,4), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \left\langle (3,4), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\rangle \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

$$a_1 = \langle x, v_1 \rangle = \left\langle (3,4), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle$$

$$= \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$a_2 = \langle x, v_2 \rangle = \left\langle (3,4), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\rangle$$

$$= \frac{3}{\sqrt{2}} - \frac{4}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

Prob

Apply Gram Schmidt process in a i.p.s and std i.p. with ~~li inde~~ subset S. Find orthonormal set & fourier coefficient

(i) $V = R^3$, li ind $S = \{(1,0,1), (0,1,1), (1,3,3)\}$,

$x = (1,1,2)$

(ii) $V = R^3$, $S = \{(1,1,1), (0,1,1), (0,0,1)\}$

$x = (1,0,1)$

(iii) $V = P_2(R)$, with inner pdt $\langle f(x), g(x) \rangle = \int_0^1 f(t)g(t)\,dt$. $S = \{1, x, x^2\}$.

$h(x) = 1 + x$

(iv) $V = C^3$, $S = \{(1, i, 0), (1-i, 2, 4i)\}$

$x = (3+i, 4i, -4)$

v) $V = R^4$, $S = \{(2,-1,-2,4), (-2,1,-5,5), (-1,3,7,11)\}$, $x = (-11, 8, -4, 18)$

vi) $V = R^4$, $S = \{(+1,-2,-1,3), (3,6,3,-1), (1,4,2,8)\}$

$x = (-1, 2, 1, 1)$

vii) $V = \text{span}(S)$ with i.p. $\langle f, g \rangle = \int_0^{\pi} f(t)g(t)\,dt$

$S = \{\sin t, \cos t, 1, t\}$ & $h(t) = 2t + 1$

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Prob

(1) $x = (2, 1+i, i)$ $y = (2-i, 2, 1+2i)$ $V = C^3$

$\langle x, y\rangle$, $\|x\|$, $\|y\|$ $\langle x, y\rangle \sum a_i \overline{b_i}$

$\langle x, y\rangle = 2(2+i) + (1+i)(2) + i(1-2i)$

$\|x\|^2 = 2(2) + (1+i)(1-i) + (i)(-i)$

$\|y\|^2 = \langle y, y\rangle = (2-i)(2+i) + (2)(2) + (1+2i)(1-2i)$

(2) In $C[0,1]$, $f(t) = t$, $g(t) = e^t$

$\langle f(t), g(t)\rangle = \int_0^1 f(t) g(t)\, dt$

$\langle f(t), g(t)\rangle = \int_0^1 t e^t\, dt = \left[t(e^t) - 1(e^t)\right]_0^1$

$= (e - e) - (0 - 1) = 1$

$\|f(t)\|^2 = \int_0^1 f(t) f(t)\, dt = \int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1$

$\|f(t)\|^2 = \frac{1}{3} \Rightarrow \|f(t)\| = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$

$\|g(t)\|^2 = \int_0^1 e^t e^t\, dt = \int_0^1 e^{2t}\, dt = \left[\frac{e^{2t}}{2}\right]_0^1 = \frac{e^2}{2} - \frac{1}{2}$

$\|g(t)\|^2 = \frac{1}{2}(e^2 - 1)$

$\|g(t)\| = \frac{1}{\sqrt{2}} \sqrt{e^2 - 1}$

Cauchy Schwarz Inequality :-

Let V be i.p.s & $x, y \in V$, then

$|\langle x, y\rangle| \leq \|x\| \|y\|$

Proof :

~~suppose~~ If $x = 0$ (or) $y = 0 \Rightarrow$ inequality holds

If not, $x \neq 0$ (or) $y \neq 0$, then

for any $c \in F$

$$0 \leq \|x - cy\|^2 = \langle x - cy, x - cy\rangle$$

$$= \cancel{\langle x, x\rangle - \bar{c}\langle x, y\rangle}$$

$$= \langle x, x\rangle - \langle x, cy\rangle - \langle cy, x\rangle + \langle cy, cy\rangle$$

$$= \langle x, x\rangle - \bar{c}\langle x, y\rangle - c\langle y, x\rangle + c\bar{c}\langle y, y\rangle$$

let $c = \dfrac{\langle x, y\rangle}{\langle y, y\rangle}$ (or) $\dfrac{\langle x, y\rangle}{\|y\|^2}$

$$0 \leq \|x\|^2 - \frac{\overline{\langle x, y\rangle}}{\langle y, y\rangle}\langle x, y\rangle - \frac{\langle x, y\rangle}{\langle y, y\rangle} + \frac{\langle x, y\rangle}{\langle y, y\rangle}\frac{\overline{\langle x, y\rangle}}{\langle y, y\rangle}\langle y, y\rangle$$

$$\Rightarrow 0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

$$\frac{|\langle x, y\rangle|^2}{\|y\|^2} \leq \|x\|^2$$

$$|\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2$$

$$|\langle x, y\rangle| = \|x\| \|y\|$$

Triangle Inequality :-

*) Let V be i.p.s and $x, y \in V$. ~~$\|x+y\|$~~ then,

$$\|x+y\| \le \|x\| + \|y\|$$

Proof :

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x = 2\,Re(z)$

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$$

$$= \|x\|^2 + 2\,Re(\langle x, y\rangle) + \|y\|^2$$

$$\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\|x+y\|^2 \le (\|x\| + \|y\|)^2$$

$$\|x+y\| \le \|x\| + \|y\|$$

Orthogonal Complement :-

→ Let V is a i.p.s, S is a subset of V. Then $S^\perp$ (or) ~~S~~ S' is

$$S^\perp = \{ y \in V / \langle x, y\rangle = 0, \forall\, x \in S \}$$

$S^\perp$ subspace of V iff

i) $0 \in S^\perp$

ii) $x, y \in S^\perp \Rightarrow x+y \in S^\perp,\ cx \in S^\perp$

→ $\langle 0, y\rangle = 0$ and $\langle x, 0\rangle = 0 \Rightarrow 0 \in S^\perp$

(ii) if $\langle x+y, v_1\rangle = 0, \ldots, \langle x+y, v_k\rangle = 0$

Then $x+y \in S^\perp$

$$\langle x+y, v_i\rangle = \langle x, v_i\rangle + \langle y, v_i\rangle$$
$$= 0$$

$$\langle cx, v_i\rangle = c\langle x, v_i\rangle$$

$\Rightarrow x+y \in S^\perp,\ cx \in S^\perp$

Example :

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$
$\quad\quad e_1 \quad\quad e_2 \quad\quad e_3$

① If $V = R^3$, $S = \{e_3\}$. Find $S^\perp$

$S^\perp = \text{Span}\{e_1, e_2\}$

② V be i.p.s

$V^\perp = \{0\}$ $\quad \{0\}^\perp = V$

Least square Approximation :-

① Find least square approximating line for the following set of points $(1,3), (2,4), (5,5), (6,10)$

Soln :

Let $y = a_0 + a_1 x$

$a_0 + a_1 x = y$

$a_0(1) + a_1 x = y$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix}$$

⇒ This is in form $AX = B$

$$(A^T A) X = A^T B$$

$$(A^T A)^{-1} (A^T A) X = (A^T A)^{-1} A^T B$$

$$\boxed{X = (A^T A)^{-1} A^T B}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 14 \\ 14 & 66 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{|A^TA|} \operatorname{adj}(A^TA)$$

$$= \frac{1}{68}\begin{pmatrix} 66 & -14 \\ -14 & 4 \end{pmatrix}$$

$$X = (A^TA)^{-1}A^TB$$

$$= \frac{1}{68}\begin{pmatrix} 66 & -14 \\ -14 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix}$$

$$= \frac{1}{68}\begin{pmatrix} 66 & -14 \\ -14 & 4 \end{pmatrix}\begin{pmatrix} 22 \\ 96 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{68}\begin{pmatrix} 108 \\ 76 \end{pmatrix} = \begin{pmatrix} \frac{27}{17} \\ \frac{19}{17} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.12 \end{pmatrix}$$

$$\therefore \boxed{y = \frac{27}{17} + \frac{19}{17}x}$$

$$E = \|y - Ax\|^2 \text{ (or) } \|Ax - y\|^2$$

$$Ax - y = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 27/17 \\ 19/17 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 46/17 \\ 65/17 \\ 122/17 \\ 141/17 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 2.7 \\ 3.8 \\ 7.1 \\ 8.7 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix}$$

$$Ax - y = \begin{pmatrix} -0.3 \\ -0.2 \\ 2.1 \\ -1.8 \end{pmatrix} \qquad Ax - y = \begin{pmatrix} -0.294 \\ -0.176 \\ +2.176 \\ -1.705 \end{pmatrix}$$

$$E = \|Ax - y\|^2$$

$$= (-0.294)^2 + (-0.176)^2 + (2.176)^2 + (-1.705)^2$$

$$= +0.0864 + 0.0309 + 4.7349 + 2.907$$

$$E = 7.7592$$

(2) Find least square approximation

for (1,2), (2,3), (3,5), (4,7)

ans → $y = 1.7x$

$a_0 = 0 \quad a_1 = 1.7$

$\boxed{E = 0.3}$

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