

Foundations of Certified Programming Language and Compiler Design

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Outline



Lecture	Logic Coq/Lean	Formalisms	PL Haskell
1	Propositional and first-order logic		
2			Functional programming
3		Syntax and Semantics	
4			The untyped lambda calculus
5		Types	
6			The typed lambda calculus
7			Polymorphism
8		Curry-Howard	
9			Higher-order types
10			Dependent types



There are two reasons for studying logic:

1. For proving theorems in Coq.
2. For understanding the deep connection between logic and programming languages.



Syntax:

A	$::=$	formulas:
	$P \mid Q \mid R$	propositional variables
	$A \Rightarrow A$	implication



- Consider the following logical formula of propositions P , Q and R :

$$(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow R) \Rightarrow (P \Rightarrow R))$$

- There exist two approaches to prove this formula.

¹Alfred Tarski. "The semantic conception of truth: and the foundations of semantics". In: *Philosophy and phenomenological research* 4.3 (1944), pp. 341–376.

²Arend Heyting. *Intuitionism: an introduction*. Vol. 41. Elsevier, 1966.



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denotational¹

- Denotation: $t \triangleq \text{true}$ and $f \triangleq \text{false}$
- If the value of the complete formula is t in all cases then the formula is said to be *valid*.
- This is referred to as *classical logic*.
- Example: truth table

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$P \Rightarrow R$	$(Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$	
f	f	f	t	t	t	t	t
f	f	t	t	t	t	t	t
f	t	f	t	f	t	t	t
f	t	t	t	t	t	t	t
t	f	f	f	t	f	f	t
t	f	t	f	t	t	t	t
t	t	f	t	f	f	t	t
t	t	t	t	t	t	t	t



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constructive²

- Replaces the question "Is P true?" with "What are the proofs of P ?"
- A proof for $P \Rightarrow Q$ is the process of constructing a proof of Q from a proof of P .
- This is called *intuitionistic logic* and was proposed in the early 20th century by Brouwer.

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- Natural deduction is a formalism for proofs due to Gentzen¹.

Definition (Formulas/Propositions)

X	\in	\mathcal{X}	fixed countable infinite set of propositional variables
A	$::=$	X	propositional variables
		$A_1 \Rightarrow A_2$	implication
		$A_1 \wedge A_2$	conjunction
		\top	truth
		$A_1 \vee A_2$	disjunction
		\perp	falsity
		$\neg A_1$	negation

¹Gerhard Gentzen. "Untersuchungen Über Das Logische Schließen. I.". In: *Mathematische Zeitschrift* 35 (1935), pp. 176–210.



Definition (Context)

A context

$$\Gamma = A_1, \dots, A_n$$

is a list of n propositions.

Interpretation:

- The comma (,) in the specification of a context can be read as a “meta” conjunction.
- Do not confuse it with the logical conjunction \wedge that appears in formulas!



Definition (Sequent/Judgement)

A *sequent* is a pair

$$\Gamma \vdash A$$

that consists of a context and a proposition.

Interpretation:

- The \vdash in the specification of a judgement can be read as a “meta” implication.
- Do not confuse it with the logical implication \Rightarrow that appears in formulas!



Definition (Inference Rule)

An inference rule

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n}{\Gamma \vdash A}$$

consists of n sequents as premises and a concluding sequent.



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deductively from the proofs for each of the premises, we can deduce the conclusion

inductively for a proof of the conclusion, we need to construct proofs for each of the premises

The NJ System

Rules for Intuitionistic Natural Deduction



$$\frac{}{\Gamma, A, \Gamma' \vdash A} (ax)$$

The NJ System

Rules for Intuitionistic Natural Deduction



Elimination rules

$$\frac{}{\Gamma, A, \Gamma' \vdash A} (ax)$$

Introduction rules

The NJ System

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explosion principle

Introduction rules

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explosion principle

(incarnation of explosion principle)

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explosion principle

principle of non-contradiction

(incarnation of explosion principle)

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explosion principle

principle of non-contradiction

(incarnation of explosion principle)

Consistency: There is at least one formula (\perp) that is not provable.



- Intuitionistic logic:

Intuitionist logic allows to extract so-called witnesses from proofs, e.g., from a proof of $A_1 \vee A_2$ we know which of the two propositions A_1 and A_2 actually holds.

- Classical logic:

$$\overline{\vdash \neg A \vee A} \text{ (excluded middle)}$$



Proofs are defined inductively:

A **sequent** $\Gamma \vdash A$ is *provable* when it is the conclusion of a proof.



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A **proof** is an inference rule (according to the definition above)

$$\frac{\frac{\pi_1}{\Gamma_1 \vdash A_1} \quad \dots \quad \frac{\pi_n}{\Gamma_n \vdash A_n}}{\Gamma \vdash A}$$

where all the premises are proofs themselves.

Examples



Proposition: $(A_1 \wedge A_2) \Rightarrow (A_1 \vee A_2)$

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$$\frac{\frac{\frac{}{A_1 \wedge A_2 \vdash A_1 \wedge A_2} (ax)}{A_1 \wedge A_2 \vdash A_1} (\wedge_E)}{A_1 \wedge A_2 \vdash A_1 \vee A_2} (\vee_I^l) \quad \frac{}{\vdash (A_1 \wedge A_2) \Rightarrow (A_1 \vee A_2)} (\Rightarrow_I)$$

Proposition: $(A_1 \vee A_2) \Rightarrow (A_2 \vee A_1)$

$$\frac{\frac{}{A_1 \vee A_2 \vdash A_1 \vee A_2} (ax)}{\frac{\frac{\frac{}{A_1 \vee A_2, A_2 \vdash A_2} (ax)}{A_1 \vee A_2, A_2 \vdash A_2 \vee A_1} (\vee_I^l) \quad A_1 \vee A_2, A_1 \vdash A_2 \vee A_1}{A_1 \vee A_2 \vdash A_2 \vee A_1} (\vee_E)}{\vdash (A_1 \vee A_2) \Rightarrow (A_2 \vee A_1)} (\Rightarrow_I)$$

Examples



Proposition: $(A_1 \wedge A_2) \Rightarrow (A_1 \vee A_2)$

$$\frac{\frac{\frac{}{A_1 \wedge A_2 \vdash A_1 \wedge A_2} (ax)}{A_1 \wedge A_2 \vdash A_1} (\wedge_E)}{A_1 \wedge A_2 \vdash A_1 \vee A_2} (\vee_I^l) \quad \frac{}{\vdash (A_1 \wedge A_2) \Rightarrow (A_1 \vee A_2)} (\Rightarrow_I)$$

Proposition: $(A_1 \vee A_2) \Rightarrow (A_2 \vee A_1)$

$$\frac{\frac{}{A_1 \vee A_2 \vdash A_1 \vee A_2} (ax) \quad \frac{\frac{}{A_1 \vee A_2, A_2 \vdash A_2} (ax)}{A_1 \vee A_2, A_2 \vdash A_2 \vee A_1} (\vee_I^l) \quad \frac{A_1 \vee A_2, A_1 \vdash A_1}{A_1 \vee A_2, A_1 \vdash A_2 \vee A_1} (\vee_I^r)}{A_1 \vee A_2 \vdash A_2 \vee A_1} (\vee_E) \quad \frac{}{\vdash (A_1 \vee A_2) \Rightarrow (A_2 \vee A_1)} (\Rightarrow_I)$$

Examples



Proposition: $(A_1 \wedge A_2) \Rightarrow (A_1 \vee A_2)$

$$\frac{\frac{\frac{}{A_1 \wedge A_2 \vdash A_1 \wedge A_2} (ax)}{A_1 \wedge A_2 \vdash A_1} (\wedge_E)}{A_1 \wedge A_2 \vdash A_1 \vee A_2} (\vee_I^l) \quad \frac{}{\vdash (A_1 \wedge A_2) \Rightarrow (A_1 \vee A_2)} (\Rightarrow_I)$$

Proposition: $(A_1 \vee A_2) \Rightarrow (A_2 \vee A_1)$

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What we have learned



- We know propositional logic and
- in particular natural deduction as a framework for proving propositions.
- We understand the difference between classical and intuitionistic logic
- and why it needs to be the foundation for theorem provers.