

Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7th Edition
Kenneth H. Rosen

References

Chapter 9

Discrete Mathematics and Its Application, 7th Edition
by Kenneth H. Rose

These slides contain material from the above resource.

Representing Relations

- Representing Relations using **Matrices**
- Representing Relations using **Digraphs**

Representing Relations Using Matrices

A **relation** between **finite sets** can be represented using a **zero–one matrix**. Suppose that R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B .) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the **zero–one matrix** representing R has a **1** as its (i, j) entry when a_i is related to b_j , and a **0** in this position if a_i is not related to b_j . (Such a representation depends on the orderings used for A and B .)

Examples of Representing Relations Using Matrices

Example: Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of Representing Relations Using Matrices (*cont.*)

Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Matrices of Relations on Sets

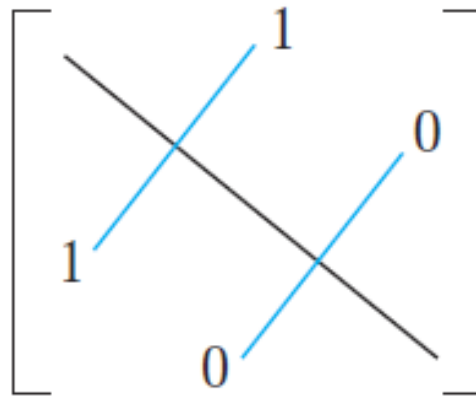
The matrix of a relation on a set, which is **a square matrix**, can be used to determine whether the **relation** has certain properties.

- If **R** is a **reflexive relation**, all the elements on the **main diagonal** of M_R are equal to **1**.

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

Note that the elements off the main diagonal can be either **0** or **1**.

□ R is a **symmetric relation**, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$



In terms of the entries of M_R , R is **symmetric** if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is **symmetric** if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

R is symmetric if and only if

$$\mathbf{M}_R = (\mathbf{M}_R)^t$$

that is, if \mathbf{M}_R is a symmetric matrix.

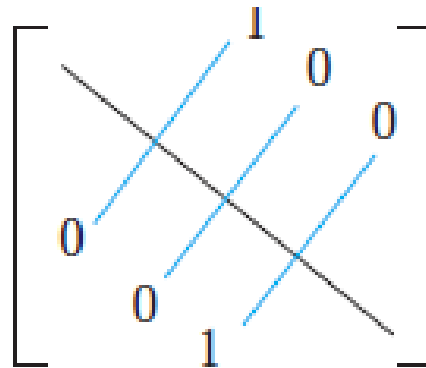
Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

The relation R is **antisymmetric** if and only if $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$.

Consequently, the matrix of an **antisymmetric relation** has the property that **if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$** .

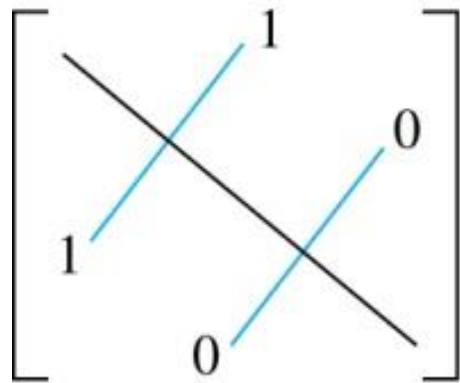
OR

R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.

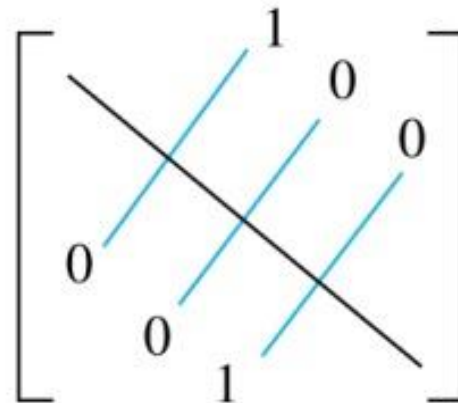


Consequently, the matrix of an **antisymmetric relation** has the property that if **$m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$** . Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when **$i \neq j$** .

Consequently, the matrix of an **antisymmetric relation** has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric



(b) Antisymmetric

Transitive

Find the non-zero entries in M_R^2 . If $M_R = 1$ then $M_R^2 = 1$ then the relation is transitive. If $M_R = 1$ then $M_R^2 = 0$ then the relation is not transitive. If $(a,b) \in M_R^2$, then $(a, b) \in M_R$

Example Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}, \text{ and}$$

$$R_3 = \{(3, 4)\}.$$

Which of these relations are *transitive*?

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R **reflexive, symmetric, and/or antisymmetric?**

Solution:

Reflexive: Because all the diagonal elements are equal to 1, R is reflexive.

Symmetric: Because M_R is symmetric, R is symmetric $m_{1,2}$ and $m_{2,1}$ are 1

Antisymmetric: It is not antisymmetric because $m_{1,2}$ and $m_{2,1}$ are 1.

If $m_{1,2} = 1$ then $m_{2,1} = 0$ in order to make it antisymmetric

Union and Intersection

The matrix representing the union of these relations has a 1 in the positions where either \mathbf{M}_{R1} or \mathbf{M}_{R2} has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both \mathbf{M}_{R1} and \mathbf{M}_{R2} have a 1. Thus, the matrices representing the union and intersection of these relations are

$$\mathbf{M}_{R1 \cup R2} = \mathbf{M}_{R1} \vee \mathbf{M}_{R2} \text{ and } \mathbf{M}_{R1 \cap R2} = \mathbf{M}_{R1} \wedge \mathbf{M}_{R2}$$

Example Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $M_{R_1 \cup R_2}$ and $M_{R_1 \cap R_2}$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R1 \cup R2} = \mathbf{M}_{R1} \vee \mathbf{M}_{R2}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R1 \cap R2} = \mathbf{M}_{R1} \wedge \mathbf{M}_{R2}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

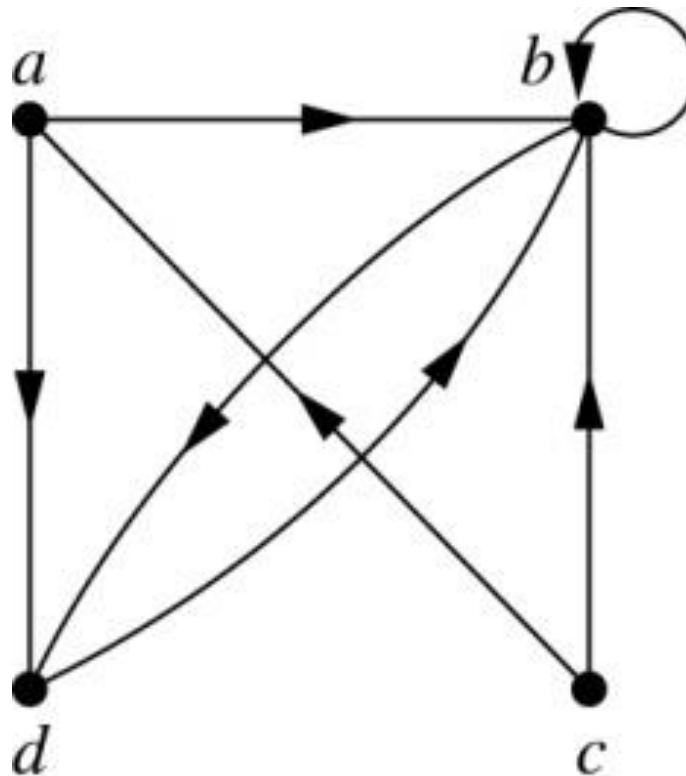
Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices (or nodes)** together with a set E of ordered pairs of elements of V called **edges (or arcs)**. The vertex **a** is called the **initial vertex** of the edge (a, b) , and the **vertex b** is called the **terminal vertex** of this edge.

- An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

Example: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), **(b, b)**, (b, d), (c, a), (c, b), and (d, b) is shown in the next slide.

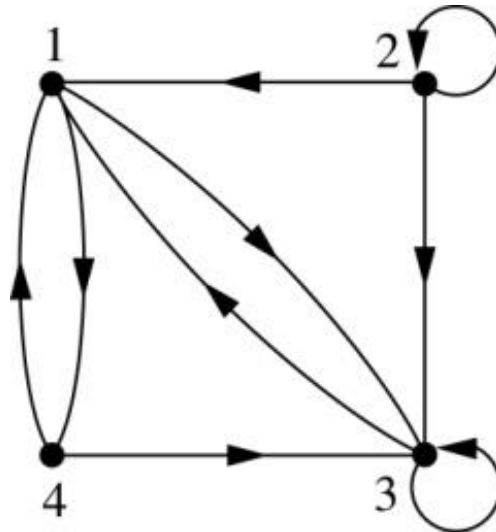
Solution

directed graph with vertices a , b , c , and d , and edges (a, b) , (a, d) , **(b, b)** , (b, d) , (c, a) , (c, b) , and (d, b)

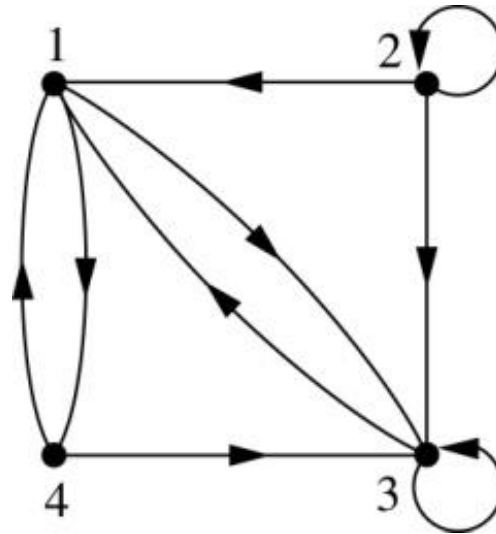


Examples of Digraphs Representing Relations

Example: What are the ordered pairs in the relation represented by this directed graph?



Solution:



The ordered pairs in the relation are

$(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

Example The directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

Solution: The directed graph of the relation $R = \{(\mathbf{1}, \mathbf{1}), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

