Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 9

Discrete Mathematics and Its Application, 7th Edition

by Kenneth H. Rose

These slides contain material from the above resource.

TOURNAMENTS

- ❑ **Round-Robin Tournaments**

- ❑ **Single-Elimination Tournaments**

Round-Robin Tournaments

A tournament where each team **plays** every other team **exactly once** and **no ties are allowed** is called a **round-robin tournament**.

- Such tournaments can be modeled using **directed graphs** where **each team** is represented by **a vertex**.

- Note that ***(a, b)*** is an **edge** if team ***a* beats team *b***

- This graph is a **simple directed graph**, containing **no loops** or **multiple directed edges** (because no two teams play each other more than once).

Example of Round-Robin Tournaments

Such a **directed graph model** is presented in **Figure 13**. We see that **Team 1** is **undefeated** in this tournament, and **Team 3** is **winless**.

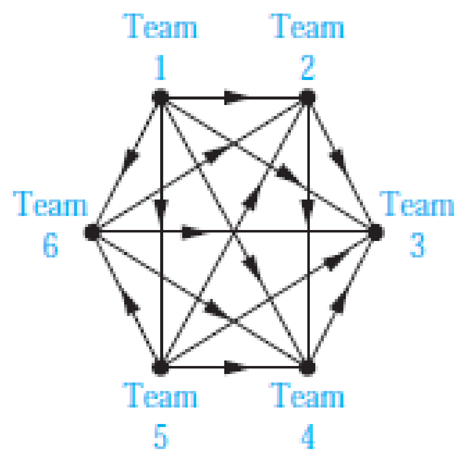


FIGURE 13 A Graph Model of a Round-Robin Tournament.

Single-Elimination Tournaments

- **Single-Elimination Tournaments** A tournament where each contestant is eliminated after one loss is called a **single-elimination tournament.**

- **Single-elimination tournaments** are often used in sports, including **tennis championships** and the yearly **NCAA basketball championship**.

- We can model such a tournament using a **vertex** to represent **each game** and **a directed edge** to connect a game to the next game the winner of this game played in.

Example of Single-Elimination Tournaments

The graph in **Figure 14** represents the games played by the final **16 teams** in the 2010 NCAA women's basketball tournament.

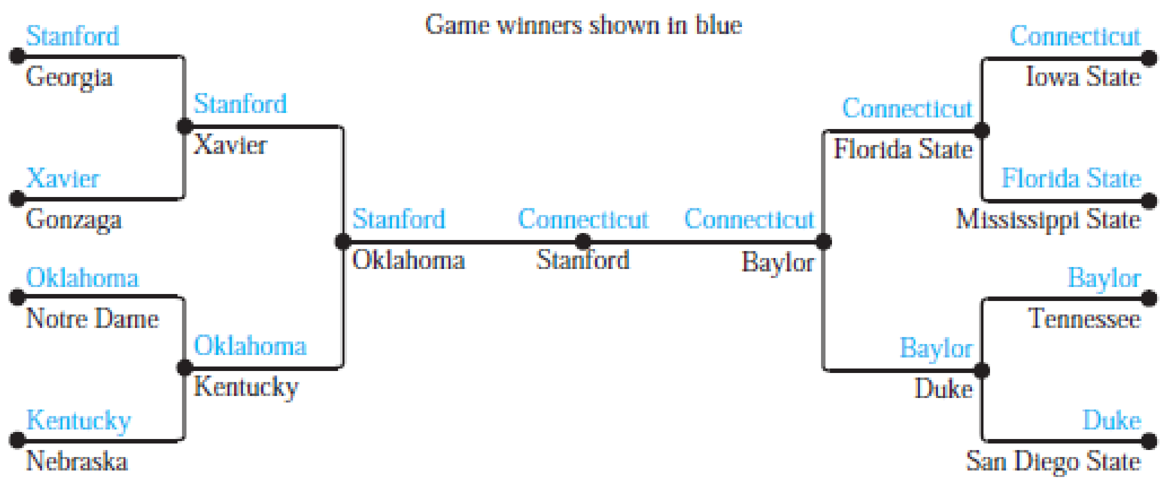


FIGURE 14 A Single-Elimination Tournament.

Adjacent (or Neighbors)

Definition: Two vertices **u** and **v** in an **undirected graph G** are called **adjacent (or neighbors)** in G if **u** and **v** are endpoints of **an edge e** of **G**. Such an **edge e** is called **incident** with the vertices **u** and **v** and **e** is said to connect **u** and **v**.

Definition: The **set of all neighbors** of a vertex **v** of G = (V, E), denoted by ***N*(v),** is called the **neighborhood** of **v**. If **A** is a subset of **V**, we denote by *N*(A) the set of all vertices in G that are adjacent to at least one vertex in A.

So, $N(A) = \bigcup_{v \in A} N(v)$

Degree of a Vertex

To keep track of how many edges are incident to a vertex, we use the following definition:

Definition: The **degree** of a vertex in an undirected graph is **the number of edges incident** with it, except **that a loop at a vertex contributes twice** to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Isolated vertex: a vertex of degree zero

Pendant vertex: a vertex of degree one

What are the **degrees** and what are the **neighborhoods** of the vertices in the graphs **G?**

deg(a) = 2

deg(b) = 4

deg(c) = 4

deg(d) = 1 **(Pendant vertex)**

deg(e) = 3

deg(f) = 4

deg(g) = 0 **(Isolated vertex)**

N(a) = {b, f}

N(b) = {a, c, e, f}

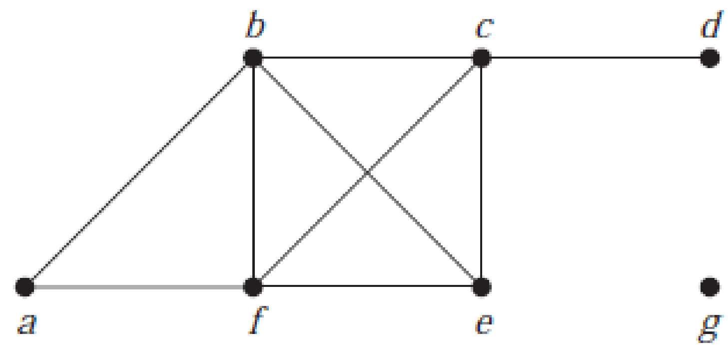
N(c) = {b, d, e, f}

N(d) = {c}

N(e) = {b, c, f}

N(f) = {a, b, c, e}

N(g) = {}



G

Example What are the **degrees** and what are the **neighborhoods of the vertices** in the graphs H

Solution

deg(a) = 4

deg(b) = 6

deg(c) = **1(Pendant vertex)**

deg(d) = 5

deg(e) = 6

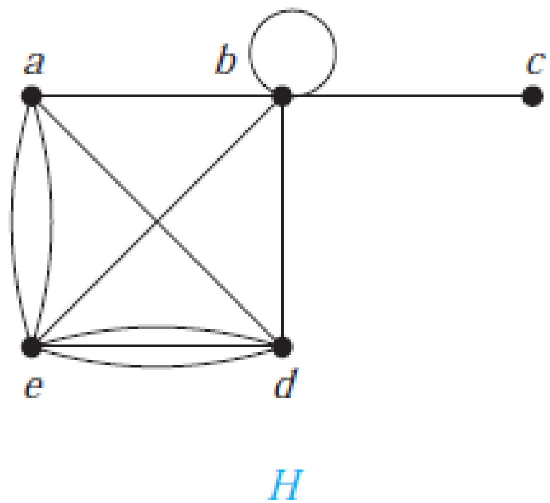
N(a) = {b, d, e}

N(b) = {a, b, c, d, e}

N(c) = {b}

N(d) = {a, b, e}

N(e) = {a, b, d}



THEOREM 1 THE HANDSHAKING THEOREM Let $G = (V, E)$ be an **undirected graph with *m* edges**. Then

$2m = \sum_{v \in V} deg(v)$

(Note that this applies even if **multiple edges** and **loops are present**.)

Example How many edges are there in a graph with 10 vertices each of degree six?

Solution:

Let $G = (V, E)$ be an undirected graph with ***m* edges**. Then

$2m = \sum_{v \in V} \deg(v)$

The sum of the degrees of the vertices is $6 \times 10 = 60$.

According to the **Handshaking theorem**

$2m = 60$ $\qquad$ $\because$ ***m* is the number of edges.**

$\Longrightarrow m = 30.$

THEOREM 1 shows that the **sum of the degrees of the vertices** of an **undirected graph is even**.

This simple fact has many consequences, one of which is given as Theorem 2.

THEOREM 2 An **undirected graph** has an **even number** of vertices of **odd degree**.

Proof: Let V_1 and V_2 be the set of **vertices of even degree** and the set of **vertices of odd degree**, respectively, in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in V_1} deg(v) + \sum_{v \in V_2} deg(v)$$

Because **$deg(v)$ is even for $v \in V_1$**, the **first term in the right-hand** side of the **last equality is even**.

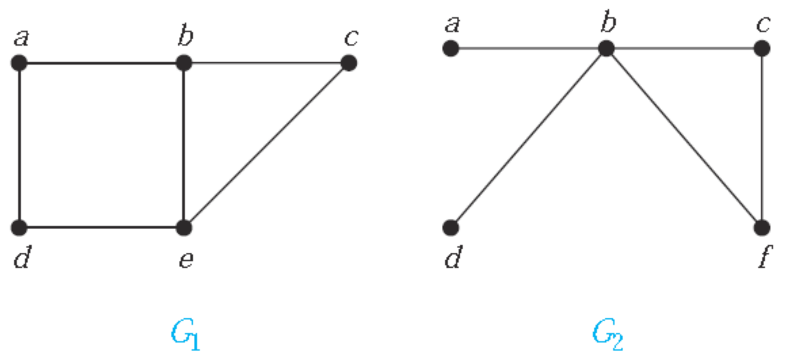
Furthermore, the sum of the **two terms on the right-hand** side of the **last equality is even**, because this sum is $2m$.

Hence, the **second term** in the **sum is also even**. Because all the terms in **this sum are odd**, there must be **an even number of such terms**.

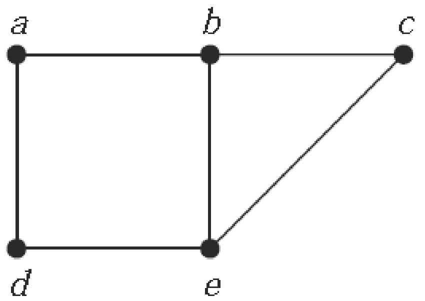
Thus, there are an **even number of vertices of odd degrees**.

Union of Graphs

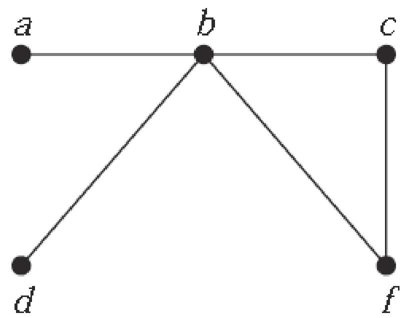
Example: Find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)



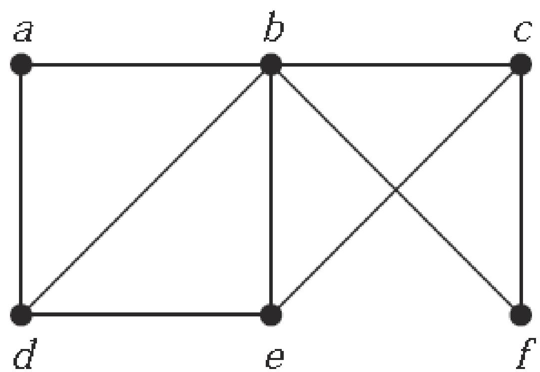
Solution



G_1



G_2



$G_1 \cup G_2$

Traversal Algorithms

Procedures for **systematically visiting** every vertex of an ordered rooted tree are called **traversal algorithms**.

We will describe three of the most commonly used such algorithms, **preorder traversal, inorder traversal,** and **postorder traversal.**

Each of these algorithms can be defined recursively

Preorder Traversal

Let ***T*** be an ordered rooted tree with root ***r***. If *T* consists only of ***r***, then ***r*** is the ***preorder traversal* of *T***. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees at *r* from **left to right in *T***. The *preorder traversal* begins by visiting *r*. It continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder

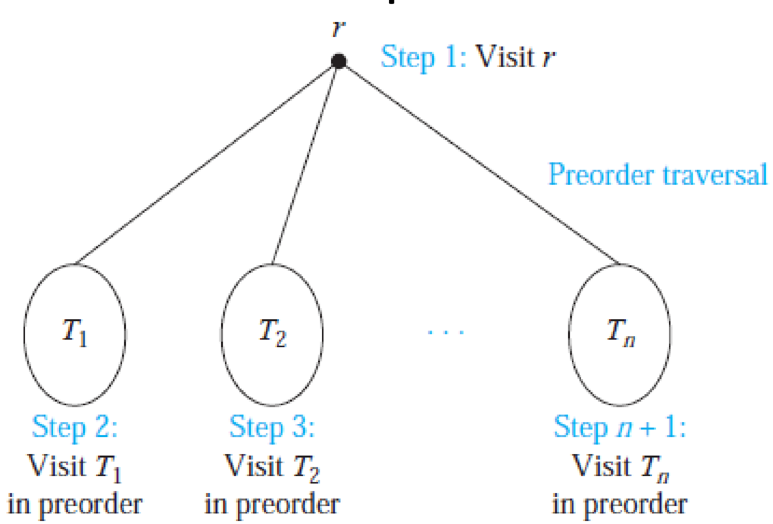


FIGURE 2 **Preorder Traversal.**

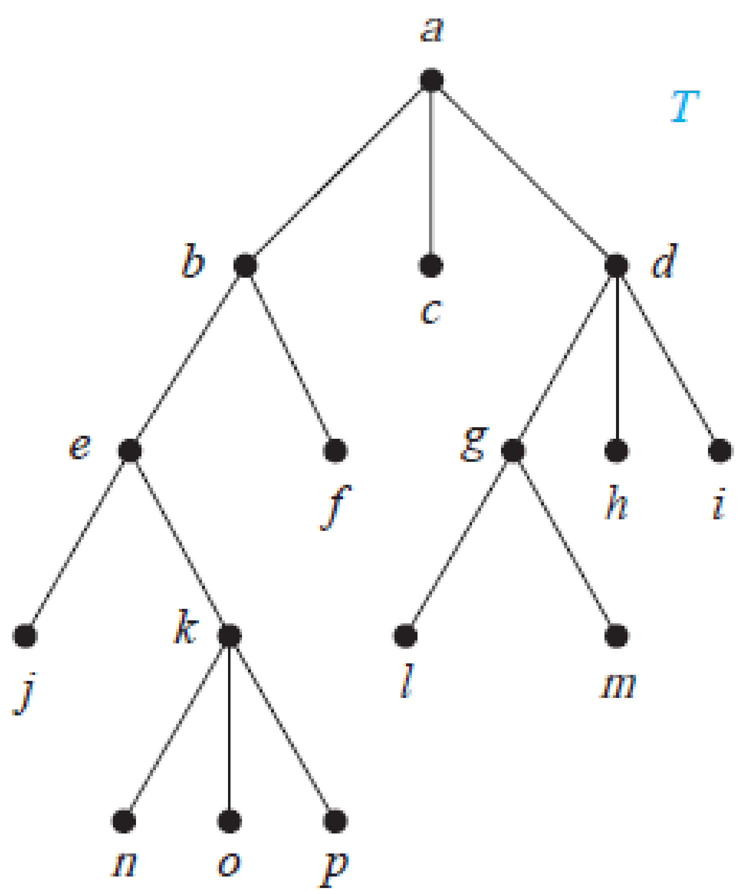


FIGURE 3 **The Ordered Rooted Tree *T*.**

Preorder Traversal

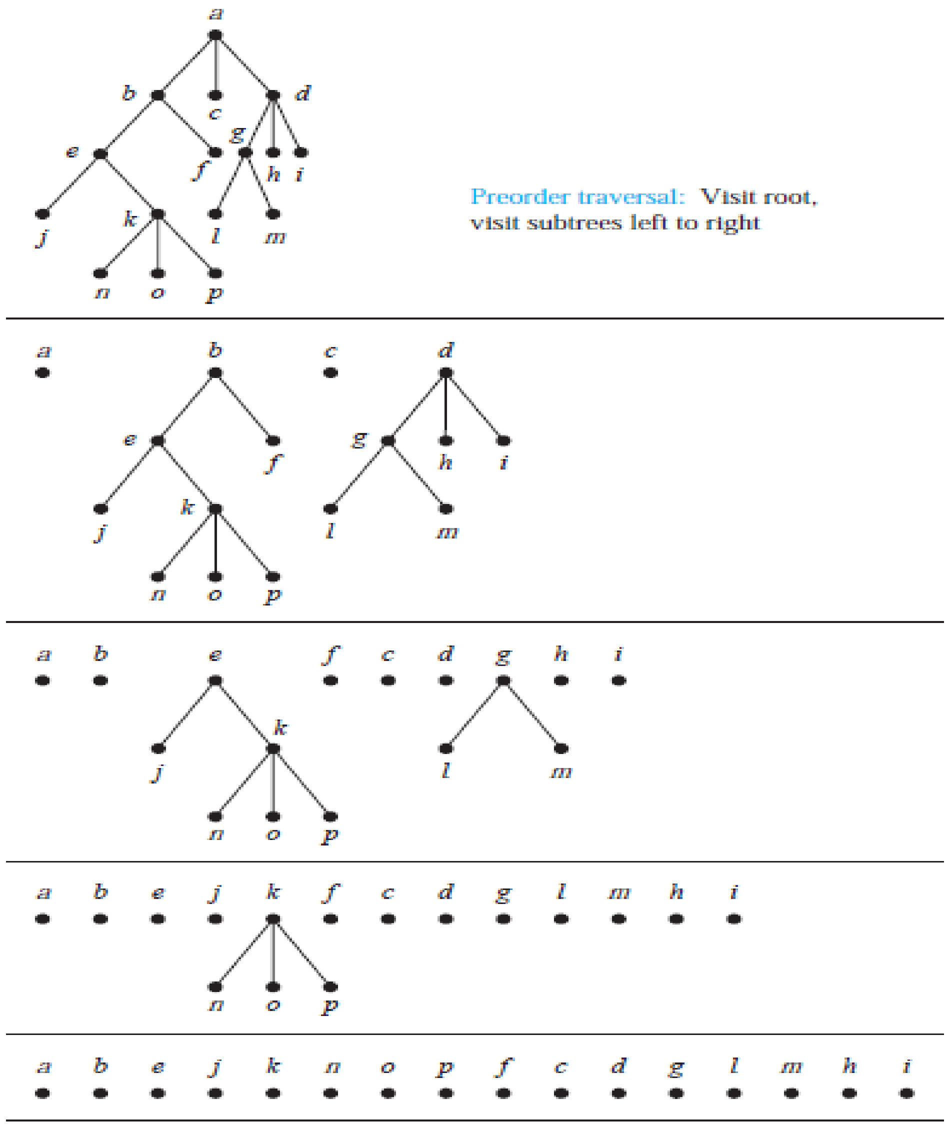


FIGURE 4 The Preorder Traversal of T.

Inorder Traversal

Let ***T*** be an ordered rooted tree with root ***r***. If T consists only of ***r***, then ***r*** is the *inorder traversal* of T . Otherwise, suppose that T_1, T_2, . . . , T_n are the subtrees at r from **left to right**. The *inorder traversal* begins by traversing T_1 in inorder, then visiting r. It continues by traversing T_2 in inorder, then T_3 in inorder, . . . , and finally T_n in inorder.

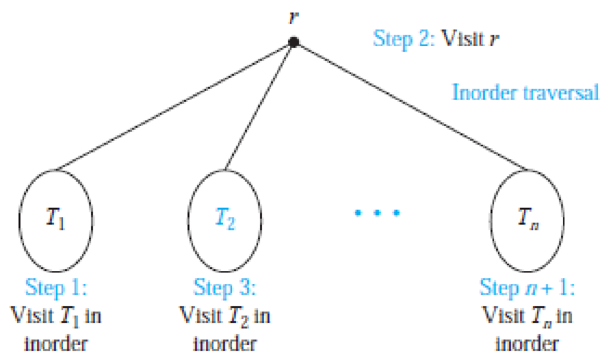


FIGURE 5 **Inorder Traversal.**

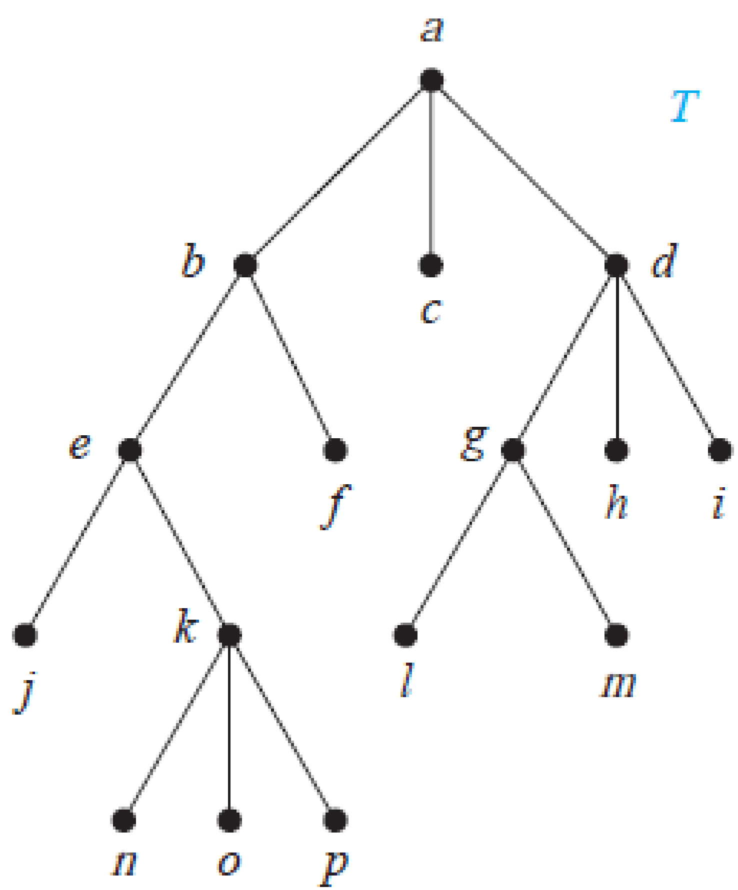


FIGURE 3 **The Ordered Rooted Tree *T*.**

Inorder Traversal

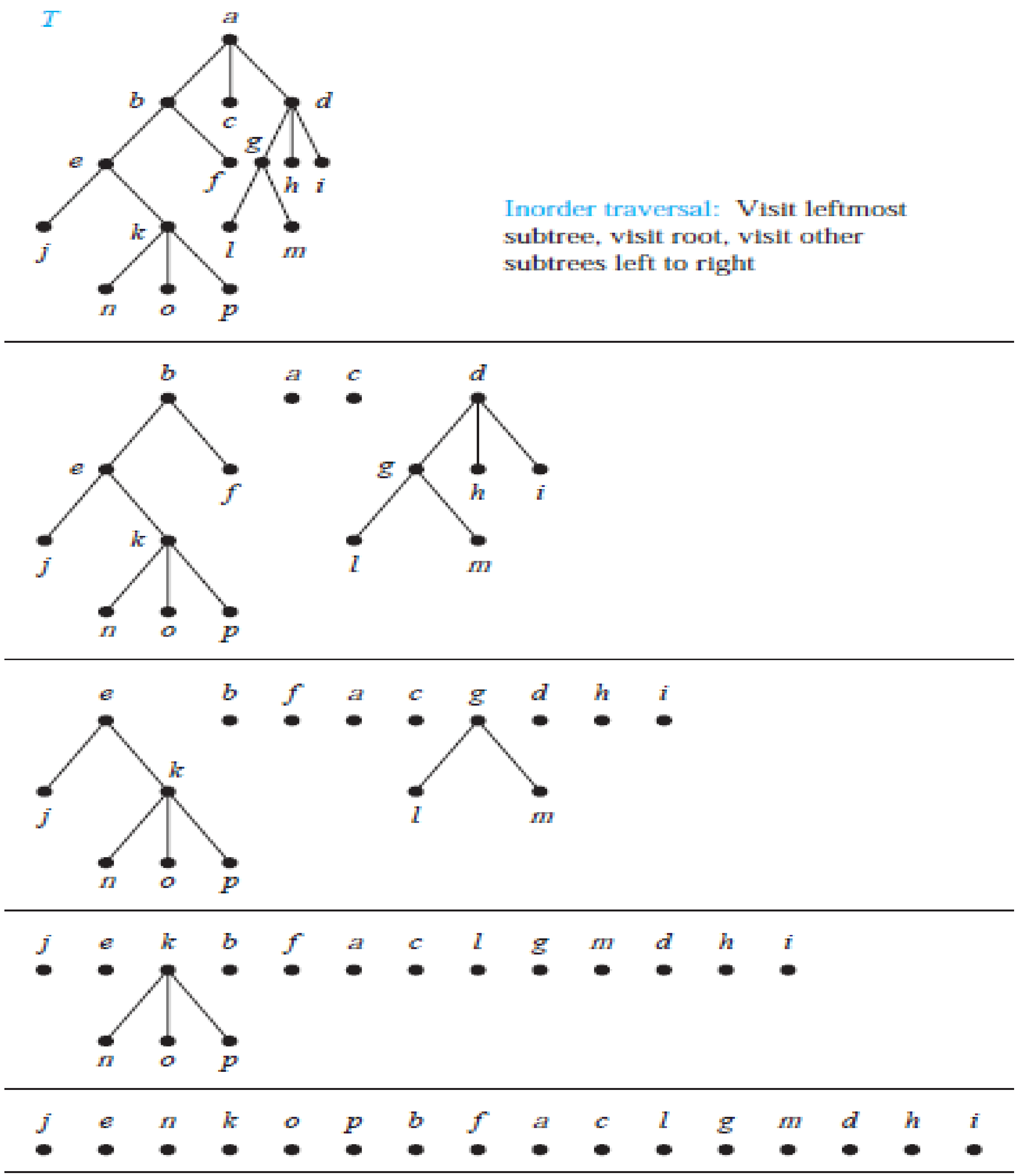


FIGURE 6 The Inorder Traversal of T.

Post order Traversal

Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T . Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees at r from left to right. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, . . . , then T_n in postorder, and ends by visiting r.

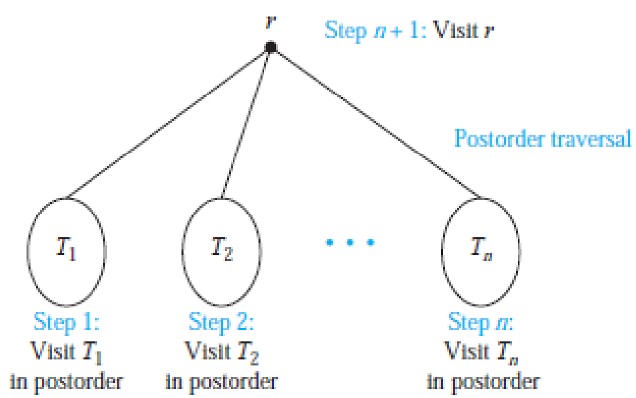


FIGURE 7 Postorder Traversal.

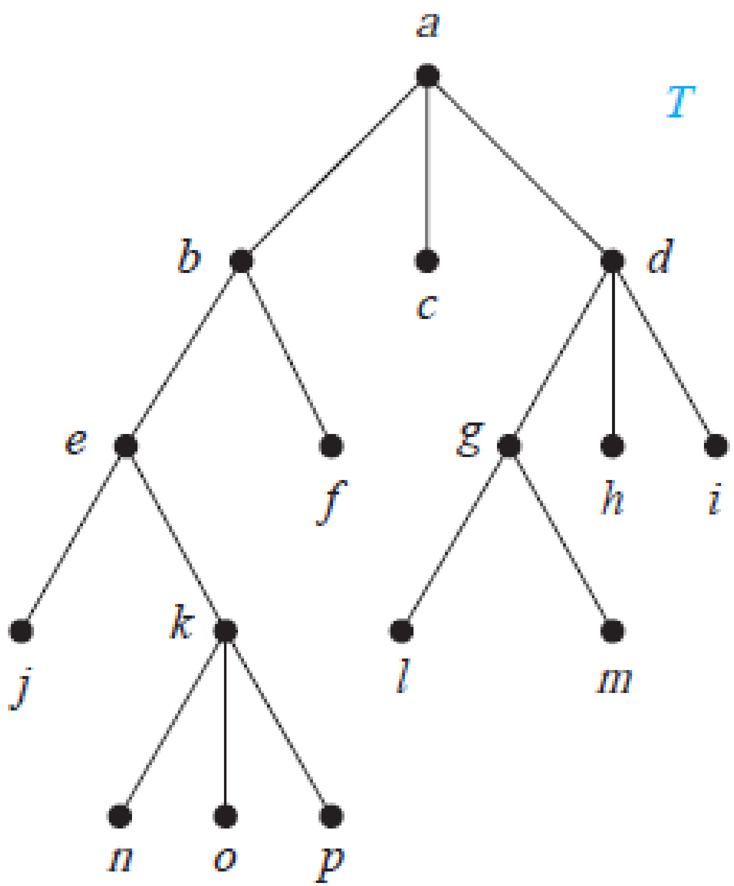


FIGURE 3 **The Ordered Rooted Tree *T*.**

Postorder

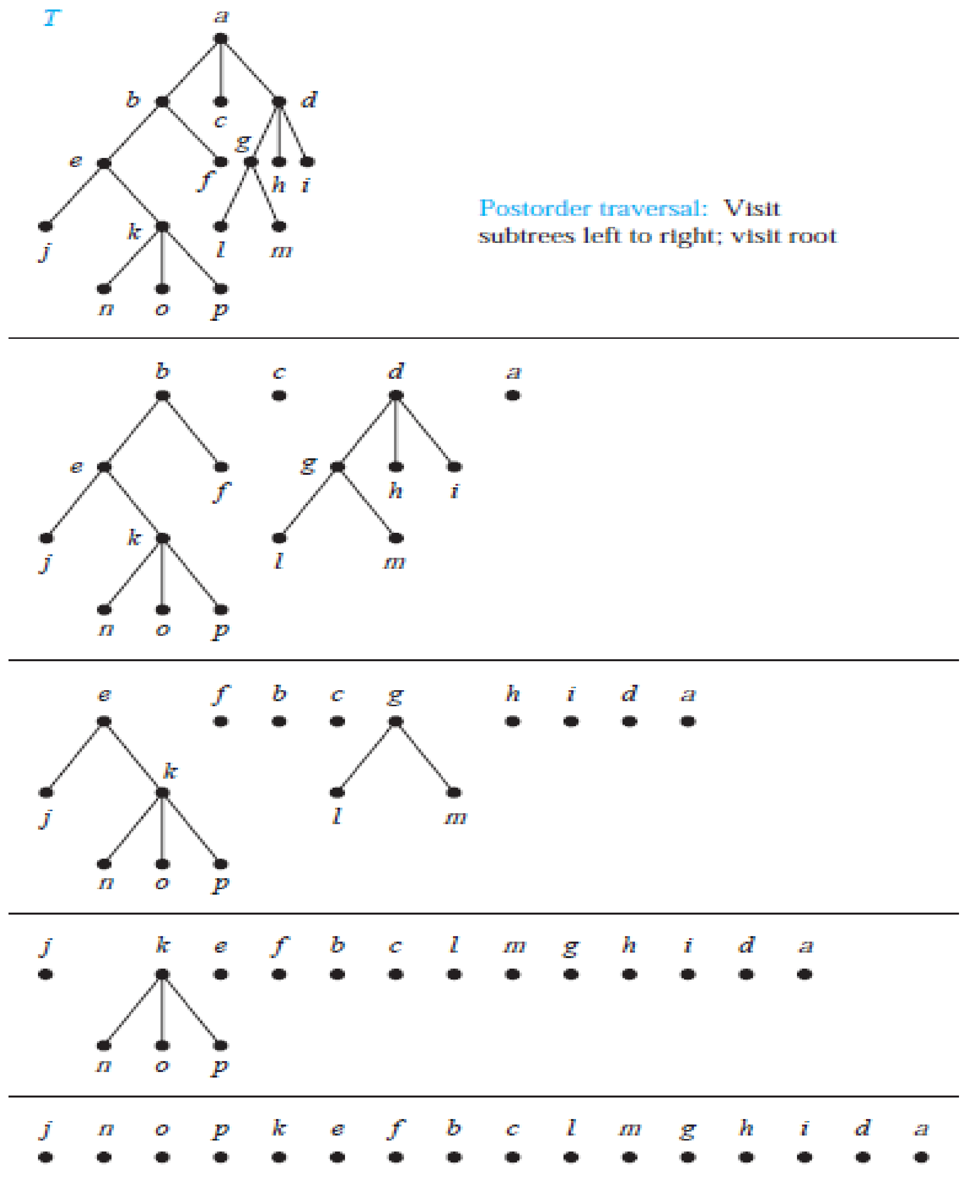


FIGURE 8 The Postorder Traversal of *T*.

Suggested Readings

10.2 Graph Terminology and Special Types of Graphs.