Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7^{th} Edition

Kenneth H. Rosen

References

Readings for these lecture notes:

- **Schaum's Outline of Probability, Second Edition (Schaum's Outlines)** by by Seymour Lipschutz, Marc Lipson
- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- **Introduction to Probability** SECOND EDITION Dimitri P. Bertsekas and John N. Tsitsiklis

These notes contain material from the above resources.

Finite Stochastic Processes And Tree Diagrams

- A (finite) sequence of experiments in which each experiment has a finite number of outcomes with given probabilities is called a **(finite) stochastic process**.

- A convenient way of describing such a process and computing the probability of any event is by a **tree diagram.**

Example:

We are given three boxes as follows:

Box 1 has **10** light bulbs of which **4** axe defective.

Box 2 has **6** light bulbs of which **1** is defective.

Box 3 has **8** light bulbs of which **3** are defective.

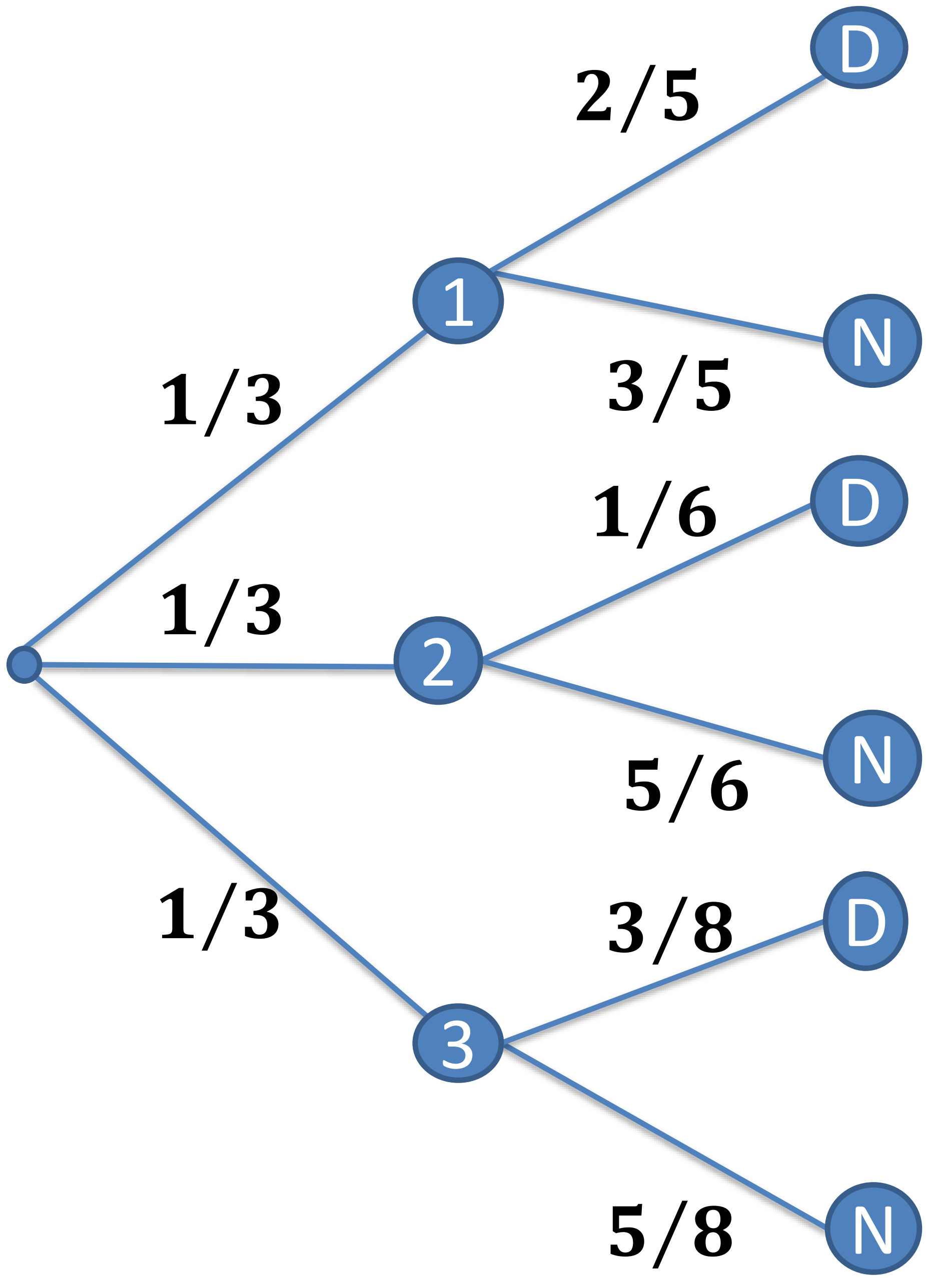
We select a box at random and then draw a bulb at random. What is the probability **p** that the **bulb** is **defective**?

Solution:

Here we perform a sequence of two experiments:

(i) select one of the three boxes;

(ii) select a bulb which is either defective ***(D)*** or nondefective ***(N).***

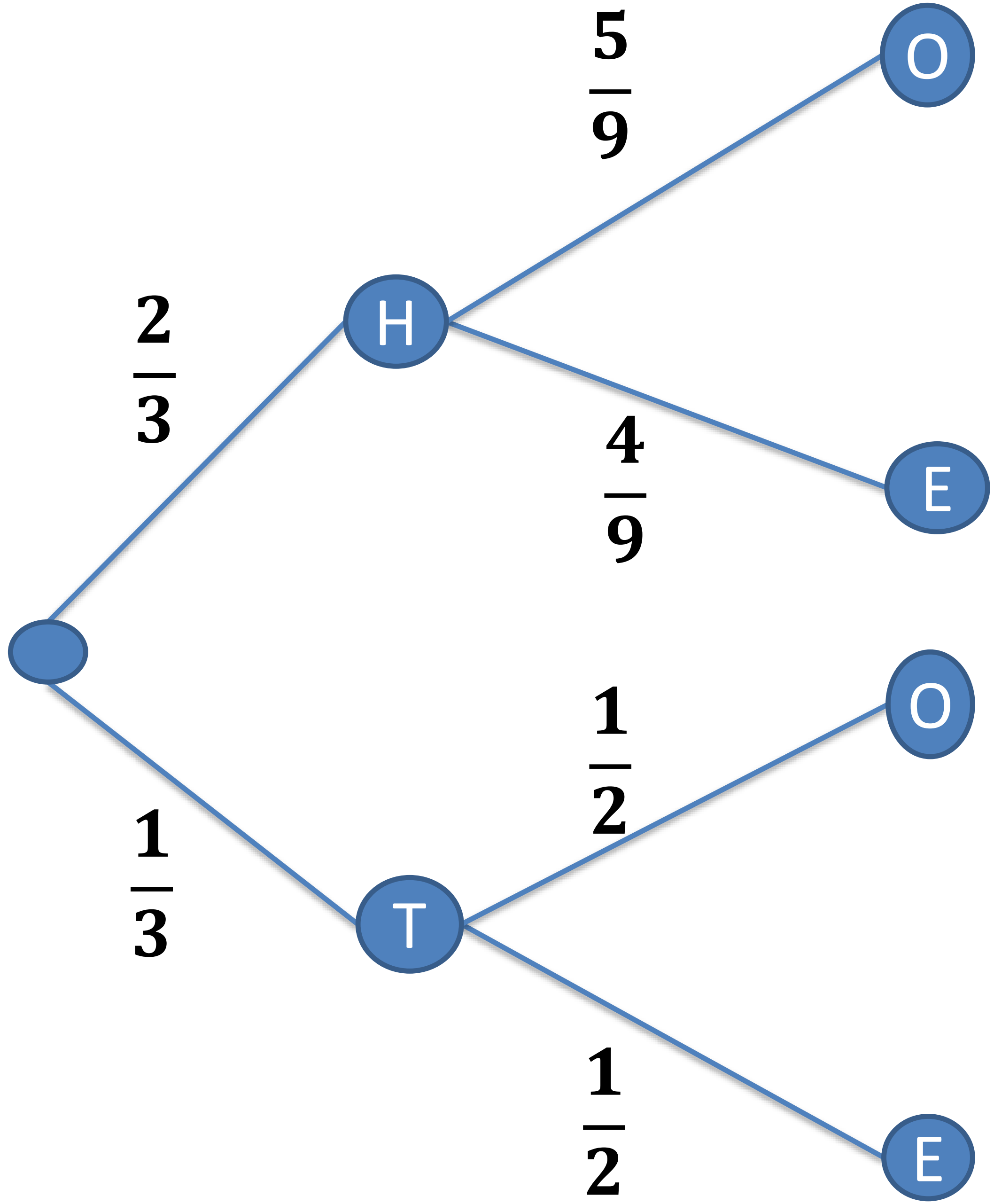

2/5
D
1
N
3/5
1/3
1/6
D
1/3
2
N
5/6
1/3
3/8
D
3
5/8
N

Thus by the multiplication theorem,

$$\mathbf{p} = \left(\frac{1}{3}\right)\left(\frac{2}{5}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)+ \left(\frac{1}{3}\right)\left(\frac{5}{8}\right) = \mathbf{\frac{113}{360}}$$

Example : A coin, weighted so that **P(H) = $\frac{2}{3}$** and **P(T) = $\frac{1}{3}$**, is tossed. If **heads** appears, then a number is selected at random from the numbers **1 through 9**; if **tails** appears, then a number is selected at random from the numbers **1 through 6.**

Find the probability p that an **even number** is selected.


5/9
O
2/3
H
4/9
E
O
1/2
1/3
T
1/2
E

Probability of even using 1 through 9:

$$P(E) = \frac{4}{9}$$

Probability of even using 1 through 6:

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

$$\mathbf{p} = \left(\frac{2}{3}\right)\left(\frac{4}{9}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \mathbf{\frac{25}{54}}$$

Recall: Mutually Exclusive or Disjoint

Two events A and B are **mutually exclusive, or disjoint,**

if **$A \cap B = \{\ \}$** or **$\emptyset$**

OR

Two events A and B are **mutually exclusive**, **or disjoint,**

if **$A \cap B = \emptyset$**, that is, if A and B have no elements in common.

OR

Events ***A*** and ***B*** are **disjoint** (or **mutually exclusive**) if they cannot occur at the same time. (That is, disjoint events do not overlap.)

Addition Rule I

Addition Rule I: When two events are **mutually exclusive**,

P(A or B) = P(A) + P(B)

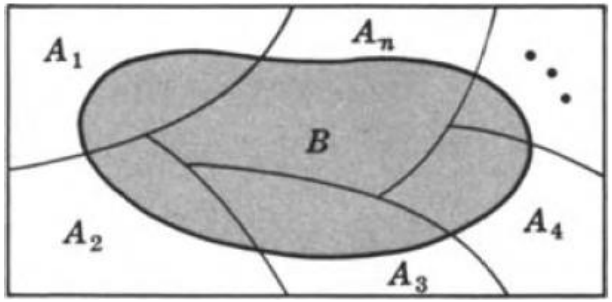
OR

$$P(A \cup B) = P(A) + P(B)$$

PARTITIONS AND BAYES' THEOREM

Suppose the events $A_1, A_2, \ldots, A_n$ form a partition of a sample space S; that is, the events A_i are **mutually exclusive** and their union is S. Now let B be any other event. Then

$$\mathbf{B} = S \cap B = (A_1 \cup A_2 \cup \cdots \cup A_n) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots (A_n \cap B)$$



where the $A_i \cap B$ are also **mutually exclusive**.

$$\therefore P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$

Using the **multiplication theorem**, we get

$$\therefore \boldsymbol{P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)} \ldots\ldots\ldots\ldots(1)$$

Total probability

$P(B) = \sum_{i=1}^{n} (A_i \cap B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$

The above formula is sometimes called the **theorem of total probability** or the **rule of elimination**.

On the other hand, for any *i*, the **conditional probability** of A_i given B is defined by

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \quad \text{...(2)}$$

Substitute values of $P(B)$ from (1) and

$P(A_i \cap B) = P(A_i)P(B|A_i)$ in (2), we get

$$\therefore P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)}$$

Bayes' Theorem

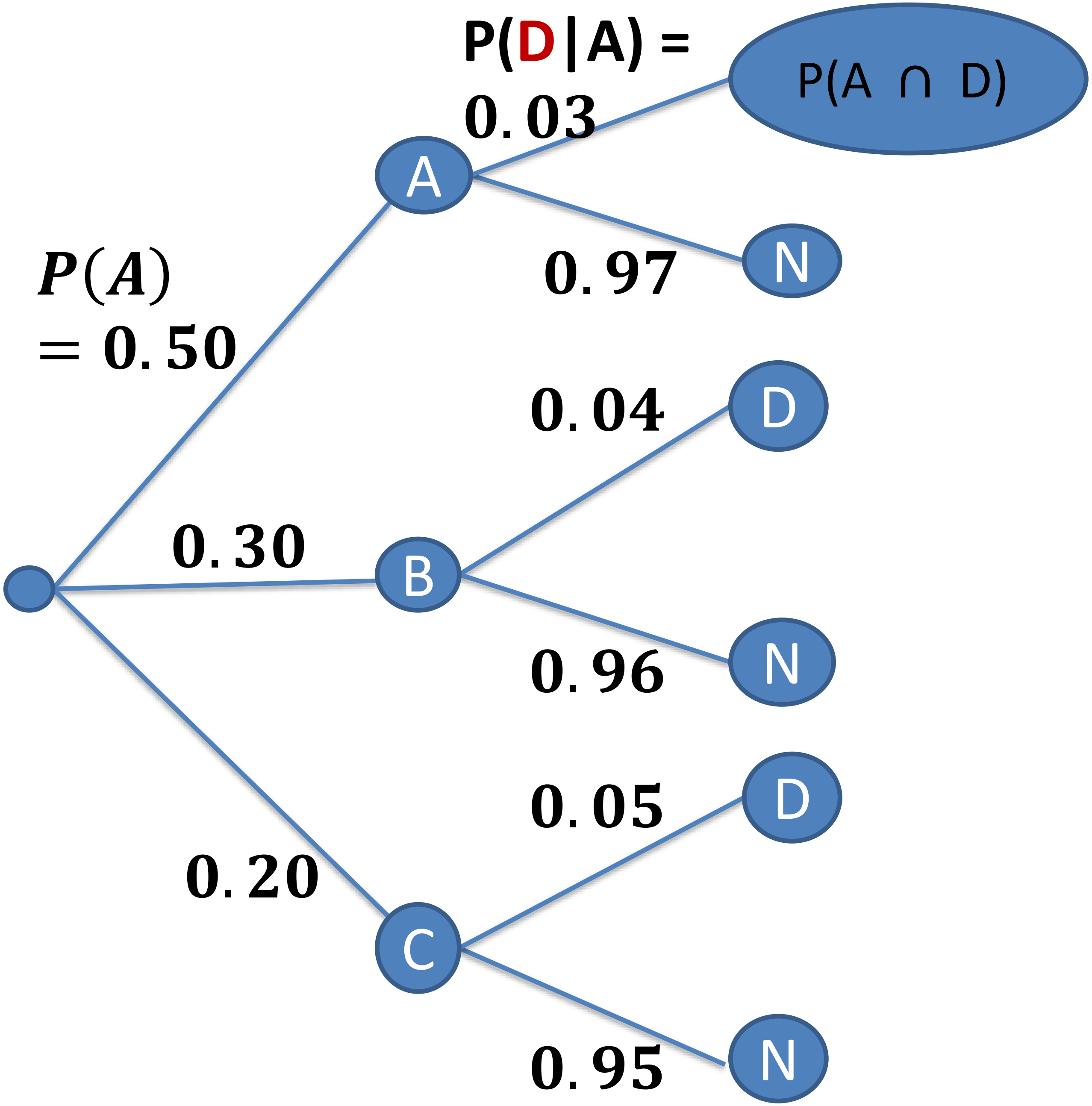
Suppose $A_1, A_2, \ldots, A_n$ is a partition of **S** and ***B*** is any event. Then for any ***i***,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)}$$

or

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

Example: Three machines ***A, B*** and ***C*** produce respectively **50%**, **30%** and **20%** of the total number of items of a factory. The percentages of **defective output** of these machines are **3%**, **4%** and **5%.** If an item is selected at random, find the probability that the item is **defective**.


P(D|A) =
0.03
P(A ∩ D)
A
0.97
N
P(A)
= 0.50
0.04
D
0.30
B
0.96
N
0.05
D
0.20
C
0.95
N

Solution:

Let **D** be the event that the item is **defective**

$\because$ $P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n)$

$\therefore$ $\boldsymbol{P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$

$= (0.50)(0.03) + (0.30)(0.04) + (0.20)(0.05)$

$= \mathbf{0.037}$ **(or 3.7%)**

Example: Consider the factory in the preceding example. Suppose an item is selected at random and is found to be **defective**. Find the probability that the item was produced by **machine A**; that is, find **$P(A|D)$.**

Solution:

By Bayes' theorem,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_j)}$$

$$P(A|D) = \frac{P(A)P(D|A)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$$

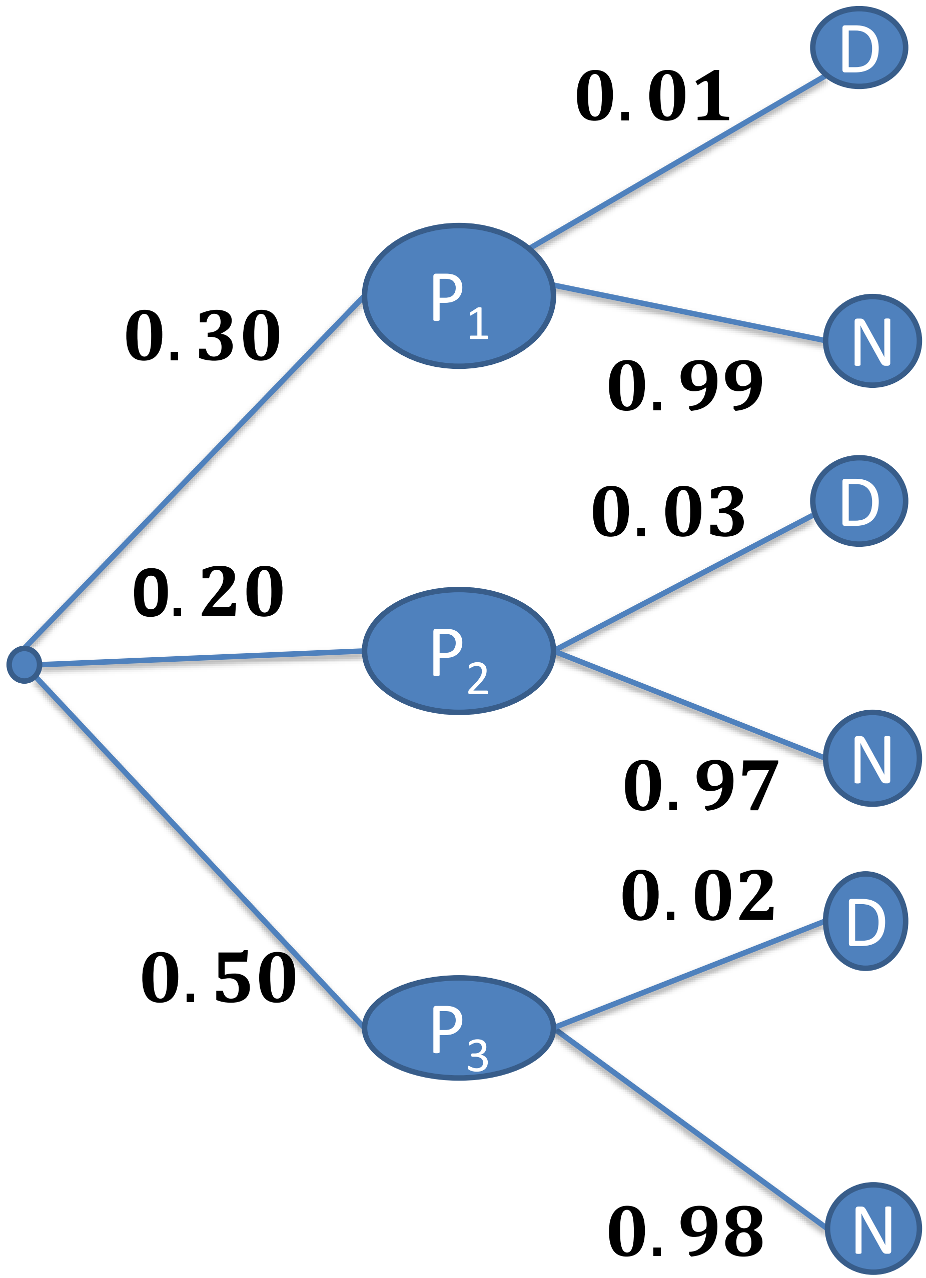
$$P(A|D) = \frac{(0.50)(0.03)}{(0.50)(0.03) + (0.30)(0.04) + (0.20)(0.05)}$$

$$= \frac{15}{37} \text{ (or 0.4054)}$$

Example: A manufacturing firm employs **three** analytical plans for the design and development of a particular product. For cost reasons, **all three** are used at varying times. In fact, plans **1**, **2**, and **3** are used for **30%**, **20%**, and **50%** of the products, respectively. The defect rate is different for the three procedures as follows:

$\mathbf{P(D|P_1) = 0.01}$, $\mathbf{P(D|P_2) = 0.03}$, $\mathbf{P(D|P_3) = 0.02}$,

where $\mathbf{P(D|P_j)}$ is the probability of a defective product, given **plan** ***j***. If a random product was observed and found to be **defective**, which **plan** was **most likely** used and thus responsible?


0.30
0.20
0.50
P_1
P_2
P_3
0.01
0.99
0.03
0.97
0.02
0.98
D
N
D
N
D
N

Solution: Given

$P(P_1) = 0.30$, $P(P_2) = 0.20$, and $P(P_3) = 0.50$,

$P(D|P_1) = 0.01$, $P(D|P_2) = 0.03$, $P(D|P_3) = 0.02$

We have to find **$P(P_j|D)$ for $j = 1, 2, 3$**.

By Bayes' theorem,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_j)}$$

$$P(P_1|D) = \frac{P(P_1)P(D|P_1)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)}$$

$$= \frac{(0.30)(0.01)}{(0.30)(0.01)+(0.20)(0.03) + (0.50)(0.02)}$$

$$= \frac{0.003}{0.019}$$

$= \mathbf{0.158}$

$$P(P_2|D) = \frac{P(P_2)P(D|P_2)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)}$$

$$= \frac{(0.20)(0.03)}{(0.30)(0.01)+(0.20)(0.03) + (0.50)(0.02)} = \mathbf{0.316}$$

$$P(P_3|D) = \frac{P(P_3)P(D|P_3)}{P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)}$$

$$= \frac{(0.50)(0.02)}{(0.30)(0.01)+(0.20)(0.03) + (0.50)(0.02)} = \mathbf{0.526.}$$

The conditional probability of a defect given **plan 3** is the largest of the **three**; thus a defective for a random product is most likely the result of the use of **plan 3**

Conditional probabilities share properties of ordinary probabilities

Since conditional probabilities satisfy all of **the probability axioms, any formula or theorem that we** ever derive for ordinary probabilities will remain true for conditional probabilities as well.

Model based on conditional probabilities

- Let us now examine what conditional probabilities are good for.
- They are used to **revise a model** when we get **new information**, but there is another way in which they arise.
- We can use **conditional probabilities** to **build a multi-stage model** of a probabilistic experiment.

Model based on conditional probabilities

Example Radar Detection. If an **aircraft is present** in a certain area, a **radar detects** it and generates **an alarm** signal with **probability 0.99**. If an **aircraft is not present** the radar generates a **(false) alarm**, with probability **0.10**. We assume that an **aircraft is present** with probability **0.05**. What is the probability of **no aircraft presence** and a **false alarm**? What is the probability of **aircraft presence** and **no detection?**

Model based on conditional probabilities

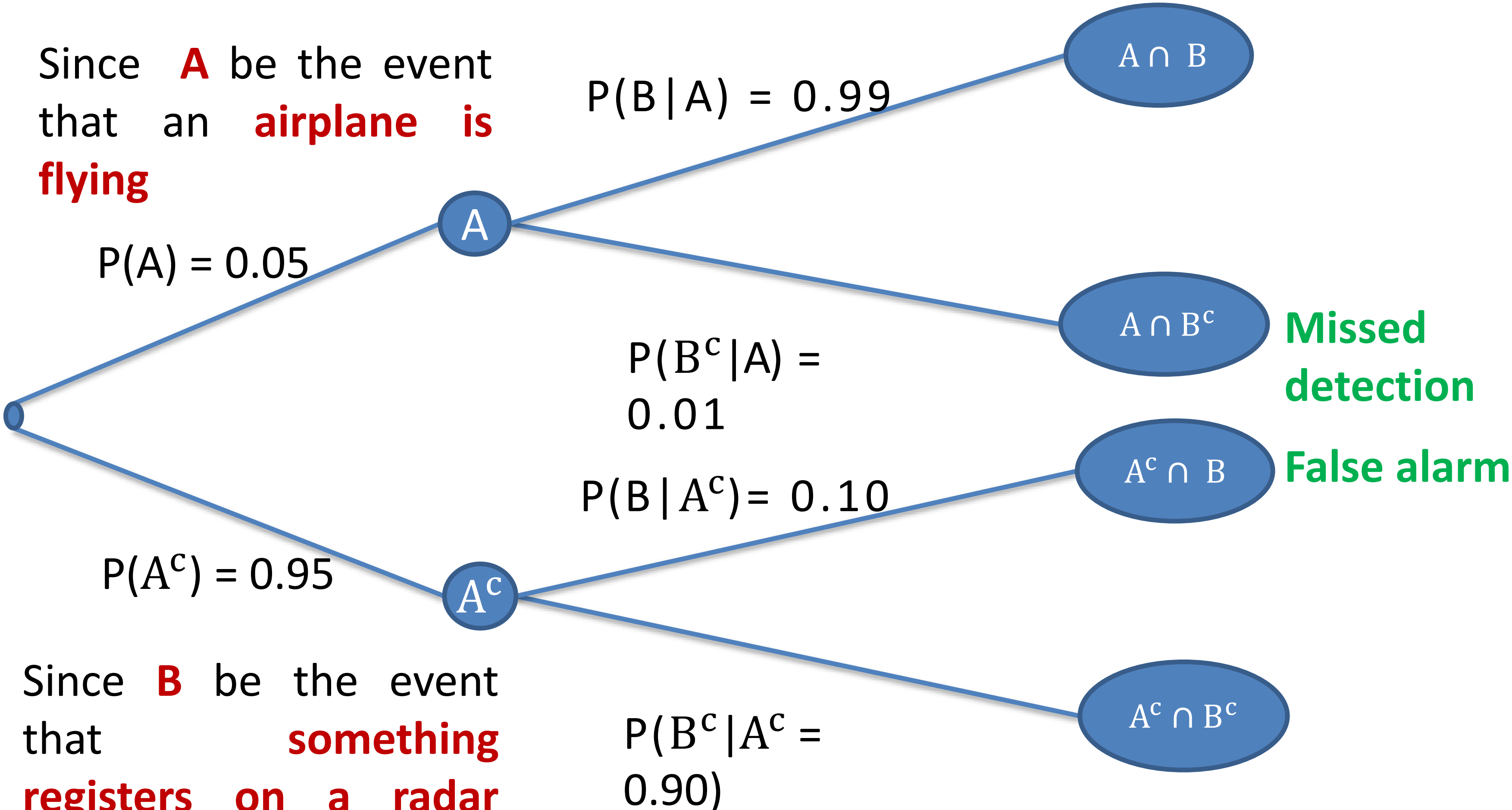
Let **A** be the event that an **airplane is flying**

Let **A^c** be the event that an **airplane is not flying**

Let **B** be the event that **something registers on a radar screen**.

Let **B^c** be the event that **something does not registers on a radar screen**.

Since **A** be the event that an **airplane is flying**



Since **B** be the event that **something registers on a radar screen**

P(not present , false alarm) = $P(A^c \cap B)$

$= P(A^c)P(B|A^c)$

$= 0.95 \times 0.10$

= **0.095 or 9.5 % ans**

P(present , no detection) = $P(A \cap B^c)$

$= P(A)P(B^c|A)$

$= 0.05 \times 0.01$

= **0.0005 or 0.05 % ans**