Discrete Structures

Dr. Syed Faisal Bukhari
Associate Professor
Department of Data Science, Faculty of Computing and Information Technology, University of the Punjab

Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 2

1. Discrete Mathematics and Its Application, 7^{th} Edition

by Kenneth H. Rose

2. Discrete Mathematics with Applications

by Thomas Koshy

These slides contain material from the above resources.

Set

- **Definition:** A set is an unordered collection of objects.

- **Definition:** The objects in a set are called the elements, or members, of the set. A set is said to contain elements.

- We write **$a \in A$** to denote that a is an element of the set A . The notation **$a \notin A$** denotes that a is not an element of the set A.

- **Note:** Lowercase letters are usually used to denote elements of sets.

- **Example** The set V of all vowels in the English alphabet can be written as V = {a , e, i, o, u} .

- **Example** The set O of odd positive integers less than 10 can be expressed by O = {1 , 3 , 5 , 7, 9}.

- **Example** {a , 2, Fred, New Jersey}

- **Note**: Although sets are usually used to group together **elements with common properties**, there is nothing that prevents a set from having **seemingly unrelated elements**.

Set builder notation

Another way to describe a set is to use **set builder notation**.

Example: The set O of all odd positive integers less than 10 can be written as

O = {x | x is an odd positive integer less than 10}

or

$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10 \}$.

Set builder notation

Note: The concept of a **datatype**, or type, in computer science is built upon the concept of a **set**.

Example: boolean is the name of the set **{0, 1}** together with operators on one or more elements of this set, such as **AND, OR,** and **NOT**

Subset

- **Definition:** The set A is said to be a subset of B if and only if every element of A is also an element of B. The notation $\mathbf{A \subseteq B}$ to indicate that A is a subset of the set B. $\mathbf{A \subseteq B}$ if and only if the quantification $\mathbf{\forall x(x \in A \rightarrow x \in B)}$ is true

Subset

- **Examples**:

1. The set of **all odd positive integers less than 10** is a subset of the set of all positive integers less than 10.

2. The **set of rational numbers** is a subset of the set of real numbers.

3. The **set of all computer science majors at your school** is a subset of the set of all students at your school.

4. The set of **all people in China** is a subset of the set of all people in China (that is, it is a subset of itself).

Subset

Theorem: For every set S,

(i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$

Proper subset

- When we wish to emphasize that a set A is a subset of the set B but that **$A \neq B$**, we write **$A \subset B$** and say that A is a **proper subset of B**. For $A \subset B$ to be true, it must be the case that **$A \subseteq B$** and there must **exist an element x of B that is not an element of A** .

- **A** is a **proper subset** of B if
  $\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \rightarrow x \notin A)$ is true.

Equal set

Equal set: Two sets are *equal* if and only if they have the same elements. Therefore, if *A* and *B* are sets, then *A* and *B* are equal if and only if **$\forall x(x \in A \leftrightarrow x \in B)$.** We write *A* = *B* if *A* and *B* are equal sets.

If A and B are sets with $A \subseteq B$ and $B \subseteq A$, then A = B

or

A = B, if and only if

$\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$

Note: Sets may have other sets as members.

A = {Ø, {a}, {b}, {a , b}}

Cardinality of a set

- Let S be a set. If there are exactly **n distinct elements** in S where **n is a nonnegative integer**, we say that **S is a finite set** and that n is the cardinality of S. The cardinality of **S** is denoted by $|S|$.

Cardinality of a set

- **Example** Let A be the set of odd positive integers less than 10. Then $|A|$= 5.

- **Example** Let S be the set of integers in the English alphabet. Then $|A|$ = 26.

- **Example** Because the null set has no elements, it follows that $|\emptyset|$= 0.

Infinite and not finite

Definition A set is said to be infinite if it is not finite.

Example: The set of positive integers is infinite

Power set

Definition Given a set S, **the power set of S** is the set of all subsets of the set S. The power set of S is denoted by P(S).

Example What is the power set of the set to {0, 1 , 2} ?

Solution: The power set P ({ 0, 1 , 2 }) is the set of all subsets of to {0, 1 , 2 } . Hence,

P ({0, 1 , 2 }) = {{Ø}, {0}, {1}, {2 }, {0, 1}, {0, 2}, { 1 , 2}, {0, 1 , 2}} .

Power set

Example What is the power set of the empty set? What is the power set of the set {Ø}?

Solution: The empty set has exactly one subset, namely, itself.

P (Ø) = {Ø} .

The set {Ø} has exactly two subsets, namely, Ø and the set {Ø} itself. Therefore,

P ({Ø}) = {Ø, {Ø}}

No of elements in a power set: If a set has n elements, then its power set has **2^n elements**.

Ordered pairs

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its nth element.

2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Note: (a, b) and (b, a) are not equal unless $a = b$.

Cartesian Products

Definition Let A and B be sets. The Cartesian product of A and B, denoted by **$A \times B$**, is the set of all ordered pairs (a , b), where $a \in A$ and $b \in B$.

$A \times B = \{(a , b) \mid a \in A \wedge b \in B \}$.

Example: What is the Cartesian product of A = { 1 , 2} and B = {a , b, c}?

The Cartesian product $A \times B$ is

A x B = {(1 , a), (1 , b) , (1 , c), (2, a), (2, b) , (2, c)} .

Relation

A **subset R** of the **Cartesian product A × B** is called a **relation** from the **set A** to the **set B**. The elements of R are ordered pairs, where the first element belongs to A and the second to B.

R = {(a , 0), (a , 1), (a , 3), (b, 1), (b, 2), (c, 0), (c, 3)} is a relation from the set {a , b , c} to the set to { 1 , 2, 3 }

The **Cartesian products A × B** and **B × A** are **not equal**,

unless A = Ø or B = Ø (so that A x B = Ø) or A = B

Using Set Notation with Quantifiers [1]

- Sometimes we **restrict the domain** of a quantified statement explicitly by making use of particular notation.

- **$\forall x \in S(P(x))$** denotes the **universal quantification** of **$P(x)$** over all elements in the **set S**.

- **$\forall x \in S(P(x))$** is shorthand for **$\forall x(x \in S \rightarrow P(x))$**

- **$\exists x \in S(P(x))$** denotes the existential quantification of **$P(x)$** over all elements in S .

- **$\exists x \in S(P(x))$** is shorthand for **$\exists x(x \in S \wedge P(x))$**

Using Set Notation with Quantifiers [2]

- **Example** What do the statements $\forall x \in R\ (x^2 \geq 0)$ and $\exists x \in Z\ (x^2 = 1)$ mean?

- The statement $\forall x \in R\ (x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as **"The square of every real number is nonnegative."** This is a true statement.

- The statement $\exists x \in Z\ (x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed **as "There is an integer whose square is 1 ."** This is also a true statement because x = 1 is such an integer (as is - 1).

Truth Sets of Quantifiers

- We will now tie together concepts from **set theory** and from **predicate logic.**
- Given a **predicate P**, and a **domain D**, we define the truth set of **P** to be the set of elements x in **D** for which **P(x)** is true. The truth set of **P(x)** is denoted by **$\{x \in D \mid P(x)\}$** .

Recall: The statement "x is greater than 3" has two parts.

Subject: (variable) x is the subject

Predicate: is greater than 3

Predicate states the property the object x has **all**, **every**, **none**, **some** and **one**.

Such words, called **quantifiers**, give us an idea about how many objects have a certain property

Truth Sets of Quantifiers

- What are the truth sets of the predicates P(x), where the domain is the set of integers and **P(x) is $|x| = 1$**

Solution:

- The truth set of **P(x)** is denoted by **$\{x \in D \mid P(x)\}$** .

- The truth set of P, $\{x \in Z \mid |x| = 1\}$, is the set of integers for which $|x| = 1$.

- **Because** $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x , we see that the truth set of P is the set **$\{-1 , 1\}$** .

Truth Sets of Quantifiers

- **Example** What are the truth sets of the predicate Q(x)where the domain is the set of integers **Q(x) is $x^2 = 2$.**

Solution:

- The truth set of **P(x)** is denoted by **$\{x \in D \mid P(x)\}$** .

- The truth set of Q, $\{x \in Z \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the **empty set because** there are no integers x for which $x^2 = 2$.

Truth Sets of Quantifiers

Example What are the truth set of the predicate R(x), where the domain is the set of integers and **R (x) is "|x| = x"**

Solution:

The truth set of **P(x)** is denoted by **{x ∈ D|P(x)}** .

The truth set of R, $\{x \in Z \mid |x| = x\}$, is the set of integers for which $|x| = x$.

Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is **N**, the set of **nonnegative integers**. **N** = {0,1,2,3,…}

Truth Sets of Quantifiers

1. **$\forall x \in$ P(x)** is true over the **domain U** if and only if the truth set of **P is the set U**.

2. **$\exists x$ P(x)** is true over the **domain U** if and only if the truth set of **P is nonempty**.

Set Operations

Definition: Let A and B be sets. The union of the sets A and B , denoted by A U B , is the set that contains those elements that are either in A or in B , or in both. An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B.

$A \cup B = \{x \mid x \in A \vee x \in B\}$.

Example The **union of the sets** {1 , 3 , 5 } and {1 , 2, 3} is the set {1 , 2 , 3 , 5}

$\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

Set Operations

Definition: Let A and B be sets. The intersection of the sets A and B , denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

Example The **intersection of the sets** {1, 3 , 5} and {1, 2 , 3} is the set {1, 3}

$\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

Disjoint and difference sets

Definition Two sets are called disjoint if their intersection is the empty set.

Example Let A = {1, 3 , 5 , 7, 9} and B = {2, 4, 6, 8 , 10}.

$A \cap B = \emptyset$, A and B are disjoint.

Definition Let A and B be sets. The difference of A and B , denoted by A - B , is the set containing those elements that are in A but not in B . The difference of A and B is also called the complement of B with respect to A.

$A - B = \{x \mid x \in A \land x \notin B\}$.

Example The difference of {1, 3 , 5} and {1, 2 , 3} is the set {5}

{1, 3 , 5} - {1, 2 , 3} = {5} .

{1, 2 , 3} - {1, 3 , 5} = {2}.

Complement of a set

Let U be the universal set. The complement of the set A, denoted by $\overline{\mathrm{A}}$, is the complement of A with respect to U. In other words, the complement of the set A is

U - A = $\overline{A}$ = {x $\in$ U | x $\notin$ A}

Example Let A = {a , e, i , 0, u } (where the universal set is the set of letters of the English alphabet).

$\overline{\mathbf{A}}$ = {b, c, d, j, g, h , j, k, I , m , n , p , q , r, S , t , v , w , x , y , z} .

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Membership tables [1]

- **Set identities** can also be proved **using membership tables**.

- We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity.

- To indicate that **an element** is in a set, a **1 is used**; to indicate that an element is **not in a set**, a **0 is used**

Membership tables [2]

- **Example** Use a membership table to show that A∩(B U C) = (A ∩ B) U (A ∩ C).

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Computer Representation of Sets [1]

- Assume that the **universal set U** is **finite** (and of reasonable size so that the number of elements of **U is not larger than the memory size of the computer being used**).

- First, specify an arbitrary ordering of the elements of U, for instance $a_1 , a_2 , . . . , a_n$.

- Represent a subset A of U with the bit string of length n , where the **i^{th} bit in this string is 1** if a_j belongs to A and is **0 if a_j does not belong to A**

Computer Representation of Sets [2]

Example Let U = {1, 2, 3 , 4, 5 , 6, 7 , 8 , 9, 10} , and the ordering of elements of U has the elements in increasing order; that is, a_j = i . What **bit strings** represent the subset of **all odd integers** in U, the subset of **all even integers** in U, and the subset of **integers not exceeding 5** in U ?

Solution

U = {1, 2, 3 , 4, 5 , 6, 7 , 8 , 9, 10}

1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

The subset of all odd integers in U

{1 , 3 , 5 , 7, 9}

1	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**

The subset of **all even integers** in U

{2, 4, 6, 8, 1 0}

0	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**

The subset of **integers not exceeding 5** in U

{ 1, 2, 3 , 4, 5}

1	**1**	**1**	**1**	**1**	**0**	**0**	**0**	**0**	**0**

Computer Representation of Sets [3]

Example We have seen that the bit string for the set {1, 3, 5 , 7, 9} (with universal set { 1, 2, 3, 4, 5 , 6, 7, 8 , 9, 10}) is 10 1 0 1 0 1 0 1 0. What is the bit string for the **complement of this set**?

Solution

U = {1, 2, 3 , 4, 5 , 6, 7 , 8 , 9, 10}

1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

The subset of all odd integers in U

{1 , 3 , 5 , 7, 9}

1	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**

The **complement odd integers**

1	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**
0	1	0	1	0	1	0	1	0	1

which corresponds to the set {2, 4, 6, 8, 10}

Example The bit strings for the sets {1 , 2 , 3 , 4, 5 } and {1 , 3 , 5 , 7, 9} are1 1 1 1 1 0 0000 and 1 0 1 0 1 0 1 0 1 0, respectively. Use bit strings to find the union and intersection of these sets.

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
{1 , 2 , 3 , 4, 5 }	1	1	1	1	1	0	0	0	0	0
{1 , 3 , 5 , 7, 9}	1	0	1	0	1	0	1	0	1	0
Union										
$\Longrightarrow$ {1, 2, 3, 4, 5, 7, 9}	1	1	1	1	1	0	1	0	1	0
Intersection										
$\Longrightarrow$ {1, 3, 5}	1	0	1	0	1	0	0	0	0	0

Suggested Readings

2.1 Sets

2.2 Set Operations