Affine Transformations

An affine transformation is a transformation $T: R^2 \to R^2$, defined by

$$T(\vec{x}) = A\vec{x} + \vec{a} \ ; \qquad \forall x \in R^2$$

For some $a \in R^2$, where A is 2×2 invertible matrix i.e. $AA^{-1} = I$.

Remarks:

1. Every orthogonal matrix is invertible but an invertible matrix may or may not be orthogonal.
2. Euclidean geometry is a subset of affine geometry or Affine transformations are the generalization of Euclidean transformation.

Types of Affine transformation:

1- Scaling, 2- Stretching, 3- Shearing

1- Scaling

A transformation $T: R^2 \to R^2$, defined by:

$$T(\vec{x}) = A\vec{x}; \qquad \forall x \in R^2$$

Where

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

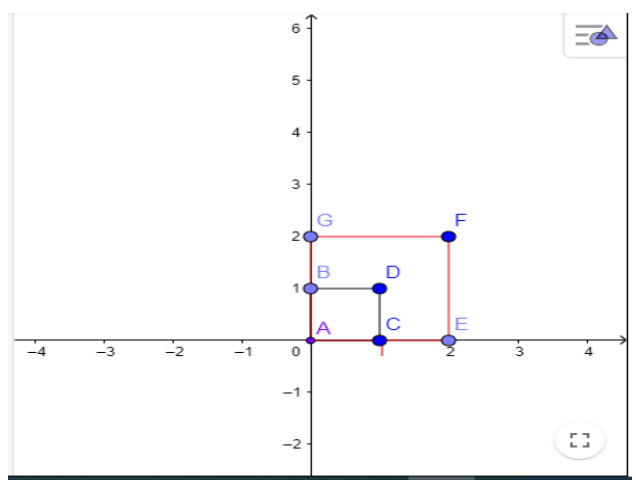
Is said to be a scaling by the factor k.

Note: **1.** If k>1, then its dilation/enlargement.

2. If $0 \le k \le 1$, then its contraction.

Example 1: (Scaling of Square)

A square with vertices $(0,0), (0,1), (1,0), (1,1)$. Scale this square by factor 2.



Solution: As

$T(\vec{x}) = A\vec{x} : \forall \ \vec{x} \in R^2$

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$$

Example 2: (Scaling of Line)

Let $y = 2x + 3$ be a line. Determine the image of this line under the scaling by factor 3.

Solution: As $T(\vec{x}) = A\vec{x} : \forall\, \vec{x} \in R^2$

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3&0\\0&3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3x\\3y\end{bmatrix}$$

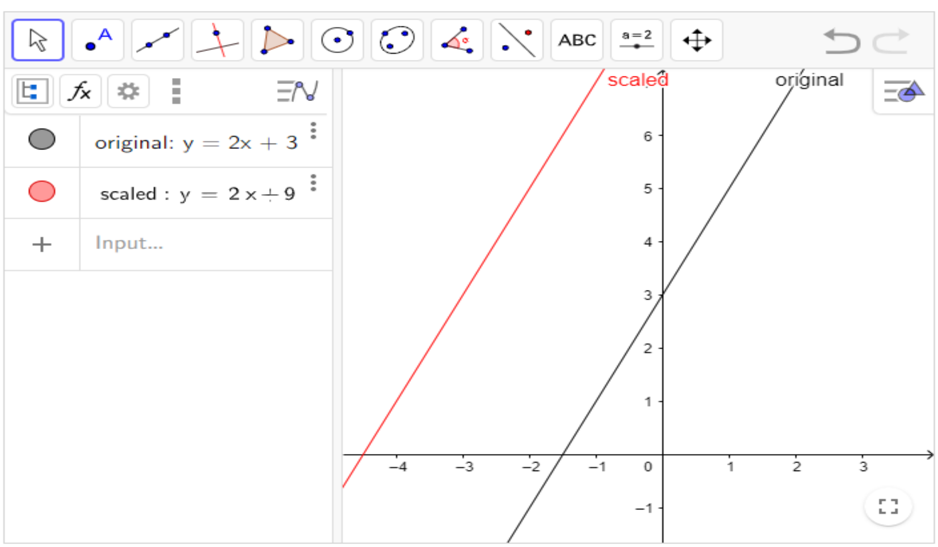
$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}3x\\3y\end{bmatrix}$$

$$\begin{cases}x' = 3x\\ y' = 3y\end{cases} \Longrightarrow \begin{cases}x = \frac{x'}{3}\\ y = \frac{y'}{3}\end{cases}$$

Put value of x and y in original equation of line: $y = 2x + 3$,

And get the equation of scaled line with factor 3.

$$y' = 2x' + 9$$



Example 3: (Scaling of Circle)

Determine the image of the circle

$$(x-2)^2 + (y-3)^2 = 4$$

under the scaling by factor 2.

Solution: As $$T(x) = Ax : \forall\, x \in R^2$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

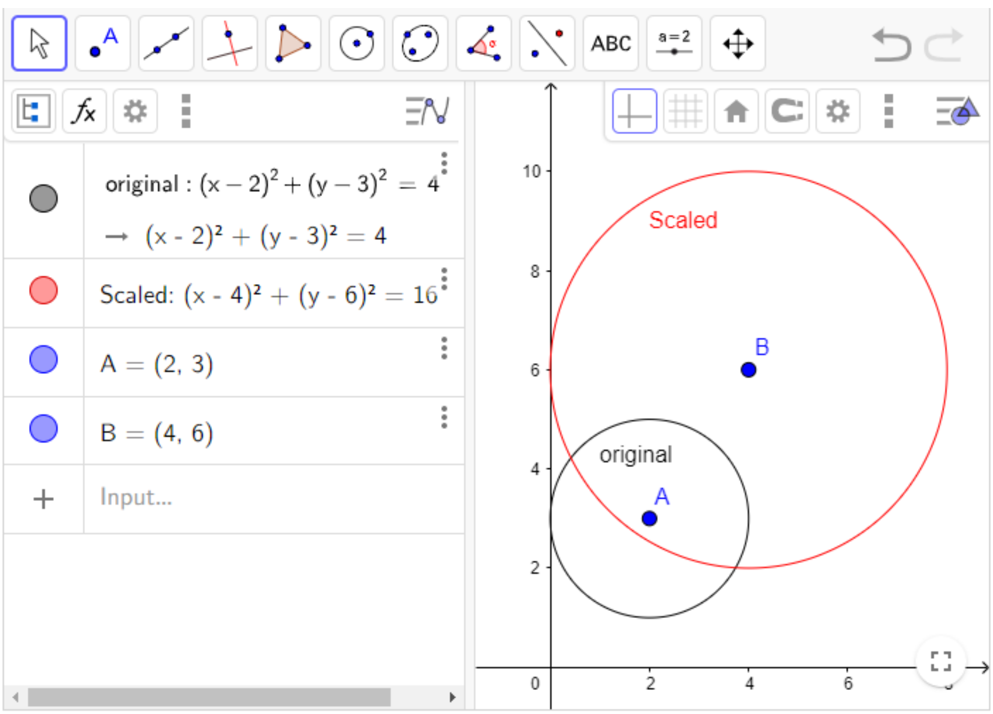
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{cases} x' = 2x \\ y' = 2y \end{cases} \Longrightarrow \begin{cases} x = \dfrac{x'}{2} \\ y = \dfrac{y'}{2} \end{cases}$$

Put value of x and y in original equation of circle:$(x-2)^2 + (y-3)^2 = 4$, implies

$$(x'-4)^2 + (y'-6)^2 = 16$$

This is the equation of circle by factor 2.



Example 5: (Scaling of an Ellipse)

Determine the image of an ellipse

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

under the transformation of scaling by factor 2.

Solution: As $$T(x) = Ax : \forall\, x \in R^2$$

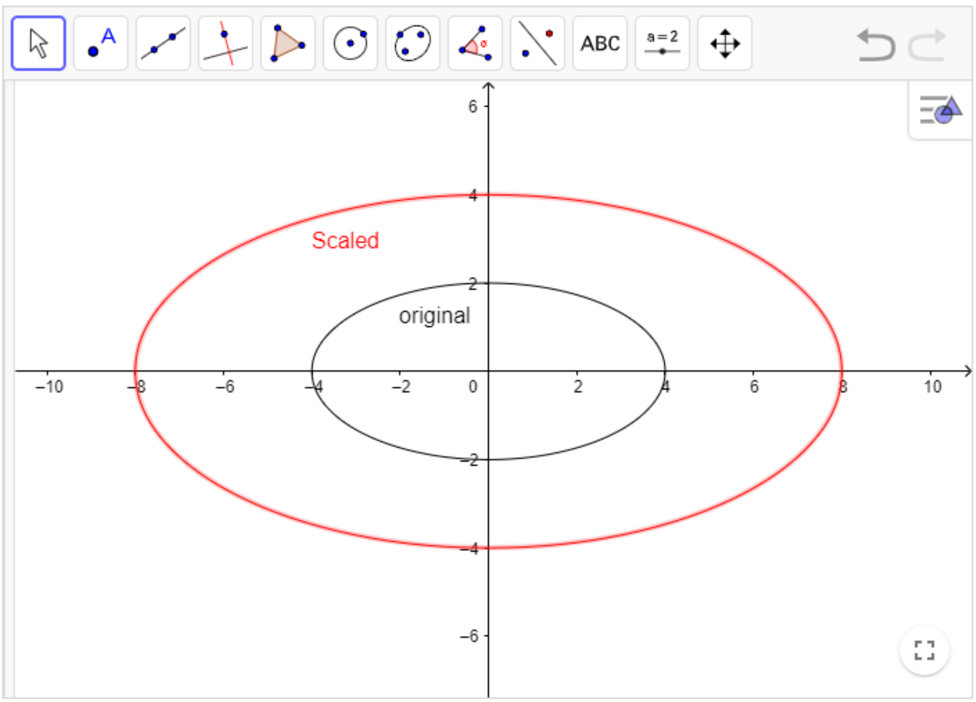
$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}2x\\2y\end{bmatrix}$$

$$\begin{cases}x' = 2x\\ y' = 2y\end{cases} \Longrightarrow \begin{cases}x = \frac{x'}{2}\\ y = \frac{y'}{2}\end{cases}$$

Put value of x and y in original equation of ellipse and get an ellipse scaled by factor 2.

$$\frac{x'^2}{64}+\frac{y'^2}{16}=1$$



Practice Problems

Q1. Find the Image of a triangle with vertices A= (2, 2), B = (4, 3), C = (5,5), when these are scaled by factor 2.

Q2. Find the Image of line$y = 2x + 5$ under the transformation of scaling by factor 3.

Q3. Find the image of an ellipse $\frac{(x-2)^2}{4}+\frac{(y-3)^2}{9}=1$under the transformation of scaling by factor 2.

2-Stretching

A transformation $T: R^2 \to R^2$ defined by,

$$T(\vec{x}) = A\vec{x}; \qquad \forall \vec{x} \in R^2$$

Is said to be stretching along x-axis by factor k if

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

And along y-axis by a factor k if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Example 1: (Stretching of a square)

Find the image of a square with vertices $A(0,0), B(1,0), C(1,1), D(0,1)$, when it is stretched along

a) X-axis with factor 3.
b) Y-axis with factor 2.

Solution: (along x-axis) As $\quad T(\vec{x}) = A\vec{x}; \qquad \forall \vec{x} \in R^2$

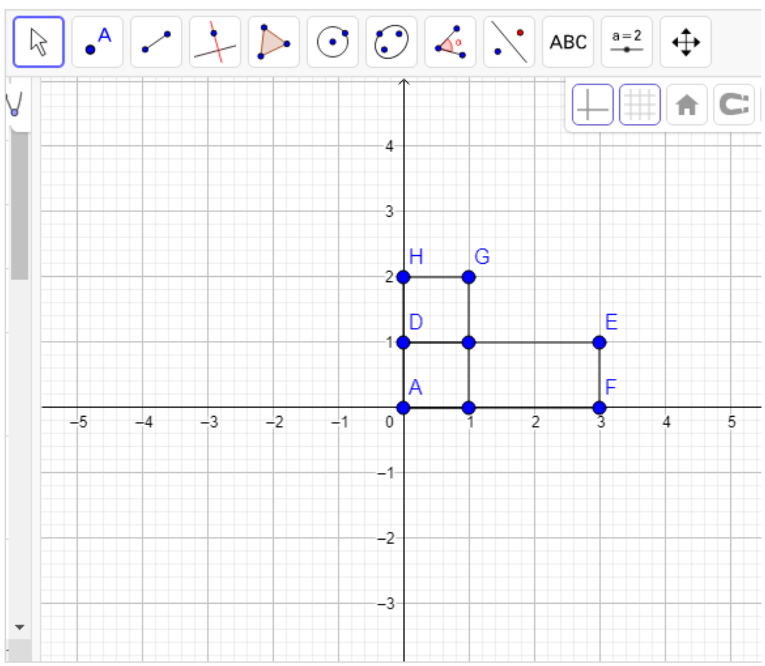
$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Along y-axis

As $\quad T(\vec{x}) = A\vec{x}; \qquad \forall \vec{x} \in R^2$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$$

Example 2: (Stretching of triangle)

Find the image of a triangle with vertices $A(1,1), B(4,1), C(3,4)$, when it is stretched along

a) X-axis with factor $\frac{3}{2}$.
b) Y-axis with factor $\frac{1}{2}$.

Solution: (along x-axis) As

$$T(\vec{x})=A\vec{x};\qquad \forall\vec{x}\in R^2$$

$$A=\begin{bmatrix}\frac{3}{2}&0\\0&1\end{bmatrix}$$

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\1\end{bmatrix}$$

$$T\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}&0\\0&1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}6\\1\end{bmatrix}$$

$$T\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}\frac{3}{2}&0\\0&1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}\frac{9}{2}\\4\end{bmatrix}$$

Along y-axis: As

$$T(\vec{x})=A\vec{x};\qquad \forall\vec{x}\in R^2$$

$$A=\begin{bmatrix}1&0\\0&\frac{3}{2}\end{bmatrix}$$

$$T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{3}{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\\frac{3}{2}\end{bmatrix}$$

$$T\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{3}{2}\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}4\\\frac{3}{2}\end{bmatrix}$$

$$T\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}$$

Example 3: (Stretching of Circle)

Let $(x-2)^2+(y-1)^2=4$ be a circle. Find its equation and image under stretching along x-axis by factor 2.

Solution: As

$$T(\vec{x})=A\vec{x};\qquad \forall\vec{x}\in R^2$$

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}2x\\y\end{bmatrix}$$

$$\begin{cases}x'=2x\\y'=y\end{cases}\Longrightarrow\begin{cases}x=\frac{x'}{2}\\y=y'\end{cases}$$

Put value of x and y in original equation of circle and get

$$\frac{(x'-4)^2}{16}+\frac{(y'-1)^2}{4}=1$$

It represents the ellipse.

Practice Problems

Q1. Let $(x-2)^2+(y-1)^2=4$ be a circle. Find its equation and image under stretching along Y-axis by factor $\frac{3}{2}$.

Q2. Find the stretching of an ellipse $\frac{(x-4)^2}{16}+\frac{(y-1)^2}{4}=1$ along x-axis by factor 2.

Q3. Find the image of a triangle with vertices $A(1,1), B(4,2), C(2,3)$, when it is stretched along

a) X-axis with factor 3
b) Y-axis with factor 1.

3- Shearing

Shearing of a point along x-axis by factor k can be defined as:

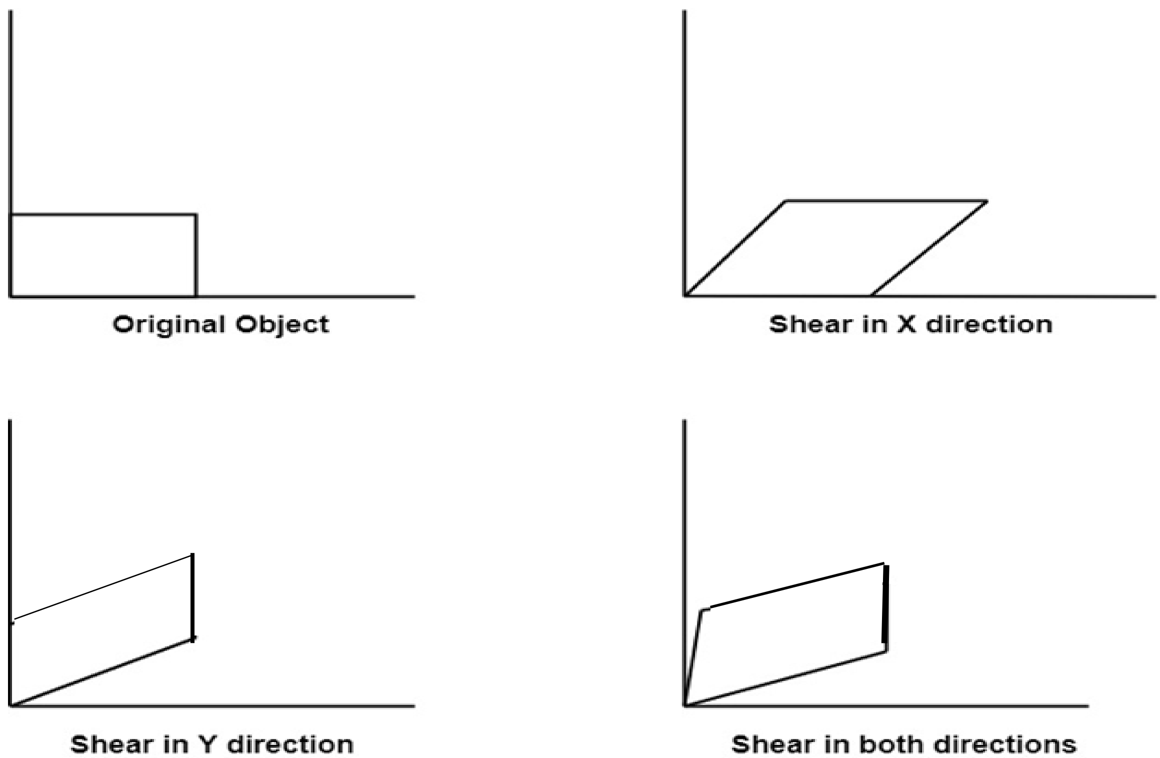
The movement of point parallel to x-axis keeping the distance of the point from the x-axis. i.e. the movement of point parallel to x-axis depending upon the y-component of the point.

- Shearing at a point along x-axis by factor k in the matrix form can be represented as

$$T(\vec{x}) = A\vec{x}; \qquad \forall \vec{x} \in R^2$$

Where

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Example 1: (Shearing of vector)

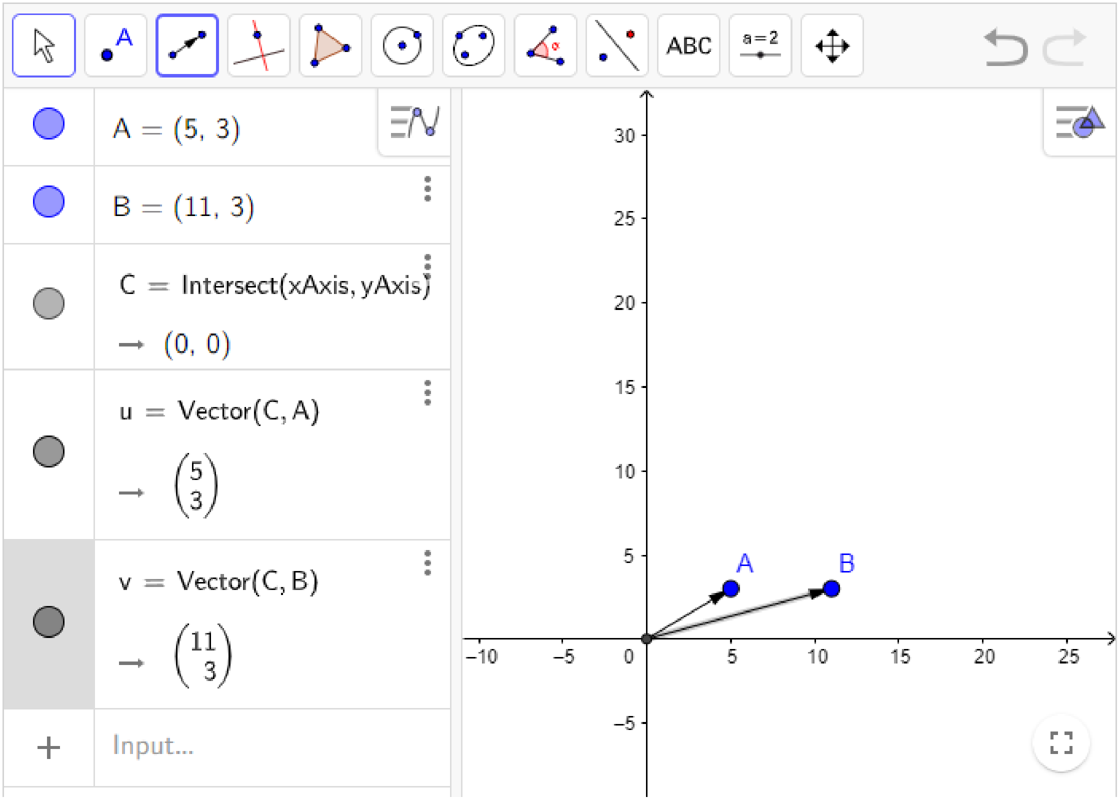
Shear a point $A = (5,3)$ along x-axis by the factor 2.

Solution: As

$$T(\vec{x}) = A\vec{x}; \qquad \forall \vec{x} \in R^2$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

$$T\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$$



Note:

1. A transformation that slants the shape of an object is called the shear transformation.
2. Shearing of an object along y-axis by factor k in matrix form can be expressed as:

$$T(\vec{x}) = A\vec{x}; \qquad \forall x \in R^2$$

Where

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Example 2: (Shearing of Triangle)

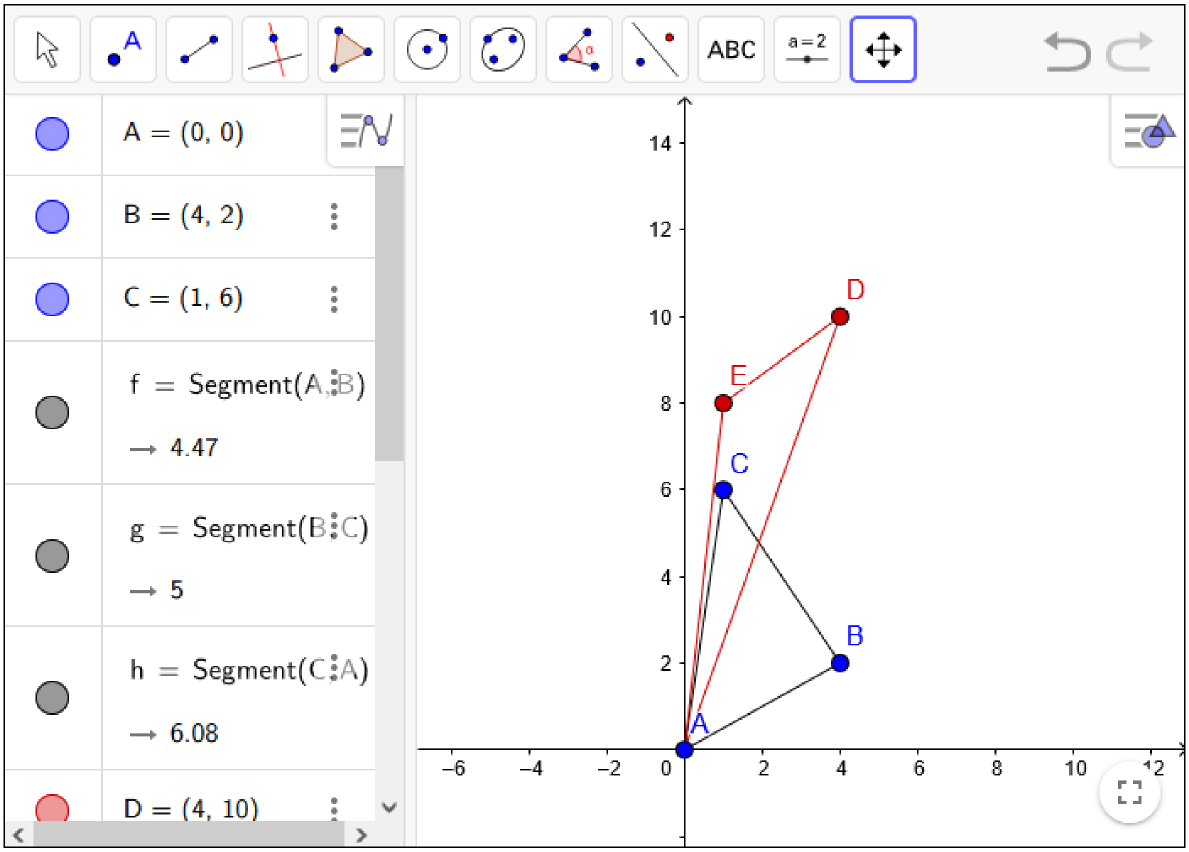
A triangle with vertices $(0,0), (4,2), (1,6)$. Find the image of triangle when it will be shear along y axis by factor 2.

Solution: As $T(\vec{x}) = A\vec{x}; \quad \forall \vec{x} \in R^2$

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$



Example 3: (Shearing of Rectangle)

Consider the rectangle with vertices $(0,0), (0,2), (4,0), (4,2)$. Shear this rectangle along x-axis by factor 2.

Example 4: (Shearing of Rectangle)

Consider the rectangle with vertices $(0,0), (0,2), (4,0), (4,2)$. Shear this rectangle along y-axis by factor -3.

Example 5:

In each part describe the matrix operator corresponding to A and show it effect on unit square

a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $c)$ $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Example 6: Let $x^2 + y^2 = 4$ be a circle. Find its equation and image under the effect of shear parallel to x-axis by factor -2.

Solution: As

$$T(\vec{x}) = A\vec{x}; \quad \forall \vec{x} \in R^2$$

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}x-2y\\y\end{bmatrix}$$

$$\begin{cases}x'=x-2y\\ y'=y\end{cases} \Rightarrow \begin{cases}x=x'+2y'\\ y=y'\end{cases}$$

Put value of x and y in original equation of circle: $x^2+y^2=4$

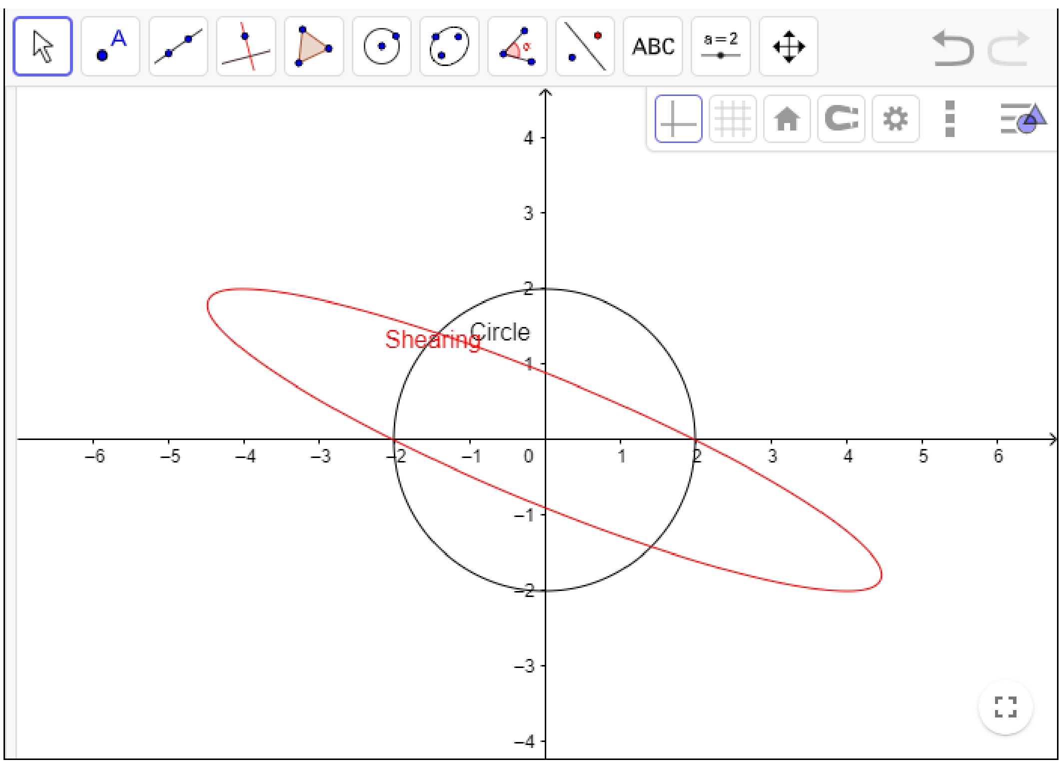
We get sheared equation of circle as (an ellipse)

$$x'^2+5y'^2+4x'y'-4=0$$

Remarks: General equation of Conics:

$$\boldsymbol{Ax^2+Bxy+cy^2+Dx+Ey+F=0}$$

1. If $B^2-4AC<0, we\ get\ ellipse.$
2. If $B^2-4AC>0, we\ get\ hyperbola.$
3. If $B^2-4AC=0, we\ get\ parabola.$



Example 6: Let $(x+2)^2+(y+1)^2=4$ be a circle. Find its equation and image under the effect of shear parallel to y-axis by factor $\frac{3}{2}$.

Solution: As

$$T(\vec{x}) = A\vec{x}; \qquad \forall \vec{x} \in R^2$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{2}x + y \end{bmatrix}$$

$$\begin{cases} x' = x \\ y' = \frac{3}{2}x + y \end{cases} \Rightarrow \begin{cases} x = x' \\ y = y' - \frac{3}{2}x' \end{cases}$$

Put value of x and y in original equation of circle:

$$(x+2)^2 + (y+1)^2 = 4$$

$$\frac{13}{4}x^2 + y^2 - 3xy - 3x + 2y + 1 = 0$$

On Comparing with General equation of Conics, we get:

$$A = \frac{13}{4}, B = -3, C = 1$$

$$B^2 - 4AC = (-3)^2 - 4\left(\frac{13}{4}\right)(1) = 9 - 13 = -4 < 0$$

So we get ellipse.

Example 7: Let $(x-2)^2 + (y-1)^2 = 4$ be a circle. Find its equation and image under the effect of shear parallel to x-axis by factor 2.