Solution Manual for Analysis I

Dean F. Valentine Jr.

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1 Beginning Chapter

2 Natural Numbers

3 Set Theory

3.1 Fundamentals

1. Reflexivity:

   (a) Equality between sets is defined as: $(\forall x \in A : x \in B) \wedge (\forall x \in B : x \in A)$

   (b) A set is defined as a collection of objects.

   (c) It is a contradiction for any object x to satisfy $x \in A$ be true but also $x \notin A$, so equality of sets is reflexive ($A = A$), meaning all objects that satisfy the claim $x \in A$ also satisfy $x \in A$, and "vice-versa"

   Symmetry:

   (a) To say that sets A and B are equal, as per our definition of equality, is to say $\forall x \in A : x \in B \wedge \forall x \in B : x \in A$.

   (b) So, if $A = B$, and by definition $\forall x \in A : x \in B \wedge \forall x \in B : x \in A$, then we can just swap the given claims around as show that $\forall x \in B : x \in A \wedge \forall x \in A : x \in B$, and see that B = A.

   Transitivity:

   (a) Suppose that among three sets A,B, and C, sets A = B, and B = C

   (b) Consider $\exists x : x \in A$

   (c) Since $A = B$, $\forall y \in A : y \in B$, and so $x \in B$

   (d) But also because $B = C$, $\forall z \in B, z \in C$, and so also $x \in C$

   (e) Thus, $\forall x \in A, x \in C$, and the same logic can be applied in reverse to find $\forall x \in C, x \in A$, and so $A = C$

2. (a) Axiom 3.2 says that there exists a set $\emptyset$, the set of no elements.

   (b) Axiom 3.1 says that if A is a set, it is also an object, and by the definition of sets (3.1.1), there exists the set X such that $A \in X$. Thus, there exists some set A_1 which contains the object $\emptyset$

   (c) Two sets are not equal if they do not share each others objects. A_1 contains $\emptyset$, but $\emptyset$ is defined as having lacking objects, so these two sets are not equal.

   (d) Similarly we can consider the set A_2 which contains only A_1, which is not equal to either A_1 or $\emptyset$ because neither of them contain A_1, and instead contain $\emptyset$ and no objects, respectively.

   (e) Lastly we think of a set $A_3 : \emptyset \in A_3 \wedge A_2 \in A_3$, and see that this does not contain the sam objects as $\emptyset$, A_1, or A_3. $\emptyset \notin A_2$, $A_1 \notin A_1$, and neither $\emptyset$ nor A_1 are in $\emptyset$ so at this point we have proven 3.2.

3. Proof $\{a, b\} = \{a\} \cup \{b\}$:

   (a) The union between sets A and B as defined in Axiom 3.4 is set whose elements are contained by A or B or both.

   (b) $\{a\} \cup \{b\}$ contains a, because $a \in \{a\}$

   (c) $\{a\} \cup \{b\}$ contains b, because $b \in \{b\}$

   (d) There can be no other objects in $\{a\} \cup \{b\}$, because all objects in $\{a\} \cup \{b\}$ must either be in $\{a\}$ or $\{b\}$ or both by the definition of a union, and the existence of a third object would imply that $\{a\}$ and $\{b\}$ contained altogether three distinct objects, a contradiction.

   Proof of commutativity:

   (a) Suppose $\exists x \in A \cup B$. That means by Axiom 3.4 $x \in A \vee x \in B$. If $x \in A$, then $x \in B \cup A$, because $B \cup A$ is the set of all elements either in B or A. If $x \in B$, then by the same logic $x \in B \cup A$ as well, and so we can say $\forall x \in A \cup B : x \in B \cup A$, and we have proven their equality.

   Proof $A \cup A = A$

   (a) For the sake of contradiction assume $A \neq A \cup A$

   (b) This implies $\exists x : x \in A \wedge x \notin A \cup A$.

   (c) But $x \notin A \cup A \rightarrow x \notin A \wedge x \notin A$.

   (d) Both of these (equivalent) implied statements contradict the earlier one, that $\exists x : x \in A \wedge x \notin A \cup A$.

   Proof $A \cup \emptyset = A$ and $\emptyset \cup A = A$

   (a) $\forall x \in A \cup \emptyset : x \in A \vee x \in \emptyset$, by Axiom 3.4

   (b) By axiom 3.2, $\neg \exists x \in \emptyset$.

(c) This means that $\forall x \in A \cup \emptyset : x \in A$. Now we must prove $\neg \exists x \in A : x \notin A \cup \emptyset$.

(d) Suppose for sake of contradiction $\exists x \in A : x \notin A \cup \emptyset$.

(e) This would mean $x \notin A \wedge x \notin \emptyset$, by Axiom 3.4.

(f) x is obviously not in the empty set, but by our assumption $x \in A$, meaning we have contradicted ourselves.

(g) The sum of the two statements we have just proved is equivalent to saying $A = A \cup \emptyset$. Because of commutitivity this also means $A = \emptyset \cup A$.

4. Proof $A \subseteq B \wedge B \subseteq C \implies A \subseteq C$:

   (a) $A \subseteq B \equiv \forall x \in A : x \in B$

   (b) $B \subseteq C \equiv \forall x \in B : x \in C$

   (c) Thus, $x \in A \implies x \in B$, and $x \in B \implies x \in C$, so $x \in A \implies x \in B \implies x \in C$ or $x \in A \implies x \in C$

   Proof $A \subseteq B \wedge B \subseteq A \implies A = B$

   (a) $A \subseteq B \equiv \forall x \in A : x \in B$

   (b) $B \subseteq A \equiv \forall x \in B : x \in A$

   (c) $\forall x \in A : x \in B \wedge \forall x \in B : x \in A \equiv A = B$, so these starting propositions combined are the definition of equality established earlier in the chapter.

   Proof $A \subset B \wedge B \subset C \implies A \subset C$:

   (a) $A \subset C \equiv \forall x \in A : x \in C \wedge A \neq C$, so we must prove two things: $A \neq C$, and $\forall x \in A : x \in C$

   (b) $A \subset B \equiv \forall x \in A : x \in B \wedge A \neq B$

   (c) $A \neq B \implies \exists x \in A : x \notin B \vee \exists x \in B : x \notin A$

   (d) Since by the definition of a subset $\neg \exists x \in A : x \notin B$, this and the relation $A \neq B$ implies $\exists x \in B : x \notin A$

   (e) $B \subset C \equiv \forall x \in B : x \in C \wedge B \neq C$

   (f) Since $\forall x \in B : x \in C$ and $\exists x \in B : x \notin A$, $\exists x \in C : x \notin A$, and $C \neq A$.

   (g) But also, $\forall x \in A : x \in B \wedge \forall x \in B : x \in C$. This means $x \in A \implies x \in B \implies x \in C$, and we see $\forall x \in A : x \in C$.

   (h) These two statements ($A \neq C$, and $\forall x \in A : x \in C$) are equivalent to saying ($A \subset C$), and so we have finished our proof.

5. Proof $A \subseteq B \implies A \cup B = B \wedge A \cap B = B$:

   (a) $A \subseteq B \equiv \forall x \in A : x \in B$

   (b) $A \cup B \equiv \{x \in A \vee x \in B\}$

(c) $\forall x \in A : x \in B$, so $\forall x \in A : x \in x \in A \vee x \in B$ and $\forall x \notin A : x \notin x \in A \vee x \in B$, proving that if $A \subseteq B \implies A \cup B = B$

(d) $A \cap B \equiv \{x \in A \wedge x \in B\}$

(e) $A \subseteq B \equiv \forall x \in A : x \in B$, so if $A \subseteq B$, then $\{x \in A \wedge x \in B\} = \{x \in B\}$

(f) $B \equiv \{x \in B\}$, so $A \subseteq B \implies A \cap B = B \wedge A \cup B = B$, and we have finished our proof.

Proof $A \cup B = B \implies A \subseteq B \wedge A \cap B = B$:

(a) $A \subseteq B \implies A \cap B = B$, as per earlier proofs, so we need only prove $A \cup B = B \implies A \subseteq B$

(b) $A \subseteq B \equiv \forall x \in A : x \in B$

(c) $B = A \cup B \equiv \forall x \in B : x \in A \cup B \wedge \forall x \in A \cup B : x \in B$

(d) $A \cup B \equiv \{x \in A \vee x \in B\}$, so

(e) $\forall x \in A \vee x \in B : x \in B$ and

(f) $\neg \exists x \in A : x \notin B$, the contrapositive of which is

(g) $\forall x \in A : x \in B$, and so we have proved our hypothesis.

Proof $A \cap B = A \implies A \subseteq B \wedge A \cup B = B$:

(a) $A \subseteq B \implies A \cup B = B$, as per earlier proofs, so we need only prove $A \cap B = A \implies A \subseteq B$

(b) $A \subseteq B \equiv \forall x \in A : x \in B$

(c) $A \cap B = A \equiv \{x \in A\} = \{x \in A \wedge x \in B\}$

(d) Thus, $A \cap B = A \implies \forall x \in A : x \in B$, or $A \cap B = B \implies A \subseteq B$, so we have already proved our hypothesis.

6. Let A, B, C be sets, and let X be a set containing A, B, C as subsets.
Proof $A \cup X = X \cap A = X$: See 3.1.5
Proof $A \cup A = A$:

(a) Suppose for sake of contradiction suppose $A \cup A \neq A$.

(b) This would mean either A contains an item not contain in $A \cup A$, or $A \cup A$ contains an item not contained in X.

(c) Suppose $\exists x \in A \cup A : x \notin A$

(d) $x \in A \cup A \equiv x \in A \vee x \in A$, which when simplified means $x \in A$.

(e) We have now already contradicted ourselves, by declaring $x \in A \wedge x \notin A$.

(f) Suppose then that $\exists x : x \in A \wedge x \notin A \cup A$

(g) We can just as easily see that $x \notin A \cup A \equiv x \notin A \wedge x \notin A$, equivalent to saying $x \notin A$.

(h) $x \notin A$ contradicts our earlier assumption that $x \in A$, and our premises must be incorrect, and $\neg\exists x : x \in A \wedge x \notin A \cup A$.

(i) Thus, $A \cup A$ has the same elements as A and is equivalent by our definition.

Proof $A \cap A = A$

(a) Suppose for sake of contradiction suppose $A \cap A \neq A$.

(b) This would mean either A contains an item not contain in $A \cap A$, or $A \cap A$ contains an item not contained in X.

(c) Suppose $\exists x : x \in A \cap A \wedge x \notin A$

(d) This means that $x \in A \wedge x \in A$, simplified $x \in A$.

(e) We have now already contradicted ourselves, by declaring $x \in A \wedge x \notin A$.

(f) Suppose then that $\exists x : x \in A \wedge x \notin A \cap A$

(g) We can just as easily see that $x \notin A \cap A \equiv x \notin A \vee x \notin A$, equivalent to saying $x \notin A$.

(h) $x \notin A$ contradicts our earlier assumption that $x \in A$, and our premises must be incorrect, and $\neg\exists x : x \in A \wedge x \notin A \cap A$.

(i) Thus, $A \cap A$ has the same elements as A and is equivalent by our definition.

Proof of Distributivity $(A \cap (B \cup C) = (A \cup B) \cup C)$

(a)

7. Proof $A \cap B \subseteq A \wedge A \cap B \subseteq B$

(a) $A \cap B = B \cap A$ because of commutativity, so we need only prove $A \cap B \subseteq A$.

(b) $(A \cap B) \equiv \{x \in A \wedge x \in B\}$

(c) Since $x \in x \in A \wedge x \in B \implies x \in A$, $\forall x \in (A \cap B) : x \in A$, which is the definition of $(A \cap B) \subseteq A$.

Proof $C \subseteq A \wedge C \subseteq B \iff C \subseteq A \cap B$

(a) This is a bidirectional equality, so we have to prove both that $C \subseteq A \cap B \implies C \subseteq A \wedge C \subseteq B$ and that $C \subseteq A \wedge C \subseteq B \implies C \subseteq A \wedge B$

(b) $C \subseteq A \cap B \equiv \forall x \in C : x \in A \wedge x \in B$

(c) $\forall x \in C : x \in A \wedge x \in B \implies \forall x \in C : x \in A$, so $C \subseteq A$. Similarly, $\forall x \in C : x \in A \wedge x \in B \implies \forall x \in C : x \in B$, so $C \subseteq B$. Thus, $C \subseteq A \cap B \implies C \subseteq A \wedge C \subseteq B$

(d) $C \subseteq A \wedge C \subseteq B \equiv \forall x \in C : x \in A \wedge \forall x \in C : x \in B$

(e) $\forall x \in C : x \in A \wedge \forall x \in C : x \in B = \forall x \in C : x \in A \wedge x \in B$, so we also have $\forall x \in C : x \in A \wedge x \in B$.

(f) $A \cap B$ is the set of elements $\forall x : x \in A \cap B$, so $\forall x \in C : x \in A \cap B$ or $C \subseteq A \cap B$, so $C \subseteq A \wedge C \subseteq B \implies C \subseteq A \cap B$. This concludes our proof that $C \subseteq A \cap B \iff C \subseteq A \wedge C \subseteq B$.

8. Proof $A \cap (A \cup B) = A$:

   (a) $\forall x \in A : x \in A \cup B$, because $A \cup B \equiv \{x \in A \vee x \in B\}$

   (b) $A \cap (A \cup B) \equiv \{x \in A \wedge x \in (A \cup B)\}$

   (c) As a result, $\forall x \in A : x \in A \cap (A \cup B)$

   (d) Also we see $\neg\exists x \in A \cap (A \cup B) : x \notin A$, because $x \in A \cap (A \cup B) \equiv x \in A \wedge x \in (A \cup B) \equiv x \in A \wedge (x \in A \vee x \in B)$. This proves $A \cap (A \cup B) = A$ and our hypothesis is correct.

   Proof $A \cup (A \cap B) = A$:

   (a) Using the properties of commutativity and associativity we can rearrange the equation like the following:

   (b) $A \cup (A \cap B) = (A \cap B) \cup A$

   (c) $(A \cap B) \cup A = A \cap (B \cup A)$

   (d) $A \cap (B \cup A) = A \cap (A \cup B)$, which was proven in an earlier exercise.

9. Let A, B, X be sets such that $A \cup B = X \wedge A \cap B = \emptyset$.
   Proof $A = X \setminus B$:

   (a) $A \cap B = \emptyset \equiv \forall x \in A \wedge x \in B : x \in \emptyset$

   (b) $\neg\exists x : x \in \emptyset$, so $\forall x \in A : x \notin B$

   (c) $X \equiv \{x : x \in A \vee x \in B\}$, and $X \setminus B \equiv \{x : x \in X \wedge x \notin B\}$, so $X \setminus B \equiv (A \cup B) \setminus B \equiv \{x : (x \in A \vee x \in B) \wedge x \notin B\}$

   (d) Since $\forall x \in X \setminus A : x \in (A \cup B) \wedge x \notin B$, $X \setminus B = \{x : (x \in A \vee x \in B) \wedge x \in B\} = \{x : (x \in A \wedge x \notin B) \vee x(\in B \wedge x \notin B)\} = \{x : x \in A \wedge x \notin B\}$

   (e) Again, $\forall x \in A : x \notin B$, so $A = x : x \in A \wedge x \notin B = X \setminus B$, and we have proven our hypothesis

   Proof $B = X \setminus A$

   (a) The $\cup$ and $\cap$ operators have the property of associativity so we need merely rearrange our starting assumptions to get:
   $A \cup B = B \cup A = X$ and
   $A \cap B = B \cap A = \emptyset$, and invoke our proof from the previous exercise.

10. Proof $A \setminus B, B \setminus A$, and $A \cap B$ are disjoint, and that their union is $A \cup B$:

(a) First we prove $A \setminus B$ is disjoint from $B \setminus A$ and $A \cap B$, and then that $A \cap B$ is disjoint from $B \setminus A$

(b) $A \setminus B \equiv \{x : x \in A \wedge x \notin B\}$

(c) $\forall x \in B \setminus A : x \notin A$, so $\forall x \in B \setminus A : x \notin A \setminus B$

(d) Similarly, $\forall x \in A \setminus B : x \notin B$, so $\forall x \in A \setminus B : x \notin B \setminus A$, and we can say that $A \setminus B$ and $B \setminus A$ are disjoint.

(e) By definition, $\forall x \in A \cap B : x \in A \wedge x \in B$

(f) This means that $\forall x \in A \cap B : x \notin B \setminus A$, because $\forall x \in B \setminus A : x \notin A$.

Proof $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B$

(a) $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) \equiv \{x : x \in A \wedge x \notin B\} \cup \{x : x \in B \wedge x \notin A\} \cup \{x : x \in A \wedge x \in B\}$

(b) $\{x : x \in A \wedge x \notin B\} \cup \{x : x \in B \wedge x \notin A\} = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\} = \{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$, thus:

(c) $\{x : x \in A \wedge x \notin B\} \cup \{x : x \in B \wedge x \notin A\} \cup \{x : x \in A \wedge x \in B\} = \{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \cup \{x : x \in A \wedge x \in B\}$, and:

(d) $\{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \cup \{x : x \in A \wedge x \in B\} = \{x : x \in A \vee x \in B\} = A \cup B$, finally proving also that $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B$

11. Proof axiom of replacement implis axiom of specification:

(a) The axiom of replacement states that for any set A and objects $x \in A$ and y "there exists a set $\{y : P(x, y)$ is true for some $x \in A\}$'.

(b) Given this axiom lets consider a possible property $P(x, y) = (y = x \wedge P(y))$, where $P(x)$ is an arbitrary property.

(c) Applied to the axiom of choice, this property proves the existence of the set $\{y : y = x \wedge P(y)$ for some $x \in A\}$.

(d) Applying the substitution property to the objects in question, we can rephrase this set as $\{x : x \in A \wedge P(x)\} \equiv \{x \in A : P(x)\}$. This is the axiom of of specification, and so we have derived it as a selective choice of $P(x, y)$.

3.2 Russell's Paradox

1. Proof Axiom of Universal Specification implies axiom 3.2

(a) There exists a set of elements $\{x : P(x)\}$, for any $P(x)$ valid for all possible objects (Axiom of Universal specification)

(b) Consider the property function $P(x) = False$

(c) This implies the existence of a set $\{x : False\}$, or a set with no elements. (Axiom 3.2)

Proof Axiom of Universal Specification implies axiom 3.3

(a) There exists a set of elements $\{x : P(x)\}$, for any $P(x)$ valid for all possible objects (Axiom of Universal specification)
(b) Consider the property function $P(x) = (x = y)$ where y is an arbitrary, unique object, either a number or a set itself.
(c) This implies the existence of a set $\{x : P(X)\}$, or $\{x : x = y\}$, the set of that one element. (Axiom 3.3 part 1)
(d) Also consider the property function $P(x) = (x = y \vee x = z)$, where again y and z are arbitrary unique objects.
(e) This implies the existence of a set $\{x : x = y \vee x = z\}$, consisting of the elements y and z, which themselves could be numbers or sets.

Proof Axiom of Universal Specification implies axiom 3.4

(a) There exists a set of elements $\{x : P(x)\}$, for any $P(x)$ valid for all possible objects (Axiom of Universal specification)
(b) Consider the existence of two arbitrary sets, A and B.
(c) Now consider the property function $P(x) = (x \in A \vee x \in B)$.
(d) This implies the existence of a set $\{x : x \in A \vee x \in B\}$, the definition of the pairwise union function in Axiom 3.4

Proof Axiom of Universal Specification implies axiom 3.5

(a) There exists a set of elements $\{x : P(x)\}$, for any $P(x)$ valid for all possible objects (Axiom of Universal specification)
(b) Consider the existence of an arbitrary set A.
(c) Now consider the property function $P(x) = (x \in A \wedge P_2(x))$, where $P_2(x)$ is some other arbitrary function.
(d) The set $\{x : x \in A \wedge P_2(x)\} \equiv \{x \in A : P_2(x)\}$, and so we can clearly see the Axiom of Specification is a special case of the Axiom of Universal Specification, tied down by the restriction that its property function must be applied to an existing set and not all possible objects.

Proof Axiom of Universal Specification implies axiom 3.6

(a) There exists a set of elements $\{x : P(x)\}$, for any $P(x)$ valid for all possible objects (Axiom of Universal specification)
(b) Consider the existence of an arbitrary set A.
(c) Now consider the property function $P(y) = (\exists x \in A : P(x, y))$, where $P(x, y)$ is a an arbitrary statement pertaining to x and y, such that for each $x \in A$ there is at most one y where $P(x, y)$ is true
(d) This implies the exxistence of the set $\{y : \exists x \in A : P(x, y)\}$, which is the claim of the axiom of Replacement.

2. Proof $A \notin A$

   (a) Suppose we have a set A.

   (b) From the singleton axiom, we can see that there exists the set $\{A\}$ of which A is an element.

   (c) From the Axiom of Regularity, we see that $\{A\}$ must contain an element that is either not a set, or is disjoint from $\{A\}$.

   (d) Since $\{A\}$ contains only one element — A — which is a set — we also find that A must not contain itself, because it is disjoint from $\{A\}$, and would otherwise violate the Axiom of Regularity.

   Proof given arbitrary sets $A, B : \neg(A \in B \wedge B \in A)$

   (a) Consider arbitrary sets A and B.

   (b) Sets do not contain themselves, as established earlier.

   (c) Consider now the set $\{A, B\}$.

   (d) Due to the Axiom of Regularity, $\{A, B\}$ must contain an element that does not itself contain either A or B.

   (e) Neither set contains itself, so A can contain B, or B can contain A, but both sets cannot contain each other at the same time, becuase this would mean $\{A, B\}$ has no elements which do not contain other elements or are not sets. So, $\neg(A \in B \wedge B \in A)$

3. Proof Axiom of Universal Specification implies the existence of the set of all objects.

   (a) There exists a set of elements $\{x : P(x)\}$, for any proposition $P(x)$ valid for all possible objects (Axiom of Universal specification)

   (b) Consider some $P(x) = True$. By Axiom of Comprehension, this shows there is a set such that $\{x : True\}$, or, simplified, the set of all possible objects, because $P(x)$ returns true no matter what x is.

   Proof set of all objects implies Axiom of Universal Specification.

   (a) Suppose there exists a set of all objects O.

   (b) By the Axiom of (Non-Universal) Specification, we know that, given a proposition $P(x)$ vaid for all objects in A, and a set of objects A, there exists some set that contains all the objects in A that satisfy $P(x)$.

   (c) This means that we can use the set of all objects along with the Axiom of Specification to declare that there exists a set of elements $\{x \in O : P(x)\}$, which is what is declared by the Axiom of Universal Speification.

3.3 Functions

1. Proof the definition of equality in 3.3.7 is reflexive.

   (a)
   (b) Any function $f(x)$ has the same domain and range X and Y of itself.
   (c) Any function $f(x)$ has the same output for all $x \in X$, or else we don't consider it a function.
   (d) By the definition of equality in the book, this means $f(x) = f(x)$ and so equality between functions is reflexive.

   Proof equality between functions is symmetric.

   (a) Suppose we have two functions $f(x)$ and $g(x)$ and $f(x) = g(x)$.
   (b) Two functions $f(x)$ and $g(x)$ are equal if and only if they have the same domain, range, and $f(x) = g(x)$ for all x in the domain of these functions. So, the domain of $f(x)$ is the same as the domain of $g(x)$, the range of $f(x)$ is the same as the range of $g(x)$, and $\forall x \in$ the domain of $f(x) : f(x) = g(x)$.
   (c) $g(x)$ and $f(x)$ have the same domain and range, because of the symmetric property of set equality.
   (d) $\forall x \in$ the domain of $g(x) : g(x) = f(x)$, because of teh symmetric property of elemental equality. So, by our definition earlier, $f(x) = g(x) \implies g(x) = f(x)$.

   Proof equality between functions is transitive.

   (a) Two functions $f(x)$ and $g(x)$ are equal if and only if they have the same domain, range, and $f(x) = g(x)$ for all x in the domain of these functions.
   (b) Consider three hypothetical functions $f(x)$, $g(x)$ and $h(x)$, where $f(x) = g(x) \wedge g(x) = h(x)$
   (c) $f(x)$ has the same domain as $g(x)$, which is the same domain as $h(x)$. By the substitution property of sets, the domain of $h(x)$ must be equal to the domain of $f(x)$.
   (d) Similarly, $f(x)$ has the same range as $g(x)$, which is the same range as $h(x)$. By the substitution property of sets, the range of $h(x)$ must be equal to the range of $f(x)$.
   (e) If all three functions have the same domain (hereafter referred to as X), and $\forall x \in X : f(x) = g(x) \wedge g(x) = h(x)$, then by the substitution property of numbers, $\forall x \in X : f(x) = h(x)$, and we have proven that $f(x) = h(x)$.

   Proof $(f = g \wedge h = j) \implies f \circ g = h \circ j$
   TBD

2. Proof for two functions f and g, where the domain of g is equal to the range of f, if f and g are injective, then $g \circ f$ is injective.

   (a) Consider the domain of f X and range of f Y, where the domain of g is the same as the range of f.

   (b) $\forall x \in X : (x \neq x') \implies (f(x) \neq f(x'))$ (Given)

   (c) $\forall y \in Y : (y \neq y') \implies (g(y) \neq g(y'))$ (Given)

   (d) Consider two unequal $x, x' \in X$. $f(x) \neq f(x')$, as given by the earlier statement.

   (e) Since $f(x) \neq f(x')$, and $f(x) \in Y \wedge f(x') \in Y$, $g(f(x)) \neq g(f(x'))$, as a result of the earlier implication. This is the definition of injectivity, and so the composition $g \circ f$ must be injective.

   Proof for two functions f and g, where the domain of g is equal to the range of f, if f and g are surjective, then $g \circ f$ is surjective.

   (a) Let's call the domain of f X and range of f Y, where the domain of g is the same as the range of f, and the range of g Z.

   (b) $\forall y \in Y : (\exists x \in X : f(x) = y)$ (Given)

   (c) $\forall z \in Z : (\exists y \in Y : g(y) = z)$ (Given)

   (d) The domain of g is the same as the range of f, so there is some value $f(x)$ equal for any value $y \in Y$.

   (e) So we can say that $\forall z \in Z : \exists y \in Y : \exists x \in X : z = g(y) = g(f(x))$

   (f) This means that for the composition function, $\forall z \in Z : \exists x \in X : z = g(f(x))$, and we have proven the surjectivity of $g \circ f$.

3. Not sure the function with an empty set domain makes a whole lot of sense, given previous definitions. Contacted Tao about it. Maybe I'm rarted.

4. Let $f_1 : X \to Y, f_2 : X \to Y, g_1 Y \to Z, g_2 \to Z$.
   Show that if $g_1 \circ f_1 = g_1 \circ f_2$ and g_1 is injective, that $f_1 = f_2$.

   (a) $\forall y, y' \in Y : y \neq y' \implies g_1(y) \neq g_1(y')$ (Given)

   (b) $\forall y, y' \in Y : g_1(y) = g_2(y') \implies y = y'$ (Contrapositive of above)

   (c) $\forall x \in X : g_1(f_1(x)) = g_1(f_2(x))$ (Given)

   (d) Thus, $\forall x \in X : f_1(x) = f_2(x)$ (Consequence of previous two statements)

   Is the same statement true if g_1 is not injective?
   Not necessarily. Consider the static function $g_1(y) = 7$. Since $\forall y \in Y : g_1(y) = 7$, even if $f_1(x) = 6$ and $f_2(x) = 23$, $g(f_1(x)) = g(f_2(x))$. So, the same statement is not guaranteed to be correct. Show that if $g_1 \circ f_1 = g_2 \circ f_1$ and f_1 is surjective, that $g_1 = g_2$.

(a) $\forall y \in Y : \exists x \in X : f_1(x) = y$ (Given)

(b) Phrased in a different way, the set of all possible outcomes for $f_1(x)$ is the set Y.

(c) $\forall x \in X : g_1(f_1(x)) = g_2(f_1(x))$ (Given)

(d) Thus, since there is some $f_1(x)$ for each value in Y, $\forall y \in Y : g_1(y) = g_2(y)$, and since g_1 and g_2 have the same domain and range, $g_1 = g_2$.

Is the same statement true if f_1 is not surjective?
Not necessarily. Consider the static function $f_1(x) = 2$, and the functions $g_1(y) = 2y$ and $g_2(y) = y^2$, each with the domain and range of all natural numbers. $g_1(f_1(x)) = g_2(f_1(x)) = 4$, but since $f_1(x)$ only outputs one number on the range of the g functions, this does not mean $g_1 = g_2$, and in this case they are not equal.

5. TBD

6. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse.
Proof $f^{-1}(f(x)) = x \wedge f(f^{-1}(y)) = y$ for all $x \in X$

   (a) $\forall x \in X : (f(x) = y) \implies (f^{-1}(y) = x)$ (Definition of inverse function)

   (b) $\forall x \in X : (f^{-1}(f(x)) = x)$ (Substitution)

   (c) $\forall x \in X : (f(x) = y) \implies (f(f^{-1}(y)) = y)$ (More substitution)

   (d) $\forall y \in Y : \exists x \in X : f(x) = y$ (Given that f is a bijective function)

   (e) $\forall y \in Y : f(f^{-1}(y)) = y$ (Combination of last two statements)

7. Let $f : X \to Y$ and $g : Y \to Z$ be bijective functions.
Prove $g \circ f$ is bijective.

   (a) $\forall y \in Y : \exists x \in X : f(x) = y$ (Given)

   (b) $\forall z \in Z : \exists y \in Y : g(y) = z$ (Given)

   (c) $\forall z \in Z : \exists x \in X : g(f(x)) = z$ (From previous two statements, proves surjectivity)

   (d) $(x \in X \wedge x' \in X \wedge x \neq x') \implies f(x) \neq f(x')$ (Given)

   (e) $(y \in Y \wedge y' \in Y \wedge y \neq Y') \implies g(y) \neq g(y')$ (Given)

   (f) $(x \in X \wedge x' \in X \wedge x \neq x') \implies g(f(x)) \neq g(f(x'))$ (From previous two statements, proves injectivity)

   Prove $(g \circ f)^{-1} = g^{-1} \circ f^{-1}$

   (a) The domain and range of these two function are the same ($Z \to X$). We must prove them equal for all inputs in their domain (X).

   (b) $\forall x \in X : (g \circ f)(x) = z \implies (g \circ f)^{-1}(z) = x$ (Definition of inverse function)

(c) $\forall x \in X : (g \circ f)^{-1}(f(x)) = x$ (Substitution)

(d) $\forall y \in Y : (g(y) = z) \implies (g^{-1}(z) = y)$ (Definition of inverse function)

(e) $\forall y \in Y : (g^{-1}(g(y)) = y)$ (Substitution)

(f) $\forall x \in X : ((g(f(x)) \equiv (g \circ f)(x)) = z) \implies (g^{-1}(z) = y = f(x))$ (Substitution)

(g) $\forall x \in X : (f(x) = y) \implies (f^{-1}(y) = x)$ (Definition of inverse function)

(h) $\forall x \in X : ((g \circ f)(x) = z) \implies (f^{-1}(g^{-1}(z)) = x)$ (Substitution)

(i) $(g \circ f)$ is bijective, so $\forall z \in Z : \exists x \in X : (g \circ f)(x) = z$, so:

(j) $\forall z \in Z : (g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z)) = x$, where $(g \circ f)(x) = z$, and we have proved these function's equality on the domain Z.

8. If X is a subset of Y, let $\iota_{X \to Y} : X \to Y$ be the inclusion map from X to Y, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) := x$ for all $x \in X$.

(a) Show that if $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$

i. $\forall y \in Y : \iota_{Y \to Z}(y) = y$ (Definition of $\iota_{Y \to Z}$)

ii. $\forall x \in X : x \in Y$ (Given)

iii. $\forall x \in X : \iota_{Y \to Z}(x) = x$ (Extension of statement i and ii)

iv. $\forall x \in X : \iota_{X \to Y}(x) = x$ (Definition of $\iota_{X \to Y}$)

v. $\forall x \in X : \iota_{Y \to Z}(\iota_{X \to Y}(x)) = x$ (Substitution for statement iii with statement iv)

vi. $\forall x \in X : \iota_{X \to Z}(x) = x$ (Definition of $\iota_{X \to Z}$)

vii. $\forall x \in X : \iota_{Y \to Z}(\iota_{X \to Y}(x)) = (\iota_{Y \to Z} \circ \iota_{X \to Y})(x) = \iota_{X \to Z}(x) = x$ (Last two statements)

viii. The domain and range of of $\iota_{X \to Z}$ are equal to the domain and range of $\iota_{X \to Y} \circ \iota_{Y \to Z}$. (Definition of both functions)

ix. As a consequence of the last two statements, by definition of functional equality, $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$

(b) Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$

i. $\forall x \in A : \iota_{A \to A}(x) = x$ (Definiton of $\iota_{A \to A}$)

ii. $\forall x \in B : \iota_{B \to B}(x) = x$ (Definiton of $\iota_{A \to A}$)

iii. f has a range of B. (Given definition of f)

iv. $\forall x \in A : f(x) = (\iota_{B \to B} \circ f)(x)$ (ii and iii)

v. f has a domain of A and $\iota_{A \to A}(x)$ is a bijective function with a domain and range of A.

vi. $\forall x \in A : f(x) = f(\iota_{A \to A}(x))$ (i and v)

vii. f has the same domain and range as $f \circ \iota_{A \to A}$ has the same domain and range as $\iota_{B \to B}$

viii. By the definitions of functional equality,

(c) Show that if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B \to B}(x) = x$ and $f^{-1} \circ f = \iota_{A \to A}$

i.