4.9 Matrix Transformations from R^n to R^m

A *matrix transformation* $T: R^n \longrightarrow R^m$ is a mapping of the form

$$\mathbf{T}(\vec{x}) = \mathbf{A}\vec{x},$$

for all $\vec{x}$ vectors in R^n and an m x n matrix A.

The matrix transformations are precisely the *linear transformations* from R^n to R^m, that is, the transformations with the linearity properties

$$T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}) \qquad \text{and} \qquad T(k\boldsymbol{u}) = kT(\boldsymbol{u})$$

We will use these two properties as the starting point for defining more general linear transformations.

Remark: It is important to note that a linear transformation is a special kind of function. The input and output are both vectors.

If we denote the output vector $\mathbf{T}(\vec{x})$ by $\vec{y}$ we can write

$$\vec{y} = \mathbf{A}\vec{x}$$

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors $e_1, e_2, \ldots, e_n$ for R^n in column form.
Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

Types of linear Transformations

There are two types of linear transformations (defining from R^2 to R^2):

1- Euclidean Transformation
2- Affine Transformation

1 – Euclidean Transformations

A Euclidean Transformation is a transformation $T: R^2 \rightarrow R^2$ defined by

$$\mathbf{T}(x) = \mathbf{A}\vec{x} + \vec{a}, \quad \forall\, \vec{x} \in R^2$$

Where A is an orthogonal 2 x 2 matrix and $\vec{a} \in R^2$. These types of transformations always preserve distance/shape.

An orthogonal matrix holds the property $\mathbf{AA^T = 1}$ **or** $\mathbf{A^T = A^{-1}}$

Types of Euclidean transformation:

1- Translation, 2- Reflection, 3- Rotation.

1- Translation

Translation is a transformation from (R^2 to R^2) or (R^3 to R^3) defined as:

$$\mathbf{T}\,(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} + \vec{\boldsymbol{a}}\,, \qquad \forall\, \vec{\boldsymbol{x}} \in R^2$$

where matrix A is the identity matrix.

Example 1: (Translation of a triangle)

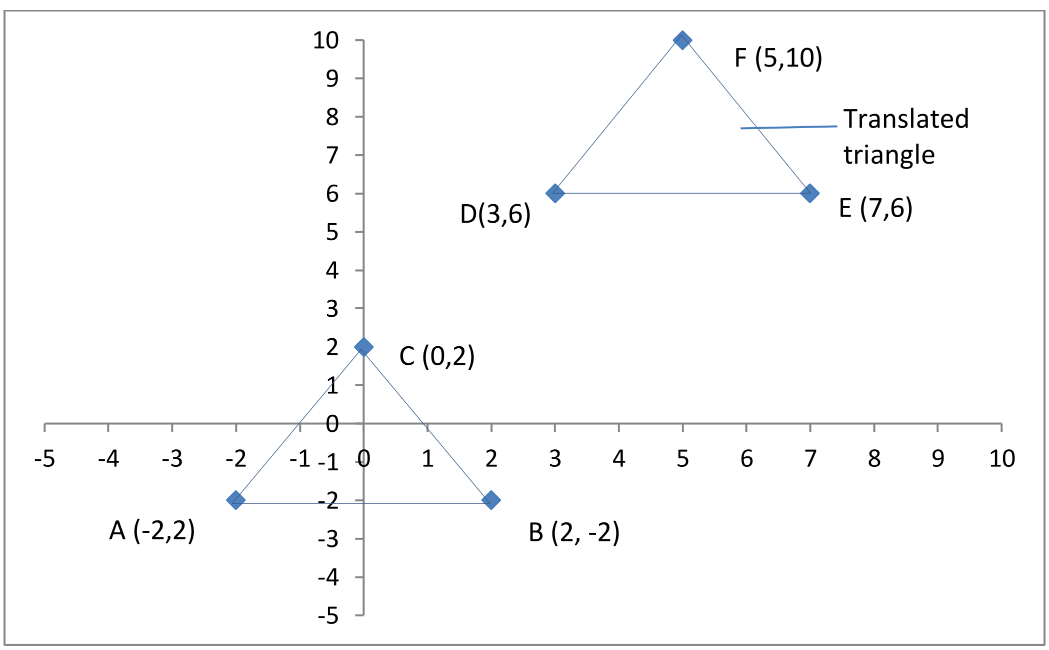
Let A = (-2, -2), B = (2, -2), C = (0, 2) form a triangle. Find the translated triangle with vector $\vec{\boldsymbol{a}} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

Solution: As the transformation of translation is $\mathbf{T}\,(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{x}} + \vec{\boldsymbol{a}}$.

For point A: $D = T\,(A) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

For point B: $E = T\,(B) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

For point C: $F = T\,(C) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$



Example 2: (Translation of a line)

For a line 3x - 4y = 2, find the equation of line translated through vector $\vec{a}$ = (2, 3).

Solution: The transformation of translation is:

$$\mathbf{T}\,(\vec{x}) = \vec{x} + \vec{a}$$

So $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **Or** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x & + & 2 \\ y & + & 3 \end{bmatrix}$

Which implies x'= x + 2 and y'= y + 3.

Then, $x = x'$ - 2 and y = y' - 3.

Put these in our given equation of line that is

$$3(x' – 2) – 4(y' – 3) = 2$$

$3x' – 4y'$ = -4 is the required translated line.

To draw original line 3x – 4y = 2

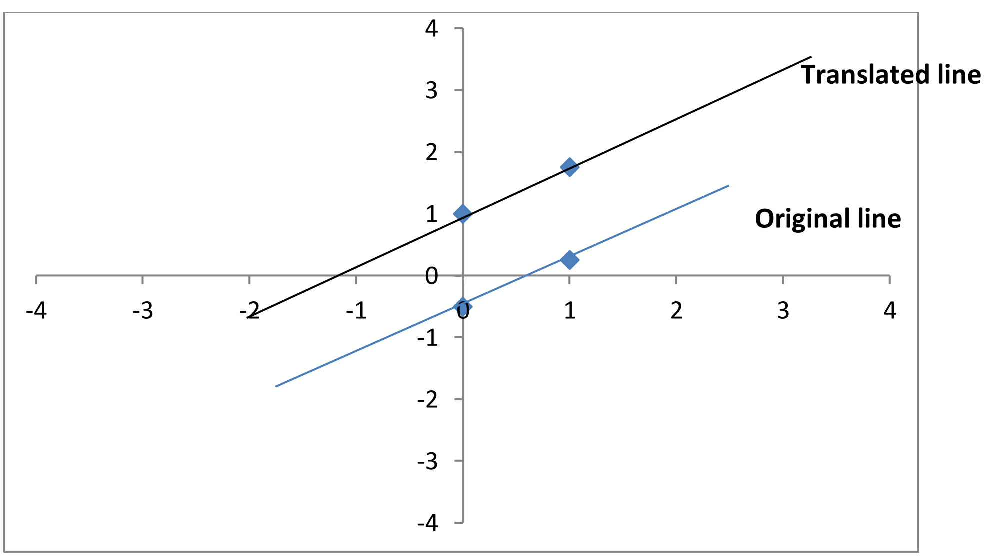
put ***x* = 0** implies y = -1/2, so **A** (0, -1/2) is a point on this line.

Similarly ***x* =1** implies y = ¼ and **B** = (1, 1/4) is another point on it.

In the same manner to draw the Translated line $3x'$ - $4y'$= -4

Putting ***x* = 0** gives y'= 1 and **C** = (0, 1).

Putting ***x* =1** provides y'= 7/4 and **D**= (1, 7/4).



Example 3: (Translation of circle)

Let $(x - 4)^2 + (y - 3)^2 = 9$ be a circle. Find the equation of the translated circle using vector (2, 3).

Note: As equation of circle: $(x - a)^2 + (y - b)^2 = r^2$ with Centre = (a, b) and Radius = r. While $x^2 + y^2 = r^2$ is circle with Center = (0, 0) and Radius = r .

Solution: The transformation of translation is

$$\mathbf{T}(\vec{x}) = \vec{x} + \vec{a}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x & + & 2 \\ y & + & 3 \end{bmatrix}$$

$x' = x + 2$ then $x = x' - 2$ and $y' = y + 3$ then $y = y' - 3$

Putting these equations in the equation of circle

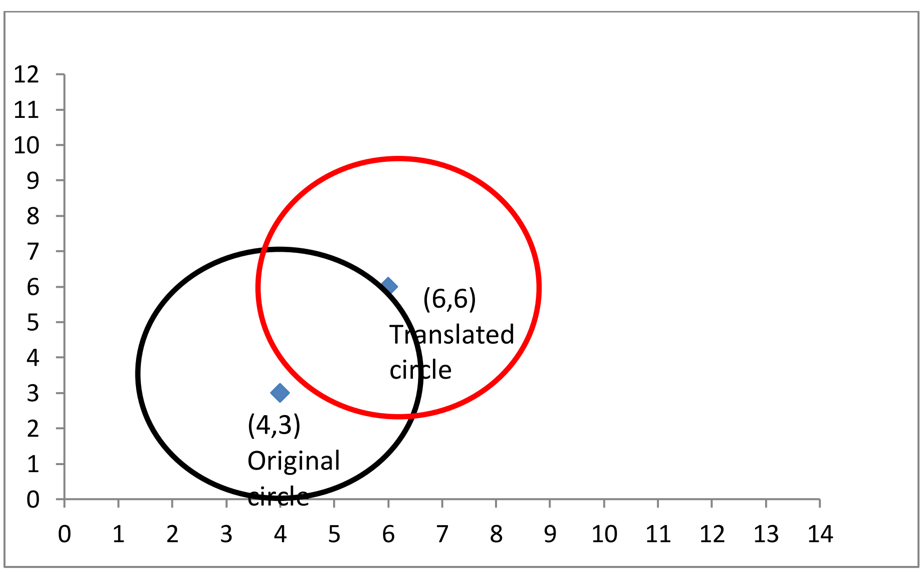
$$(x - 4)^2 + (y - 3)^2 = 9$$

$$(x' - 2 - 4)^2 + (y' - 3 - 3)^2 = 9$$

$$(x' - 6)^2 + (y' - 6)^2 = 9$$

Hence, Original circle is $(x - 4)^2 + (y - 3)^2 = 9$ with Center = (4, 3), Radius = 3.

While Translated circle is $(x' - 6)^2 + (y' - 6)^2 = 9$ with Center = (6, 6), Radius = 3



Note: As we said earlier that **Euclidean transformations** are distance/shape preserving. So in all above examples we can see that translation transformation being a Euclidean transformation preserves the shape of each object and just translated or moved the object.

Work to do:

Q1. Consider the letter L in figure, made up of the vectors (1, 0) or $\begin{bmatrix}1\\0\end{bmatrix}$ and (0, 2) or $\begin{bmatrix}0\\2\end{bmatrix}$, show that the effect of the linear transformation on the matrices and describe each of the transformation in words.

$$A = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \qquad B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},$$

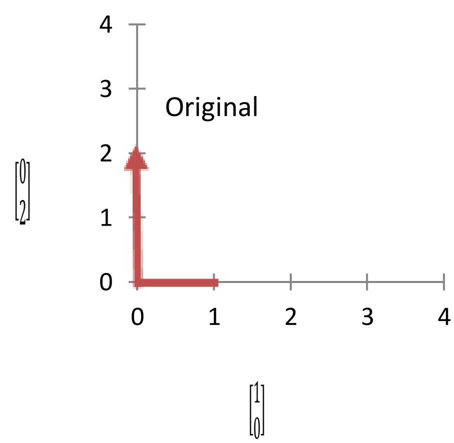
$$C = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \qquad D = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix},$$

$$E = \begin{bmatrix}0 & 0.2\\0 & 1\end{bmatrix}, \qquad F = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

Solution: As, $\mathbf{T}(\vec{x}) = \mathbf{A}\vec{x}$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$



For $\mathbf{T}(\vec{x}) = \mathbf{C}\vec{x}$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$$

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Q2. Let A = (3, 4), B = (3, 2), C = (6, 2) and D = (6, 4) form a rectangle. Find its translation thorough vector (3, 5) and verify your translated rectangle from the figure below.

Original rectangle