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Foundations of Certified Programming Language and Compiler Design

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Outline

Lecture	Logic	Formalisms	PL
1	Propositional and first-order logic		
2			Functional programming
3		Syntax and Semantics	
4			The untyped lambda calculus
5		Types	
6			The typed lambda calculus
7			Polymorphism
8		Curry-Howard	
9			Higher-order types
10			Dependent types

Goals

- At first, we have a look at an important second-order type system.
- Then, we resolve all long-standing mysteries of data types and
- introduce type-level computation to
- finally express “more interesting” propositions
- (and prove them).

From First to Second-order

- If we remove the top-level restriction for quantification, we
- enter second-order logic ,i.e.,
- we allow to quantify over predicates.

- The system that we arrive when doing so is called *System F/Polymorphic Lambda Calculus*.
- System F is the foundation for the rich and powerful type system of Haskell.

System F/Polymorphic Lambda Calculus

Syntax:

t	$::=$		terms:
	$\mid$	x	variable
	$\mid$	$\lambda x : T.t$	abstraction
	$\mid$	$t\ t$	application
	$\mid$	$\lambda X.t$	type abstraction
	$\mid$	$t\ [T]$	type application
v	$::=$		values:
	$\mid$	$\lambda x : T.t$	abstraction value
	$\mid$	$\lambda X.t$	type abstraction value

T	$::=$		types:
	$\mid$	X	type variable
	$\mid$	$T \to T$	type of functions
	$\mid$	$\forall X.T$	universal type
Γ	$::=$		contexts:
	$\mid$	$\varnothing$	empty context
	$\mid$	$\Gamma, x : T$	term variable binding
	$\mid$	Γ, X	type variable binding

System F

Evaluation

$$\boxed{t \longrightarrow t'}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2}\ \text{E-App1} \qquad \frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'}\ \text{E-App2} \qquad \frac{}{(\lambda x : T.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12}}\ \text{E-AppAbs}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ [T_2] \longrightarrow t_1'\ [T_2]}\ \text{E-TApp} \qquad \frac{}{(\lambda X.t_{12})\ [T_2] \longrightarrow [X \mapsto T_2]t_{12}}\ \text{E-TAppTAbs}$$

Typing

$$\boxed{\Gamma \vdash t : T}$$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \text{T-Var} \qquad \frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1.t_2 : T_1 \to T_2}\ \text{T-Abs} \qquad \frac{\Gamma, t_1 : T_{11} \to T_{12} \quad t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \text{T-App}$$

$$\frac{\Gamma, X \vdash t_2 : T_2}{\Gamma \vdash \lambda X.t_2 : \forall X.T_2 \to T_2}\ \text{T-TAbs} \qquad \frac{\Gamma, t_1 : \forall X.T_{12}}{\Gamma \vdash t_1\ [T_2] : [X \mapsto T_2]T_{12}}\ \text{T-TApp}$$

System F/Polymorphic Lambda Calculus

History

1972 – Jean-Yves Girard , a logician, discovers *System F* in the context of proof theory.

1974 – John Reynolds , a computer scientist, discovers the *polymorphic lambda calculus* that has a type system with the same expressive power.

Encoding Existentials

- With 2nd-order types, we can encode:

$$\{\exists X, T\} \stackrel{\text{def}}{=} \forall Y.\, (\forall X.\, T \to Y) \to Y$$

- ...and we can remove the explicit forms for existentials from the language definition.
- Pack: – A package is an abstraction that needs a result type Y and a continuation f.

$$\begin{aligned}
\{*S, t\}\ \texttt{as}\ \{\exists X, T\} &\stackrel{\text{def}}{=} \lambda Y.\, \lambda f : (\forall X.\, T \to Y).\ \ldots && \text{A package can only be an abstraction.}\\
&\stackrel{\text{def}}{=} \lambda Y.\, \lambda f : (\forall X.\, T \to Y).\, f\ [S]\ \ldots && \text{The continuation needs the type of the existential. (See T-PACK)}\\
&\stackrel{\text{def}}{=} \lambda Y.\, \lambda f : (\forall X.\, T \to Y).\, f\ [S]\ t && \text{The continuation needs the existential value.}
\end{aligned}$$

- Unpack: – Unpack with application to a continuation.

$$\begin{aligned}
\texttt{let}\ \{X, x\} = t_1\ \texttt{in}\ t_2 &\stackrel{\text{def}}{=} t_1\ \ldots && \text{Remember: an existential is an abstraction.}\\
&\stackrel{\text{def}}{=} t_1\ [T_2]\ \ldots && \text{The result type of the continuation.}\\
&\stackrel{\text{def}}{=} t_1\ [T_2]\ (\lambda X.\, \lambda x : T_{11}.\, t_2) && \text{The continuation itself.}
\end{aligned}$$

Impredicativity

- System F implements *impredicative* polymorphism.
- $T = \forall X.X \to X$ ranges over all types, even T itself!

System F in Haskell

- Haskell implements System FC[1], an extension of system F.
- Haskell has all the described features, even visible type applications, . . .
- . . . and still provides type inference based on an extension of HM![2]

[1] Martin Sulzmann et al. “System F with type equality coercions”. In: *Proceedings of the 2007 ACM SIGPLAN international workshop on Types in languages design and implementation*. 2007.

[2] Richard A. Eisenberg, Stephanie Weirich, and Hamidhasan G. Ahmed. “Visible Type Application”. In: *Programming Languages and Systems*. Springer Berlin Heidelberg, 2016.

Higher-Order Types

Intuitions

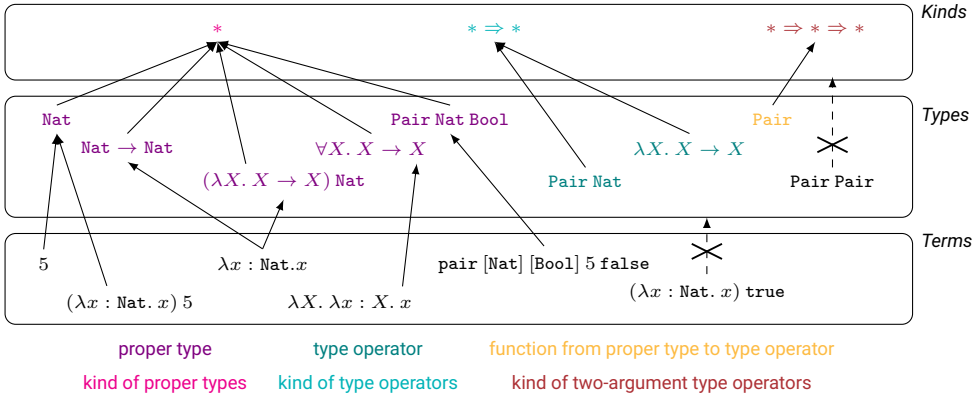
- We have seen (both in the lecture and in the exercise) types such as
  - `Bool`, `Pair a b`, `List a`, ...
- Typing them in System F and Haskell (and also in Coq, Lean, etc.) would yield:
  - `CBool` $= \forall X. X \to X \to X$
  - `tru` $= \lambda X.\ \lambda x : X.\ \lambda y : X.\ x$: `CBool` `fls` $= \lambda X.\ \lambda x : X.\ \lambda y : X.\ y$: `CBool`
  - `data Bool = True | False`
  - `Pair` $X\ Y = \forall Z. (X \to Y \to Z) \to Z$
  - `data Pair a b = Pair a b`
- Abbreviations:

  Simple `CBool`

  Parametric `Pair` $= \lambda X.\ \lambda Y. \forall Z. (X \to Y \to Z) \to Z$

- `Pair` is essentially a type-level function, called a *type operator*.
- To make sure that these type-level functions are well-typed, we will lift STLC into the type-level!
- The types of types are referred to as *kinds*.

From Types to Kinds



- Note the uninhabited types in this figure!

The Systems of Todays Lecture

λ_ω Monomorphic kinds (,i.e., kinding without quantifiers)

System F_ω Polymorphic kinds

λ_ω
Syntax

- We extend STLC with type operators:

t	$::=$		terms:
	$\mid$	x	variable
	$\mid$	$\lambda x : T.t$	abstraction
	$\mid$	$t\ t$	application
v	$::=$		values:
	$\mid$	$\lambda x : T.t$	abstraction value

T	$::=$		types:
	$\mid$	X	type variable
	$\mid$	$\lambda X :: K.T$	operator abstraction
	$\mid$	$T\ T$	operator application
	$\mid$	$T \to T$	type of functions
Γ	$::=$		contexts:
	$\mid$	$\varnothing$	empty context
	$\mid$	$\Gamma, x : T$	term variable binding
	$\mid$	$\Gamma, X :: K$	type variable binding
K	$::=$		kinds:
	$\mid$	$*$	kind of proper types
	$\mid$	$K \Rightarrow K$	kind of operators

- For conciseness, we will not assume base kinds. But we will kinds later.

Evaluation

$$\boxed{t \longrightarrow t'}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2}\ \text{E-App1} \qquad \frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'}\ \text{E-App2}$$

$$\frac{}{(\lambda x : T.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12}}\ \text{E-AppAbs}$$

Typing and Kinding

$\boxed{\Gamma \vdash t : T}$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \text{T-VAR}$$

$$\frac{\Gamma \vdash T_1 :: * \quad \Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1 . t_2 : T_1 \to T_2}\ \text{T-ABS}$$

$$\frac{\Gamma, t_1 : T_{11} \to T_{12} \quad t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \text{T-APP}$$

$$\frac{\Gamma \vdash t : S \quad S \equiv T \quad \Gamma \vdash T :: *}{\Gamma \vdash t : T}\ \text{T-EQ}$$

$\boxed{\Gamma \vdash T :: K}$

$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K}\ \text{T-TVAR}$$

$$\frac{\Gamma, X :: K_1 \vdash T_2 :: K_2}{\Gamma \vdash \lambda X :: K_1 . T_2 :: K_1 \Rightarrow K_2}\ \text{K-ABS}$$

$$\frac{\Gamma, T_1 :: K_{11} \Rightarrow K_{12} \quad T_2 : K_{11}}{\Gamma \vdash T_1\ T_2 :: K_{12}}\ \text{K-APP}$$

$$\frac{\Gamma \vdash T_1 :: * \quad \Gamma \vdash T_2 :: *}{\Gamma \vdash T_1 \to T_2 :: *}\ \text{K-ARROW}$$

Type Equivalence

Intuition

- In a type checker for STLC, we would implement:

```
typeOf ctxt (TmApp t1 t2) =
  let tyT1 = typeOf ctxt t1
      tyT2 = typeOf ctxt t2
  in case tyT1 of
       (TyArr tyT11 tyT12) ->
         if tyT2 == tyT11
         then tyT2
         else error "type mismatch"
```

Axioms

Congruence

$$\boxed{\Gamma \vdash T \equiv T}$$

$$\frac{}{\Gamma \vdash \texttt{Nat} \equiv \texttt{Nat}}\ \text{QT-NAT}$$

$$\frac{}{\Gamma \vdash \texttt{Bool} \equiv \texttt{Bool}}\ \text{OT-BOOL}$$

$$\frac{\Gamma \vdash T_1 \equiv S_1 \quad \Gamma \vdash T_2 \equiv S_2}{\Gamma \vdash T_1 \to T_2 \equiv S_1 \to S_2}\ \text{OT-ABS}$$

- (Remember, in HM, we would not do this comparison directly.)
- We want to *define* that $\texttt{Nat} \to \texttt{Nat} \equiv (\lambda X.\ X \to X)\ \texttt{Nat}$.

- *Definitional equivalence*:

$$\boxed{S \equiv T}$$

$$\frac{}{T \equiv T}\ \text{Q-Refl} \qquad \frac{T \equiv S}{S \equiv T}\ \text{Q-Symm} \qquad \frac{S \equiv U \quad U \equiv T}{S \equiv T}\ \text{Q-Trans} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \to S_2 \equiv T_1 \to T_2}\ \text{Q-Arrow}$$

$$\frac{S_2 \equiv T_2}{\lambda X :: K_1.S_2 \equiv \lambda X :: K_1.T_2}\ \text{Q-Abs} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1\ S_2 \equiv T_1\ T_2}\ \text{Q-App}$$

$$\frac{}{(\lambda X :: K_{11}.T_{12})\ T_2 \equiv [X \mapsto T_2]T_{12}}\ \text{Q-AppAbs}$$

No more mysteric data types

- We were mumbling about the syntax of these two data type definitions (to encode relations):

(recursive) ADT	`data Nat = O | S Nat`	`type Nat :: Type`

GADT	`data Nat where`
	`  O :: Nat`
	`  S :: Nat -> Nat`
		`type Nat :: Type`

parameterized	`data List a :: Type where`
	`  Nil :: List a`
	`  Cons :: a -> List a -> List a`
		`type List :: Type -> Type`

annotated/indexed	`data Ev :: Nat -> Type where`
	`  EvO :: Ev 'O`
	`  EvSS :: Ev n -> Ev ('S ('S n))`
		`type Ev :: Nat -> Type`

- Now that we know what kinds are and what type operators are, we see that:
  - There is nothing special about their syntax!
  - They just have a different kind!
  - (See code!)
- (Don't worry, we will unlock the rest of the mysteries in a couple of slides.)

F_ω

- F_ω is a combination of System F and λ_ω.
- To directly read System F terms in System F_ω, we abbreviate $\forall X :: *.T$ as $\forall X.T$.

F_ω
Syntax

t	$::=$		terms:
	$\mid$	x	variable
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$$\frac{t_1 \longrightarrow t_1'}{t_1\ [T_2] \longrightarrow t_1'\ [T_2]}\ \text{E-TApp} \qquad \frac{}{(\lambda X :: K_{11}.t_{12})\ [T_2] \longrightarrow [X \mapsto T_2]t_{12}}\ \text{E-TAppTAbs}$$

F_ω
Typing

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$$\frac{\Gamma, X :: K_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda X :: K_1 . t_2 : \forall X :: K_1 . T_2}\ \text{T-TABS} \qquad \frac{\Gamma, t_1 : \forall X :: K_{11} . T_{12} \quad \Gamma \vdash T_2 :: K_{11}}{\Gamma \vdash t_1\ [T_2] : [X \mapsto T_2] T_{12}}\ \text{T-TAPP}$$

$$\frac{\Gamma \vdash t : S \quad S \equiv T \quad \Gamma \vdash T :: *}{\Gamma \vdash t : T}\ \text{T-EQ}$$

F_ω
Kinding

$$\boxed{\Gamma \vdash T :: K}$$

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$$\frac{\Gamma \vdash T_1 :: * \quad \Gamma \vdash T_2 :: *}{\Gamma \vdash T_1 \to T_2 :: *} \text{ K-ARROW}$$

$$\frac{\Gamma, t_1 : T_{11} \to T_{12} \quad t_2 : T_{11}}{\Gamma \vdash T_1\ T_2 :: K_{12}} \text{ K-APP}$$

$$\frac{\Gamma, X :: K_1 \vdash T_2 :: *}{\Gamma \vdash \forall X :: K_1 . T_2 :: *} \text{ K-ALL}$$

$$\boxed{S \equiv T}$$

$$\frac{}{T \equiv T}\ \text{Q-Refl} \qquad \frac{T \equiv S}{S \equiv T}\ \text{Q-Symm} \qquad \frac{S \equiv U \quad U \equiv T}{S \equiv T}\ \text{Q-Trans} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \to S_2 \equiv T_1 \to T_2}\ \text{Q-Arrow}$$

$$\frac{S_2 \equiv T_2}{\forall X :: K_1.S_2 \equiv \forall X :: K_1.T_2}\ \text{Q-All} \qquad \frac{S_2 \equiv T_2}{\lambda X :: K_1.S_2 \equiv \lambda X :: K_1.T_2}\ \text{Q-Abs} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1\ S_2 \equiv T_1\ T_2}\ \text{Q-App}$$

$$\frac{}{(\lambda X :: K_{11}.T_2)\ T_{11} \equiv [X \mapsto T_{11}]T_2}\ \text{Q-AppAbs}$$

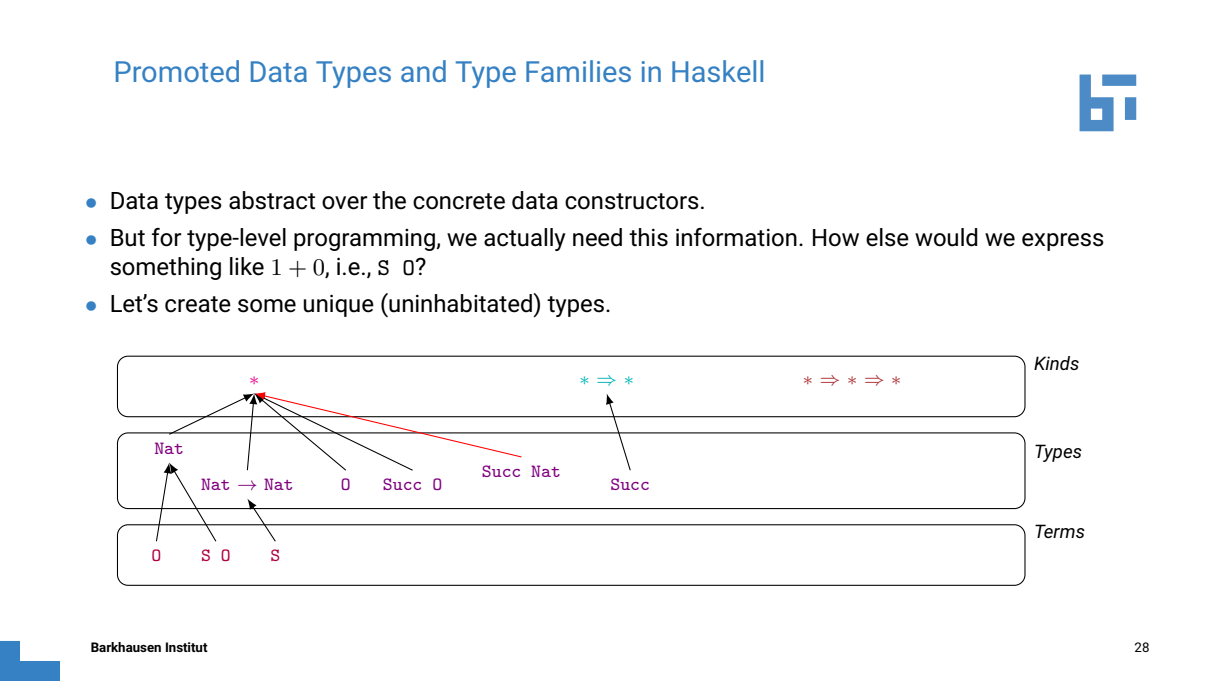
Dependently-typed Programming in Haskell

- We have worked so hard this semester to reach this point. Congratulations!
- Let's see the Curry-Howard Correspondence in dependently-typed Haskell in action and prove some theorems! (See code!)

Language Extensions in Haskell

- To perform type-level programming, we need to be able to
- create new types that are well-kinded and
- type-level functions that are total.

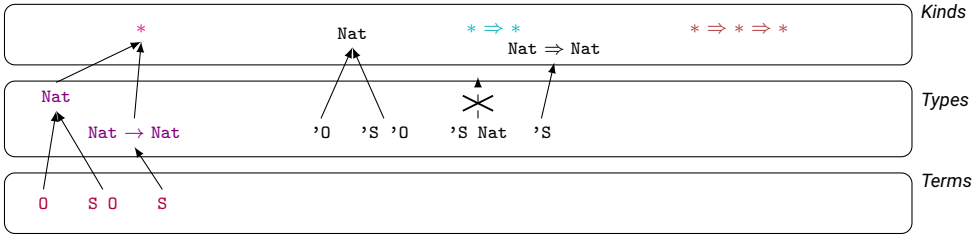
Promoted Data Types and Type Families in Haskell



- Data types abstract over the concrete data constructors.
- But for type-level programming, we actually need this information. How else would we express something like $1 + 0$, i.e., `S 0`?
- Let's create some unique (uninhabitated) types.

Promoted Data Types and Type Families in Haskell

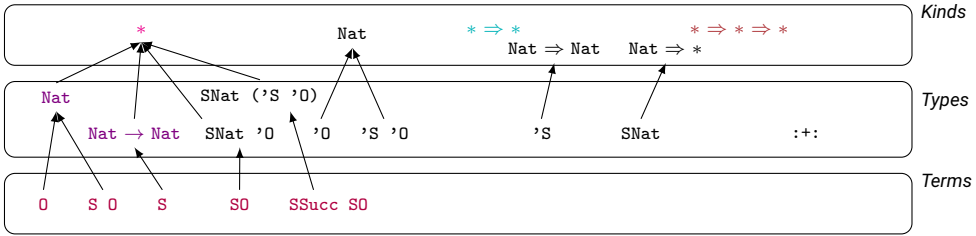
- The language extension `{# LANGUAGE DataKinds #}` ...
- promotes (data) types to the kind-level and
- constructors to the type-level.[1]



[1] Brent A Yorgey et al. "Giving Haskell a promotion". In: *Proceedings of the 8th ACM SIGPLAN Workshop on Types in Language Design and Implementation*. 2012, pp. 53–66.

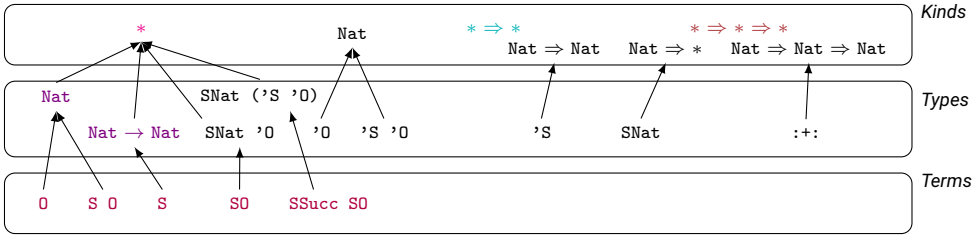
Promoted Data Types and Type Families in Haskell

- Remember: proofs are programs!
- Hence, we need to connect terms and types.
- We need a term-level representation for every type: `'O`, `'S 'O`, `'S ('S 'O)`, ...
- We do this by creating a *singleton* data type, i.e., a type that is index by the types of kind `Nat`.



Promoted Data Types and Type Families in Haskell

- Language extension: `{# LANGUAGE TypeFamilies #}`
- Promoting *closed* data types, i.e., ADTs and GADTs, enables
- closed type families.[1]
- (Where “closed” here means: not extensible.)



[1] Richard A Eisenberg et al. “Closed type families with overlapping equations”. In: *ACM SIGPLAN Notices* (2014).

Further Reading on Dependently-Typed Programming in Haskell

Hasochism describes the benefits and pain points of dependent programming in Haskell.[1]

Singletons a library-based approach to create singleton types and promote term-level functions to the type-level.[2]

Stitch describes a very elegant implementation of a language and its compiler in Haskell that promotes all the type-checking of the implemented language into GHCs type system.[3]

DH Make sure to stay tuned for Dependently-typed Haskell.[4]

[1] Sam Lindley and Conor McBride. "Hasochism: The Pleasure and Pain of Dependently Typed Haskell Programming". In: *SIGPLAN Not.* (2013).

[2] Richard A Eisenberg and Stephanie Weirich. "Dependently typed programming with singletons". In: *ACM SIGPLAN Notices* (2012).

[3] Richard A Eisenberg. "Stitch: the sound type-indexed type checker (functional pearl)". In: *Proceedings of the 13th ACM SIGPLAN International Symposium on Haskell*. 2020, pp. 39–53.

[4] Stephanie Weirich, Justin Hsu, and Richard A Eisenberg. "Towards dependently typed Haskell: System FC with kind equality". In: *Proceedings of the 18th ACM SIGPLAN International Conference on Functional Programming, ICFP*.

What we have learned

- This lecture marks the end of our path towards understanding the type system of Haskell.
- We have learned that
  - kinds are types of the types and
  - most importantly that proofs are just programs (that produce a proof witness for their result type, i.e., their proposition)!
- Type-level functions provide the possibility to
  - express powerful propositions and
  - in Haskell translate them into adding equations for the HM type inference algorithm.