FCPLCD – Exercise 3

This exercise sheet is about operational semantics. We focus on big-step and small-step operational semantics and establish a fundamental connection between the two. We define these semantics over a language we call `FCPLang`. `FCPLang` contains natural numbers, booleans and conditionals:

$$
\begin{array}{lll}
t ::= & & \text{FCPLang terms:} \\
\quad | & n & \text{natural number, i.e., } n \in \mathbb{N} \\
\quad | & b & \text{boolean, i.e., } b \in \mathbb{B} \\
\quad | & \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 & \text{conditional}
\end{array}
$$

Here are some `FCPLang` terms:

```
true
if true then 1 else false
if (if 1 then true else false) then 2 else 3
```

Please define this language in Coq/Lean.

1 Big-Step Semantics

Please implement a big-step evaluation function

$$\texttt{eval}_{\texttt{BS}} :: \texttt{FCPLang} \rightarrow \texttt{Option FCPLang}$$

The following set of equalities specifies the semantics of $\texttt{eval}_{\texttt{BS}}$:

$$
\begin{array}{rlcl}
\forall\, n \in \mathbb{N}, & \texttt{eval}_{\texttt{BS}}\ n & = & \texttt{Some}\ n \\
\forall\, b \in \mathbb{B}, & \texttt{eval}_{\texttt{BS}}\ b & = & \texttt{Some}\ b \\
\forall\, t_1\ t_2\ t_3, & \texttt{eval}_{\texttt{BS}}\ t_1 = \texttt{None} \Rightarrow & & \\
 & \texttt{eval}_{\texttt{BS}}\ (\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3) & = & \texttt{None} \\
\forall\, t_1\ t_2\ t_3, & \texttt{eval}_{\texttt{BS}}\ t_1 = \texttt{Some true} \Rightarrow & & \\
 & \texttt{eval}_{\texttt{BS}}\ (\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3) & = & \texttt{eval}_{\texttt{BS}}\ t_2 \\
\forall\, t_1\ t_2\ t_3, & \texttt{eval}_{\texttt{BS}}\ t_1 = \texttt{Some false} \Rightarrow & & \\
 & \texttt{eval}_{\texttt{BS}}\ (\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3) & = & \texttt{eval}_{\texttt{BS}}\ t_3 \\
\forall\, t_1\ t_2\ t_3, \forall\, n \in \mathbb{N}, & \texttt{eval}_{\texttt{BS}}\ t_1 = \texttt{Some}\ n \Rightarrow & & \\
 & \texttt{eval}_{\texttt{BS}}\ (\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3) & = & \texttt{None}
\end{array}
$$

Please prove these equalities for your definition of $\texttt{eval}_{\texttt{BS}}$.

2 Small-Step Semantics

Please implement a small-step evaluation function

$$\texttt{eval}_{\texttt{S}} :: \texttt{FCPLang} \rightarrow \texttt{Option FCPLang}$$

The following set of equalities specifies the semantics of $\texttt{eval}_{\texttt{S}}$:

$$
\begin{aligned}
\forall\, n \in \mathbb{N},\quad & \texttt{eval}_{\texttt{S}}\ n & = \ & \texttt{Some}\ n \\
\forall\, b \in \mathbb{B},\quad & \texttt{eval}_{\texttt{S}}\ b & = \ & \texttt{Some}\ b \\
\forall\, t_2\ t_3,\quad & \texttt{eval}_{\texttt{S}}\ (\texttt{if true then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{Some}\ t_2 \\
\forall\, t_2\ t_3,\quad & \texttt{eval}_{\texttt{S}}\ (\texttt{if false then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{Some}\ t_3 \\
\forall\, t_1\ t_2\ t_3,\quad & \texttt{eval}_{\texttt{S}}\ t_1 = \texttt{None} \Rightarrow & & \\
& \texttt{eval}_{\texttt{S}}\ (\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{None} \\
\forall\, t_1\ t_2\ t_3\ s_1\ s_2\ s_3,\quad & \texttt{eval}_{\texttt{S}}\ (\texttt{if}\ s_1\ \texttt{then}\ s_2\ \texttt{else}\ s_3) = \texttt{Some}\ t_1 \Rightarrow & & \\
& \texttt{eval}_{\texttt{S}}\ (\texttt{if}\ (\texttt{if}\ s_1\ \texttt{then}\ s_2\ \texttt{else}\ s_3)\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{Some}\ (\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) \\
\forall n \in \mathbb{N},\quad & \texttt{eval}_{\texttt{S}}\ (\texttt{if}\ n\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{None}
\end{aligned}
$$

Please prove these equalities for your definition of $\texttt{eval}_{\texttt{S}}$.

3 Multi-Step Semantics

Please use $\texttt{eval}_{\texttt{S}}$ to implement a multi-step evaluation function

$$\texttt{eval}_{\texttt{MS}} :: \texttt{FCPLang} \rightarrow \texttt{Option FCPLang}$$

The following set of equalities specifies the sematics of $\texttt{eval}_{\texttt{S}}$:

$$
\begin{aligned}
\forall\, n \in \mathbb{N},\quad & \texttt{eval}_{\texttt{MS}}\ n & = \ & \texttt{Some}\ n \\
\forall\, b \in \mathbb{B},\quad & \texttt{eval}_{\texttt{MS}}\ b & = \ & \texttt{Some}\ b \\
\forall\, t_2\ t_3,\quad & \texttt{eval}_{\texttt{MS}}\ (\texttt{if true then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{eval}_{\texttt{MS}}\ t_2 \\
\forall\, t_2\ t_3,\quad & \texttt{eval}_{\texttt{MS}}\ (\texttt{if false then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{eval}_{\texttt{MS}}\ t_3 \\
\forall\, t_1\ t_2\ t_3,\quad & \texttt{eval}_{\texttt{S}}\ t_1 = \texttt{None} \Rightarrow & & \\
& \texttt{eval}_{\texttt{MS}}\ (\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{None} \\
\forall t_1\ t_1'\ t_2\ t_3,\quad & \texttt{eval}_{\texttt{S}}\ t_1 = \texttt{Some}\ t_1' \Rightarrow & & \\
& \texttt{eval}_{\texttt{MS}}\ (\texttt{if}\ t_1\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{eval}_{\texttt{MS}}\ (\texttt{if}\ t_1'\ \texttt{then}\ t_2\ \texttt{else}\ t_3) \\
\forall n \in \mathbb{N},\quad & \texttt{eval}_{\texttt{MS}}\ (\texttt{if}\ n\ \texttt{then}\ t_2\ \texttt{else}\ t_3) & = \ & \texttt{None}
\end{aligned}
$$

Please prove these equalities for your definition of $\texttt{eval}_{\texttt{MS}}$. If you run into problems with Coq/Lean complaining that it can't show termination of this function, message us on Slack for hints on how to solve this.

4 Big-step = Multi-step

To show the relationship between these two semantics, prove:

$$\forall\, t, \texttt{eval}_{\texttt{BS}}\ t = \texttt{eval}_{\texttt{MS}}\ t.$$

Proving this theorem is not trivial. You will need to prove multiple smaller lemmas first. We will give you an outline of how to approach this proof in the exercise session on Dec. 30. If you want to get started on the proof before then, message us on Slack. – Happy proving!