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Foundations of Certified Programming Language and Compiler Design

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Theorem Proving at TUD

- **There is an internship (“Komplexpraktikum”) about LEAN at TUD!**
- Please have a look at https://iccl.inf.tu-dresden.de/web/Theorem_Proving_with_LEAN_(WS2023)
- Meet Stephan and Lukas and join their research!

Formally-verified your dreams!

Outline

Lecture	Logic Coq/Lean	Formalisms	PL Haskell
1	Propositional and first-order logic		
2			Functional programming
3		Syntax and Semantics	
4			The untyped lambda calculus
5		Types	
6			The typed lambda calculus
7			Polymorphism
8		Curry-Howard	
9			Higher-order types
10			Dependent types

Goals

Let’s enter a more powerful logic system that allows us to express

- (in-)equalities and
- arithmetics.

Motivating examples

- Natural numbers with addition:

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  $0 + 1 = 1$

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- Verification of the correctness of a compiler transformation `trans` for a program p:

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- Verification of the correctness of a compiler transformation $\texttt{trans}$ for a program p:

  $\texttt{exec}(\texttt{trans}\ p) = \texttt{exec}\ p$

From Propositional to First-Order Logic

Syntax A ::=		formulas/propositions:
$\mid$	X	propositional variables
$\mid$	$A \Rightarrow A$	implication
$\mid$	$A \wedge A$	conjunction
$\mid$	$\top$	truth
$\mid$	$A \vee A$	disjunction
$\mid$	$\bot$	falsity
$\mid$	$\neg A$	negation

Examples: Propositional Logic

$$X_1 \vee X_2 \Rightarrow X_3$$

From Propositional to First-Order Logic

Syntax A	$::=$	formulas/propositions:
$\mid$	$P(t_1, \ldots, t_n)$	predicates on terms
$\mid$	$A \Rightarrow A$	implication
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$\mid$	$\top$	truth
$\mid$	$A \vee A$	disjunction
$\mid$	$\bot$	falsity
$\mid$	$\neg A$	negation
$\mid$	$\forall x. A$	universally quantified term variables
$\mid$	$\exists x. A$	existentially quantified term variables

Examples:

Propositional Logic	First-Order Logic
$X_1 \vee X_2 \Rightarrow X_3$	$\forall x. \exists y. (x \times y = 1 \wedge y \times x = 1)$

Predicates

Syntax P	$::=$	predicates:
	$\mid \quad P \in \mathcal{P}$	predicate symbol

- ... with $a :: \mathcal{P} \to \mathbb{N}$ defining the arity of predicates.
- Let $\mathcal{P} = \mathcal{X}$ where $\mathcal{X}$ is the set of propositional variables such that $\forall X \in \mathcal{X}.a(X) = 0$,
- then

Propositional formula	First-order formula
$X \vee \neg Y$	$X() \vee \neg Y()$

Terms

Syntax	t	$::=$	terms:
		$\mid \quad x$	variables
		$\mid \quad f(t_1, \ldots, t_n)$	function application
	f	$::= \quad f \in \Sigma$	function symbols

- ... with an associated arity: $a :: \Sigma \to \mathbb{N}$
- and $\mathcal{T}$ as the smallest set of terms.

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Example:

Predicates	Terms	Formula
$\mathcal{P} = \{=: 2\}$	$\Sigma = \{\times : 2, 1 : 0\}$	$\forall x. \exists y. (x \times y = 1 \wedge y \times x = 1)$

Bound and Free Variables

Syntax	A	$::=$	$\dots$	formulas/propositions:
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- We say: “Term variable x is *bound* in formula A.

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- A variable that is not bound is *free*.
- A formula A is *closed* when it does not contain free variables, i.e., $FV(A) = \emptyset$.

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Definitions and Notation

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- Notation: $A(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are free variable of A

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- Notation:

  $A(t_1, \dots t_n)$ instead of $A[t_1/x_1, \dots, t_n/x_n]$

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consistent	when $\perp$ is not provable in the theory
complete	when for every formula A, **either** A **or** $\neg A$ is provable in the theory
decidable	when there is an algorithm that decides whether a given formula is provable or not

The decidable theory of equality with uninterpreted functions

$$\Theta := (\mathcal{P}, \Sigma) \quad \text{where} \quad \begin{aligned} \mathcal{P} &= \{= :\ 2\} \\ \Sigma &= \{f\ :\ n \mid f \in \text{Strings}, n \in \mathbb{N}\} \end{aligned}$$

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$$\frac{P(A_1, \dots, A_m) \quad A_1 = A_1' \quad \dots \quad A_m = A_m'}{P(A_1', \dots, A_m')} \ \text{Predicate Congruence}$$

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$$\frac{0 = S(x)}{\perp}\ (\mathsf{ax}_{=\perp})$$

Presburger Arithmetic

- Axiomatizes the addition over natural numbers.
- Theory of equality over the signature $\Sigma = \{0 : 0, S : 1, + : 2\}$
- with the following axioms:

$$\frac{0 = S(x)}{\perp}\ (\mathsf{ax}_{=\perp}) \qquad \frac{S(x) = S(y)}{x = y}\ (S_{\mathsf{inj}})$$

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- with the following axioms:

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- Axiomatizes the addition over natural numbers.
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Recurrence Principle

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$$\frac{x = 0 \vdash x + 0 = x \quad x = S(y) \vdash S(y) + 0 = S(y)}{x + 0 = x} \text{ (cases)}$$

Recurrence Principle

$$\frac{\pi_0 \qquad \pi_S}{\forall x . x + 0 = x} \text{ (cases)}$$

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Recurrence Principle

$$\pi_S \quad :=$$

$$\pi_0 \quad :=$$

$$\pi_{\text{Left Identity}} \quad := \quad \frac{\pi_0 \qquad \pi_S}{\forall x. x + 0 = x} \text{ (cases)}$$

```
fn pi_left_identity(x:Nat) { }
```

Recurrence Principle

$$\pi_S \quad :=$$

$$\pi_0 \quad := \quad 0 + 0 = 0 \qquad =: \quad (x + 0 = x)[0/x]$$

$$\pi_{\text{Left Identity}} \quad := \quad \frac{\pi_0 \qquad \pi_S}{\forall x. x + 0 = x} \text{ (cases)}$$

Recurrence Principle

$$\pi_S \quad :=$$

$$\pi_0 \quad := \quad \frac{}{0 + 0 = 0}\ (\mathsf{ax}_{+1})$$

$$\pi_{\text{Left Identity}} \quad := \quad \frac{\pi_0 \qquad \pi_S}{\forall x. x + 0 = x}\ (\text{cases})$$

Recurrence Principle

$$\pi_S \quad := \quad \frac{\forall x. S(x) + 0 = S(x)}{S(x) + 0 = S(x)} \; (\forall_E)$$

$$\pi_0 \quad := \quad \frac{}{0 + 0 = 0} \; (\mathrm{ax}_{+1})$$

$$\pi_{\text{Left Identity}} \quad := \quad \frac{\pi_0 \qquad \pi_S}{\forall x. x + 0 = x} \; (\text{cases})$$

Recurrence Principle

$$\pi_S := \dfrac{\forall x.S(x)+0=S(x+0) \qquad \forall x.S(x+0)=S(x)}{\dfrac{\forall x.S(x)+0=S(x)}{S(x)+0=S(x)}\,(\forall_E)}\,(=_{\text{transitivity}})$$

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Recurrence Principle

$$\pi_S := \dfrac{\dfrac{}{\forall x. S(x) + 0 = S(x+0)}\,(\mathsf{ax}_{+_2}) \quad \dfrac{S = S \qquad \forall x. x + 0 = x}{\forall x. S(x+0) = S(x)}\,(\text{fun cong})}{\dfrac{\forall x. S(x) + 0 = S(x)}{S(x) + 0 = S(x)}\,(\forall_E)}\,(=_{\text{transitivity}})$$

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Recurrence Principle

$$\pi_S \quad := \quad \dfrac{\dfrac{\dfrac{}{\forall x. S(x) + 0 = S(x+0)}\ (\text{ax}_{+_2}) \qquad \dfrac{\dfrac{}{S = S}\ (\text{ax}_{S-\text{cong}}) \qquad \forall x. x + 0 = x}{\forall x. S(x+0) = S(x)}\ (\text{fun cong})}{\dfrac{\forall x. S(x) + 0 = S(x)}{S(x) + 0 = S(x)}\ (\forall_E)}\ (=_{\text{transitivity}})$$

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$$\pi_0 := \dfrac{}{0+0=0}\,(\mathsf{ax}_{+_1})$$

$$f(n) := \begin{cases} \end{cases}$$

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$$\pi_0 := \dfrac{}{0 + 0 = 0}\,(\mathsf{ax}_{+_1})$$

$$f(n) := \begin{cases} \pi_0 & \text{if } n \text{ is } 0 \\ & \end{cases}$$

$$\pi_{\text{Left Identity}} := \dfrac{\pi_0 \qquad \pi_S}{\forall x. x + 0 = x}\,(\text{cases})$$

Recurrence Principle

$$\pi_S := \dfrac{\dfrac{}{\forall x. S(x) + 0 = S(x+0)}\,(\text{ax}_{+_2}) \qquad \dfrac{\dfrac{}{S = S}\,(\text{ax}_{S-\text{cong}}) \qquad \dfrac{\forall x. f(x)}{\forall x. x + 0 = x}}{\forall x. S(x+0) = S(x)}\,(\text{fun cong})}{\dfrac{\forall x. S(x) + 0 = S(x)}{S(x) + 0 = S(x)}\,(\forall_E)}\,(=_{\text{transitivity}})$$

$$\pi_0 := \dfrac{}{0 + 0 = 0}\,(\text{ax}_{+_1})$$

$$f(n) := \begin{cases} \pi_0 & \text{if } n \text{ is } 0 \\ \pi_S[y/x] & \text{if } n \text{ is } S(y) \end{cases}$$

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Recurrence Principle

$$\pi_S \quad := \quad \dfrac{\dfrac{\dfrac{}{\forall x. S(x) + 0 = S(x+0)}\ (\mathsf{ax}_{+_2}) \qquad \dfrac{\dfrac{}{S = S}\ (\mathsf{ax}_{S-\mathsf{cong}}) \qquad \dfrac{\forall x. f(x)}{\forall x. x + 0 = x}}{\forall x. S(x+0) = S(x)}\ (\text{fun cong})}{\dfrac{\forall x. S(x) + 0 = S(x)}{S(x) + 0 = S(x)}\ (\forall_E)}\ (=_{\text{transitivity}})$$

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```
fn f(x:Nat) {
  match x {
    0    => pi_0,
    S(y) => pi_S(y)
  }
}
```

Recurrence Principle

$$\pi_S := \dfrac{\dfrac{\dfrac{}{\forall x. S(x) + 0 = S(x+0)}\ (\mathsf{ax}_{+_2}) \qquad \dfrac{\dfrac{}{S = S}\ (\mathsf{ax}_{S-\mathsf{cong}}) \qquad \dfrac{\forall x. f(x)}{\forall x. x + 0 = x}}{\forall x. S(x+0) = S(x)}\ (\text{fun cong})}{\forall x. S(x) + 0 = S(x)}\ (=_{\text{transitivity}})}{S(x) + 0 = S(x)}\ (\forall_E)$$

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axiom scheme: $\boxed{\dfrac{A(0) \qquad (\forall x. A(x) \rightarrow A(S(x))}{\forall x. A\, x}\ (\text{Rec})}$

Induction Principle

$$\pi_S \quad := \quad \cfrac{\cfrac{}{S(x)+0 = S(x+0)}\;(\mathsf{ax}_{+_2}) \qquad \cfrac{\cfrac{}{S = S}\;(\mathsf{ax}_{S-\mathsf{cong}}) \qquad x+0=x \vdash x+0=x}{x+0=x \vdash S(x+0) = S(x)}\;(\text{fun cong})}{\cfrac{x+0=x \vdash S(x)+0 = S(x)}{x+0=x \Rightarrow S(x)+0=S(x)}\;(\Rightarrow_I)}\;(=_{\text{transitivity}})$$

$$\pi_0 \quad := \quad \cfrac{}{0+0=0}\;(\mathsf{ax}_{+_0})$$

$$\pi_{\text{Left Identity}} \quad := \quad \cfrac{\pi_0 \qquad \pi_S}{\forall x. x+0 = x}\;(\textbf{Ind})$$

$$\boxed{\cfrac{A(0) \qquad (\forall x. A(x) \Rightarrow A(S(x)))}{\forall x. A\ x}\;(\text{Ind})}$$

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$$\boxed{\dfrac{A(0) \qquad (\forall x. A(x) \Rightarrow A(S(x)))}{\forall x. A\ x}\ (\text{Ind})}$$

Induction Principle

Induction Hypothesis

$$\pi_S \quad := \quad \dfrac{\dfrac{}{S(x)+0 = S(x+0)}\,(\mathsf{ax}_{+_2}) \qquad \dfrac{\dfrac{}{S = S}\,(\mathsf{ax}_{S-\mathsf{cong}}) \quad \dfrac{}{x+0=x \vdash x+0=x}\,(\mathsf{ax})}{x+0=x \vdash S(x+0)=S(x)}\,(\text{fun cong})}{\dfrac{x+0=x \vdash S(x)+0=S(x)}{x+0=x \Rightarrow S(x)+0=S(x)}\,(\Rightarrow_I)}\,(=_{\text{transitivity}})$$

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Induction vs. Cases

$$\frac{x = 0 \vdash A(x) \qquad (\forall x. A(x) \Rightarrow A(S(x))}{\forall x. A\ x}\ \text{(Ind)} \qquad \frac{x = 0 \vdash A(x) \quad x = S(y) \vdash A(S(y))}{\forall x. A(x)}\ \text{(cases)}$$

- Important point to remember:
  - Cases does not provide a induction hypothesis, i.e.,
  - it does not cover recursion!

Induction vs. Recurrence

$$\frac{A(0) \qquad (\forall x. A(x) \Rightarrow A(S(x))}{\forall x. A\ x}\ \text{(Ind)} \qquad\qquad \frac{A(0) \qquad (\forall x. A(x) \rightarrow A(S(x))}{\forall x. A\ x}\ \text{(Rec)}$$

- Look what just happened:
  - We used a proof just like a function!
  - There seems to be a deeper connection between implication and a function (type)?
- Let's explore this connection in this lecture!