The Trapezium Rule

Saad Mehmood

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1 Introduction

The Trapezium Rule, also known as the trapezoidal rule, is a numerical method used to approximate the definite integral of a function. Unlike other methods, such as the midpoint or rectangular rule that use rectangles, the trapezium rule divides the area under the curve into trapezoidal segments. Summing the areas of these trapezoids generally provides a more accurate approximation, particularly when the function being integrated is smooth. This method is often employed when an exact analytical solution is either complex or unattainable.

2 Problem Setup

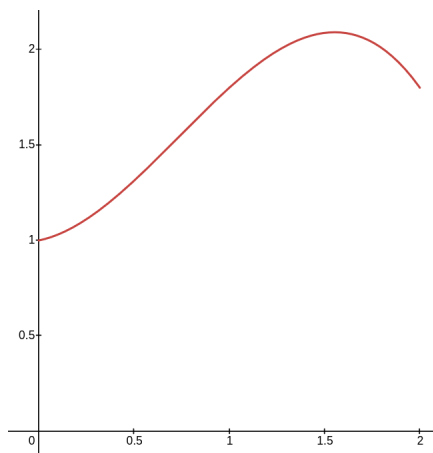Consider a curve described by the function $f(x)$:

Figure 1: An arbitrary curve.

To compute the area under the curve, one would typically evaluate the definite integral:

$$I = \int_a^b f(x)\,dx \tag{2.1}$$

However, exact integration is not always feasible. In such cases, numerical methods such as the trapezoidal rule are employed. This method approximates the area by dividing it into trapezoidal sections, which is conceptually similar to the Riemann sum but replaces rectangles with trapezoids. Figure 2 illustrates this process.
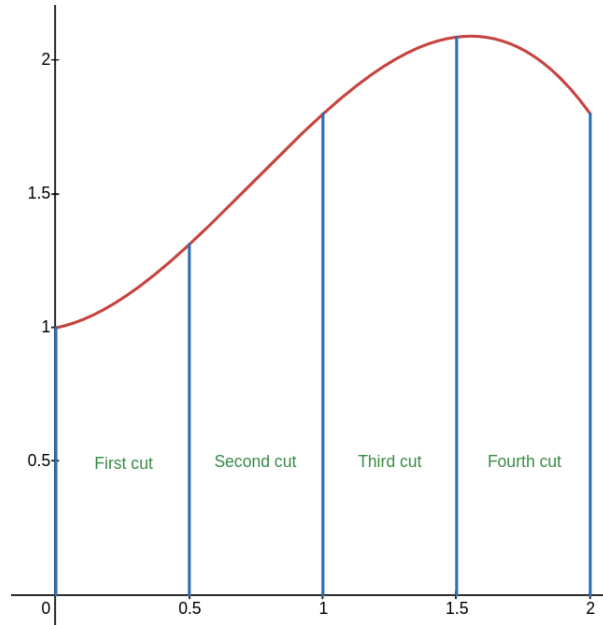

Figure 2: Dividing the area into trapezoidal segments.

As shown in Figure 2, the segments can be approximated as trapezoids. While some segments, such as the second cut, closely resemble trapezoids, others, like the first and fourth cuts, may deviate slightly. These deviations result in small errors. Nevertheless, reducing the step size h can significantly decrease the error. For instance, halving the step size from 0.5 to 0.25 results in shapes that more closely resemble ideal trapezoids. This refinement is illustrated in Figure 5.

To compute the total area, the areas of the trapezoidal segments are summed. For accurate results, it is essential to use a sufficiently small step size. Although smaller steps result in a greater number of trapezoids, possibly numbering in the hundreds or thousands, the process can be simplified by employing a generalized mathematical expression.
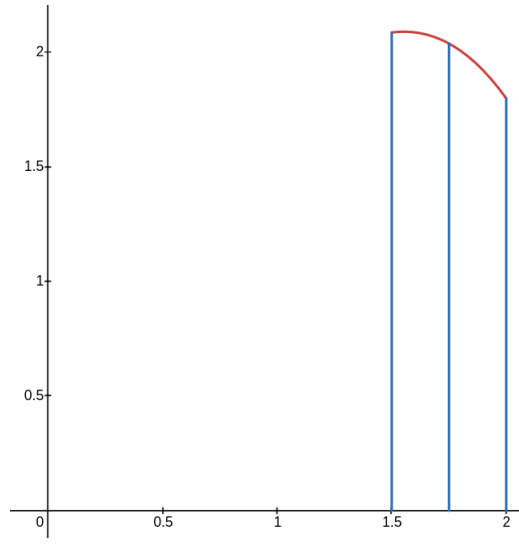
Figure 3: Refining the fourth trapezoidal segment.

3   Generalized Equation

The area of a single trapezoid is given by:

$$A = \left[\frac{a + b}{2}\right] h \tag{3.1}$$

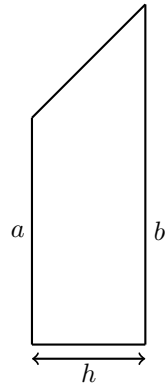where a and b are the lengths of the parallel sides, and h is the height.

Figure 4: A right-angled trapezoid.

For the case illustrated in Figure 2, where each trapezoid shares one parallel side with an adjacent trapezoid, the total area can be expressed as:

$$A_{\text{total}} = \left[\frac{a_0 + b_1}{2}\right] h + \left[\frac{a_1 + b_2}{2}\right] h + \left[\frac{a_2 + b_3}{2}\right] h + \left[\frac{a_3 + b_4}{2}\right] h \qquad (3.2)$$

where a_n and b_n represent the lengths of the parallel sides at specific points. If the function is known, these lengths can be expressed as function values at those points, resulting in:

$$A_{\text{total}} = \left[\frac{f(0) + f(0.5)}{2}\right] h + \left[\frac{f(0.5) + f(1)}{2}\right] h + \left[\frac{f(1) + f(1.5)}{2}\right] h + \left[\frac{f(1.5) + f(2)}{2}\right] h \quad (3.3)$$

By factoring out $\frac{h}{2}$, we obtain the following:

$$A_{\text{total}} = \frac{h}{2} \left[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)\right] \qquad (3.4)$$

This can be generalized to:

$$A_{\text{total}} = \frac{h}{2} \left[(f(0) + f(2)) + 2(f(0.5) + f(1) + f(1.5))\right] \qquad (3.5)$$

Thus, the general form of the trapezium rule for any domain is:

$$A_{\text{total}} = \frac{h}{2} \left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \qquad (3.6)$$

4 Algorithm

Using Eq. (3.6), the area under a curve for a specific domain can be computed by following these steps:

1. Define the domain, x_i and x_f.

2. Discretize the domain into N intervals.

3. Compute the step size:
$$h = \frac{x_f - x_i}{N}$$

4. Sum the function values from x_1 to x_{n-1}.

5. Apply Eq. (3.6) to compute the total area.

5   Example Problem

Consider the function $f(x) = e^{-2x}$:
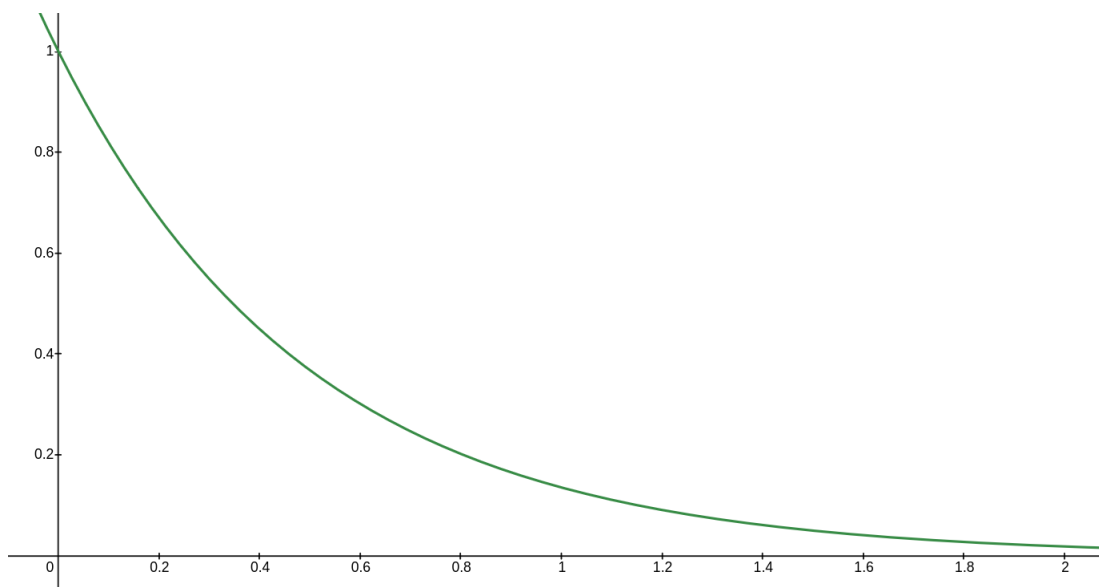
$$f(x) = e^{-2x} \tag{5.1}$$

Figure 5: Plot of the function e^{-2x}.

The exact area under the curve for $f(x)$ over the domain $[0, 2]$ is 0.49084. Applying the trapezium rule with 10 intervals yields an area of 0.49190, resulting in a percentage error of 0.217%, which is well within acceptable limits. As shown in Table 1, increasing the number of intervals further reduces the error.

Number of Intervals	Exact Area	Computed Area	% Error
10	0.49084	0.49190	0.217
50	0.49084	0.49031	0.108
100	0.49084	0.49053	0.064
500	0.49084	0.49077	0.014
1000	0.49084	0.49081	0.007

Table 1: Comparison of the number of intervals and percentage error.

6 Computational Problems

The accuracy of the trapezium rule is directly tied to the step size, h. A larger step size can result in significant errors. To achieve a higher degree of accuracy, the step size must be reduced, leading to a larger number of trapezoids. However, smaller step sizes increase the computational cost, as more function evaluations are required.

Furthermore the trapezium rule works best for smooth and continuous functions. If the function has discontinuities, sharp peaks, or highly oscillatory behavior, the trapezium rule may struggle to provide accurate results, even with a reduced step size. In such cases, other numerical integration methods may perform better.