Solution Manual for Analysis I

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April 26, 2019

1 Beginning Chapter

2 Natural Numbers

3 Set Theory

3.1 Fundamentals

1. Reflexivity:

   (a) Equality between sets is defined as: $(\forall x \in A : x \in B) \wedge (\forall x \in B : x \in A)$

   (b) A set is defined as a collection of objects.

   (c) It is a contradiction for any object x to satisfy $x \in A$ be true but also $x \notin A$, so equality of sets is reflexive ($A = A$), meaning all objects that satisfy the claim $x \in A$ also satisfy $x \in A$, and "vice-versa"

   Symmetry:

   (a) To say that sets A and B are equal, as per our definition of equality, is to say $\forall x \in A : x \in B \wedge \forall x \in B : x \in A$.

   (b) So, if $A = B$, and by definition $\forall x \in A : x \in B \wedge \forall x \in B : x \in A$, then we can just swap the given claims around as show that $\forall x \in B : x \in A \wedge \forall x \in A : x \in B$, and see that B = A.

   Transitivity:

   (a) Suppose that among three sets A,B, and C, sets A = B, and B = C

   (b) Consider $\exists x : x \in A$

   (c) Since $A = B$, $\forall y \in A : y \in B$, and so $x \in B$

   (d) But also because $B = C$, $\forall z \in B, z \in C$, and so also $x \in C$

   (e) Thus, $\forall x \in A, x \in C$, and the same logic can be applied in reverse to find $\forall x \in C, x \in A$, and so $A = C$

2. (a) Axiom 3.2 says that there exists a set $\emptyset$, the set of no elements.
   (b) Axiom 3.1 says that if A is a set, it is also an object, and by the definition of sets (3.1.1), there exists the set X such that $A \in X$. Thus, there exists some set A_1 which contains the object $\emptyset$
   (c) Two sets are not equal if they do not share each others objects. A_1 contains $\emptyset$, but $\emptyset$ is defined as having lacking objects, so these two sets are not equal.
   (d) Similarly we can consider the set A_2 which contains only A_1, which is not equal to either A_1 or $\emptyset$ because neither of them contain A_1, and instead contain $\emptyset$ and no objects, respectively.
   (e) Lastly we think of a set $A_3 : \emptyset \in A_3 \wedge A_2 \in A_3$, and see that this does not contain the sam objects as $\emptyset$, A_1, or A_3. $\emptyset \notin A_2$, $A_1 \notin A_1$, and neither $\emptyset$ nor A_1 are in $\emptyset$ so at this point we have proven 3.2.

3. Proof $\{a, b\} = \{a\} \cup \{b\}$:
   (a) The union between sets A and B as defined in Axiom 3.4 is set whose elements are contained by A or B or both.
   (b) $\{a\} \cup \{b\}$ contains a, because $a \in \{a\}$
   (c) $\{a\} \cup \{b\}$ contains b, because $b \in \{b\}$
   (d) There can be no other objects in $\{a\} \cup \{b\}$, because all objects in $\{a\} \cup \{b\}$ must either be in $\{a\}$ or $\{b\}$ or both by the definition of a union, and the existence of a third object would imply that $\{a\}$ and $\{b\}$ contained altogether three distinct objects, a contradiction.

   Proof of commutativity:
   (a) Suppose $\exists x \in A \cup B$. That means by Axiom 3.4 $x \in A \vee x \in B$. If $x \in A$, then $x \in B \cup A$, because $B \cup A$ is the set of all elements either in B or A. If $x \in B$, then by the same logic $x \in B \cup A$ as well, and so we can say $\forall x \in A \cup B : x \in B \cup A$, and we have proven their equality.

   Proof $A \cup A = A$
   (a) For the sake of contradiction assume $A \neq A \cup A$
   (b) This implies $\exists x : x \in A \wedge x \notin A \cup A$.
   (c) But $x \notin A \cup A \rightarrow x \notin A \wedge x \notin A$.
   (d) Both of these (equivalent) implied statements contradict the earlier one, that $\exists x : x \in A \wedge x \notin A \cup A$.

   Proof $A \cup \emptyset = A$ and $\emptyset \cup A = A$
   (a) $\forall x \in A \cup \emptyset : x \in A \vee x \in \emptyset$, by Axiom 3.4
   (b) By axiom 3.2, $\neg \exists x \in \emptyset$.

(c) This means that $\forall x \in A \cup \emptyset : x \in A$. Now we must prove $\neg \exists x \in A : x \notin A \cup \emptyset$.

(d) Suppose for sake of contradiction $\exists x \in A : x \notin A \cup \emptyset$.

(e) This would mean $x \notin A \wedge x \notin \emptyset$, by Axiom 3.4.

(f) x is obviously not in the empty set, but by our assumption $x \in A$, meaning we have contradicted ourselves.

(g) The sum of the two statements we have just proved is equivalent to saying $A = A \cup \emptyset$. Because of commutitivity this also means $A = \emptyset \cup A$.

4. Proof $A \subseteq B \wedge B \subseteq C \implies A \subseteq C$:

   (a) $A \subseteq B \equiv \forall x \in A : x \in B$

   (b) $B \subseteq C \equiv \forall x \in B : x \in C$

   (c) Thus, $x \in A \implies x \in B$, and $x \in B \implies x \in C$, so $x \in A \implies x \in B \implies x \in C$ or $x \in A \implies x \in C$

   Proof $A \subseteq B \wedge B \subseteq A \implies A = B$

   (a) $A \subseteq B \equiv \forall x \in A : x \in B$

   (b) $B \subseteq A \equiv \forall x \in B : x \in A$

   (c) $\forall x \in A : x \in B \wedge \forall x \in B : x \in A \equiv A = B$, so these starting propositions combined are the definition of equality established earlier in the chapter.

   Proof $A \subset B \wedge B \subset C \implies A \subset C$:

   (a) $A \subset C \equiv \forall x \in A : x \in C \wedge A \neq C$, so we must prove two things: $A \neq C$, and $\forall x \in A : x \in C$

   (b) $A \subset B \equiv \forall x \in A : x \in B \wedge A \neq B$

   (c) $A \neq B \implies \exists x \in A : x \notin B \vee \exists x \in B : x \notin A$

   (d) Since by the definition of a subset $\neg \exists x \in A : x \notin B$, this and the relation $A \neq B$ implies $\exists x \in B : x \notin A$

   (e) $B \subset C \equiv \forall x \in B : x \in C \wedge B \neq C$

   (f) Since $\forall x \in B : x \in C$ and $\exists x \in B : x \notin A$, $\exists x \in C : x \notin A$, and $C \neq A$.

   (g) But also, $\forall x \in A : x \in B \wedge \forall x \in B : x \in C$. This means $x \in A \implies x \in B \implies x \in C$, and we see $\forall x \in A : x \in C$.

   (h) These two statements ($A \neq C$, and $\forall x \in A : x \in C$) are equivalent to saying ($A \subset C$), and so we have finished our proof.

5. Proof $A \subseteq B \implies A \cup B = B \wedge A \cap B = B$:

   (a) $A \subseteq B \equiv \forall x \in A : x \in B$

   (b) $A \cup B \equiv \{x \in A \vee x \in B\}$

(c) $\forall x \in A : x \in B$, so $\forall x \in A : x \in x \in A \vee x \in B$ and $\forall x \notin A : x \notin x \in A \vee x \in B$, proving that if $A \subseteq B \implies A \cup B = B$

(d) $A \cap B \equiv \{x \in A \wedge x \in B\}$

(e) $A \subseteq B \equiv \forall x \in A : x \in B$, so if $A \subseteq B$, then $\{x \in A \wedge x \in B\} = \{x \in B\}$

(f) $B \equiv \{x \in B\}$, so $A \subseteq B \implies A \cap B = B \wedge A \cup B = B$, and we have finished our proof.

Proof $A \cup B = B \implies A \subseteq B \wedge A \cap B = B$:

(a) $A \subseteq B \implies A \cap B = B$, as per earlier proofs, so we need only prove $A \cup B = B \implies A \subseteq B$

(b) $A \subseteq B \equiv \forall x \in A : x \in B$

(c) $B = A \cup B \equiv \forall x \in B : x \in A \cup B \wedge \forall x \in A \cup B : x \in B$

(d) $A \cup B \equiv \{x \in A \vee x \in B\}$, so

(e) $\forall x \in A \vee x \in B : x \in B$ and

(f) $\neg \exists x \in A : x \notin B$, the contrapositive of which is

(g) $\forall x \in A : x \in B$, and so we have proved our hypothesis.

Proof $A \cap B = A \implies A \subseteq B \wedge A \cup B = B$:

(a) $A \subseteq B \implies A \cup B = B$, as per earlier proofs, so we need only prove $A \cap B = A \implies A \subseteq B$

(b) $A \subseteq B \equiv \forall x \in A : x \in B$

(c) $A \cap B = A \equiv \{x \in A\} = \{x \in A \wedge x \in B\}$

(d) Thus, $A \cap B = A \implies \forall x \in A : x \in B$, or $A \cap B = B \implies A \subseteq B$, so we have already proved our hypothesis.

6. Let A, B, C be sets, and let X be a set containing A, B, C as subsets.
Proof $A \cup X = X \cap A = X$: See 3.1.5
Proof $A \cup A = A$:

(a) Suppose for sake of contradiction suppose $A \cup A \neq A$.

(b) This would mean either A contains an item not contain in $A \cup A$, or $A \cup A$ contains an item not contained in X.

(c) Suppose $\exists x \in A \cup A : x \notin A$

(d) $x \in A \cup A \equiv x \in A \vee x \in A$, which when simplified means $x \in A$.

(e) We have now already contradicted ourselves, by declaring $x \in A \wedge x \notin A$.

(f) Suppose then that $\exists x : x \in A \wedge x \notin A \cup A$

(g) We can just as easily see that $x \notin A \cup A \equiv x \notin A \wedge x \notin A$, equivalent to saying $x \notin A$.

(h) $x \notin A$ contradicts our earlier assumption that $x \in A$, and our premises must be incorrect, and $\neg\exists x : x \in A \wedge x \notin A \cup A$.

(i) Thus, $A \cup A$ has the same elements as A and is equivalent by our definition.

Proof $A \cap A = A$

(a) Suppose for sake of contradiction suppose $A \cap A \neq A$.

(b) This would mean either A contains an item not contain in $A \cap A$, or $A \cap A$ contains an item not contained in X.

(c) Suppose $\exists x : x \in A \cap A \wedge x \notin A$

(d) This means that $x \in A \wedge x \in A$, simplified $x \in A$.

(e) We have now already contradicted ourselves, by declaring $x \in A \wedge x \notin A$.

(f) Suppose then that $\exists x : x \in A \wedge x \notin A \cap A$

(g) We can just as easily see that $x \notin A \cap A \equiv x \notin A \vee x \notin A$, equivalent to saying $x \notin A$.

(h) $x \notin A$ contradicts our earlier assumption that $x \in A$, and our premises must be incorrect, and $\neg\exists x : x \in A \wedge x \notin A \cap A$.

(i) Thus, $A \cap A$ has the same elements as A and is equivalent by our definition.

Proof of Distributivity $(A \cap (B \cup C) = (A \cup B) \cup C)$

(a)

7. Proof $A \cap B \subseteq A \wedge A \cap B \subseteq B$

(a) $A \cap B = B \cap A$ because of commutativity, so we need only prove $A \cap B \subseteq A$.

(b) $(A \cap B) \equiv \{x \in A \wedge x \in B\}$

(c) Since $x \in x \in A \wedge x \in B \implies x \in A$, $\forall x \in (A \cap B) : x \in A$, which is the definition of $(A \cap B) \subseteq A$.

Proof $C \subseteq A \wedge C \subseteq B \iff C \subseteq A \cap B$

(a) This is a bidirectional equality, so we have to prove both that $C \subseteq A \cap B \implies C \subseteq A \wedge C \subseteq B$ and that $C \subseteq A \wedge C \subseteq B \implies C \subseteq A \wedge B$

(b) $C \subseteq A \cap B \equiv \forall x \in C : x \in A \wedge x \in B$

(c) $\forall x \in C : x \in A \wedge x \in B \implies \forall x \in C : x \in A$, so $C \subseteq A$. Similarly, $\forall x \in C : x \in A \wedge x \in B \implies \forall x \in C : x \in B$, so $C \subseteq B$. Thus, $C \subseteq A \cap B \implies C \subseteq A \wedge C \subseteq B$

(d) $C \subseteq A \wedge C \subseteq B \equiv \forall x \in C : x \in A \wedge \forall x \in C : x \in B$

(e) $\forall x \in C : x \in A \wedge \forall x \in C : x \in B = \forall x \in C : x \in A \wedge x \in B$, so we also have $\forall x \in C : x \in A \wedge x \in B$.

(f) $A \cap B$ is the set of elements $\forall x : x \in A \cap B$, so $\forall x \in C : x \in A \cap B$ or $C \subseteq A \cap B$, so $C \subseteq A \wedge C \subseteq B \implies C \subseteq A \cap B$. This concludes our proof that $C \subseteq A \cap B \iff C \subseteq A \wedge C \subseteq B$.

8. Proof $A \cap (A \cup B) = A$:

   (a) $\forall x \in A : x \in A \cup B$, because $A \cup B \equiv \{x \in A \vee x \in B\}$

   (b) $A \cap (A \cup B) \equiv \{x \in A \wedge x \in (A \cup B)\}$

   (c) As a result, $\forall x \in A : x \in A \cap (A \cup B)$

   (d) Also we see $\neg \exists x \in A \cap (A \cup B) : x \notin A$, because $x \in A \cap (A \cup B) \equiv x \in A \wedge x \in (A \cup B) \equiv x \in A \wedge (x \in A \vee x \in B)$. This proves $A \cap (A \cup B) = A$ and our hypothesis is correct.

   Proof $A \cup (A \cap B) = A$:

   (a) Using the properties of commutativity and associativity we can rearrange the equation like the following:

   (b) $A \cup (A \cap B) = (A \cap B) \cup A$

   (c) $(A \cap B) \cup A = A \cap (B \cup A)$

   (d) $A \cap (B \cup A) = A \cap (A \cup B)$, which was proven in an earlier exercise.

9. Let A, B, X be sets such that $A \cup B = X \wedge A \cap B = \emptyset$.
   Proof $A = X \setminus B$:

   (a) $A \cap B = \emptyset \equiv \forall x \in A \wedge x \in B : x \in \emptyset$

   (b) $\neg \exists x : x \in \emptyset$, so $\forall x \in A : x \notin B$

   (c) $X \equiv \{x : x \in A \vee x \in B\}$, and $X \setminus B \equiv \{x : x \in X \wedge x \notin B\}$, so $X \setminus B \equiv (A \cup B) \setminus B \equiv \{x : (x \in A \vee x \in B) \wedge x \notin B\}$

   (d) Since $\forall x \in X \setminus A : x \in (A \cup B) \wedge x \notin B$, $X \setminus B = \{x : (x \in A \vee x \in B) \wedge x \in B\} = \{x : (x \in A \wedge x \notin B) \vee x (\in B \wedge x \notin B)\} = \{x : x \in A \wedge x \notin B\}$

   (e) Again, $\forall x \in A : x \notin B$, so $A = x : x \in A \wedge x \notin B = X \setminus B$, and we have proven our hypothesis

   Proof $B = X \setminus A$

   (a) The $\cup$ and $\cap$ operators have the property of associativity so we need merely rearrange our starting assumptions to get:
   $A \cup B = B \cup A = X$ and
   $A \cap B = B \cap A = \emptyset$, and invoke our proof from the previous exercise.

10. Proof $A \setminus B, B \setminus A$, and $A \cap B$ are disjoint, and that their union is $A \cup B$:

(a) First we prove $A \setminus B$ is disjoint from $B \setminus A$ and $A \cap B$, and then that $A \cap B$ is disjoint from $B \setminus A$

(b) $A \setminus B \equiv \{x : x \in A \wedge x \notin B\}$

(c) $\forall x \in B \setminus A : x \notin A$, so $\forall x \in B \setminus A : x \notin A \setminus B$

(d) Similarly, $\forall x \in A \setminus B : x \notin B$, so $\forall x \in A \setminus B : x \notin B \setminus A$, and we can say that $A \setminus B$ and $B \setminus A$ are disjoint.

(e) By definition, $\forall x \in A \cap B : x \in A \wedge x \in B$

(f) This means that $\forall x \in A \cap B : x \notin B \setminus A$, because $\forall x \in B \setminus A : x \notin A$.

Proof $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B$

(a) $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) \equiv \{x : x \in A \wedge x \notin B\} \cup \{x : x \in B \wedge x \notin A\} \cup \{x : x \in A \wedge x \in B\}$

(b) $\{x : x \in A \wedge x \notin B\} \cup \{x : x \in B \wedge x \notin A\} = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\} = \{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$, thus:

(c) $\{x : x \in A \wedge x \notin B\} \cup \{x : x \in B \wedge x \notin A\} \cup \{x : x \in A \wedge x \in B\} = \{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \cup \{x : x \in A \wedge x \in B\}$, and:

(d) $\{x : (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} \cup \{x : x \in A \wedge x \in B\} = \{x : x \in A \vee x \in B\} = A \cup B$, finally proving also that $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B$

11. Proof axiom of replacement implis axiom of specification:

(a) The axiom of replacement states that for any set A and objects $x \in A$ and y "there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$'.

(b) Given this axiom lets consider a possible property $P(x, y) = (y = x \wedge P(y))$, where $P(x)$ is an arbitrary property.

(c) Applied to the axiom of choice, this property proves the existence of the set $\{y : y = x \wedge P(y) \text{ for some } x \in A\}$. In other words: $z \in \{y : y = x \wedge P(y) \text{ for some } x \in A\} \iff (\exists x \in A : x = z \wedge P(z))$

(d)