Chain Rule

\# Given $f(z) = \log_e (1+z)$ where $z = x^T x$, $x \in \mathbb{R}^d$

Sol^n:

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$

Then $x^T = [x_1 \; x_2 \; \cdots \; x_d]$

$x^T x = [x_1^2 + x_2^2 + \cdots + x_d^2]$

Applying Chain Rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{\partial}{\partial z} (\log(1+z)) \cdot \frac{\partial}{\partial x} (x^T . x)$$

$$= \frac{1}{1+z} \cdot \frac{\partial}{\partial z} (z) \frac{\partial}{\partial x} (x_1^2 + x_2^2 + \cdots + x_d^2)$$

$$= \frac{1}{1+z} (2x_1 + 2x_2 + \cdots + 2x_d)$$

$$= \frac{1}{1+z} \cdot 2 \, (x_1 + x_2 + \cdots + x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

(Ans.)

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2. $f(z) = e^{-z/2}$; where $z = g(x)$, $\varphi(y) = y^T S^{-1} y$, $y = h(x)$

$h(x) = x - \mu$

$\Rightarrow$ Using Chain Rule :

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

here, $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(e^{-z/2}) = -\frac{e^{-z/2}}{2}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(y^T S^{-1} y)$$

$$= \lim_{h \to 0} \frac{\varphi(y+h) - \varphi(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h \to 0} (y^T S^{-1} + S^{-1} y + h S^{-1}) = y^T S^{-1} + S^{-1} y$$

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$$\frac{\partial y}{\partial x} = \frac{\partial (x-\mu)}{\partial x} = 1$$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= -\frac{e^{-z/2}}{2} (y^T S^{-1} + S^{-1} y) \cdot 1$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{S} (y^T + y)$$

(Ans.)

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