FCPLCD – Exercise 4

When working in Coq/Lean so far, we've used tactics to avoid the details of how proofs are represented. But last lecture you learned that propositions are types and proofs are terms (according to the Curry-Howard Isomorphism). In this exercise sheet, you'll construct proofs explicitly as terms. *That is, for the rest of this exercise sheet the use of tactics is forbidden!*

Many of the exercises are sourced from Exercise Sheet 1 (the one using Holbert). So if you ever get stuck on a proof or definition you should be able to find solutions in that exercise sheet.

1 Propositional Logic

1.1 Connectives

Define the following propositional connectives without using Coq/Lean's built-in implementations: $\top$, $\bot$, $\wedge$, $\vee$, $\leftrightarrow$, $\neg$

1.2 Proofs

Use your connectives to define the following types and write the according proof term:

- $a \wedge b \rightarrow a \vee b$
- $a \vee b \rightarrow b \vee a$
- $(a \rightarrow b \rightarrow c) \rightarrow (a \wedge b \rightarrow c)$
- $(a \wedge b \rightarrow c) \rightarrow (a \rightarrow b \rightarrow c)$

2 First-Order Logic

For the sake of simplicity, we use Coq's/Lean's universal quantifier and focus on natural numbers and equality.

2.1 Natural Arithmetic

Define a type of natural numbers and its addition and multiplication functions.

2.2 Equality

Define the equality relation on natural numbers. This relation bascially states the following: "For any natural number x, the equality relation is indexed by exactly this natural number x." Subsequently, define the following types and write their according proof terms (where $=$ denotes to aforementioned equality relation):

Symmetry $\forall x, y : (x = y) \to (y = x)$

Transitivity $\forall x, y, z : (x = y) \to (y = z) \to (x = z)$

2.3 Definitional Equality

The implementations of the two functions, addition and multiplication, are defined over the following equalities:

$\mathbf{add_0}$ :: $\forall x : 0 + x = x$

$\mathbf{add_S}$ :: $\forall x, y : (succ\ x) + y = succ\ (x + y)$

$\mathbf{mul_0}$ :: $\forall x : 0 \times x = 0$

$\mathbf{mul_S}$ :: $\forall x, y : (succ\ x) \times y = y + (x \times y)$

Define these types and write the according proof terms.
Note: If you can't solve the last theorem trivially, adjust your definition of multiplication so that it becomes trivial.

2.4 Congruence

Prove the following theorems:

$\mathbf{cong_S}$ :: $\forall x, y : (x = y) \to (succ\ x = succ\ y)$

$\mathbf{cong_{add}}$ :: $\forall x_1, x_2, y_1, y_2 : (x_1 = x_2) \to (y_1 = y_2) \to (x_1 + y_1 = x_2 + y_2)$

2.5 Holbert Revisited

Define the following types and write their according proof terms, which you've proven already in Exercise Sheet 1. Note that the proofs need not be identical to your previous solutions.

right_identity$_+$ $\forall x : (x + 0) = x$

left_identity$_\times$ $\forall x : (1 \times x) = x$

3 Bonus Exercise

Formalize the relations G_1, L_4 and R_{14} from Exercise Sheet 1 and prove the accompanying theorems:

- $R_{14}\ 3$
- $\neg(G_1\ 0) \to \neg(R_{14}\ 0)$