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Foundations of Certified Programming Language and Compiler Design

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Outline

Lecture	Logic	Formalisms	PL
1	Propositional and first-order logic		
2			Functional programming
3		Syntax and Semantics	
4			The untyped lambda calculus
5		Types	
6			The typed lambda calculus
7			Polymorphism
8		Curry-Howard	
9			Higher-order types
10			Dependent types

The simply typed lambda calculus

Syntax:

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t	$::=$		terms:
	$\mid$	x	variable
	$\mid$	$\lambda x:T.t$	abstraction
	$\mid$	$t\ t$	application
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Γ	$::=$		contexts:
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	$\mid$	$\Gamma, x : T$	term variable binding

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$$\boxed{t \longrightarrow t'}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2}\ \text{E-App1}$$

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$$\boxed{\Gamma \vdash t : T}$$

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$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \quad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \text{T-APP}$$

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$$\dfrac{\dfrac{\dfrac{x : \texttt{Bool} \in x : \texttt{Bool}}{x : \texttt{Bool} \vdash x : \texttt{Bool}}\ \text{T-Var}}{\lambda x : \texttt{Bool}.x : \texttt{Bool} \to \texttt{Bool}}\ \text{T-Abs} \qquad \dfrac{}{\vdash \texttt{true} : \texttt{Bool}}\ \text{T-True}}{\vdash (\lambda x : \texttt{Bool}.x)\ \texttt{true} : \texttt{Bool}}\ \text{T-App}$$

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- Check: Show the derivation tree for
  $f : \texttt{Bool} \to \texttt{Bool} \vdash f\ (\texttt{if false then true else false}) : \texttt{Bool}$

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$$\Gamma = \quad f : \texttt{Bool} \to \texttt{Bool} \to \texttt{Bool},\ x : \texttt{Bool}, y : \texttt{Bool} \qquad f : \texttt{Nat} \to \texttt{Nat} \to \texttt{Bool},\ x : \texttt{Nat}, y : \texttt{Nat} \qquad f : \texttt{T} \to \texttt{T} \to \texttt{Bool},\ x : \texttt{T}, y : \texttt{T}$$

Properties

Lemmas:

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Inversion of the typing relation.

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What is the type of $(\lambda x.x\ x)\ (\lambda x.x\ x)$?

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Theorem (Preservation)

If $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$

Theorem (Normalization)

If $\vdash t : T$, then t is normalizable.

What is the type of $(\lambda x.x\ x)\ (\lambda x.x\ x)$?

Extensions

- Currently, we cannot implement the STLC because we are missing the base case for our types.
- Let’s study extensions:

Extensions
Base Types

- Base types: `Nat`, `Bool`, `String`, `Float`, ...

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  New syntactic forms:

T	$::=$	...	types:
	$\mid$	`A`	base type

- Consider: $\lambda x : A.\ x : A \to A$
- We could assume that $A = \texttt{Nat}$ is some number: $(\lambda x : A.\ x : A \to A)\ 5$

Extensions

- But where do these values come from and what actually is `Nat`?
- We do not want to add unknown values/types?!
- We would like to have a closed system to reason about it.

Unit

New syntactic forms:

Unit

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	`unit`	constant `unit`		$\mid$	`unit`	constant `unit`

T	$::=$	$\ldots$	types:
	$\mid$	`Unit`	unit type

Unit

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	`unit`	constant `unit`		$\mid$	`unit`	constant `unit`

T	$::=$	$\ldots$	types:
	$\mid$	`Unit`	unit type

New Typing Rules:

Unit

New syntactic forms:

t	$::=$	$\ldots$	terms:		
	$\mid$	`unit`	constant `unit`		
v	$::=$	$\ldots$	values:		
	$\mid$	`unit`	constant `unit`		

T	$::=$	$\ldots$	types:
	$\mid$	`Unit`	unit type

New Typing Rules:

$$\boxed{\Gamma \vdash t : T}$$

$$\frac{}{\texttt{unit} : \texttt{Unit}}\ \text{T-Unit}$$

Ascription

- Document your code . . . with types!

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\begin{array}{lcll}
t & ::= & \ldots & \text{terms:} \\
 & | & t \text{ as } T & \text{ascription}
\end{array}
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New Evaluation Rules:

$$t \longrightarrow t'$$

$$\frac{}{v_1 \text{ as } T_1 \longrightarrow v_1} \text{ E-AscribE1}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1 \text{ as } T_1 \longrightarrow t_1' \text{ as } T_1} \text{ E-AscribE2}$$

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New Evaluation Rules:

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New Typing Rules:

$$\Gamma \vdash t : T$$

$$\frac{\Gamma \vdash t_1 : T_1}{\Gamma \vdash t_1 \text{ as } T_1 : T_1} \quad \text{T-AScribe}$$

Product Types a.k.a. Pairs a.k.a. Tuples (with two elements)

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New syntactic forms:

t	$::=$	$\ldots$	terms:
	$\mid$	$\{t, t\}$	pair
	$\mid$	$t.1$	first projection
	$\mid$	$t.2$	second projection

v	$::=$	$\ldots$	values:
	$\mid$	$\{v, v\}$	pair

T	$::=$	$\ldots$	types:
	$\mid$	$T_1 \times T_2$	product type

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	$\mid$	$t.2$	second projection	T	$::=$	$\ldots$	types:
					$\mid$	$T_1 \times T_2$	product type

New Evaluation Rules:

$t \longrightarrow t'$

$$\frac{}{\{v_1, v_2\}.1 \longrightarrow v_1}\ \text{E-PairBeta1}$$

$$\frac{}{\{v_1, v_2\}.2 \longrightarrow v_2}\ \text{E-PairBeta2}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.1 \longrightarrow t_1'.1}\ \text{E-Proj1}$$

$$\frac{t_2 \longrightarrow t_2'}{t_2.2 \longrightarrow t_2'.2}\ \text{E-Proj2}$$

$$\frac{t_1 \longrightarrow t_1'}{\{t_1, t_2\} \longrightarrow \{t_1', t_2\}}\ \text{E-Pair1}$$

$$\frac{t_2 \longrightarrow t_2'}{\{v_1, t_2\} \longrightarrow \{v_1, t_2'\}}\ \text{E-Pair2}$$

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\end{array}
\qquad
\begin{array}{lcll}
v & ::= & \ldots & \text{values:} \\
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T & ::= & \ldots & \text{types:} \\
 & | & T_1 \times T_2 & \text{product type}
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New Typing Rules:

$$\boxed{\Gamma \vdash t : T}$$

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \{t_1, t_2\} : T_1 \times T_2} \text{ T-PAIR}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash t.1 : T_1} \text{ T-PROJ1}$$

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash t.2 : T_2} \text{ T-PROJ2}$$

Sum Types

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t	$::=$	$\ldots$	terms:
	$\mid$	$\texttt{inl}\ t$	tagging (left)
	$\mid$	$\texttt{inr}\ t$	tagging (right)
	$\mid$	$\texttt{case}\ t\ \texttt{of}\ \texttt{inl}\ x \Rightarrow t \mid \texttt{inr}\ x \Rightarrow t$	case

v	$::=$	$\ldots$	values:
	$\mid$	$\texttt{inl}\ v$	tagged value (left)
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T	$::=$	$\ldots$	types:
	$\mid$	$T + T$	sum type

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Sum Types

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$$
\begin{array}{rcll}
v & ::= & \ldots & \text{values:} \\
 & | & \texttt{inl}\ v & \text{tagged value (left)} \\
 & | & \texttt{inr}\ v & \text{tagged value (right)}
\end{array}
$$

$$
\begin{array}{rcll}
T & ::= & \ldots & \text{types:} \\
 & | & T + T & \text{sum type}
\end{array}
$$

New Evaluation Rules:

$$\boxed{t \longrightarrow t'}$$

$$\frac{t \longrightarrow t'}{\texttt{inl}\ t \longrightarrow \texttt{inl}\ t'}\ \text{E-InL} \qquad \frac{t \longrightarrow t'}{\texttt{inr}\ t \longrightarrow \texttt{inr}\ t'}\ \text{E-InR}$$

Sum Types

New syntactic forms:

terms:

$$
\begin{array}{rcll}
t & ::= & \ldots & \\
 & | & \texttt{inl}\ t & \text{tagging (left)} \\
 & | & \texttt{inr}\ t & \text{tagging (right)} \\
 & | & \texttt{case}\ t\ \texttt{of inl}\ x \Rightarrow t \mid \texttt{inr}\ x \Rightarrow t & \text{case}
\end{array}
$$

values:

$$
\begin{array}{rcll}
v & ::= & \ldots & \\
 & | & \texttt{inl}\ v & \text{tagged value (left)} \\
 & | & \texttt{inr}\ v & \text{tagged value (right)}
\end{array}
$$

types:

$$
\begin{array}{rcll}
T & ::= & \ldots & \\
 & | & T + T & \text{sum type}
\end{array}
$$

New Evaluation Rules:

$t \longrightarrow t'$

$$\frac{}{\texttt{case inl}\ v_1\ \texttt{of inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2 \longrightarrow [x_1 \mapsto v_1]t_1}\ \text{E-CaseInL}$$

$$\frac{}{\texttt{case inr}\ v_2\ \texttt{of inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2 \longrightarrow [x_2 \mapsto v_2]t_2}\ \text{E-CaseInR}$$

$$\frac{t_0 \longrightarrow t_0'}{\texttt{case}\ t_0\ \texttt{of inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2 \longrightarrow \texttt{case}\ t_0'\ \texttt{of inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2}\ \text{E-Case}$$

Sum Types

New syntactic forms:

terms:

$$
\begin{array}{rcll}
t & ::= & \ldots & \\
 & \mid & \texttt{inl}\ t & \text{tagging (left)} \\
 & \mid & \texttt{inr}\ t & \text{tagging (right)} \\
 & \mid & \texttt{case}\ t\ \texttt{of inl}\ x \Rightarrow t \mid \texttt{inr}\ x \Rightarrow t & \text{case}
\end{array}
$$

values:

$$
\begin{array}{rcll}
v & ::= & \ldots & \\
 & \mid & \texttt{inl}\ v & \text{tagged value (left)} \\
 & \mid & \texttt{inr}\ v & \text{tagged value (right)}
\end{array}
$$

types:

$$
\begin{array}{rcll}
T & ::= & \ldots & \\
 & \mid & T + T & \text{sum type}
\end{array}
$$

New Typing Rules:

$\boxed{\Gamma \vdash t : T}$

$$\frac{\Gamma \vdash t : T_1}{\Gamma \vdash \texttt{inl}\ t : T_1 + T_2}\ \text{T-InL} \qquad \frac{\Gamma \vdash t : T_2}{\Gamma \vdash \texttt{inl}\ t : T_1 + T_2}\ \text{T-InR}$$

$$\frac{\Gamma \vdash t_0 : T_1 + T_2 \quad \Gamma, x_1 : T_1 \vdash t_1 : T \quad \Gamma, x_2 : T_2 \vdash t_2 : T}{\Gamma \vdash \texttt{case}\ t_0\ \texttt{of inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2 : T}\ \text{T-Case}$$

Labeled Product Types a.k.a. Records

New syntactic forms:

Labeled Product Types a.k.a. Records

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\{l_i{=}t_i\ ^{i\in 1..n}\}$	record		$\mid$	$\{l_i{=}v_i\ ^{i\in 1..n}\}$	record value
	$\mid$	$t.l$	projection	T	$::=$	$\ldots$	types:
					$\mid$	$\{l_i{:}T_i\ ^{i\in 1..n}\}$	type of records

Labeled Product Types a.k.a. Records

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\{l_i{=}t_i\ ^{i\in 1..n}\}$	record		$\mid$	$\{l_i{=}v_i\ ^{i\in 1..n}\}$	record value
	$\mid$	$t.l$	projection	T	$::=$	$\ldots$	types:
					$\mid$	$\{l_i{:}T_i\ ^{i\in 1..n}\}$	type of records

New Evaluation Rules:

Labeled Product Types a.k.a. Records

New syntactic forms:

$$
\begin{array}{lcll}
t & ::= & \ldots & \text{terms:} \\
 & | & \{l_i{=}t_i\ ^{i\in 1..n}\} & \text{record} \\
 & | & t.l & \text{projection}
\end{array}
\qquad
\begin{array}{lcll}
v & ::= & \ldots & \text{values:} \\
 & | & \{l_i{=}v_i\ ^{i\in 1..n}\} & \text{record value} \\
 & & & \\
T & ::= & \ldots & \text{types:} \\
 & | & \{l_i{:}T_i\ ^{i\in 1..n}\} & \text{type of records}
\end{array}
$$

New Evaluation Rules:

$t \longrightarrow t'$

$$\frac{}{\{l_i{=}v_i\ ^{i\in 1..n}\}.l_j \longrightarrow v_j}\ \text{E-ProjRcd}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.l \longrightarrow t_1'.l}\ \text{E-Proj}$$

Labeled Product Types a.k.a. Records

New syntactic forms:

$$
\begin{array}{rcll}
t & ::= & \ldots & \text{terms:} \\
 & | & \{l_i{=}t_i\ ^{i\in 1..n}\} & \text{record} \\
 & | & t.l & \text{projection}
\end{array}
$$

$$
\begin{array}{rcll}
v & ::= & \ldots & \text{values:} \\
 & | & \{l_i{=}v_i\ ^{i\in 1..n}\} & \text{record value}
\end{array}
$$

$$
\begin{array}{rcll}
T & ::= & \ldots & \text{types:} \\
 & | & \{l_i{:}T_i\ ^{i\in 1..n}\} & \text{type of records}
\end{array}
$$

New Evaluation Rules:

$t \longrightarrow t'$

$$
\frac{t_j \longrightarrow t'_j}{\{l_i{=}v_i\ ^{i\in 1..j-1}, l_j{=}t_j, l_k{=}t_k\ ^{k\in j+1..n}\} \longrightarrow \{l_i{=}v_i\ ^{i\in 1..j-1}, l_j{=}t'_j, l_k{=}t_k\ ^{k\in j+1..n}\}}\ \text{E-RCD}
$$

Labeled Product Types a.k.a. Records

New syntactic forms:

$$
\begin{array}{rcll}
t & ::= & \ldots & \text{terms:} \\
 & | & \{l_i{=}t_i\ ^{i\in 1..n}\} & \text{record} \\
 & | & t.l & \text{projection}
\end{array}
$$

$$
\begin{array}{rcll}
v & ::= & \ldots & \text{values:} \\
 & | & \{l_i{=}v_i\ ^{i\in 1..n}\} & \text{record value}
\end{array}
$$

$$
\begin{array}{rcll}
T & ::= & \ldots & \text{types:} \\
 & | & \{l_i{:}T_i\ ^{i\in 1..n}\} & \text{type of records}
\end{array}
$$

New Typing Rules:

$$\boxed{\Gamma \vdash t : T}$$

$$\frac{\text{for each } i \quad \Gamma \vdash t_i : T_i}{\Gamma \vdash \{l_i{=}t_i\ ^{i\in 1..n}\} : \{l_i{:}T_i\ ^{i\in 1..n}\}}\ \text{T-RCD}$$

$$\frac{\Gamma \vdash t_1 : \{l_i{:}T_i\ ^{i\in 1..j-1}, l_j{:}T_j, l_k{:}T_k\ ^{k\in j+1..n}\}}{\Gamma \vdash \{t_1.l_j\} : T_j}\ \text{T-PROJ}$$

Sum Types (again)

Uniqueness of Typing

Valid types for `inl 5` are $\texttt{Nat} + \texttt{Nat}$, $\texttt{Nat} + \texttt{Bool}$ etc.

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New syntactic forms:

Sum Types (again)

Uniqueness of Typing

Valid types for `inl 5` are $\texttt{Nat} + \texttt{Nat}$, $\texttt{Nat} + \texttt{Bool}$ etc.

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\texttt{inl}\ t\ \texttt{as}\ T$	tagging (left)		$\mid$	$\texttt{inl}\ v\ \texttt{as}\ T$	tagged value (left)
	$\mid$	$\texttt{inr}\ t\ \texttt{as}\ T$	tagging (right)		$\mid$	$\texttt{inr}\ v\ \texttt{as}\ T$	tagged value (right)

Sum Types (again)

Uniqueness of Typing

Valid types for `inl 5` are $\texttt{Nat} + \texttt{Nat}$, $\texttt{Nat} + \texttt{Bool}$ etc.

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\texttt{inl}\ t\ \texttt{as}\ T$	tagging (left)		$\mid$	$\texttt{inl}\ v\ \texttt{as}\ T$	tagged value (left)
	$\mid$	$\texttt{inr}\ t\ \texttt{as}\ T$	tagging (right)		$\mid$	$\texttt{inr}\ v\ \texttt{as}\ T$	tagged value (right)

New Evaluation Rules:

Sum Types (again)

Uniqueness of Typing

Valid types for `inl 5` are $\texttt{Nat} + \texttt{Nat}$, $\texttt{Nat} + \texttt{Bool}$ etc.

New syntactic forms:

t	$::=$	$\ldots$	terms:		
	$\mid$	$\texttt{inl}\ t\ \texttt{as}\ T$	tagging (left)		
	$\mid$	$\texttt{inr}\ t\ \texttt{as}\ T$	tagging (right)		
v	$::=$	$\ldots$	values:		
	$\mid$	$\texttt{inl}\ v\ \texttt{as}\ T$	tagged value (left)		
	$\mid$	$\texttt{inr}\ v\ \texttt{as}\ T$	tagged value (right)		

New Evaluation Rules:

$t \longrightarrow t'$

$$\frac{}{\texttt{case}\ (\texttt{inl}\ v_1\ \texttt{as}\ T)\ \texttt{of}\ \texttt{inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2 \longrightarrow [x_1 \mapsto v_1]t_1}\ \text{E-CaseInL}$$

$$\frac{}{\texttt{case}\ (\texttt{inr}\ v_2\ \texttt{as}\ T)\ \texttt{of}\ \texttt{inl}\ x_1 \Rightarrow t_1 \mid \texttt{inr}\ x_2 \Rightarrow t_2 \longrightarrow [x_2 \mapsto v_2]t_2}\ \text{E-CaseInR}$$

Sum Types (again)

Uniqueness of Typing

Valid types for `inl 5` are $\texttt{Nat} + \texttt{Nat}$, $\texttt{Nat} + \texttt{Bool}$ etc.

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\texttt{inl}\ t\ \texttt{as}\ T$	tagging (left)		$\mid$	$\texttt{inl}\ v\ \texttt{as}\ T$	tagged value (left)
	$\mid$	$\texttt{inr}\ t\ \texttt{as}\ T$	tagging (right)		$\mid$	$\texttt{inr}\ v\ \texttt{as}\ T$	tagged value (right)

New Typing Rules:

$\boxed{\Gamma \vdash t : T}$

$$\frac{\Gamma \vdash t : T_1}{\Gamma \vdash \texttt{inl}\ t\ \texttt{as}\ T_1 + T_2 : T_1 + T_2}\ \text{T-InL} \qquad \frac{\Gamma \vdash t : T_2}{\Gamma \vdash \texttt{inl}\ t\ \texttt{as}\ T_1 + T_2 : T_1 + T_2}\ \text{T-InR}$$

Labeled Sum Types a.k.a. Variants a.k.a. Algebraic Datatypes

New syntactic forms:

Labeled Sum Types a.k.a. Variants a.k.a. Algebraic Datatypes

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\langle l{=}t \rangle$ as T	tagging		$\mid$	$\langle l{=}v \rangle$ as T	tagged value
	$\mid$	case t of $\langle l_i{=}t_i \rangle \Rightarrow t_i \;^{i \in 1..n}$	case	T	$::=$	$\ldots$	types:
					$\mid$	$\langle l_i : T_i \;^{i \in 1..n} \rangle$	type of variants

Labeled Sum Types a.k.a. Variants a.k.a. Algebraic Datatypes

New syntactic forms:

t	$::=$	$\ldots$	terms:	v	$::=$	$\ldots$	values:
	$\mid$	$\langle l{=}t \rangle$ as T	tagging		$\mid$	$\langle l{=}v \rangle$ as T	tagged value
	$\mid$	case t of $\langle l_i{=}t_i \rangle \Rightarrow t_i \; {}^{i \in 1..n}$	case	T	$::=$	$\ldots$	types:
					$\mid$	$\langle l_i : T_i \; {}^{i \in 1..n} \rangle$	type of variants

New Evaluation Rules:

Labeled Sum Types a.k.a. Variants a.k.a. Algebraic Datatypes

New syntactic forms:

$$
\begin{array}{lcll}
t & ::= & \ldots & \text{terms:} \\
 & | & \texttt{<}l\texttt{=}t\texttt{>}\ \texttt{as}\ T & \text{tagging} \\
 & | & \texttt{case}\ t\ \texttt{of}\ \texttt{<}l_i\texttt{=}t_i\texttt{>}\ \Rightarrow\ t_i\ ^{i\in 1..n} & \text{case}
\end{array}
$$

$$
\begin{array}{lcll}
v & ::= & \ldots & \text{values:} \\
 & | & \texttt{<}l\texttt{=}v\texttt{>}\ \texttt{as}\ T & \text{tagged value}
\end{array}
$$

$$
\begin{array}{lcll}
T & ::= & \ldots & \text{types:} \\
 & | & \texttt{<}l_i\texttt{:}T_i\ ^{i\in 1..n}\texttt{>} & \text{type of variants}
\end{array}
$$

New Evaluation Rules:

$t \longrightarrow t'$

$$
\frac{t_i \longrightarrow t_i'}{\texttt{<}l_i\texttt{=}t_i\texttt{>}\ \texttt{as}\ T \longrightarrow \texttt{<}l_i\texttt{=}t_i'\texttt{>}\ \texttt{as}\ T}\ \text{E-VARIANT}
$$

Labeled Sum Types a.k.a. Variants a.k.a. Algebraic Datatypes

New syntactic forms:

t	$::=$	$\ldots$	terms:	v $::=$ $\ldots$	values:
	$\mid$	$\langle l{=}t \rangle$ as T	tagging	$\mid$ $\langle l{=}v \rangle$ as T	tagged value
	$\mid$	case t of $\langle l_i{=}t_i \rangle \Rightarrow t_i^{\,i \in 1..n}$	case	T $::=$ $\ldots$	types:
				$\mid$ $\langle l_i : T_i^{\,i \in 1..n} \rangle$	type of variants

New Evaluation Rules:

$t \longrightarrow t'$

$$\frac{}{\text{case } (\langle l_j{=}t_j \rangle \text{ as } T) \text{ of } \langle l_i{=}x_i \rangle \Rightarrow t_i^{\,i \in 1..n} \longrightarrow [x_j \mapsto v_j]t_j} \quad \text{E-CaseVariant}$$

$$\frac{t_0 \longrightarrow t_0'}{\text{case } t_0 \text{ of } \langle l_i{=}x_i \rangle \Rightarrow t_i^{\,i \in 1..n} \longrightarrow \text{case } t_0' \text{ of } \langle l_i{=}x_i \rangle \Rightarrow t_i^{\,i \in 1..n}} \quad \text{E-Case}$$

Labeled Sum Types a.k.a. Variants a.k.a. Algebraic Datatypes

New syntactic forms:

terms:

$$t ::= \ldots$$
$$\mid \texttt{<}l\texttt{=}t\texttt{>}\ \texttt{as}\ T \quad \text{tagging}$$
$$\mid \texttt{case}\ t\ \texttt{of}\ \texttt{<}l_i\texttt{=}t_i\texttt{>} \Rightarrow t_i\ ^{i\in 1..n} \quad \text{case}$$

values:

$$v ::= \ldots$$
$$\mid \texttt{<}l\texttt{=}v\texttt{>}\ \texttt{as}\ T \quad \text{tagged value}$$

types:

$$T ::= \ldots$$
$$\mid \texttt{<}l_i : T_i\ ^{i\in 1..n}\texttt{>} \quad \text{type of variants}$$

New Typing Rules:

$\Gamma \vdash t : T$

$$\frac{\Gamma \vdash t_j : T_j}{\Gamma \vdash \texttt{<}l_j\texttt{=}t_j\texttt{>}\ \texttt{as}\ \texttt{<}l_i : T_i\ ^{i\in 1..n}\texttt{>} : \texttt{<}l_i : T_i\ ^{i\in 1..n}\texttt{>}} \text{ T-VARIANT}$$

$$\frac{\Gamma \vdash t_0 : \texttt{<}l_i : T_i\ ^{i\in 1..n}\texttt{>} \qquad \texttt{for each}\ i \quad \Gamma, x_i : T_i \vdash t_i : T}{\Gamma \vdash \texttt{case}\ t_0\ \texttt{of}\ \texttt{<}l_i\texttt{=}x_i\texttt{>} \Rightarrow t_i\ ^{i\in 1..n} : T} \text{ T-CASE}$$

Examples

STLC++

```
Bool = <false:Unit, true:Unit>
false = <false=unit> as Bool
true = <true=unit> as Bool
```

```
Week = <weekday:Unit, weekend:Unit>
weekday = <weekday=unit> as Week
weekend = <weekend=unit> as Week
weekendYet = λ x:Week.
  case x of
    <weekday=unit> ⇒ false
    <weekend=unit> ⇒ true
```

```
OptionNat = <none:Unit, some:Nat>
NatList = <nil:Unit, cons:{Nat, NatList}>
Nat =  <zero:Unit, succ:Nat>
```

Coq

```
Inductive bool := False | True.
```

```
Inductive week = Weekday | Weekend.
```

```
Inductive optionNat = None | Some (_:nat_
Inductive natList = Nil | Cons (_:nat) (_:natList).
Inductive nat = Zero | Succ (_:nat).
```