Discrete Strutrues

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

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by Kenneth H. Rose

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http://raider.mountunion.edu/ma/MA125/Fall2011/Chapter7/IntroToFunctions.html

Functions

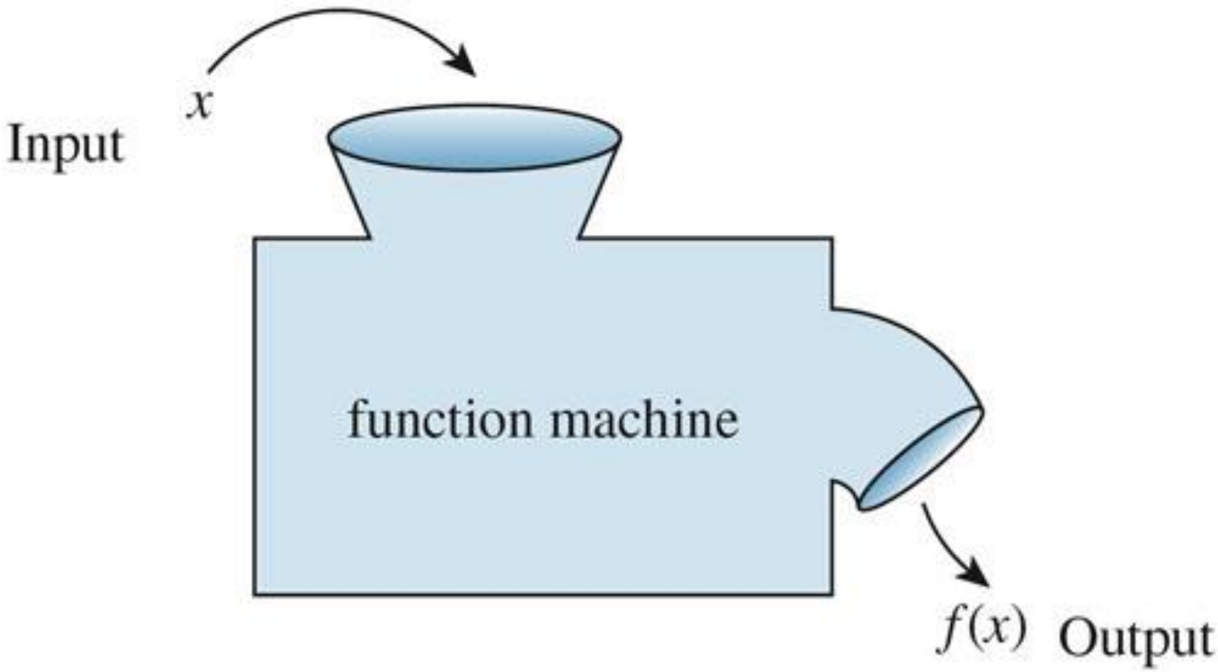
- The concept of a function is extremely important in mathematics and computer science.

- For example, in **discrete mathematics functions** are used in the definition of such **discrete structures as sequences** and **strings**.

- Functions are also used to represent how long it takes a **computer to solve problems of a given size**.

- Many computer programs and subroutines are designed to calculate **values of functions**.

Function Machines



Function Machines

One way to think of a **function is as a machine**. You drop a **domain element** into the input hopper, and it produces a **codomain** element from the output chute.

There is a rule or **formula hiding inside** the machine.

We do have to specify what the **domain** is for the rule, so we **don't drop things into the machine that might "break"** it. (It may not know how to handle certain inputs.)

Functions Defined by Formulas

1. **$f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$**

This is a well-defined function, since the rule inside the function machine can **handle all possible values of x** from the domain real numbers

2. **$f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$**

This is not a **well-defined function** (that is, it is not a function!), since the rule inside the function machine can not handle all possible values of x from the domain real numbers

What if we changed the domain to $\mathbb{R}^+$?

Functions

- **Definition 1** Let *A* and *B* be nonempty sets. A *function* ***f*** from ***A*** to ***B*** is an assignment of exactly **one element** of ***B*** to **each element of *A***. We write ***f (a) = b*** if *b* is the unique element of *B* assigned by the function *f* to the element *a* of *A*. If *f* is a function from *A* to *B*, we write $f : A \rightarrow B$.

- **Remark:** Functions are sometimes also called **mappings** or **transformations.**

Functions

- In many instances **we assign to each element of a set a particular element of a second set** (which may be the same as the first).
- For example, suppose that each student in a **discrete mathematics class** is assigned a letter grade from the set **{A, B, C, D, F}.** And suppose that the grades are A for Adams, C for Chou, B for Good friend, A for Rodriguez, and F for Stevens.

Functions

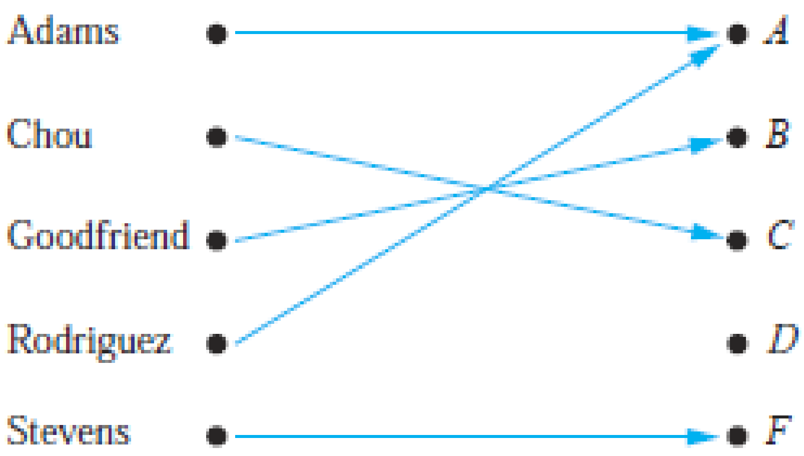


FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

Functions

- Functions are specified in many different ways. Sometimes we explicitly state the assignments, as in Figure 1.

- Often we give a **formula**, such as $f(x) = x + 1$, to define a function.

- Other times we use a **computer program** to specify a **function**.

Functions

- A function $f : A \to B$ can also be defined in terms of a **relation from *A* to *B***.

- Recall from that a **relation from *A* to *B*** is just a subset of ***A* × *B***.

- A relation from A to B that contains one, and **only one, ordered pair *(a, b)*** for **every element $a \in A$**, defines a function f from A to B.

- This function is defined by the assignment $f(a) = b$, where (a, b) is the **unique ordered pair** in the relation that has a as its first element

Functions

- **Definition 2** If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f. If f (a) = b, we say that **b** is the **image** of **a** and **a** is a **preimage** of b. The **range**, of f is **the set of all images of elements** of A. Also, if f is a function from A to B, we say that f maps A to B.

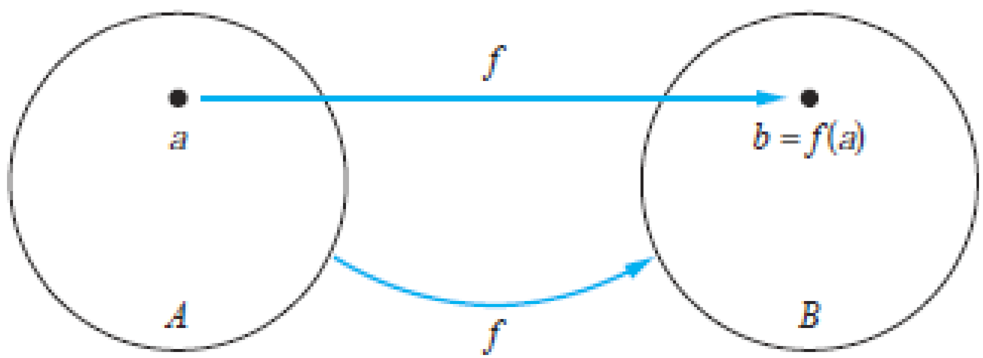


FIGURE 2 The Function f Maps A to B.

- When we define a function we specify its **domain**, its **codomain**, and the mapping of elements of the **domain to elements in the codomain**.

- Two functions are **equal** when they have the **same domain**, have **the same codomain**, and map each element of their common domain to the same element in their common codomain.

Note that if we change either **the domain or the codomain** of a function, then we obtain a **different function**. If we change the mapping of elements, then we also obtain a different function.

Domain, codomain, and range of a function

Example What are the domain, codomain, and range of the function that assigns grades to students described in Figure 1?

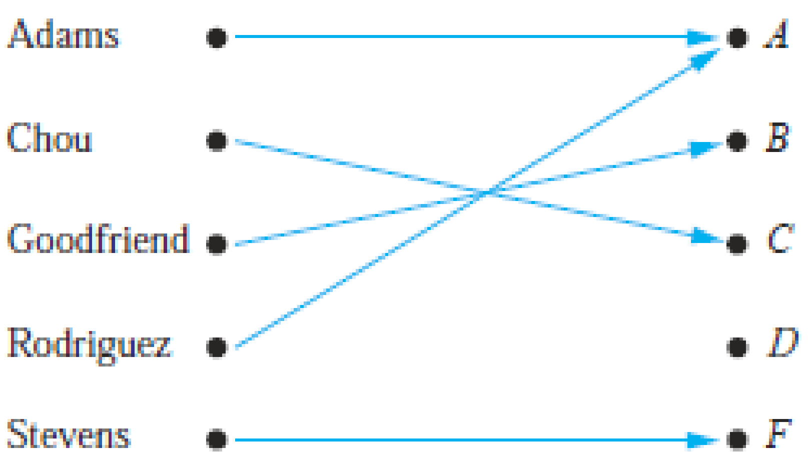


FIGURE 1 Assignment of Grades in a Discrete Mathematics Class.

Solution: Let **G** be the function that assigns a grade to a student in our discrete mathematics class.

Note that G(Adams) = A

domain of G = {Adams, Chou, Goodfriend, Rodriguez, Stevens}

codomain = {A, B, C, D, F}.

range of G = {A, B, C, F}, because each grade **except D** is assigned to some student.

Domain, codomain, and range of a function

Example Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution:

f (x) is the age of x, where x is a student.

If f is a **function** specified by R, then

f (Abdul) = 22, **f (Brenda) = 24, f (Carla) = 21, f (Desire) = 22, f (Eddie) = 24**, and **f (Felicia) = 22**.

domain of F = {Abdul, Brenda, Carla, Desire, Eddie, Felicia}

codomain of F = contain all possible ages of students

Or

⇒ **codomain** of F = the set of positive integers less than 100

range of F = { 21, 22, 24}

Domain, codomain, and range of a function

Example: Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example,

$f(11010) = 10$.

Solution

domain of f = the set of all bit strings of length 2 or greater

codomain of f = {00, 01, 10, 11}.

range of f = {00, 01, 10, 11}.

Domain, codomain, and range of a function

Example: Let f: **Z** $\rightarrow$ **Z** assign the square of an integer to this integer. Then, $f(x) = x^2$. Find domain, codomain, and range of f.

Solution

domain of f = set of all integers

codomain of f = set of integers

range of f = the set of all integers that are perfect squares

or

range of f = {0, 1, 4, 9, . . . }.

Note: A function is called **real-valued** if its codomain is the set of real numbers, and it is called **integer-valued** if its codomain is the set of integers.

Domain, codomain, and range of a function

Example The domain and codomain of functions are often specified in programming languages. For instance, the **Java** statement

```
int floor(float real){. . .}
```

and the **C++** function statement

```
int function (float x){. . .}
```

both tell us that the **domain** of the **floor function** is the set of **real numbers** (represented by floating point numbers) and its **codomain** is **the set of integers**.

One-to-One and Onto Functions

Some functions never assign **the same value** to **two different domain elements**. These functions are said to be **one-to-one**.

.

One-to-One and Onto Functions

Definition 5 function f is said to be **one-to-one**, or an **injunction**, if and only if **f (a) = f (b)** implies that **a = b** for all a and b in the domain of f. A function is said to be **injective** if it is **one-to-one**

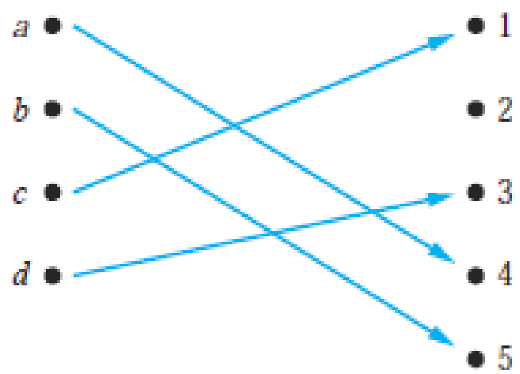


FIGURE 3 A One-to-One Function.

Note: A function *f* is one-to-one if and **only if *f (a)* ≠ *f (b)* whenever *a* ≠ *b***. This way of expressing that *f* is one-to-one is obtained by taking the contrapositive of the implication in the definition.

Example Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution

The function *f* is one-to-one because *f* takes on different values at the four elements of its domain

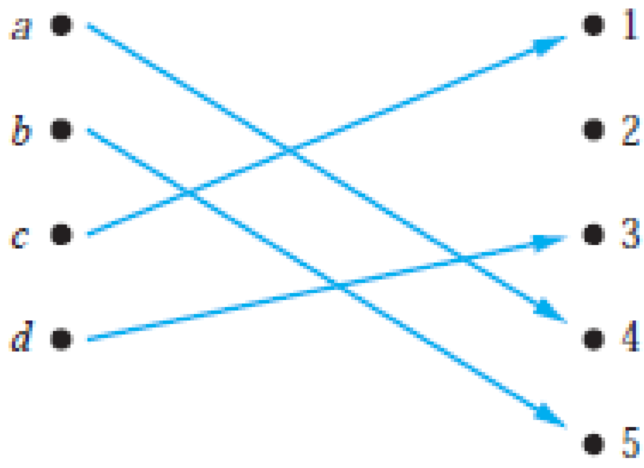


FIGURE 3 A One-to-One Function.

Addition and multiplication of functions

Two real-valued functions or **two integer valued functions** with the **same domain** can be added, as well as multiplied.

Definition 3 Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and f_1f_2 are also functions from A to **R** defined for all $x \in A$ by

1. $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
2. $(f_1f_2)(x) = f_1(x)f_2(x)$.

Example Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and f_1f_2?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$$

When f is a function from A to B, the image of a subset of A can also be defined.

Image

Example Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $\boldsymbol{f(a) = 2}$, $\boldsymbol{f(b) = 1, f(c) = 4, f(d) = 1}$, and $\boldsymbol{f(e) = 1}$.

Solution:

The **image** of the **subset** $\boldsymbol{S = \{b, c, d\}}$ is the set $\boldsymbol{f(S) = \{1, 4\}}$.

Definition function f is said to be **one-to-one**, or an **injunction**, if and only if f (a) = f (b) implies that a = b for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one

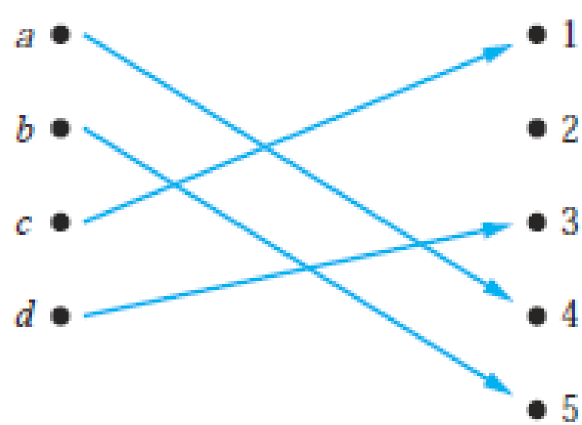


FIGURE 3 A One-to-One Function.

Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with $f(a) = 4, f(b) = 5, f(c) = 1$, and $f(d) = 3$ is **one-to-one**.

Solution: The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in Figure 3.

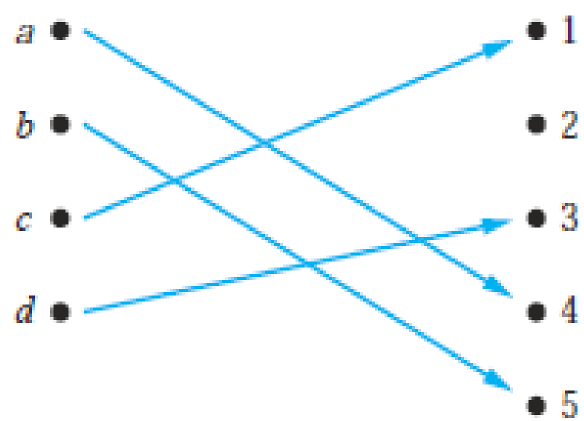


FIGURE 3 A One-to-One Function.

Example Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution:

The function $f(x) = x^2$ is **not one-to-one because**, for instance, $f(1) = f(-1) = 1$, but $1 = -1$.

Note that the function $f(x) = x^2$ with its domain restricted to $\mathbf{Z}^+$ is one-to-one.

Remark: We can express that f is one-to-one using quantifiers as

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

or $\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$,

where the universe of discourse is the domain of the function.

Example Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

The function $f(x) = x + 1$ is a one-to-one function. To demonstrate this, note that

$x + 1 \neq y + 1$ when $x \neq y$.

Recall:

$\forall a \forall b(f(a) = f(b) \to a = b)$

or $\forall a \forall b(a \neq b \to f(a) \neq f(b))$

Increasing and Decreasing Functions

Definition A function f whose domain and codomain are subsets of the set of **real numbers** is called

1. **increasing if f (x) ≤ f (y),**
2. **strictly increasing** if f(x) < f(y), whenever x < y and x and y are in the domain of f.
3. **decreasing if f (x) ≥ f (y)**
4. **strictly decreasing** if f(x) > f(y), whenever x < y and x and y are in the domain of f.

Increasing and Decreasing Functions

- A function f is **increasing** if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$

- **Strictly increasing** if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$

- **Decreasing** if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$

- **Strictly decreasing** if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

Onto Function

Definition A function f from A to B is called ***onto***, or a *surjection*, if and only if for **every element *b*** $\in$ ***B*** there is an **element *a*** $\in$ ***A*** with $f(a) = b$. A function f is called *surjective* if it is onto.

Example Let *f* be the function from {*a, b, c, d*} to {1, 2, 3} defined by *f (a)* = 3, *f (b)* = 2, *f (c)* = 1, and *f (d)* = 3. Is *f* an onto function?

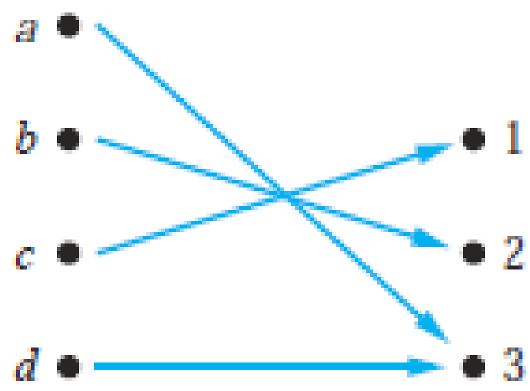


FIGURE 4 An Onto Function.

Because all **three elements of the codomain** are images of elements in the domain, we see that *f* is onto

Example of different types of correspondences

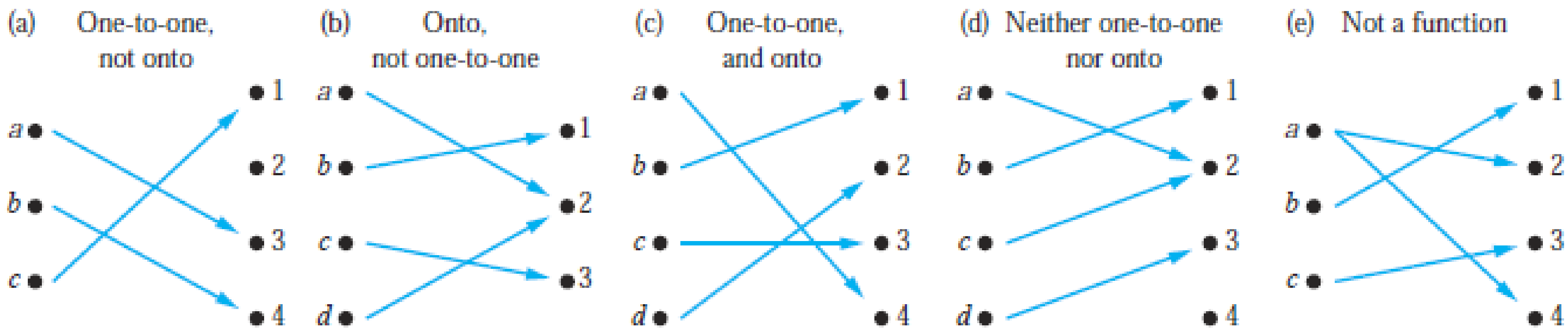


FIGURE 5 Examples of Different Types of Correspondences.

Inverse function

Definition Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b. The inverse function of f is denoted by f^{-1}

Hence, f

f^{-1} (a) = b when

f (a) = b.

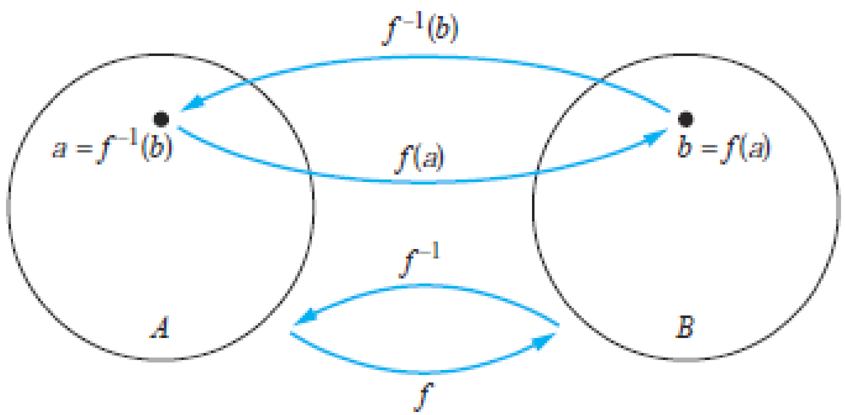


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

Inverse function

- If a function f is not a **one-to-one correspondence**, we cannot define an inverse function of f . When f is not a one-to-one correspondence, either it is **not one-to-one** or it is **not onto.**

- Note: A function f is a **one-to-one correspondence** if it is one-to-one and onto

Example Let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible because it is a **one-to-one correspondence**. The inverse function f^{-1} reverses the correspondence given by f ,

$f^{-1}(1) = c$

$f^{-1}(2) = a$

$f^{-1}(3) = b$

Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

The function f has an inverse because it is a one-to-one correspondence, as follows from the previous examples.

$y = x + 1.$

$\Rightarrow x = y - 1$

$f^{-1}(y) = y - 1$

Example Let f be the function from **R** to **R** with $f(x) = x^2$. Is f invertible?

$f(-2) = f(2) = 4$,

f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible

Composition of the functions

Definition Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all a ∈ A by f ∘ g, is defined by (f ∘ g)(a) = f (g(a)).

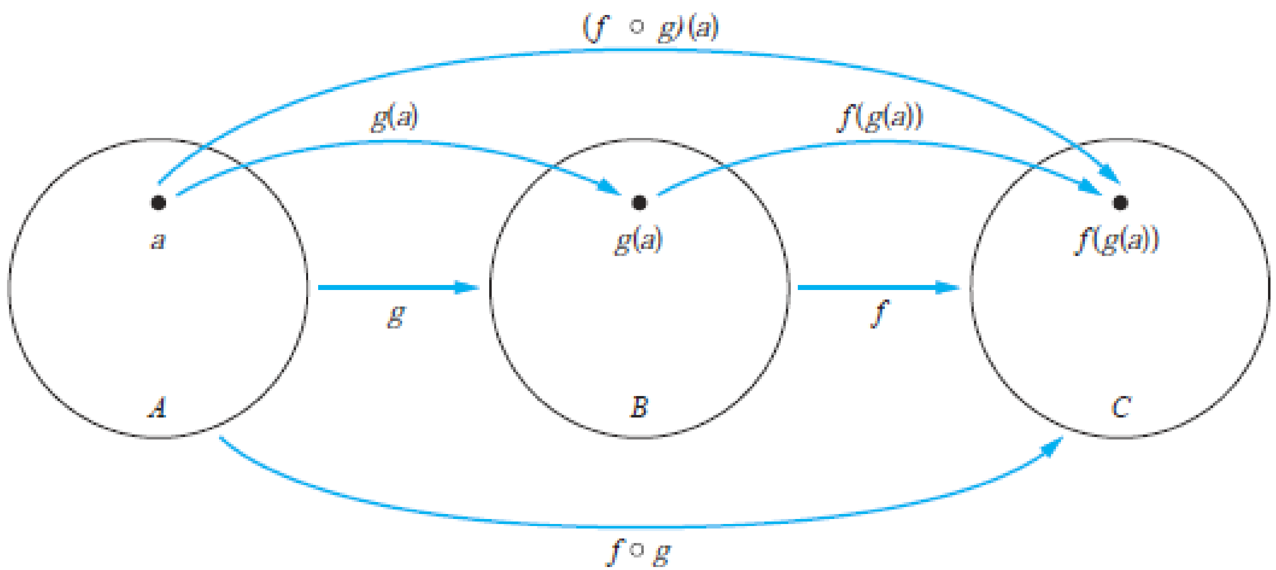


FIGURE 7 The Composition of the Functions *f* and *g*.

Composition of the functions

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to g(a). That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain g(a) and then we apply the function f to the result g(a) to obtain $(f \circ g)(a) = f(g(a))$.

Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f

Example Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f ?

Solution

Given

g be the function from the set {a, b, c} to itself

$g(a) = b$, $g(b) = c$, and $g(c) = a$

f be the function from the set {a, b, c} to the set {1, 2, 3}

$f(a) = 3$, $f(b) = 2$, and $f(c) = 1$

$f \circ g = (f \circ g)(a) = f(g(a)) = f(b) = 2$

$f \circ g = (f \circ g)(b) = f(g(b)) = f(c) = 1$

$f \circ g = (f \circ g)(c) = f(g(c)) = f(a) = 3$

$g \circ f = (g \circ f)(a) = g(f(a)) = g(3) = ?$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Example Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

Both the compositions $f \circ g$ and $g \circ f$ are defined.

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

Floor and ceiling functions

Definition The **floor function** assigns to the real number x the largest integer that is **less than or equal to x**. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The **ceiling function** assigns to the real number x the smallest integer that is **greater than or equal to x**. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Floor and ceiling functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Example These are some values of the floor and ceiling functions:

$\left\lfloor \frac{1}{2} \right\rfloor = \lfloor 0.5 \rfloor$ = 0 and $\left\lceil \frac{1}{2} \right\rceil = \lceil 0.5 \rceil$ = 1

$\left\lfloor -\frac{1}{2} \right\rfloor = \lfloor -0.5 \rfloor$ = -1 and $\left\lceil -\frac{1}{2} \right\rceil = \lceil -0.5 \rceil$ = 0

$\lfloor 3.1 \rfloor$ = 3 and $\lceil 3.1 \rceil$ = 4

$\lfloor 7 \rfloor$ = 7 and $\lceil 7 \rceil$ = 7

Example: Data stored on a computer disk or transmitted over a data network are usually represented as a string of **bytes**. Each **byte** is made up of **8 bits**. How many **bytes** are required to encode **100 bits** of data?

Solution

$\left\lceil \frac{100}{8} \right\rceil = \lceil 12.5 \rceil = 13$ bytes

Reasons for not using floor function:

Since data is represented as a string of bytes, if we take **the floor function** of it then we might **lose some bits** while transmitted over a data network.

Example In **asynchronous transfer mode (ATM)** (a communications protocol used on backbone networks), data are organized into **cells of 53 bytes**. How many ATM cells can be transmitted in **1 minute** over a connection that transmits data at the rate of **500 kilobits per second**.

Solution

- In **1 minute**, this connection can transmit **500,000** × **60** = **30,000,000 bits**

- Each ATM cell is **53 bytes** long, which means that it is 53 × 8 = **424 bits** long.

- $\left\lfloor \frac{30,000,000}{424} \right\rfloor$ = 70,754 ATM cells can be transmitted in 1 minute over a **500 kilobit per second connection**.

Reasons for not using ceiling function:

- Data is organized into **cells of 53 bytes**. But connection transmits data at the rate of 500 kilobits per second (500kb per second bandwidth). It **means the maximum bandwidth** (of the channel) is 500 kb per second.

- We cannot send more data than the bandwidth of the channel. That is the reason we take the ceiling function

Suggested Readings

2.3 Functions