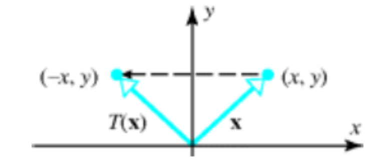
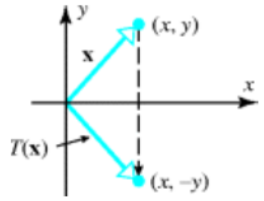
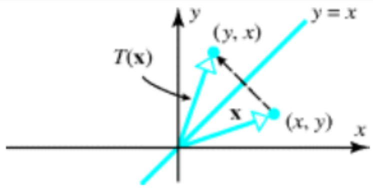
2- Reflection

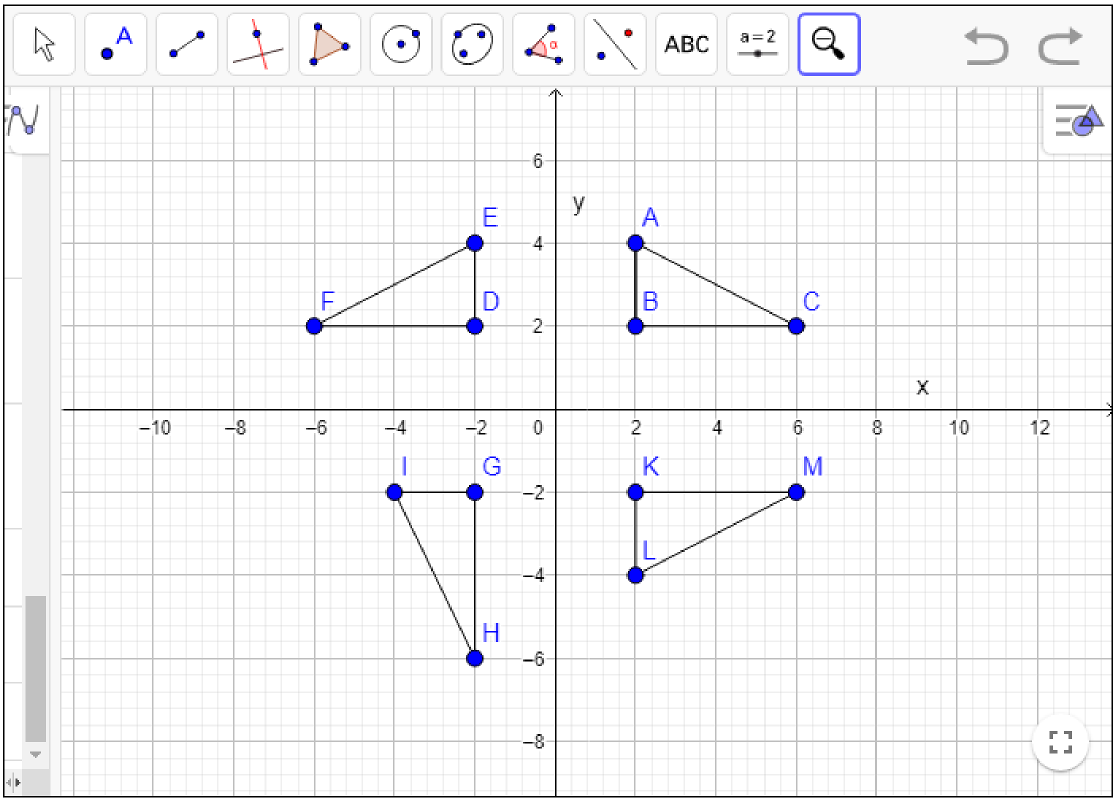
Let us find the images of the standard basis vectors e_1, e_2 for R^2 in column form.

Table 1

Operator	**Illustration**	**Images of $\mathbf{e}_1$ and $\mathbf{e}_2$**	**Standard Matrix**
Reflection about the y-acis $T(x, y) = (-x, y)$	 	$T(e_1) = T(1, 0) = (-1, 0)$ $T(e_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis $T(x, y) = (x, -y)$	 	$T(e_1) = T(1, 0) = (1, 0)$ $T(e_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$	 	$T(e_1) = T(1, 0) = (0, 1)$ $T(e_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 1. (Reflection of Triangle)

Reflect the triangle with vertices A= (2, 4), B = (2, 2), C = (6, 2) along x-axis, y-axis and y = -x.



Example 2. (Reflection of a line)

Let y = 2x +1 be a line. Find the reflection of that line along the line y = x.

Solution. Here

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, $\mathbf{T}\,(\vec{x}) = \mathbf{A}\vec{x}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

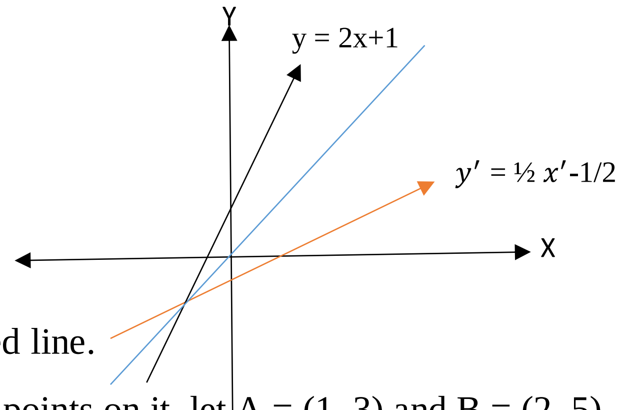
So, $x = y', \; y = x'$

Put x and y in original line y = 2x + 1

$x' = 2y' + 1$

Or $2y' = x' - 1$

So $y' = 1/2\, x' - 1/2$ is the reflected line.



To draw original line y = 2x + 1 take two points on it, let A = (1, 3) and B = (2, 5).

And to draw the Reflected line $y' = ½\, x' - ½$, A' = (2, 1/2) and B' = (4, 3/2).

Reflection of circle

Let $(x-2)^2 + (y-3)^2 = 4$ be a circle. Find its reflection along the line y = -x

Solution. The transformation of reflection is

$$\mathbf{T}\,(\vec{x}) = \mathbf{A}\vec{x}, \quad \forall\, \vec{x} \in \mathbf{R}^2$$

Where

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

So,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x' = -y \\ y' = -x \end{matrix}$$

Putting $x = -y', \; y = -x'$ in the original circle $(x - 2)^2 + (y - 3)^2 = 4$, we get

$(x' + 3)^2 + (y' + 2)^2 = 4$, reflected circle.

As original circle $(x - 2)^2 + (y - 3)^2 = 4$ is with Centre = (2, 3) and Radius = 2

While Reflected circle $(x' + 3)^2 + (y' + 2)^2 = 4$ has Centre = (-3, -2), Radius = 2. We can draw both circles easily.

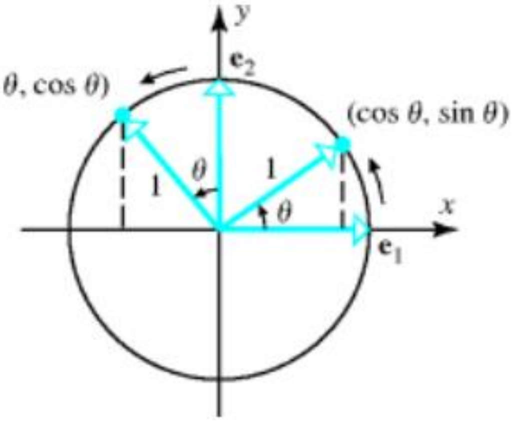
3-Rotation

Rotation about origin through an angle θ is a transformation $T: R^2 \to R^2$ defined as:

$$T(\vec{x}) = A\vec{x} \quad : \forall \vec{x} \in R^2$$

$$T(\mathbf{e}_1) = T(1, 0) = (\cos\theta, \sin\theta) \text{ and } T(\mathbf{e}_2) = T(0, 1) = (-\sin\theta, \cos\theta)$$

Where



$$A = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

- If the direction of θ is not defined, then it is understood to be in anticlockwise direction.
- If θ is in clockwise direction, then replace θ by $-\theta$ in the above definition as:

$$A = \begin{bmatrix} Cos(\theta) & Sin(\theta) \\ -Sin(\theta) & Cos(\theta) \end{bmatrix}$$

Example 1: Sketch the image of given rectangle with vertices A(0,0), B(3,0), C(3,2), D(0,2) under the rotation of 30^0 (anticlockwise).

Solution: As the transformation of rotation is

$$T(\vec{x}) = A\vec{x} \quad : \forall \vec{x} \in R^2$$

Where

$$A = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

As $\theta = 30^0$, so

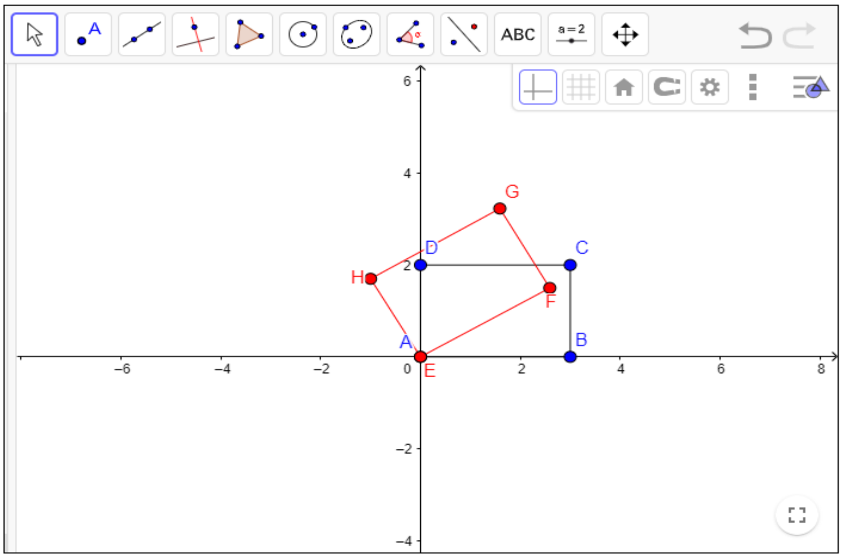
$$A = \begin{bmatrix} Cos(30^0) & -Sin(30^0) \\ Sin(30^0) & Cos(30^0) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

For point A: $$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For point B: $$T\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}\frac{3\sqrt{3}}{2}\\ \frac{3}{2}\end{bmatrix} = \begin{bmatrix}2.598\\1.5\end{bmatrix}$$

For point C: $$T\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}\frac{3\sqrt{3}}{2} - 1\\ \frac{3}{2} + \sqrt{3}\end{bmatrix} = \begin{bmatrix}1.599\\3.23\end{bmatrix}$$

For point D: $$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-1\\1.732\end{bmatrix}$$



Work to do:

Q1. Sketch the image of given parallelogram with vertices A(0,1), B(3,0), C(5,-2), D(2,-1) under the rotation of 90^0 (anticlockwise) .

Q2. Sketch the image of given triangle with vertices A(2,4), B(2,2), C(4,2) under the rotation of 90^0 (clockwise) .

Example 2. Let y = 2x+5 be a line. Find the equation of line after rotating it through an angle of $\frac{\pi}{2}$ clockwise direction about origin.

Solution: The matrix of rotation in clockwise direction is

$$A = \begin{bmatrix}Cos(\theta) & Sin(\theta)\\ -Sin(\theta) & Cos(\theta)\end{bmatrix}$$

As $\theta = \frac{\pi}{2}$, so

$$A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

and $$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

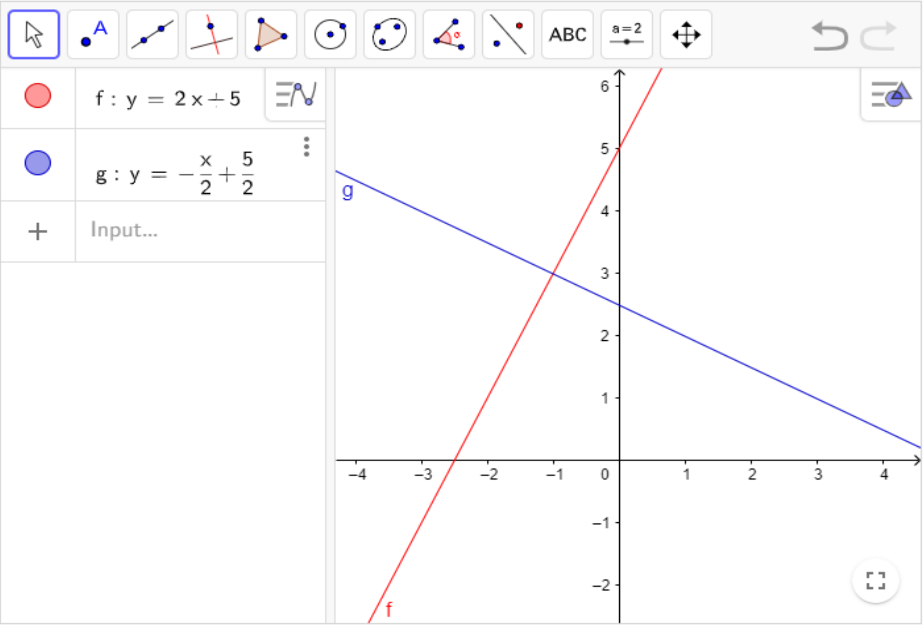
or $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

or $$\begin{cases} x' = y \\ y' = -x \end{cases} \Rightarrow \begin{cases} x = -y' \\ y = x' \end{cases}$$

Put value of x and y in original equation of line $y = 2x + 5$ and obtain

$$y' = -\frac{x'}{2} + \frac{5}{2}$$

This is the rotated line with angle $\frac{\pi}{2}$ in clockwise direction.



Work to do:

Q3. Let $y = -2x + 7$ be a line. Find the equation of line after rotating it through an angle of $180^0, 270^0$ clockwise direction about origin.

Example 3. Let $(x-4)^2 + (y-3)^2 = 9$ be a circle. Find the equation of circle after rotating it through an angle of 90^0 in anticlockwise direction about origin.

Solution: The matrix of rotation in anticlockwise direction is

$$A = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

As $\theta = 90^0$, so

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore, for $$T(\vec{x}) = A\vec{x} \ : \forall \ \vec{x} \in R^2$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

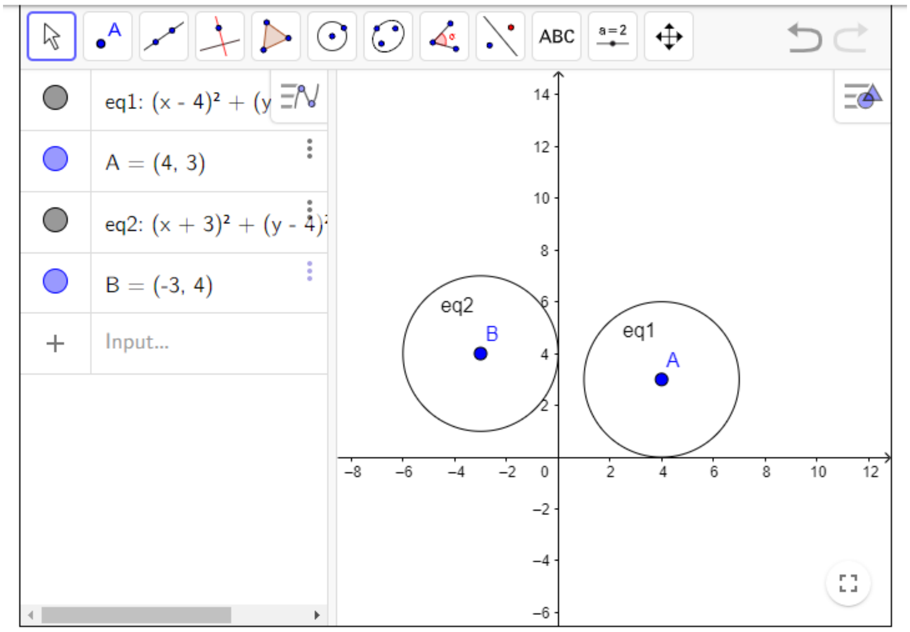
Or $\begin{cases} x' = -y \\ y' = x \end{cases} \Rightarrow \begin{cases} x = y' \\ y = -x' \end{cases}$

Putting these values of x and y in original equation of circle

$$(x-4)^2 + (y-3)^2 = 9$$

We get $(x'+3)^2 + (y'-4)^2 = 9$

This is the equation of rotated circle with angle $\frac{\pi}{2}$ in anticlockwise direction.



Work to do:

Q4. Let $(x-4)^2 + (y-3)^2 = 9$ be a circle. Find the equation of circle after rotating it through an angle of $180^0, 270^0$ in clockwise direction about origin.

Example 4: Let $\frac{x^2}{(4)^2} + \frac{y^2}{(3)^2} = 1$ be an ellipse. Find the equation of ellipse after rotating it through an angle of 90^0 in anticlockwise direction about origin.

Solution: The transformation of rotation in anticlockwise direction is

$$T(\vec{x}) = A\vec{x} \quad : \forall \vec{x} \in R^2$$

$$T(\vec{x}) = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix} \vec{x}$$

As $\theta = 90^0$, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

Hence, $\begin{cases} x' = -y \\ y' = x \end{cases}$ or $\begin{cases} x = y' \\ y = -x' \end{cases}$

Put these values of x and y in original equation of ellipse, we get the rotated ellipse with angle $\frac{\pi}{2}$ in anticlockwise direction as

$$\frac{x'^2}{9} + \frac{y'^2}{16} = 1$$

For plotting we can neglect the dash (ʻ) from our rotated equation of ellipse.

Original Ellipse

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

Major axis is along x-axis

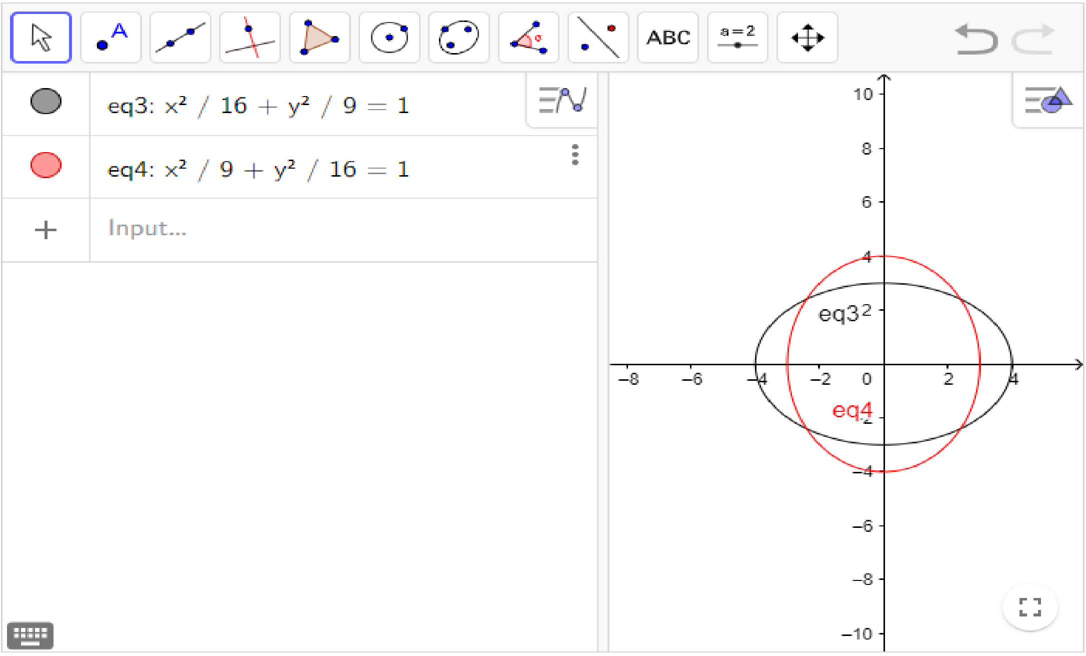
$a = \pm 4, \quad b = \pm 3$

Rotated Ellipse

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

Major axis is along- yaxis

$a = \pm 3, \quad b = \pm 4$



Q 5. Let $\frac{x^2}{9} + \frac{y^2}{16} = 1$ be an ellipse. Find the equation of ellipse after rotating it through an angle of 180^0 in anticlockwise direction about origin.