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Foundations of Certified Programming Language and Compiler Design

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Outline

Lecture	Logic	Formalisms	PL
1	Propositional and first-order logic		
2			Functional programming
3		Syntax and Semantics	
4			The untyped lambda calculus
5		Types	
6			The typed lambda calculus
7			Polymorphism
8		Curry-Howard	
9			Higher-order types
10			Dependent types

Goals

- At first, we have a look at an important second-order type system.
- Then, we resolve all long-standing mysteries of data types and
- introduce type-level computation to
- finally express “more interesting” propositions
- (and prove them).

From First to Second-order

- If we remove the top-level restriction for quantification, we
- enter second-order logic ,i.e.,
- we allow to quantify over predicates.

- The system that we arrive when doing so is called *System F/Polymorphic Lambda Calculus*.
- System F is the foundation for the rich and powerful type system of Haskell.

System F/Polymorphic Lambda Calculus

Syntax:

t	$::=$		terms:
	$\vert$	x	variable
	$\vert$	$\lambda x : T.t$	abstraction
	$\vert$	$t\ t$	application
	$\vert$	$\lambda X.t$	type abstraction
	$\vert$	$t\ [T]$	type application
v	$::=$		values:
	$\vert$	$\lambda x : T.t$	abstraction value
	$\vert$	$\lambda X.t$	type abstraction value

T	$::=$		types:
	$\vert$	X	type variable
	$\vert$	$T \to T$	type of functions
	$\vert$	$\forall X.T$	universal type
Γ	$::=$		contexts:
	$\vert$	$\varnothing$	empty context
	$\vert$	$\Gamma, x : T$	term variable binding
	$\vert$	Γ, X	type variable binding

System F

Evaluation $\boxed{t \longrightarrow t'}$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \longrightarrow t_1'\ t_2}\ \text{E-App1}$$

$$\frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \longrightarrow v_1\ t_2'}\ \text{E-App2}$$

$$\frac{}{(\lambda x : T.t_{12})\ v_2 \longrightarrow [x \mapsto v_2]t_{12}}\ \text{E-AppAbs}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ [T_2] \longrightarrow t_1'\ [T_2]}\ \text{E-TApp}$$

$$\frac{}{(\lambda X.t_{12})\ [T_2] \longrightarrow [X \mapsto T_2]t_{12}}\ \text{E-TAppTAbs}$$

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$$\frac{t_1 \longrightarrow t_1'}{t_1\ [T_2] \longrightarrow t_1'\ [T_2]} \quad \text{E-TAPP}$$

$$\frac{}{(\lambda X.t_{12})\ [T_2] \longrightarrow [X \mapsto T_2]t_{12}} \quad \text{E-TAPPTABS}$$

Typing $\boxed{\Gamma \vdash t : T}$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \text{T-VAR}$$

$$\frac{\Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1.t_2 : T_1 \to T_2} \quad \text{T-ABS}$$

$$\frac{\Gamma, t_1 : T_{11} \to T_{12} \quad t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \quad \text{T-APP}$$

$$\frac{\Gamma, X \vdash t_2 : T_2}{\Gamma \vdash \lambda X.t_2 : \forall X.T_2 \to T_2} \quad \text{T-TABS}$$

$$\frac{\Gamma, t_1 : \forall X.T_{12}}{\Gamma \vdash t_1\ [T_2] : [X \mapsto T_2]T_{12}} \quad \text{T-TAPP}$$

System F/Polymorphic Lambda Calculus

History

1972 – Jean-Yves Girard , a logician, discovers *System F* in the context of proof theory.

1974 – John Reynolds , a computer scientist, discovers the *polymorphic lambda calculus* that has a type system with the same expressive power.

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$$\texttt{let}\ \{X, x\} = t_1\ \texttt{in}\ t_2 \stackrel{\text{def}}{=} t_1\ \ldots$$
Remember: an existential is an abstraction.

$$\stackrel{\text{def}}{=} t_1\ [T_2]\ \ldots$$
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Impredicativity

- System F implements *impredicative* polymorphism.
- $T = \forall X. X \to X$ ranges over all types, even T itself!

System F in Haskell

- Haskell implements System FC[1], an extension of system F.
- Haskell has all the described features, even visible type applications, ...
- ...and still provides type inference based on an extension of HM![2]

[1] Martin Sulzmann et al. "System F with type equality coercions". In: *Proceedings of the 2007 ACM SIGPLAN international workshop on Types in languages design and implementation*. 2007.

[2] Richard A. Eisenberg, Stephanie Weirich, and Hamidhasan G. Ahmed. "Visible Type Application". In: *Programming Languages and Systems*. Springer Berlin Heidelberg, 2016.

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$$\begin{array}{rl} \text{Simple} & \texttt{CBool} \\ \text{Parametric} & \texttt{Pair} = \lambda X.\, \lambda Y.\, \forall Z.\, (X \to Y \to Z) \to Z \end{array}$$

- `Pair` is essentially a type-level function, called a *type operator*.
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- The types of types are referred to as *kinds*.

From Types to Kinds

Terms

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$(\lambda x : \texttt{Nat}.\, x)\ 5$

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proper type

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$\texttt{Nat}$

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$\forall X.\, X \to X$

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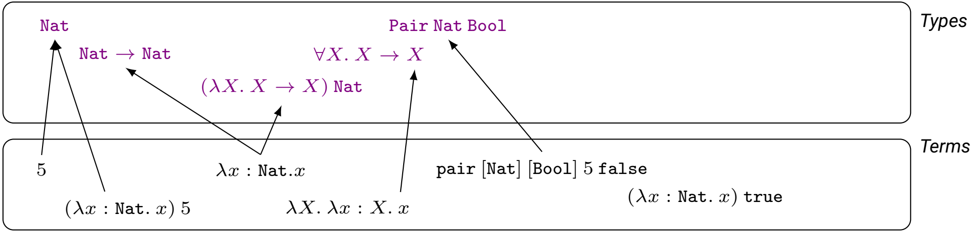

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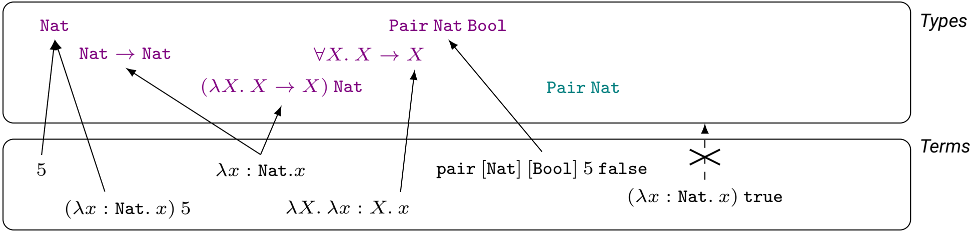

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type operator

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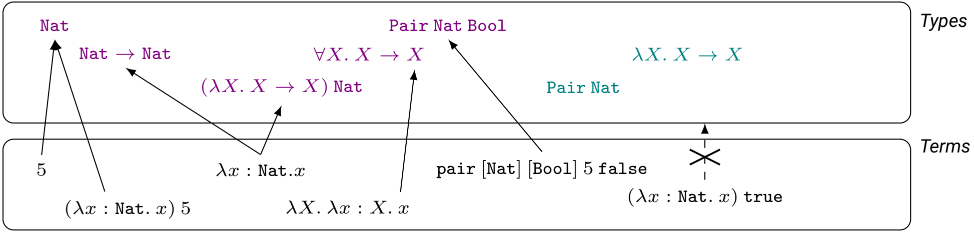

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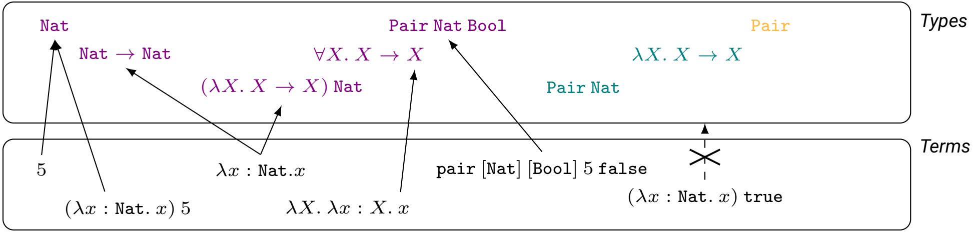

proper type   type operator   function from proper type to type operator

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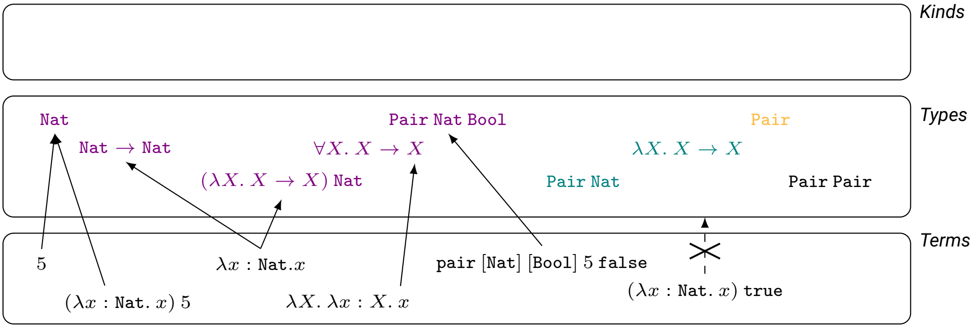

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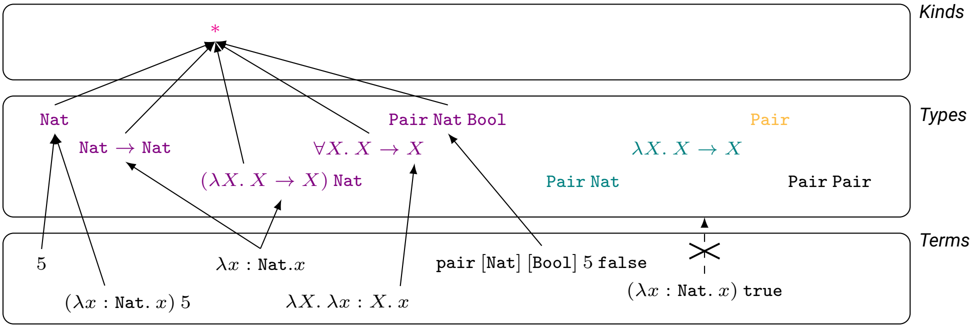

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type operator

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kind of proper types

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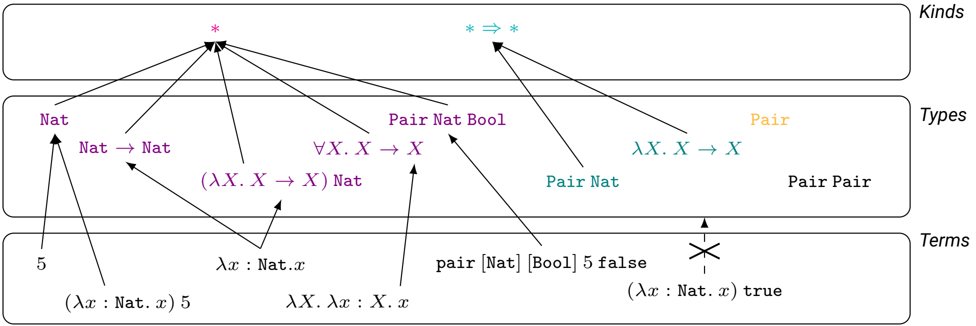

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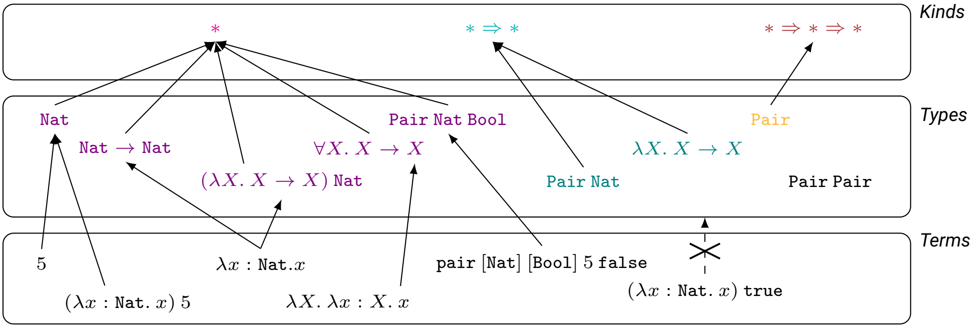
kind of type operators

From Types to Kinds



proper type type operator function from proper type to type operator

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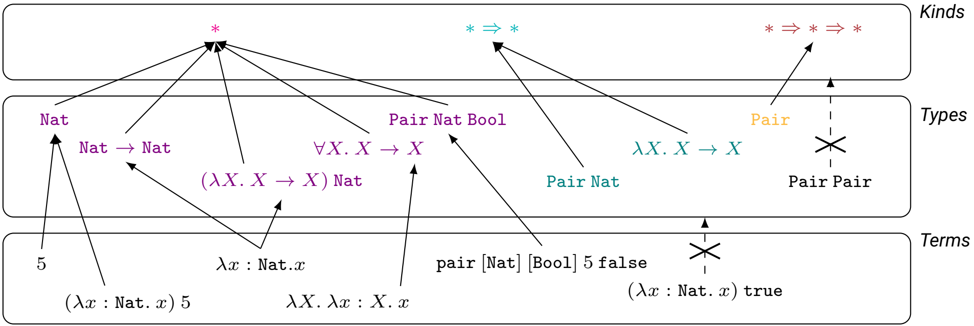

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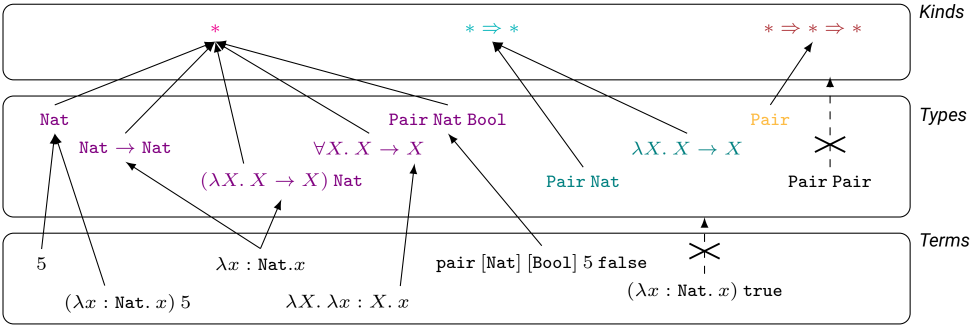

proper type | type operator | function from proper type to type operator

kind of proper types | kind of type operators | kind of two-argument type operators

- Note the uninhabited types in this figure!

The Systems of Todays Lecture

λ_ω	Monomorphic kinds (,i.e., kinding without quantifiers)
System F_ω	Polymorphic kinds

Syntax

- We extend STLC with type operators:

t	$::=$		terms:
	$\vert$	x	variable
	$\vert$	$\lambda x : T.t$	abstraction
	$\vert$	$t\ t$	application
v	$::=$		values:
	$\vert$	$\lambda x : T.t$	abstraction value

T	$::=$		types:
	$\vert$	X	type variable
	$\vert$	$\lambda X :: K.T$	operator abstraction
	$\vert$	$T\ T$	operator application
	$\vert$	$T \to T$	type of functions
Γ	$::=$		contexts:
	$\vert$	$\varnothing$	empty context
	$\vert$	$\Gamma, x : T$	term variable binding
	$\vert$	$\Gamma, X :: K$	type variable binding
K	$::=$		kinds:
	$\vert$	$*$	kind of proper types
	$\vert$	$K \Rightarrow K$	kind of operators

- For conciseness, we will not assume base kinds. But we will kinds later.

Evaluation

$$\boxed{t \longrightarrow t'}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1\ t_2 \ \longrightarrow\ t_1'\ t_2} \text{ E-APP1} \qquad \frac{t_2 \longrightarrow t_2'}{v_1\ t_2 \ \longrightarrow\ v_1\ t_2'} \text{ E-APP2}$$

$$\frac{}{(\lambda x : T.t_{12})\ v_2 \ \longrightarrow\ [x \mapsto v_2]t_{12}} \text{ E-APPABS}$$

Typing and Kinding

$\boxed{\Gamma \vdash t : T}$

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$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T}\ \textsc{T-Var} \qquad \frac{\Gamma \vdash T_1 :: * \qquad \Gamma, x : T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x : T_1 . t_2 : T_1 \to T_2}\ \textsc{T-Abs} \qquad \frac{\Gamma, t_1 : T_{11} \to T_{12} \qquad t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \textsc{T-App}$$

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Axioms

$$\frac{}{\Gamma \vdash \texttt{Nat} \equiv \texttt{Nat}} \quad \text{QT-NAT}$$

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Congruence

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- (Remember, in HM, we would not do this comparison directly.)
- We want to *define* that $\texttt{Nat} \to \texttt{Nat} \equiv (\lambda X.\, X \to X)\ \texttt{Nat}$.

λ_ω

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```
data Nat where
    O :: Nat
    S :: Nat -> Nat
```

parameterized

```
data List a :: Type where
    Nil :: List a
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annotated/indexed

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data Ev :: Nat -> Type where
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  - There is nothing special about their syntax!
  - They just have a different kind!
  - (See code!)
- (Don't worry, we will unlock the rest of the mysteries in a couple of slides.)

F_ω

- F_ω is a combination of System F and λ_ω.
- To directly read System F terms in System F_ω, we abbreviate $\forall X :: *.T$ as $\forall X.T$.

F_ω

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	$\mid$	X	type variable
	$\mid$	$T \to T$	type of functions
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$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K} \textsc{T-TVar}$$

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$$\frac{\Gamma, t_1 : T_{11} \to T_{12} \quad t_2 : T_{11}}{\Gamma \vdash T_1\ T_2 :: K_{12}} \textsc{K-App}$$

$$\frac{\Gamma \vdash T_1 :: * \quad \Gamma \vdash T_2 :: *}{\Gamma \vdash T_1 \to T_2 :: *} \textsc{K-Arrow}$$

Kinding

$$\boxed{\Gamma \vdash T :: K}$$

$$\frac{X :: K \in \Gamma}{\Gamma \vdash X :: K} \ \text{T-TVar}$$

$$\frac{\Gamma, X :: K_1 \vdash T_2 :: K_2}{\Gamma \vdash \lambda X :: K_1 . T_2 :: K_1 \Rightarrow K_2} \ \text{K-Abs}$$

$$\frac{\Gamma, t_1 : T_{11} \to T_{12} \quad t_2 : T_{11}}{\Gamma \vdash T_1\ T_2 :: K_{12}} \ \text{K-App}$$

$$\frac{\Gamma \vdash T_1 :: * \quad \Gamma \vdash T_2 :: *}{\Gamma \vdash T_1 \to T_2 :: *} \ \text{K-Arrow}$$

$$\frac{\Gamma, X :: K_1 \vdash T_2 :: *}{\Gamma \vdash \forall X :: K_1 . T_2 :: *} \ \text{K-All}$$

F_ω

Type Equivalence

$$\boxed{S \equiv T}$$

$$\frac{}{T \equiv T}\ \text{Q-Refl} \qquad \frac{T \equiv S}{S \equiv T}\ \text{Q-Symm} \qquad \frac{S \equiv U \quad U \equiv T}{S \equiv T}\ \text{Q-Trans} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \to S_2 \equiv T_1 \to T_2}\ \text{Q-Arrow}$$

$$\frac{S_2 \equiv T_2}{\lambda X :: K_1.S_2 \equiv \lambda X :: K_1.T_2}\ \text{Q-Abs} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1\ S_2 \equiv T_1\ T_2}\ \text{Q-App}$$

$$\frac{}{(\lambda X :: K_{11}.T_2)\ T_{11} \equiv [X \mapsto T_{11}]T_2}\ \text{Q-AppAbs}$$

F_ω
Type Equivalence

$$\boxed{S \equiv T}$$

$$\frac{}{T \equiv T}\ \text{Q-Refl} \qquad \frac{T \equiv S}{S \equiv T}\ \text{Q-Symm} \qquad \frac{S \equiv U \quad U \equiv T}{S \equiv T}\ \text{Q-Trans} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1 \to S_2 \equiv T_1 \to T_2}\ \text{Q-Arrow}$$

$$\frac{S_2 \equiv T_2}{\forall X :: K_1.S_2 \equiv \forall X :: K_1.T_2}\ \text{Q-All} \qquad \frac{S_2 \equiv T_2}{\lambda X :: K_1.S_2 \equiv \lambda X :: K_1.T_2}\ \text{Q-Abs} \qquad \frac{S_1 \equiv T_1 \quad S_2 \equiv T_2}{S_1\ S_2 \equiv T_1\ T_2}\ \text{Q-App}$$

$$\frac{}{(\lambda X :: K_{11}.T_2)\ T_{11} \equiv [X \mapsto T_{11}]T_2}\ \text{Q-AppAbs}$$

Dependently-typed Programming in Haskell

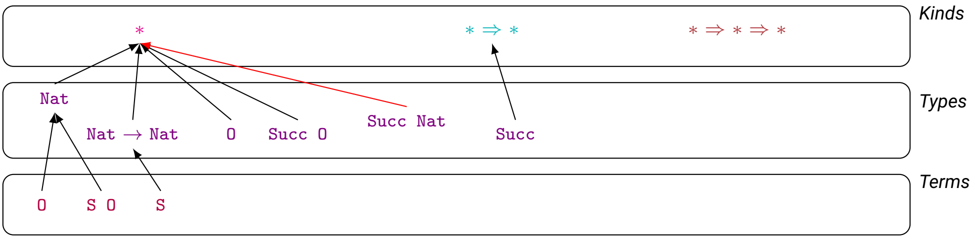
- We have worked so hard this semester to reach this point. Congratulations!
- Let's see the Curry-Howard Correspondence in dependently-typed Haskell in action and prove some theorems! (See code!)

Language Extensions in Haskell

- To perform type-level programming, we need to be able to
- create new types that are well-kinded and
- type-level functions that are total.

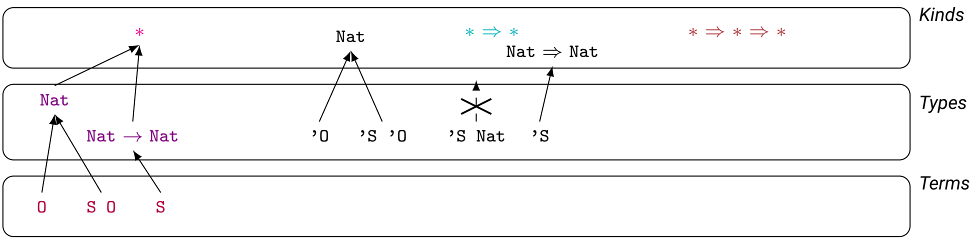
Promoted Data Types and Type Families in Haskell

- Data types abstract over the concrete data constructors.
- But for type-level programming, we actually need this information. How else would we express something like $1 + 0$, i.e., `S 0`?
- Let's create some unique (uninhabitated) types.



Promoted Data Types and Type Families in Haskell

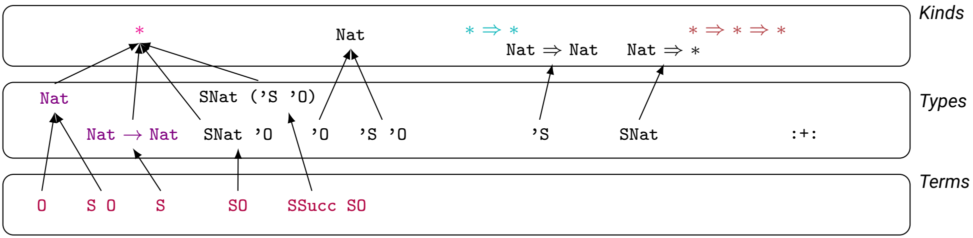
- The language extension `{# LANGUAGE DataKinds #}` ...
- promotes (data) types to the kind-level and
- constructors to the type-level.[1]



[1] Brent A Yorgey et al. "Giving Haskell a promotion". In: *Proceedings of the 8th ACM SIGPLAN Workshop on Types in Language Design and Implementation*. 2012, pp. 53–66.

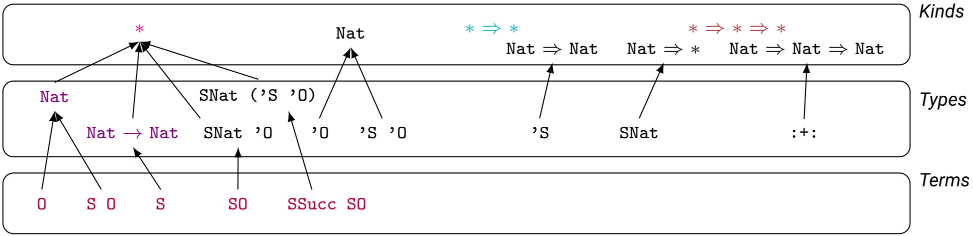
Promoted Data Types and Type Families in Haskell

- Remember: proofs are programs!
- Hence, we need to connect terms and types.
- We need a term-level representation for every type: `'O`, `'S 'O`, `'S ('S 'O)`, ...
- We do this by creating a *singleton* data type, i.e., a type that is index by the types of kind `Nat`.



Promoted Data Types and Type Families in Haskell

- Language extension: `{# LANGUAGE TypeFamilies #}`
- Promoting *closed* data types, i.e., ADTs and GADTs, enables
- closed type families.[1]
- (Where "closed" here means: not extensible.)



[1] Richard A Eisenberg et al. "Closed type families with overlapping equations". In: *ACM SIGPLAN Notices* (2014).

Further Reading on Dependently-Typed Programming in Haskell

Hasochism describes the benefits and pain points of dependent programming in Haskell.[1]

Singletons a library-based approach to create singleton types and promote term-level functions to the type-level.[2]

Stitch describes a very elegant implementation of a language and its compiler in Haskell that promotes all the type-checking of the implemented language into GHCs type system.[3]

DH Make sure to stay tuned for Dependently-typed Haskell.[4]

[1] Sam Lindley and Conor McBride. "Hasochism: The Pleasure and Pain of Dependently Typed Haskell Programming". In: *SIGPLAN Not.* (2013).

[2] Richard A Eisenberg and Stephanie Weirich. "Dependently typed programming with singletons". In: *ACM SIGPLAN Notices* (2012).

[3] Richard A Eisenberg. "Stitch: the sound type-indexed type checker (functional pearl)". In: *Proceedings of the 13th ACM SIGPLAN International Symposium on Haskell*. 2020, pp. 39–53.

[4] Stephanie Weirich, Justin Hsu, and Richard A Eisenberg. "Towards dependently typed Haskell: System FC with kind equality". In: *Proceedings of the 18th ACM SIGPLAN International Conference on Functional Programming, ICFP*.

What we have learned

- This lecture marks the end of our path towards understanding the type system of Haskell.
- We have learned that
  - kinds are types of the types and
  - most importantly that proofs are just programs (that produce a proof witness for their result type, i.e., their proposition)!
- Type-level functions provide the possibility to
  - express powerful propositions and
  - in Haskell translate them into adding equations for the HM type inference algorithm.