Trees

Chapter 11

Chapter Summary

- Introduction to Trees
- Applications of Trees
- Tree Traversal
- Spanning Trees
- Minimum Spanning

Introduction to Trees

Section 11.1

Section Summary

- Introduction to Trees
- Rooted Trees
- Trees as Models
- Properties of Trees

Trees

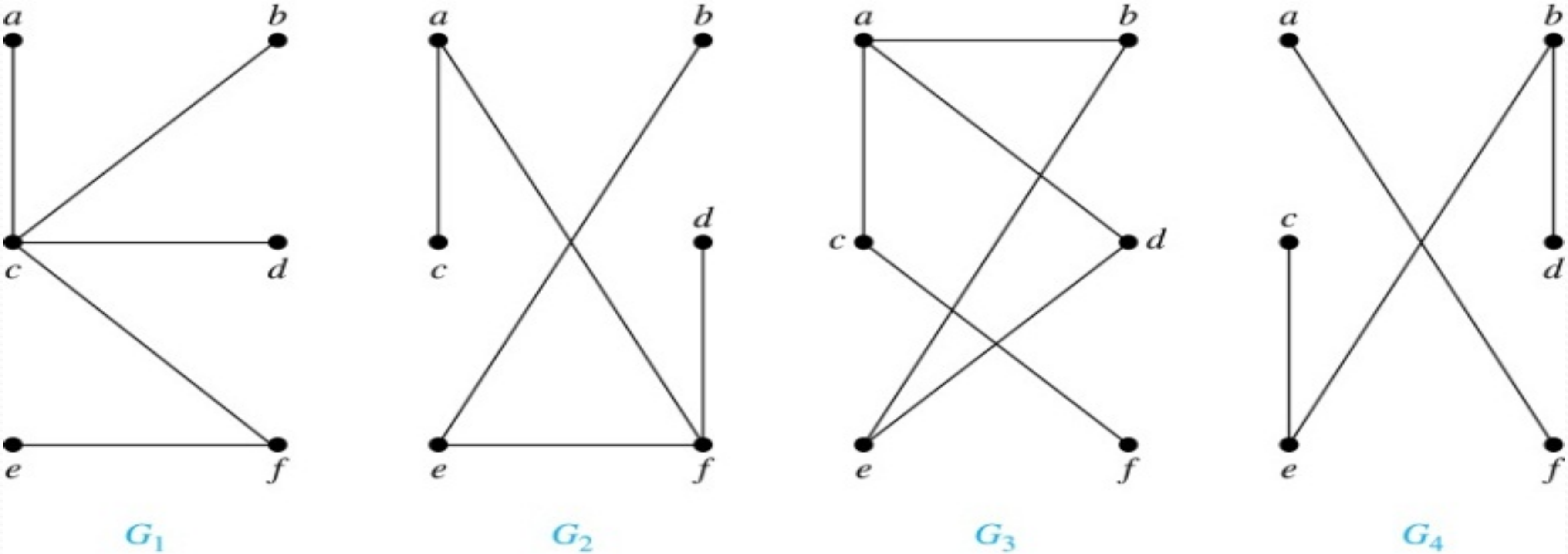
Definition: A *tree* is a connected undirected graph with no simple circuits.

Definition: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices. A tree cannot contain multiple edges or loops.

Definition: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Trees

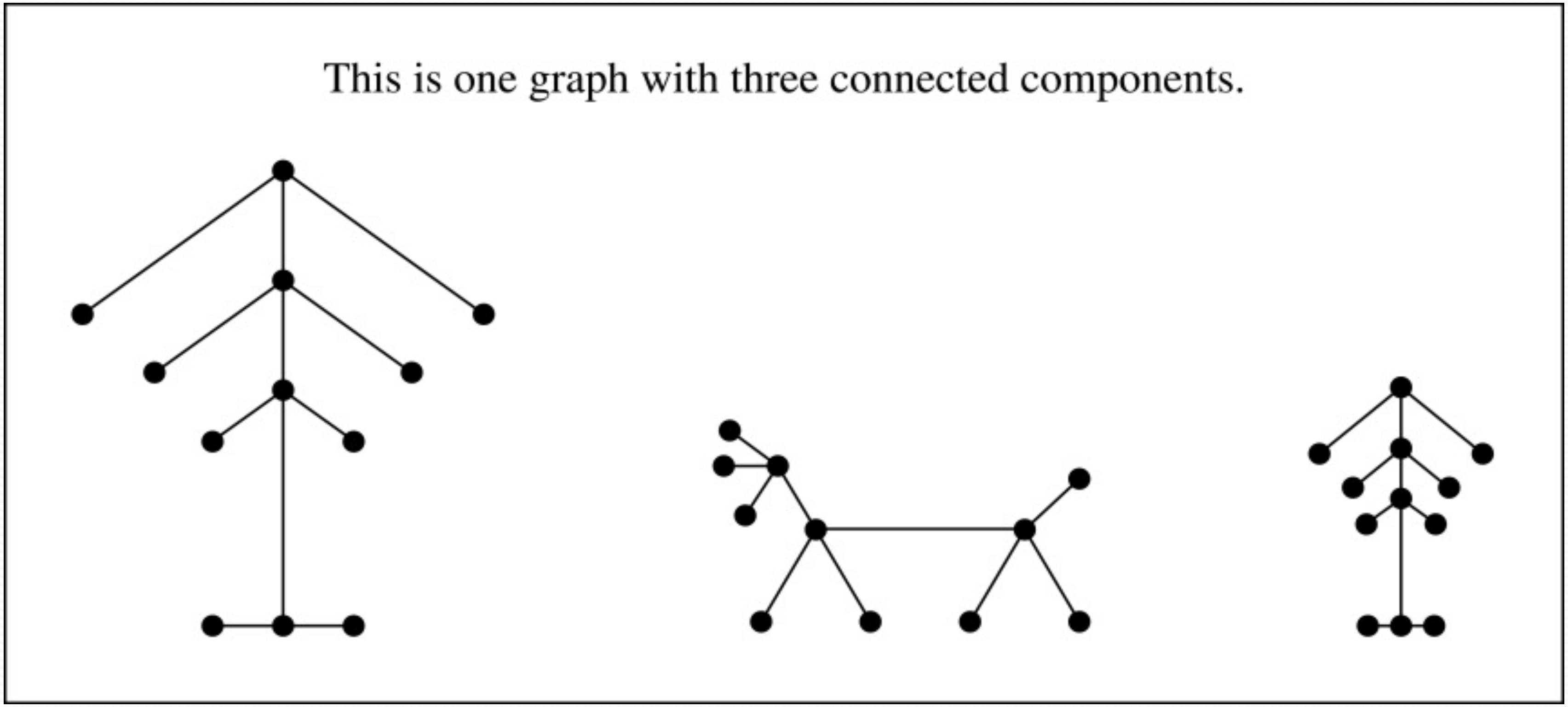
Example: Which of these graphs are trees?



Solution: G_1 and G_2 are trees - both are connected and have no simple circuits. G_3 is not a tree because e, b, a, d, e is a simple circuit,. G_4 is not a tree because it is not connected.

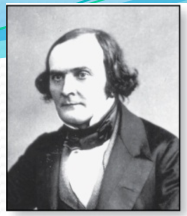
FOREST

Definition: A *forest* is a graph that has no simple circuit, but is not connected. Each of the connected components in a forest is a tree.



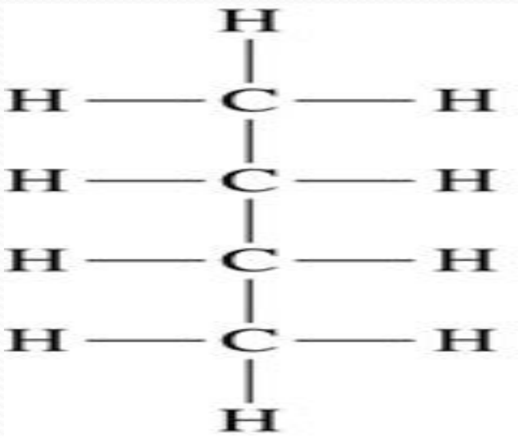
Trees as Models

Arthur Cayley
(1821-1895)

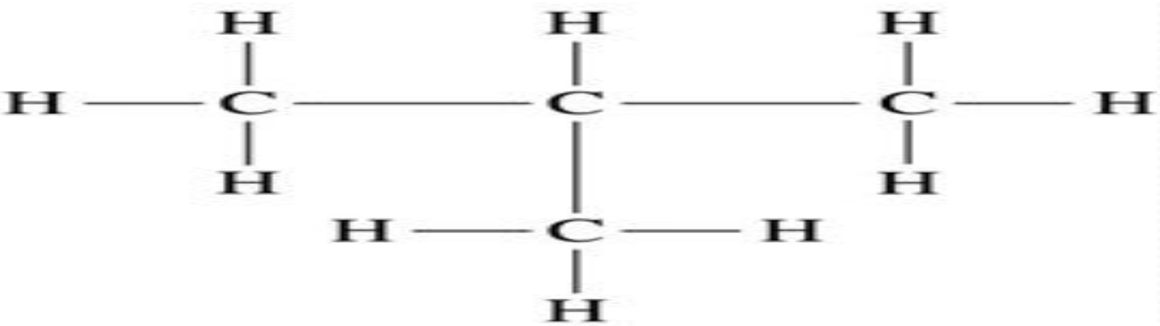


- Trees are used as models in computer science, chemistry, geology, botany, psychology, and many other areas.

- Trees were introduced by the mathematician Cayley in 1857 in his work counting the number of isomers of saturated hydrocarbons. The two isomers of butane are:



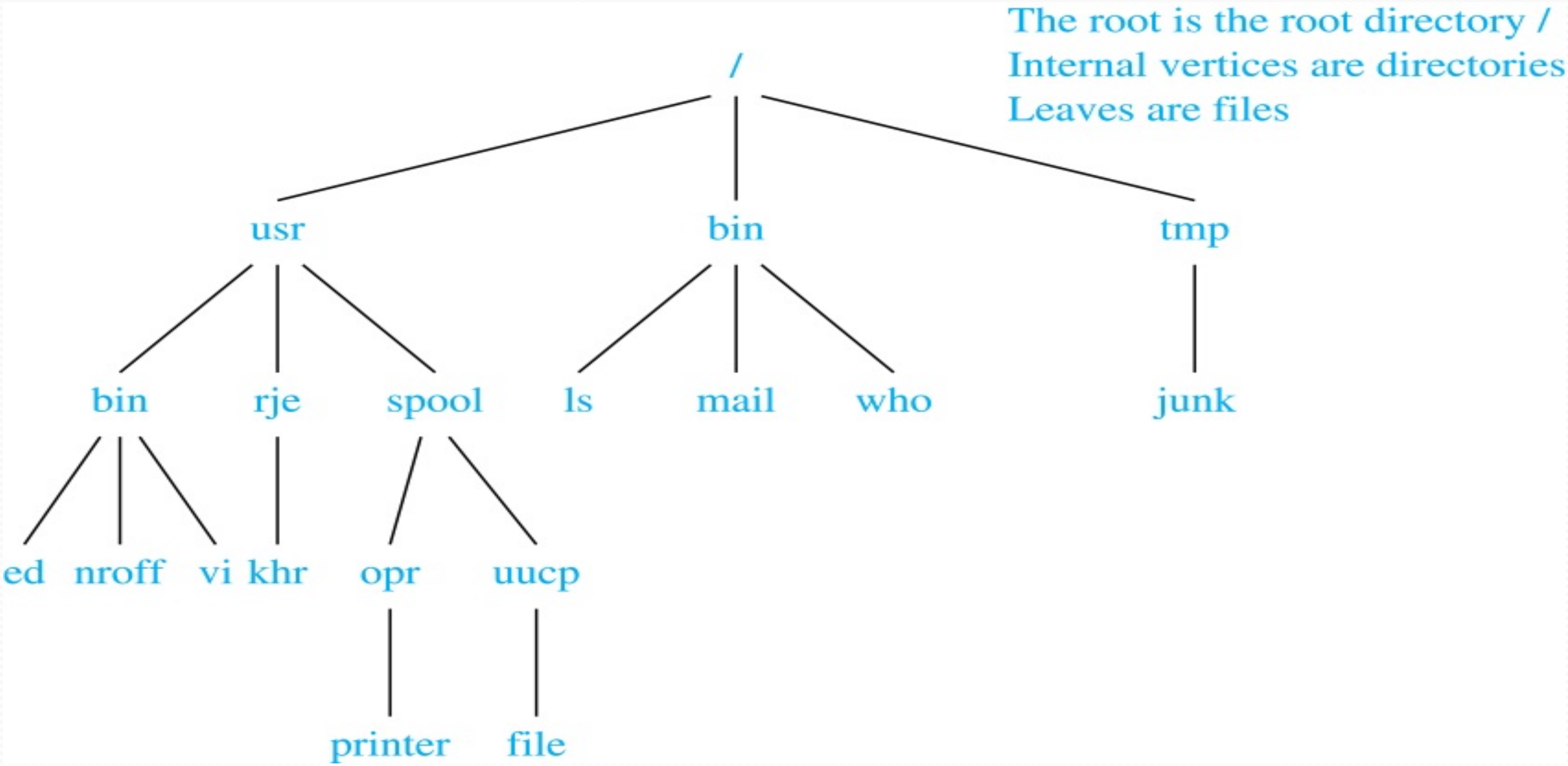
Butane



Isobutane

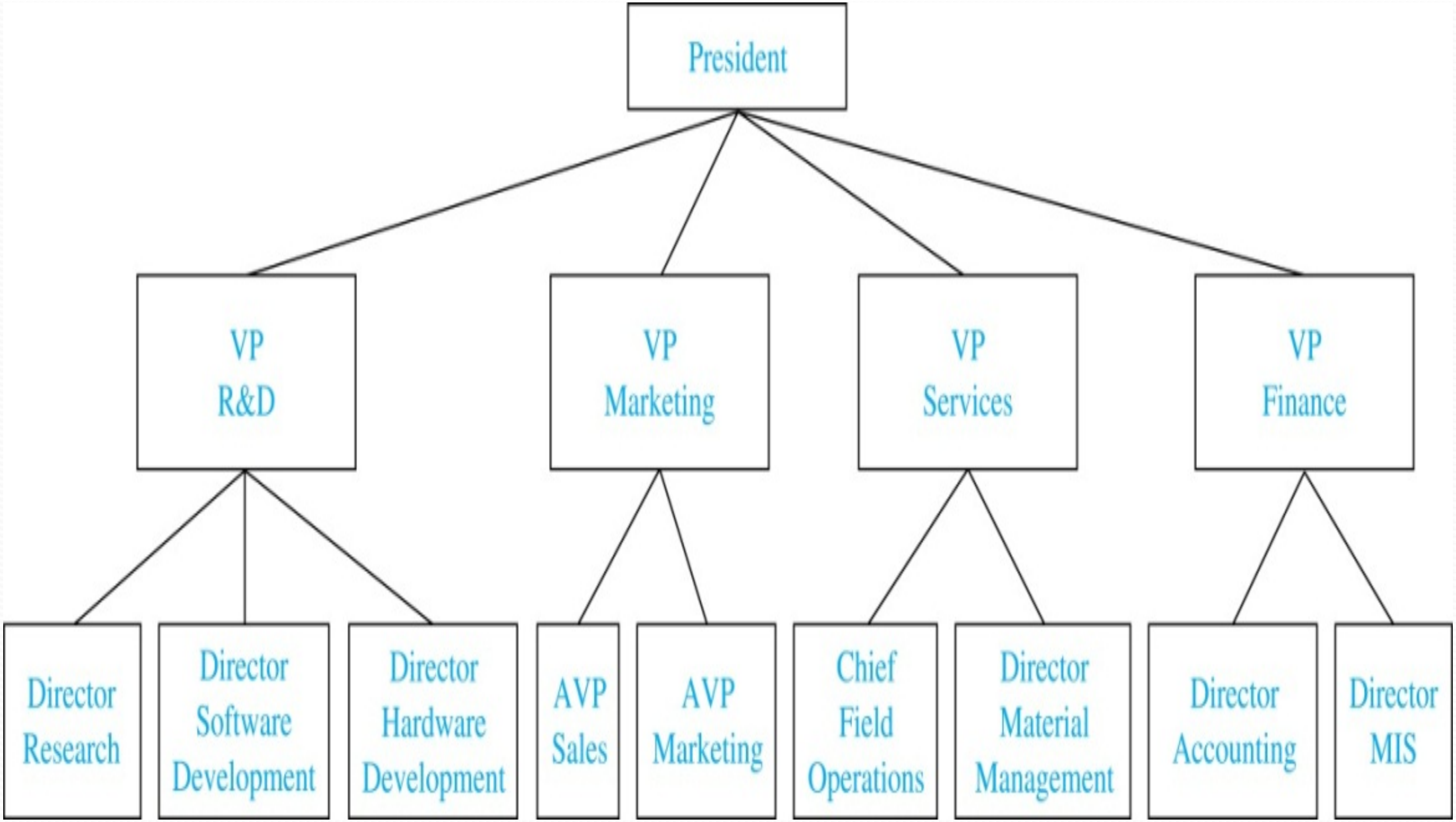
Trees as Models

- The organization of a computer file system into directories, subdirectories, and files is naturally represented as a tree.



Trees as Models

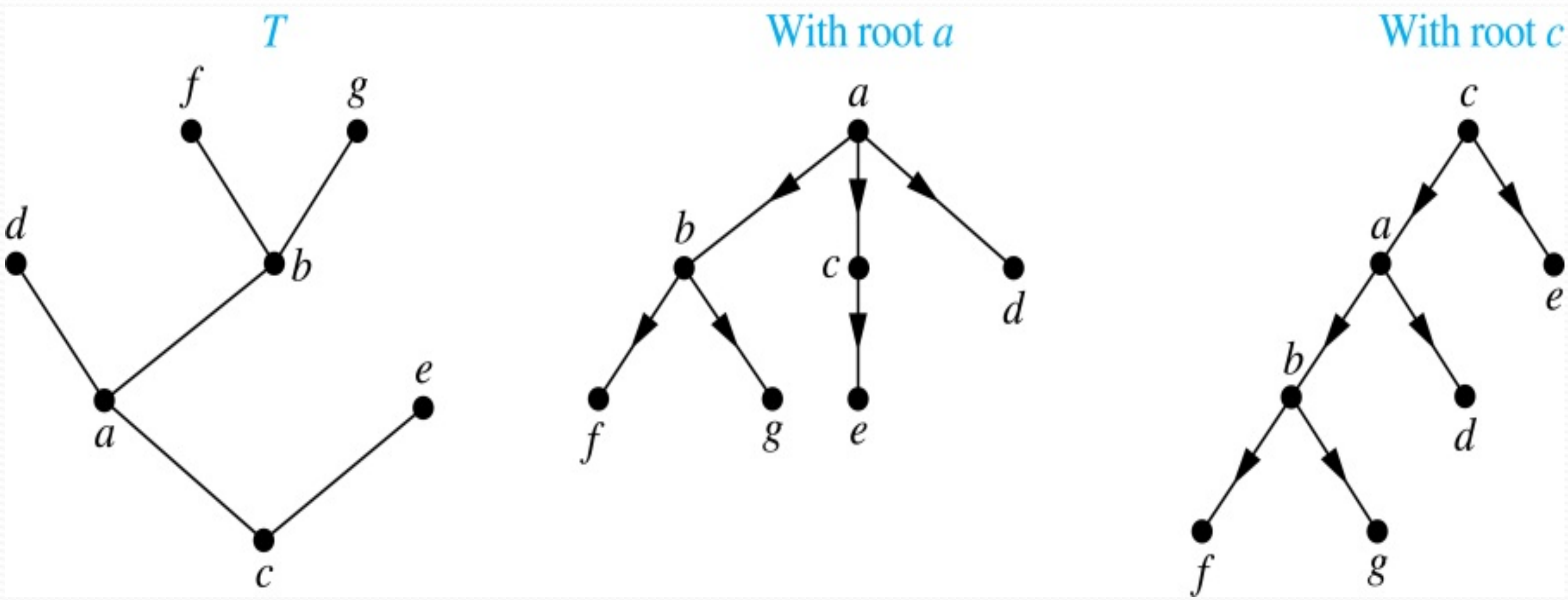
- Trees are used to represent the structure of organizations.



Rooted Trees

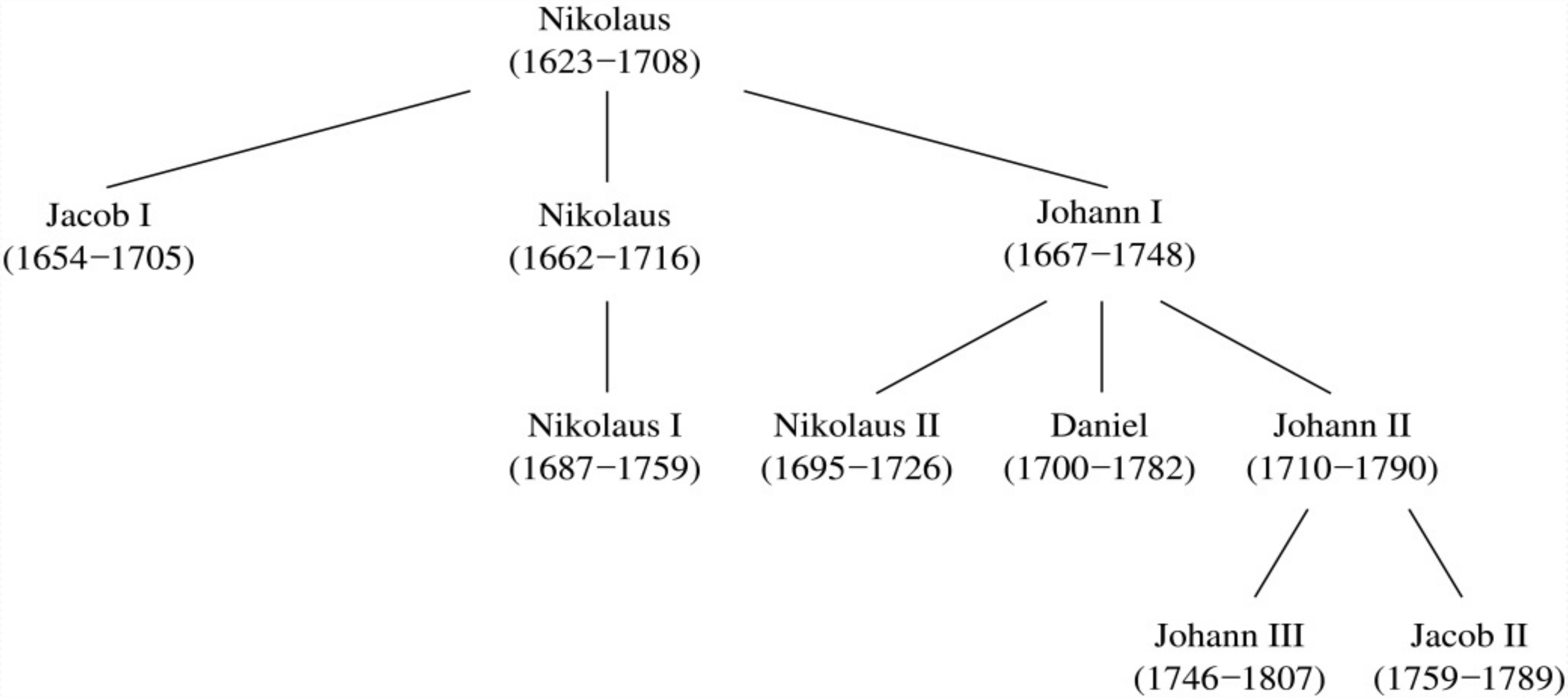
Definition: A *rooted tree* is a tree in which one vertex has been designated as the *root* and every edge is directed away from the root.

- An unrooted tree is converted into different rooted trees when different vertices are chosen as the root.



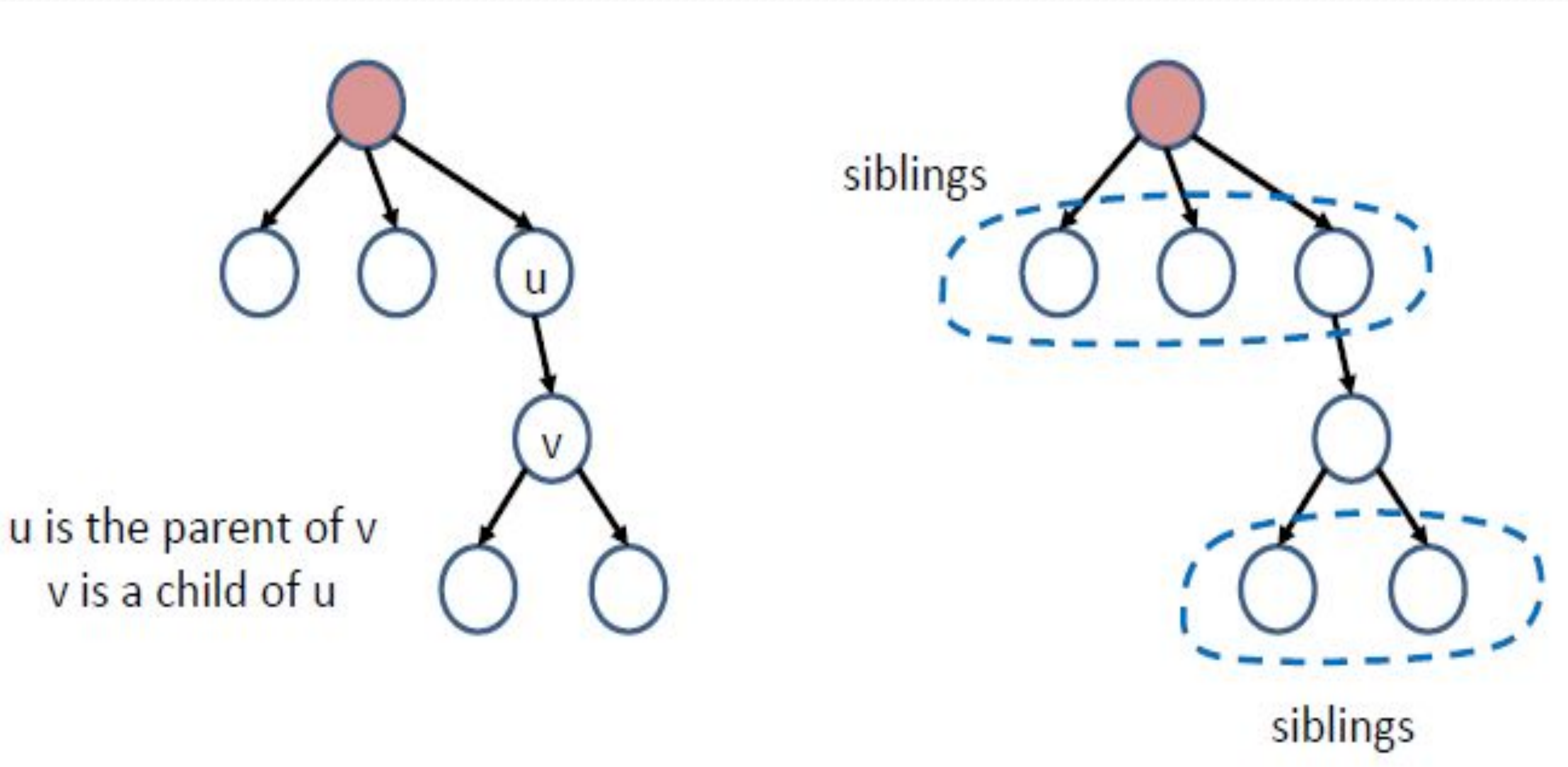
Rooted Tree Terminology

- Terminology for rooted trees is a mix from botany and genealogy (such as this family tree of the Bernoulli family of mathematicians).



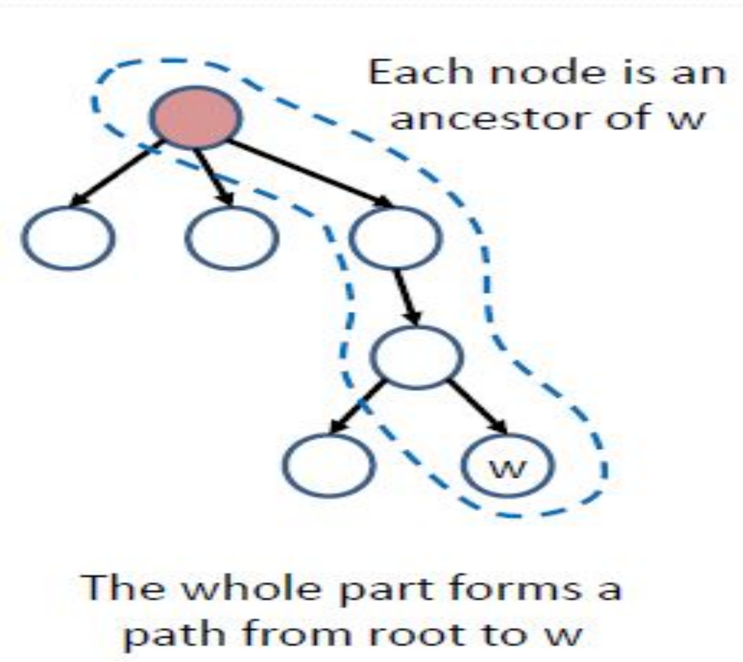
Rooted Tree Terminology

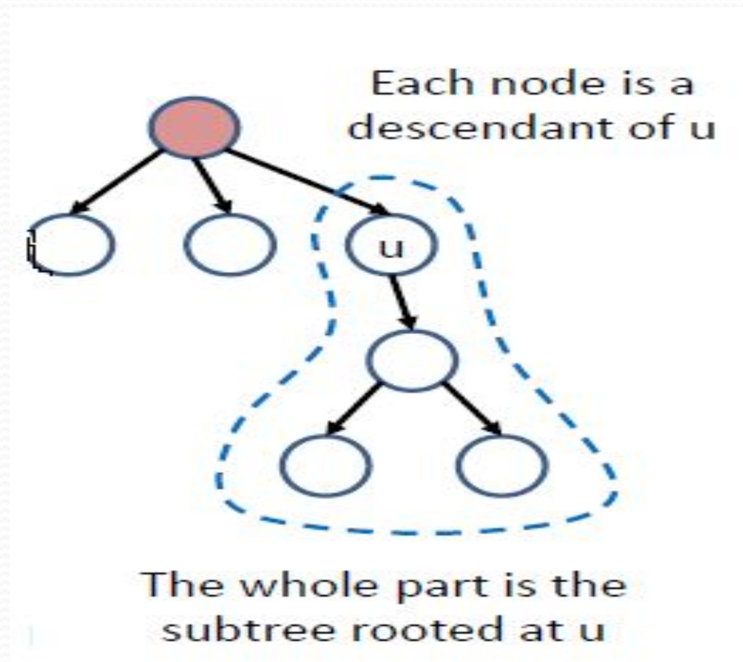
- If v is a vertex of a rooted tree other than the root, the *parent* of v is the unique vertex u such that there is a directed edge from u to v. When u is a parent of v, v is called a *child* of u. Vertices with the same parent are called *siblings*.



Rooted Tree Terminology

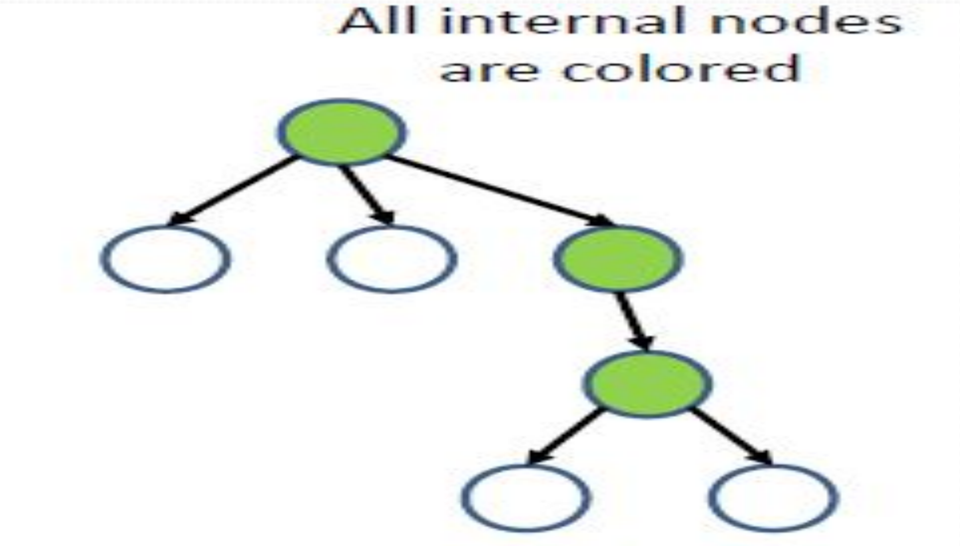
- The *ancestors* of a vertex are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.
- The *descendants* of a vertex *v* are those vertices that have *v* as an ancestor. The subtree rooted at u includes all the descendants of u, and all edges that connect between them.





Rooted Tree Terminology

- A vertex of a rooted tree with no children is called a *leaf*. Vertices that have children are called *internal vertices*.



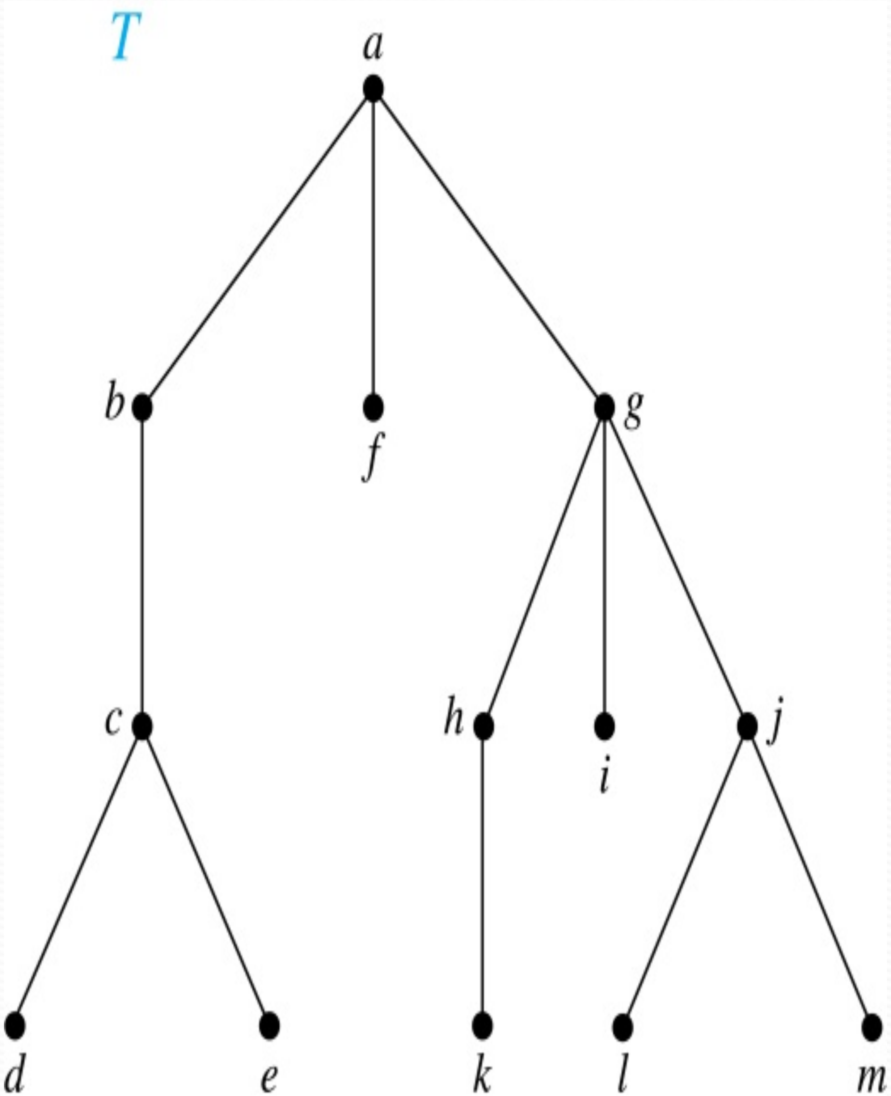
Terminology for Rooted Trees

Example: In the rooted tree T (with root a):

(i) Find the parent of c, the children of g, the siblings of h, the ancestors of e, and the descendants of b.

Solution:

(i) The parent of c is b. The children of g are h, i, and j. The siblings of h are i and j. The ancestors of e are c, b, and a. The descendants of b are c, d, and e.



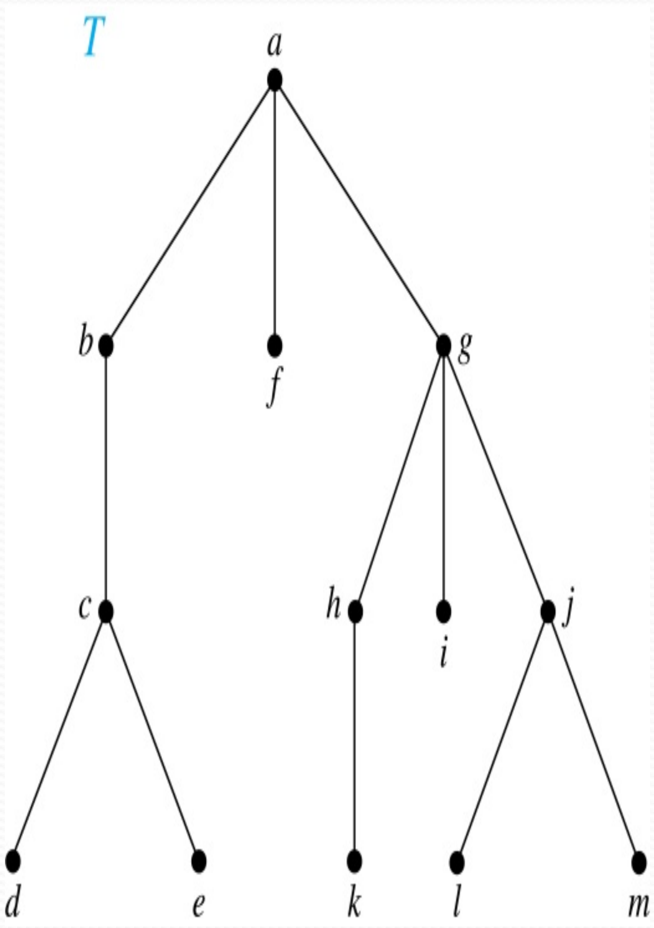
Terminology for Rooted Trees

Example: In the rooted tree T (with root a):

(i) Find all internal vertices and all leaves.

Solution:

(i) The internal vertices are a, b, c, g, h, and j. The leaves are d, e, f, i, k, l, and m.

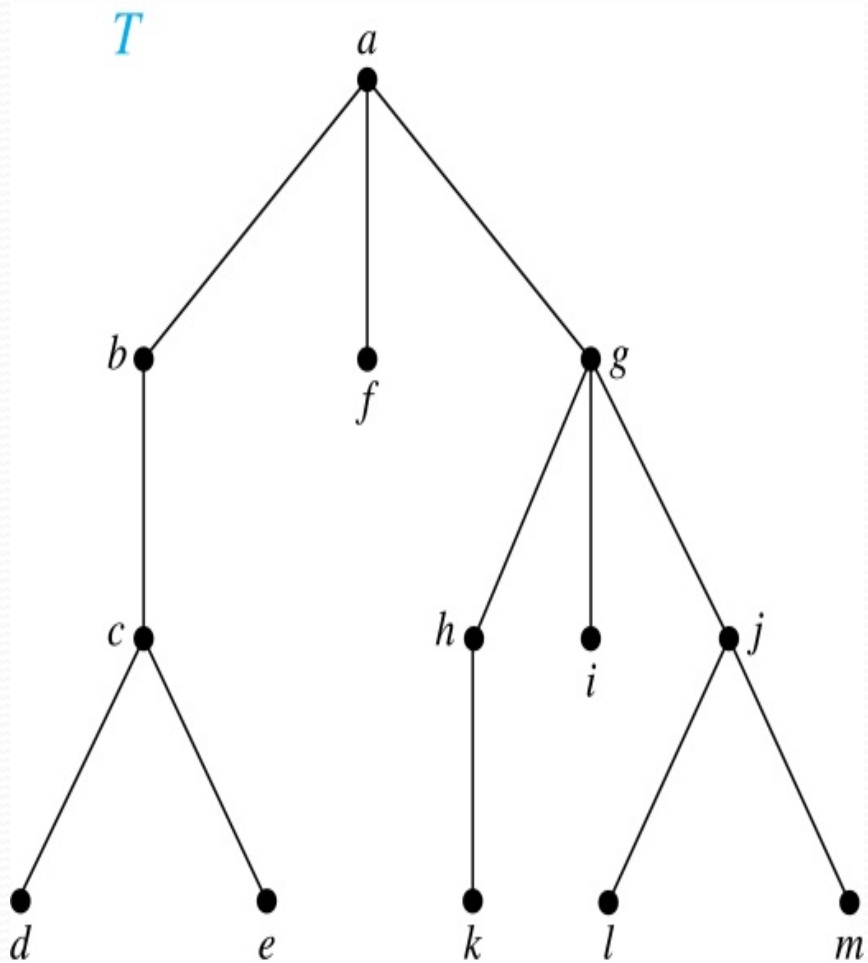


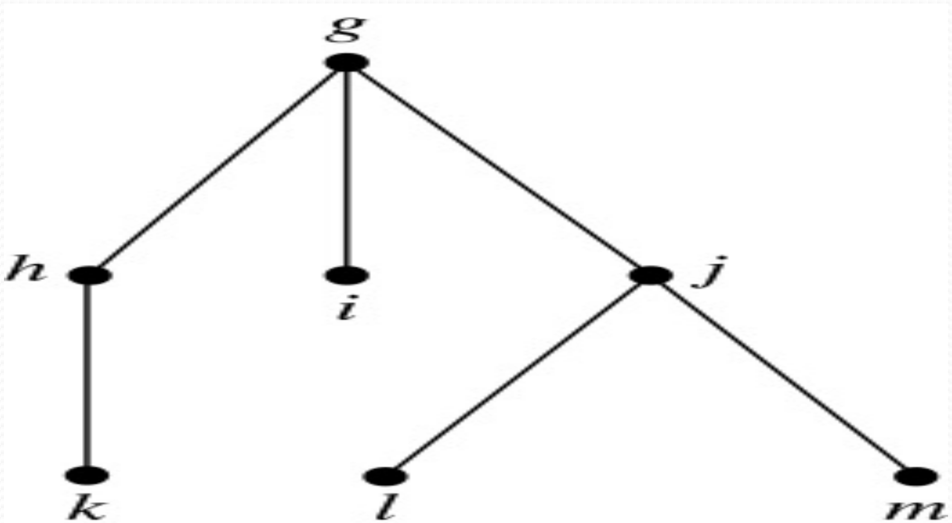
Terminology for Rooted Trees

(i) What is the subtree rooted at g?

Solution:

(i) We display the subtree rooted at g.

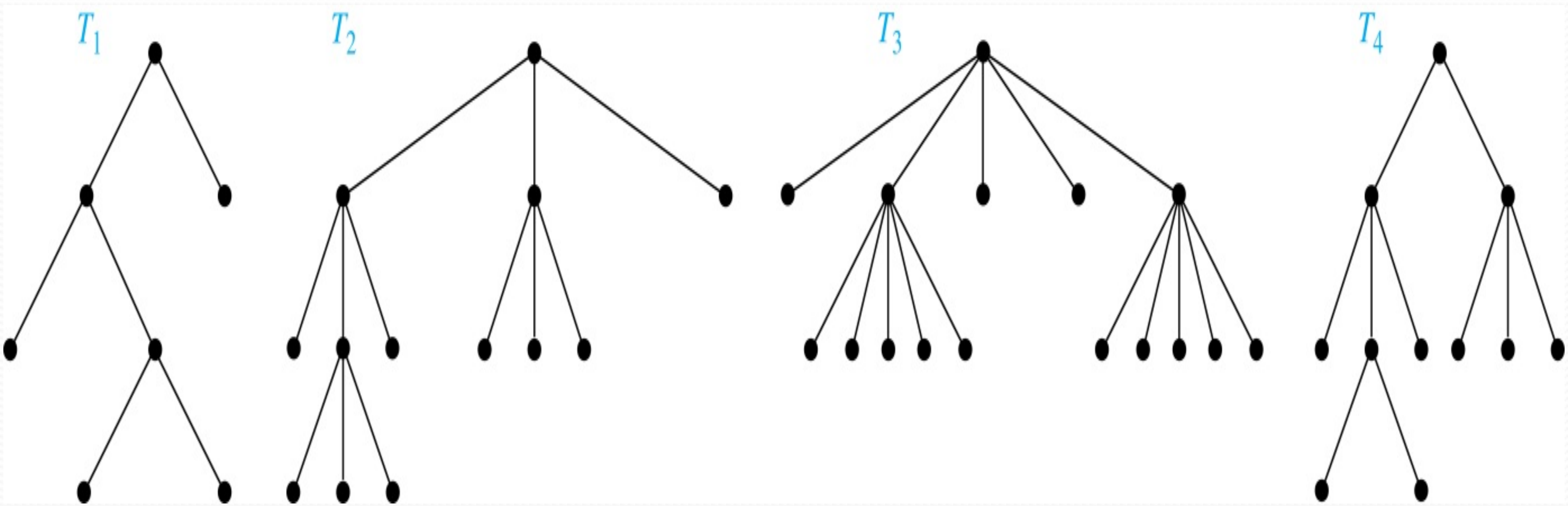


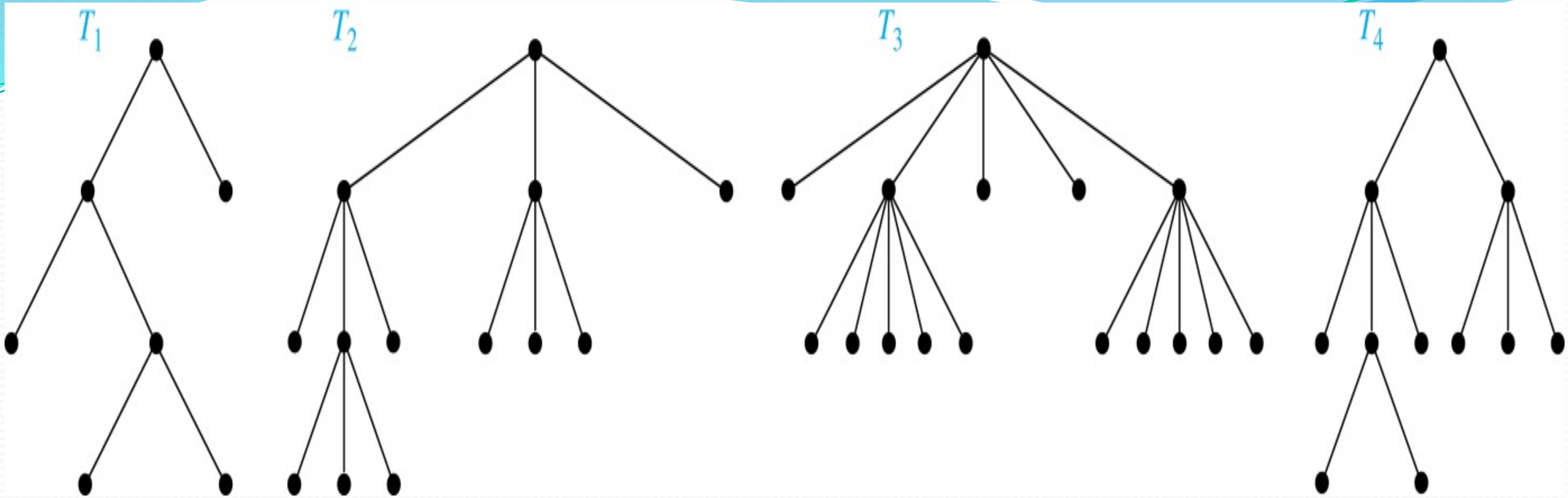


m-ary Rooted Trees

Definition: A rooted tree is called an *m-ary tree* if every internal vertex has no more than m children. The tree is called a *full m-ary tree* if every internal vertex has exactly m children. An m-ary tree with $m = 2$ is called a *binary* tree.

Example: Are the following rooted trees full m-ary trees for some positive integer m?





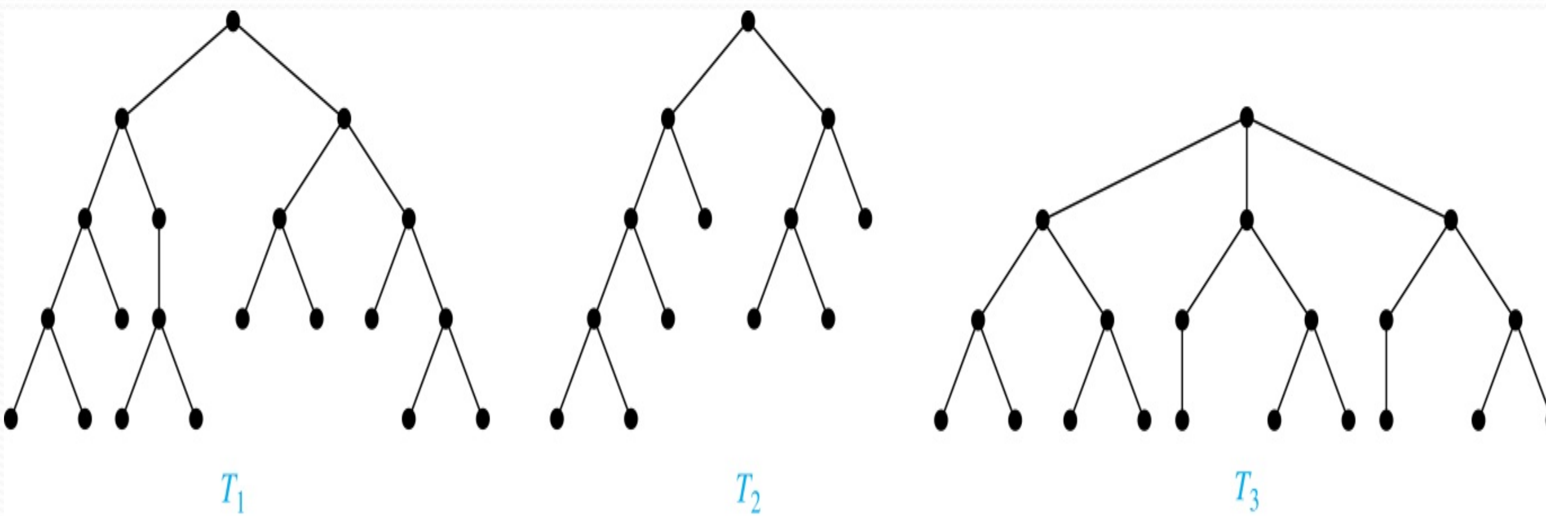
Solution:

- T_1 is a full binary tree because each of its internal vertices has two children.
- T_2 is a full 3-ary tree because each of its internal vertices has three children.
- In T_3 each internal vertex has five children, so T_3 is a full 5-ary tree.
- T_4 is not a full *m*-ary tree for any m because some of its internal vertices have two children and others have three children.

Balanced *m*-Ary Trees

Definition: A rooted m-ary tree of height h is *balanced* if all leaves are at levels h or $h - 1$.

Example: Which of the rooted trees shown below is balanced?

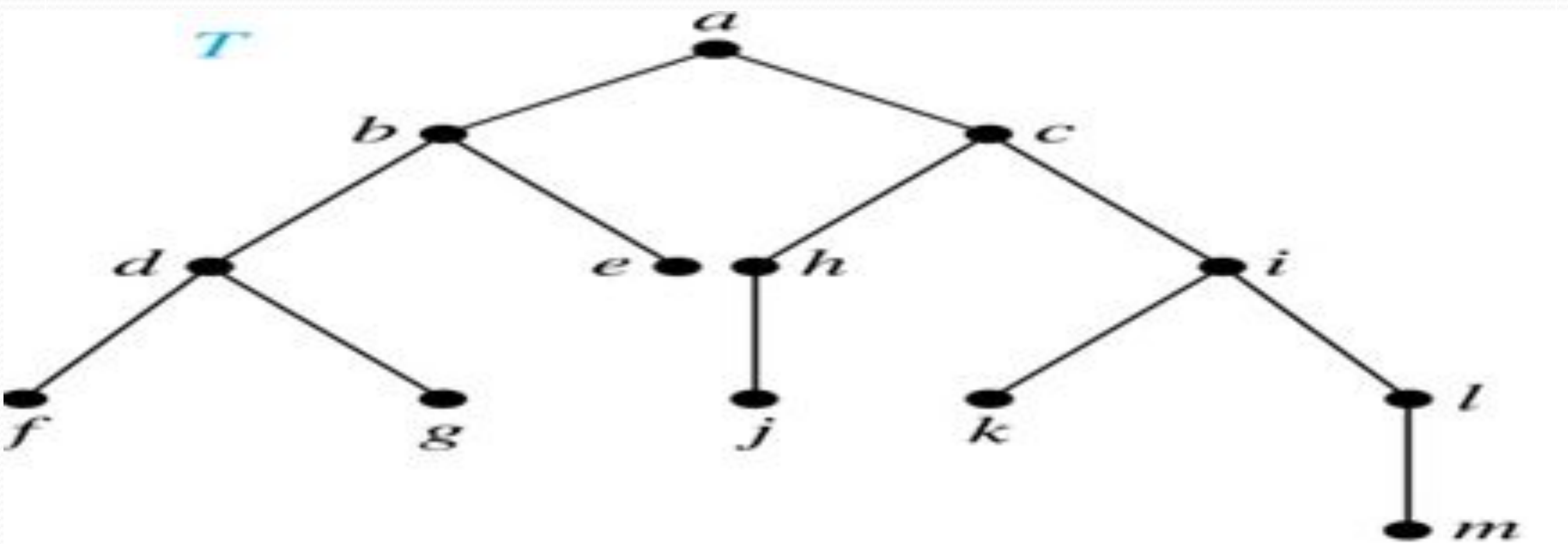


Solution: T_1 and T_3 are balanced, but T_2 is not because it has leaves at levels 2, 3, and 4.

Ordered Rooted Trees

Definition: An *ordered rooted tree* is a rooted tree where the children of each internal vertex are ordered.

- We draw ordered rooted trees so that the children of each internal vertex are shown in order from left to right.



Binary Trees

Definition: A *binary tree* is an ordered rooted where each internal vertex has at most two children. If an internal vertex of a binary tree has two children, the first is called the *left child* and the second the *right child*. The tree rooted at the left child of a vertex is called the *left subtree* of this vertex, and the tree rooted at the right child of a vertex is called the *right subtree* of this vertex.

Example:

Consider the binary tree T.

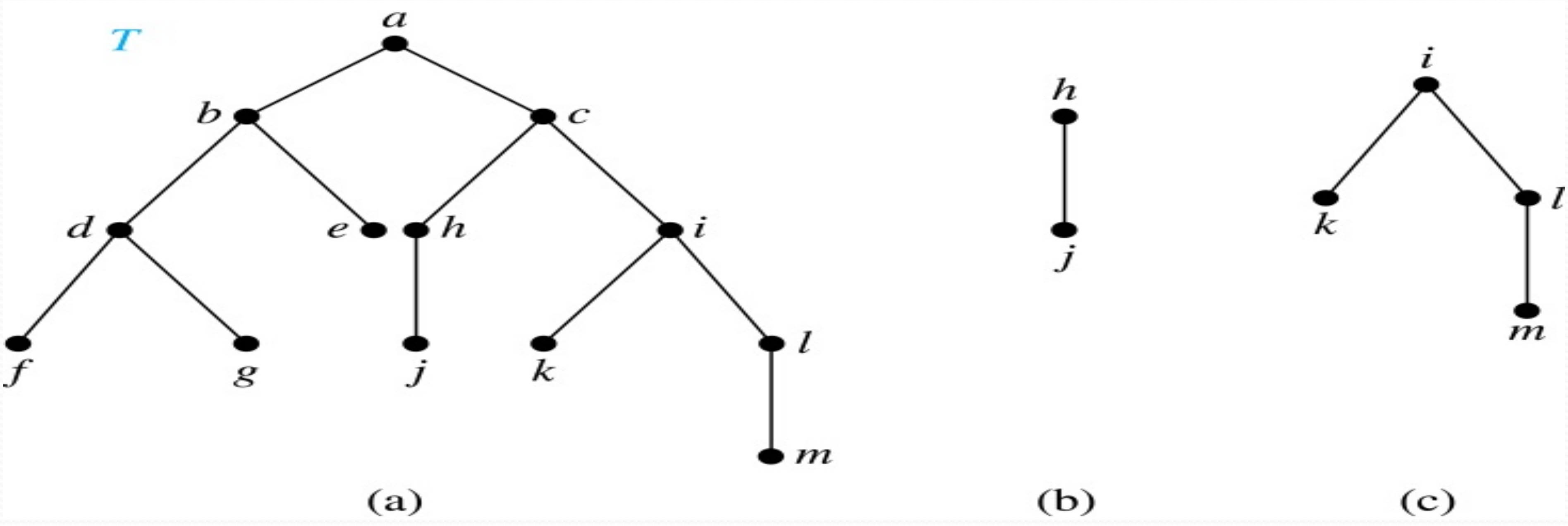
(*i*) What are the left and right children of d?

(*ii*) What are the left and right subtrees of c?

Solution:

(*i*) The left child of d is f and the right child is g.

(*ii*) The left and right subtrees of c are displayed in (b) and (c).



Properties of Trees

- A tree with n vertices has $n - 1$ edges.
- A full m-ary tree with i internal vertices has $n = mi + 1$ vertices.
- A full m-ary tree with:

(i) n *vertices has* $i = (n - 1)/m$ *internal vertices and* $l = [(m - 1)n + 1]/m$ *leaves,*

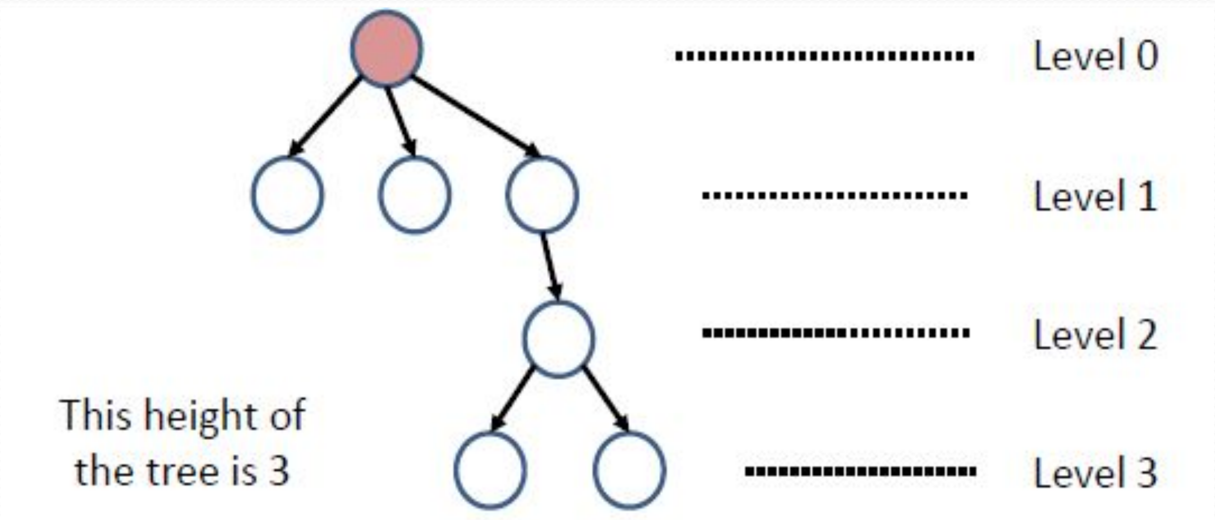
(ii) i internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves,

(iii) l leaves has $n = (ml - 1)/(m - 1)$ vertices and $i = (l - 1)/(m - 1)$ internal vertices.

- There are at most m^h leaves in an m-ary tree of height h.

Level of vertices and height of trees

- When working with trees, we often want to have rooted trees where the sub trees at each vertex contain paths of approximately the same length.
- To make this idea precise we need some definitions:
  - The *level* of a vertex *v* in a rooted tree is the length of the unique path from the root to this vertex.
  - The *height* of a rooted tree is the maximum of the levels of the vertices.

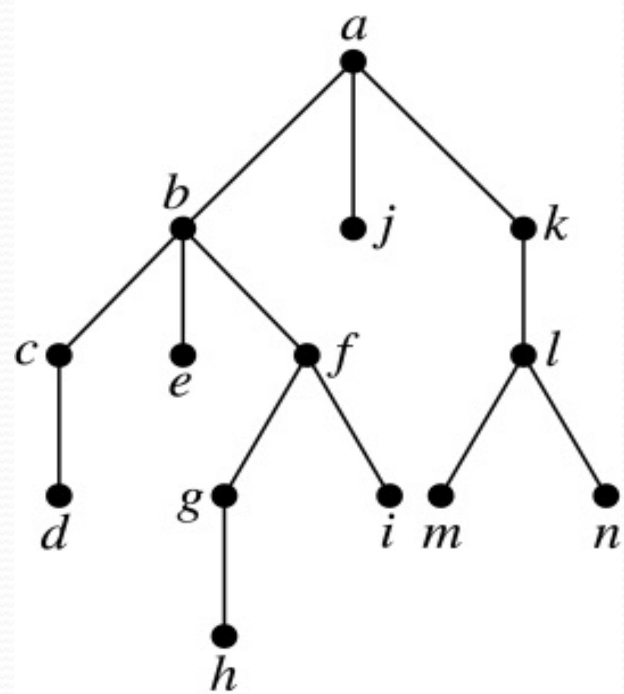


Level of vertices and height of trees

Example:

(i) Find the level of each vertex in the tree to the right.

(ii) What is the height of the tree?



Solution:

(i) The root a is at level 0.

Vertices b, j, and k are at level 1.

Vertices c, e, f, and l are at level 2.

Vertices d, g, i, m, and n are at level 3.

Vertex h is at level 4.

(ii) The height is 4, since 4 is the largest level of any vertex.

Applications of Trees

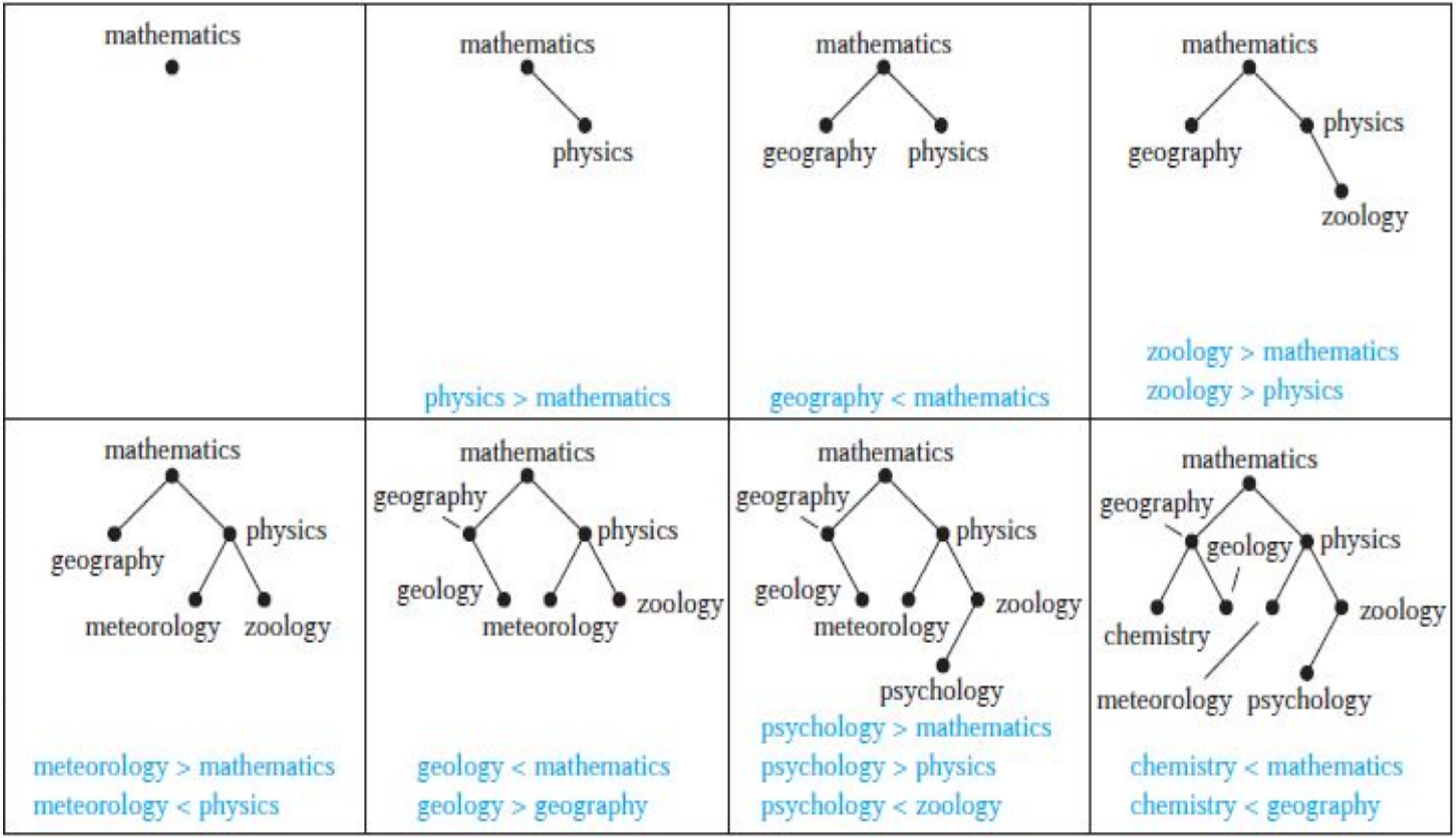
Section 11.2

Binary Search Tree

Definition: A binary tree in which the vertices are labeled with items so that a label of a vertex is greater than the labels of all vertices in the left subtree of this vertex and is less than the labels of all vertices in the right subtree of this vertex.

- Searching for items in a list is one of the most important tasks that arises in computer science.
- Our primary goal is to implement a searching algorithm that finds items efficiently when the items are totally ordered. This can be accomplished through the use of a binary search tree

Example : Form a binary search tree for the words mathematics, physics, geography, zoology, meteorology, geology, psychology, and chemistry (using alphabetical order).

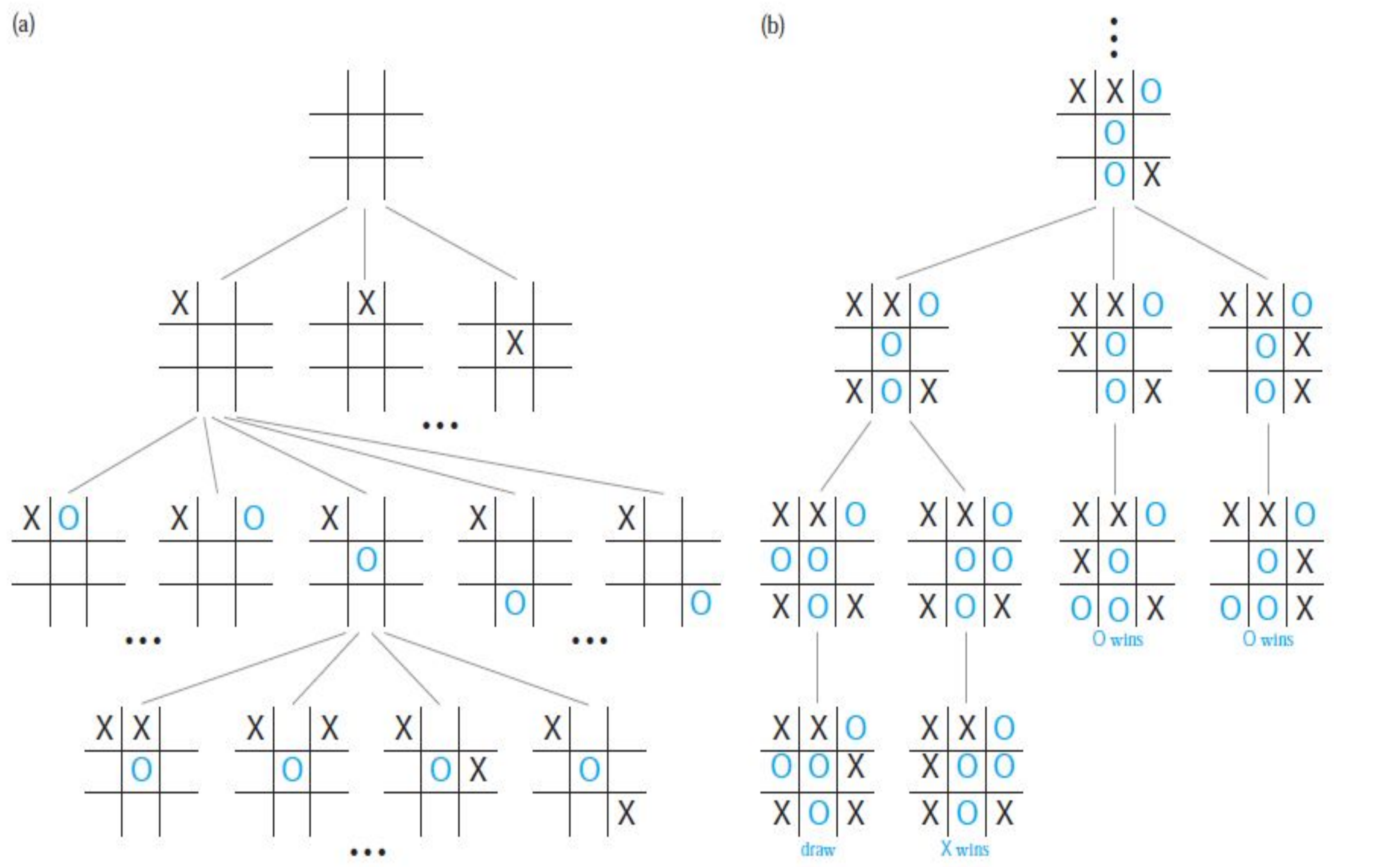


Applications of Trees

- **Game Trees**

Trees can be used to analyze certain types of games such as tic-tac-toe, nim, checkers, and chess.

Game Tree for Tic-Tac-Toe



Universal Address Systems

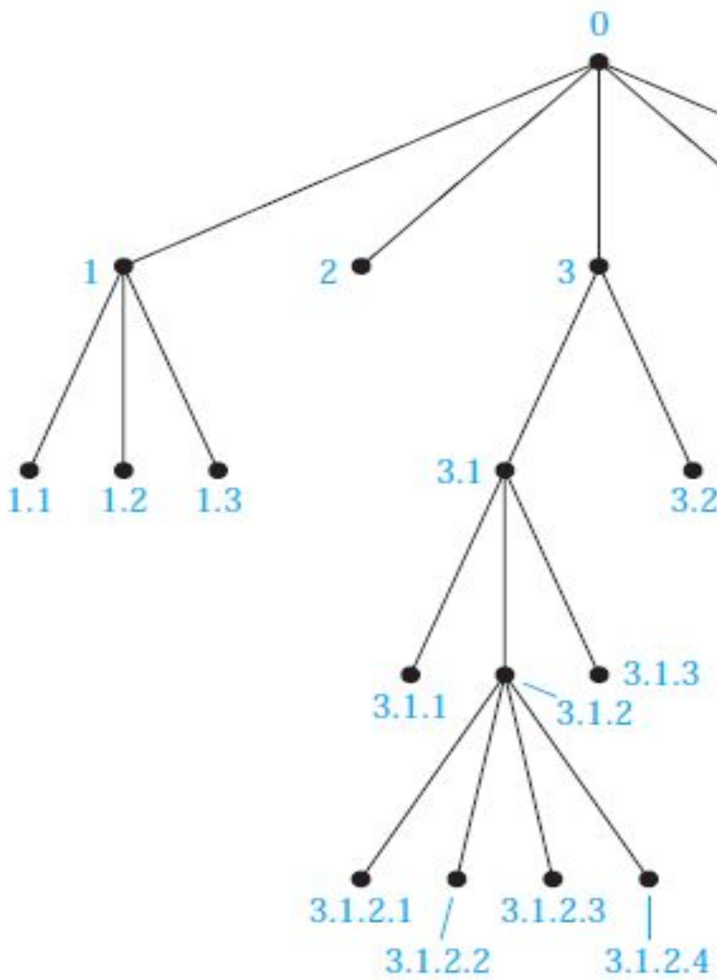


FIGURE 1 The Universal Address System of an Ordered Rooted Tree.

Prefix code

Definition: A code that has the property that the code of a character is never a prefix of the code of another character.

- A prefix code can be represented using a binary tree, where the characters are the labels of the leaves in the tree.
- The edges of the tree are labeled so that an edge leading to a left child is assigned a 0 and an edge leading to a right child is assigned a 1.
- The bit string used to encode a character is the sequence of labels of the edges in the unique path from the root to the leaf that has this character as its label.
- For instance, the tree in Figure 5 represents the encoding of *e* by 0, *a* by 10, *t* by 110, *n* by 1110, and *s* by 1111.

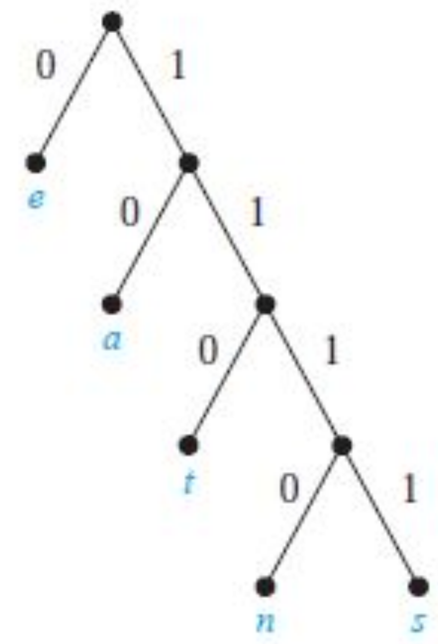


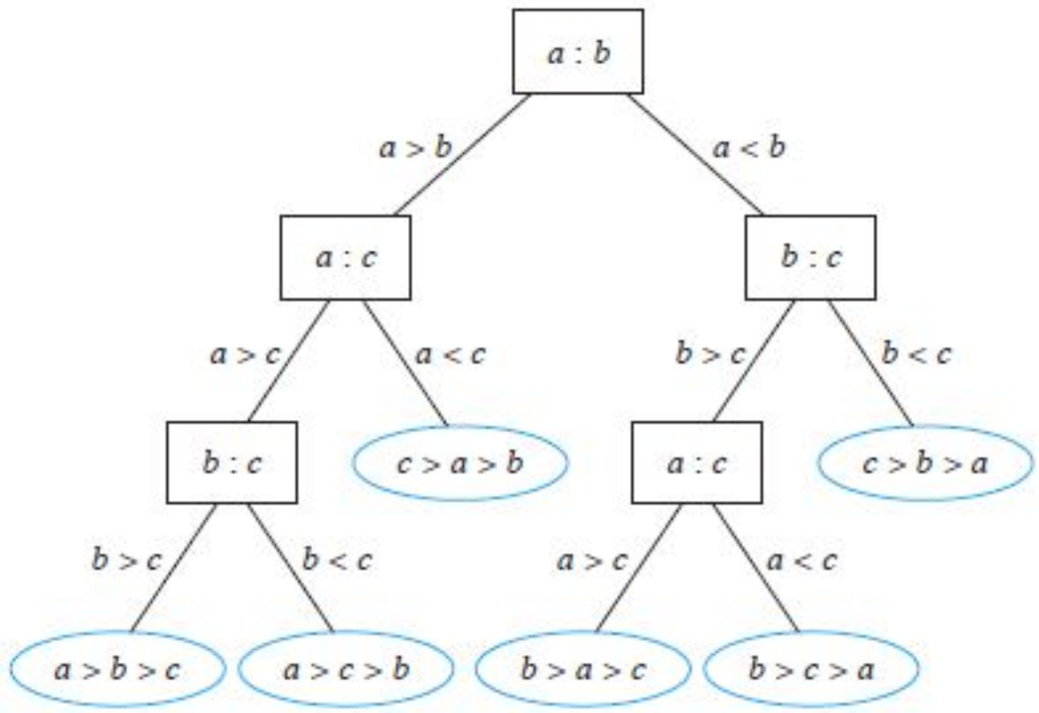
FIGURE 5 A Binary Tree with a Prefix Code.

Decision Trees

Definition: A rooted tree where each vertex represents a possible outcome of a decision and the leaves represent the possible solutions of a problem.

- Rooted trees can be used to model problems in which a series of decisions leads to a solution.
- The possible solutions of the problem correspond to the paths to the leaves of this rooted tree.

Example : A decision tree that orders the elements of the list *a*, *b*, *c*.



A Decision Tree for Sorting Three Distinct Elements.

Tree Traversal

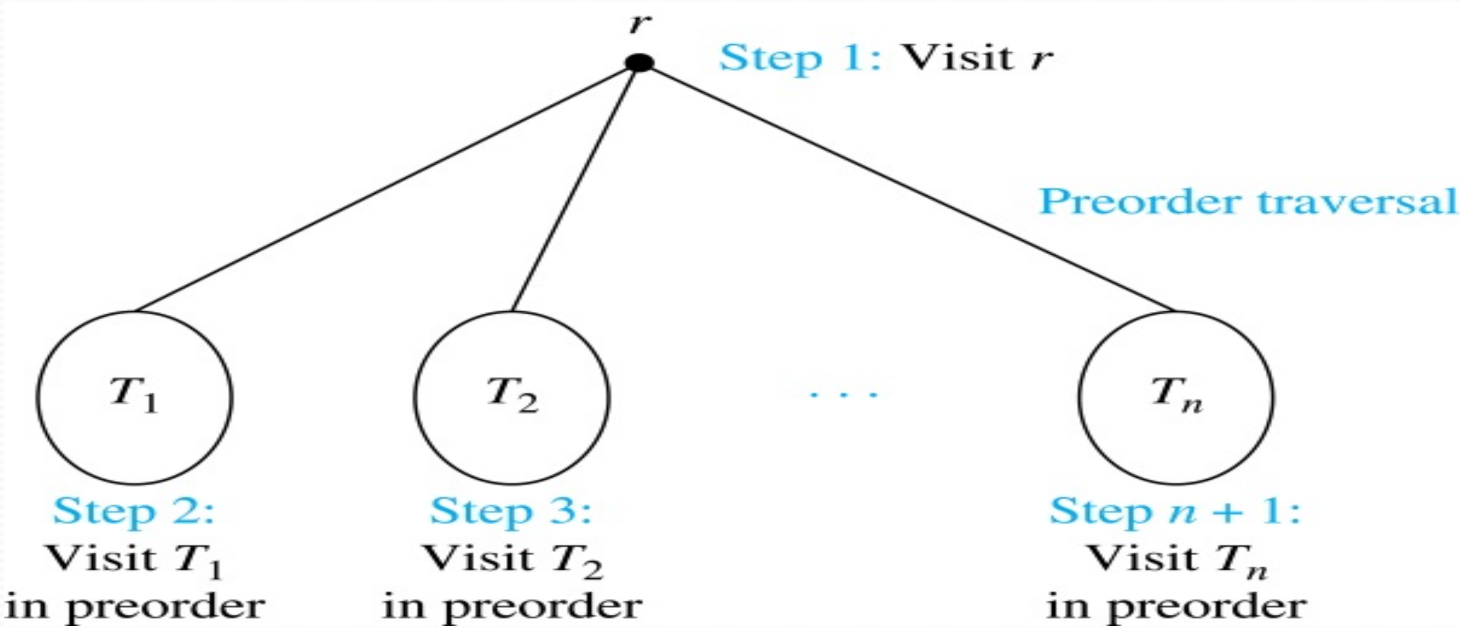
Section 11.3

Tree Traversal

- Procedures for systematically visiting every vertex of an ordered tree are called *traversals*.
- The three most commonly used *traversals* are *preorder traversal*, *inorder traversal*, and *postorder traversal*.

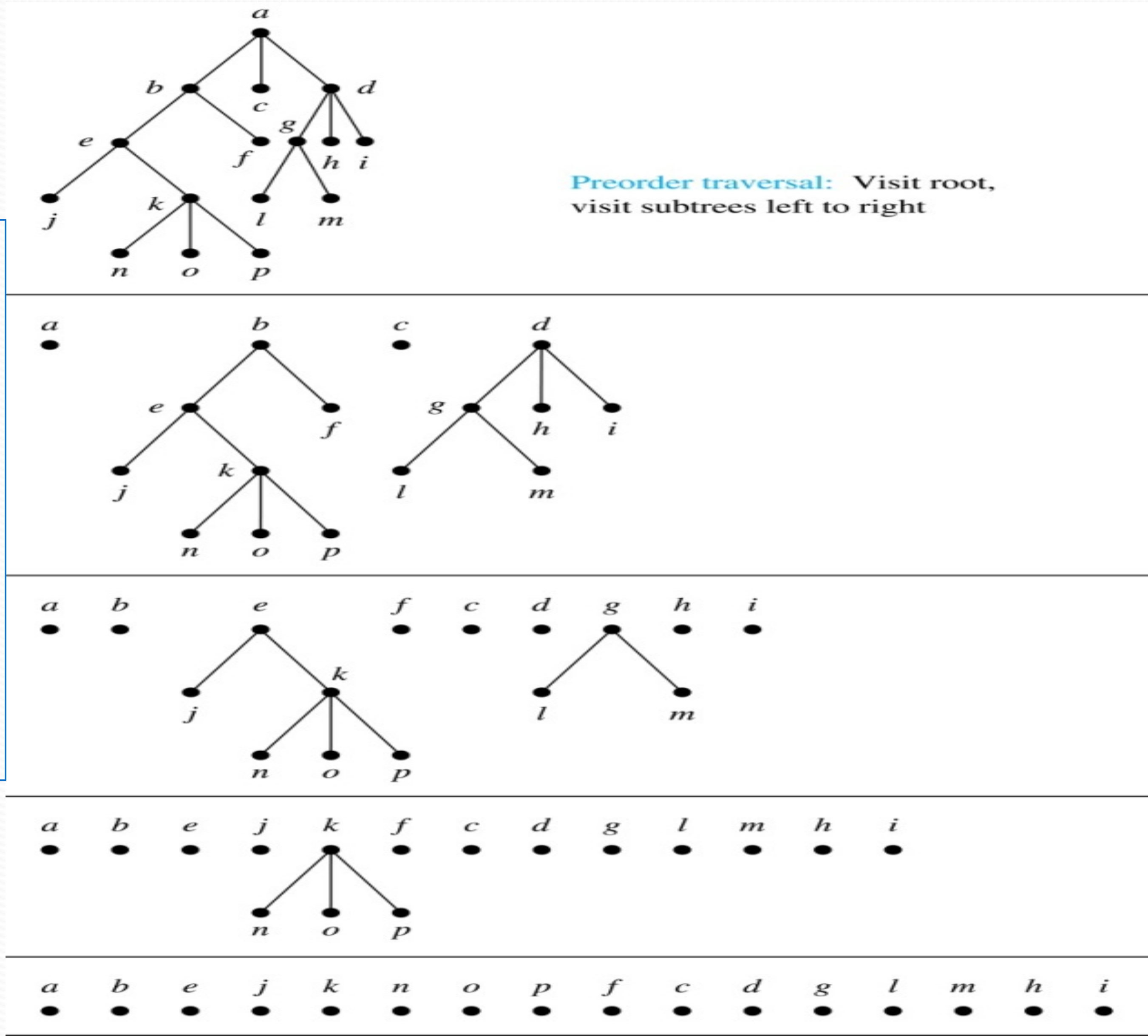
Preorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The preorder traversal begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.



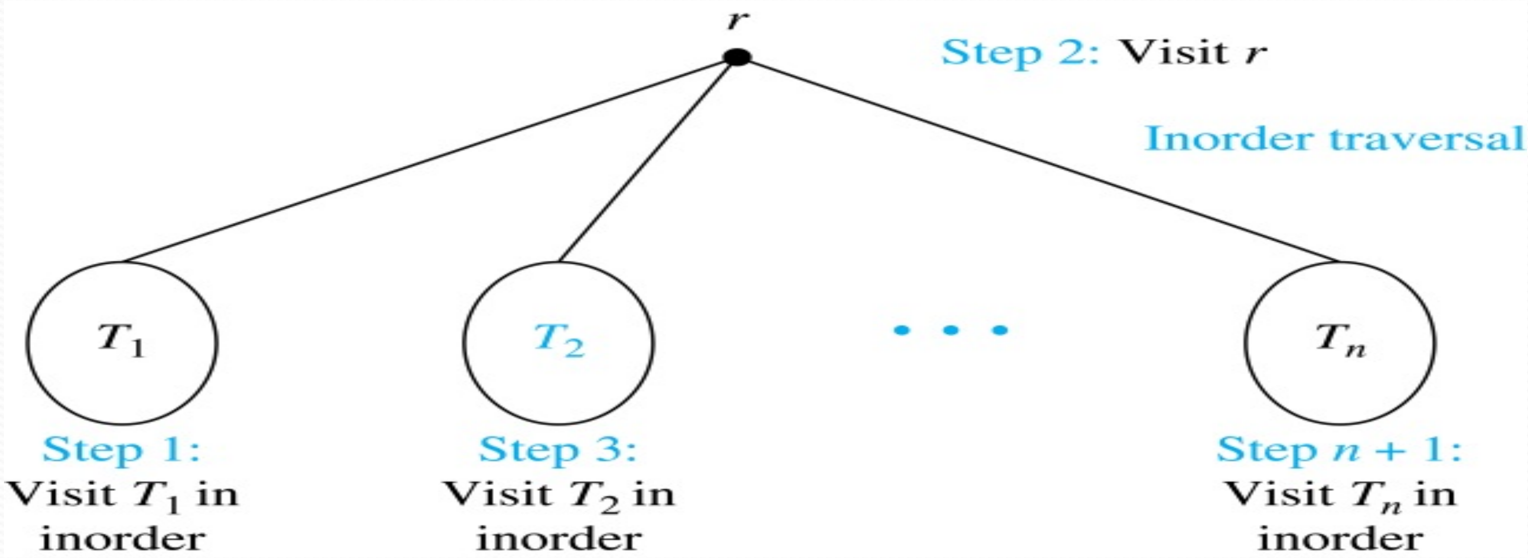
Preorder Traversal (*continued*)

procedure *preorder* (T: ordered rooted tree)
r := root of T
list r
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    preorder($T(c)$)



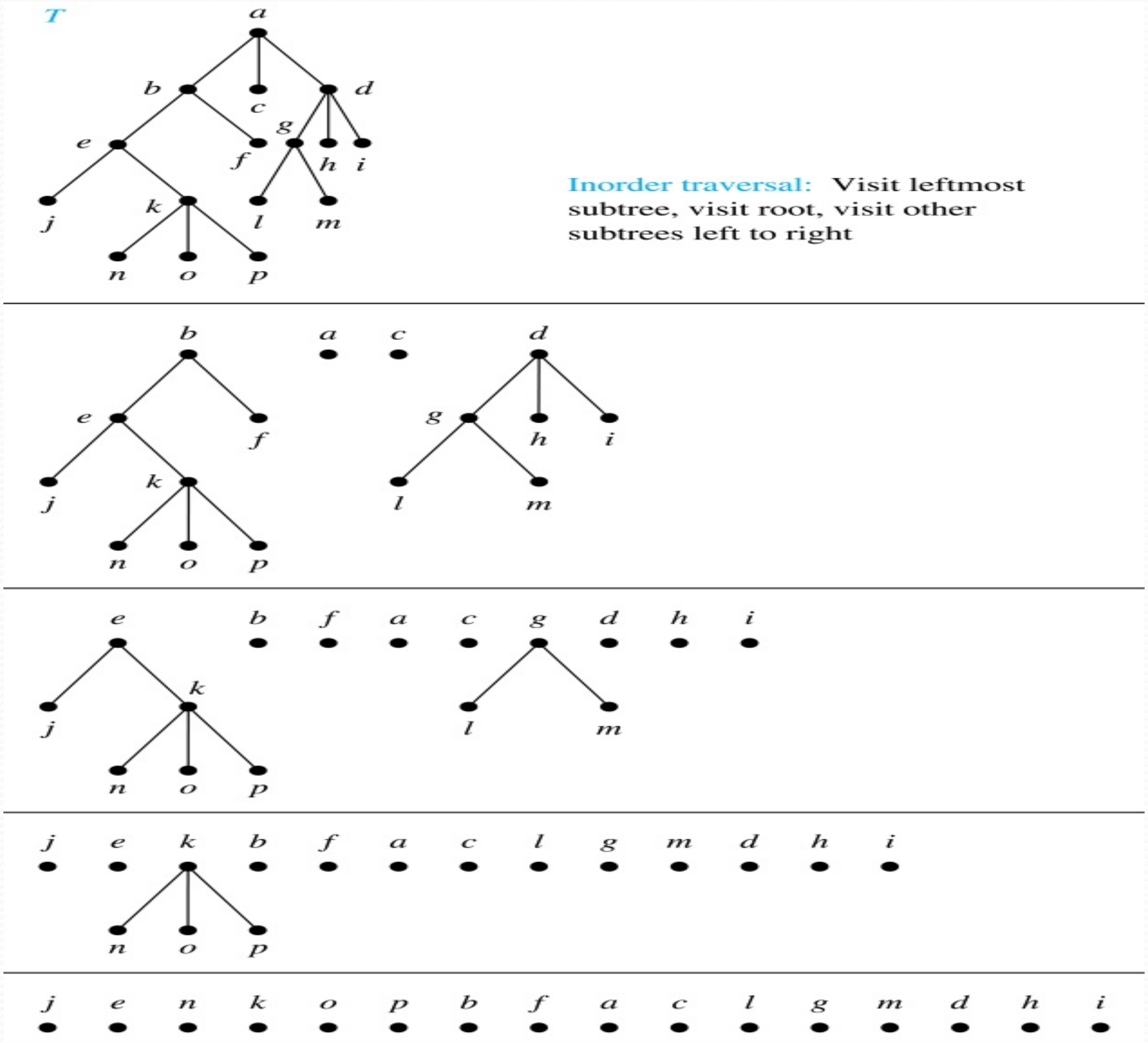
Inorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *inorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The inorder traversal begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.



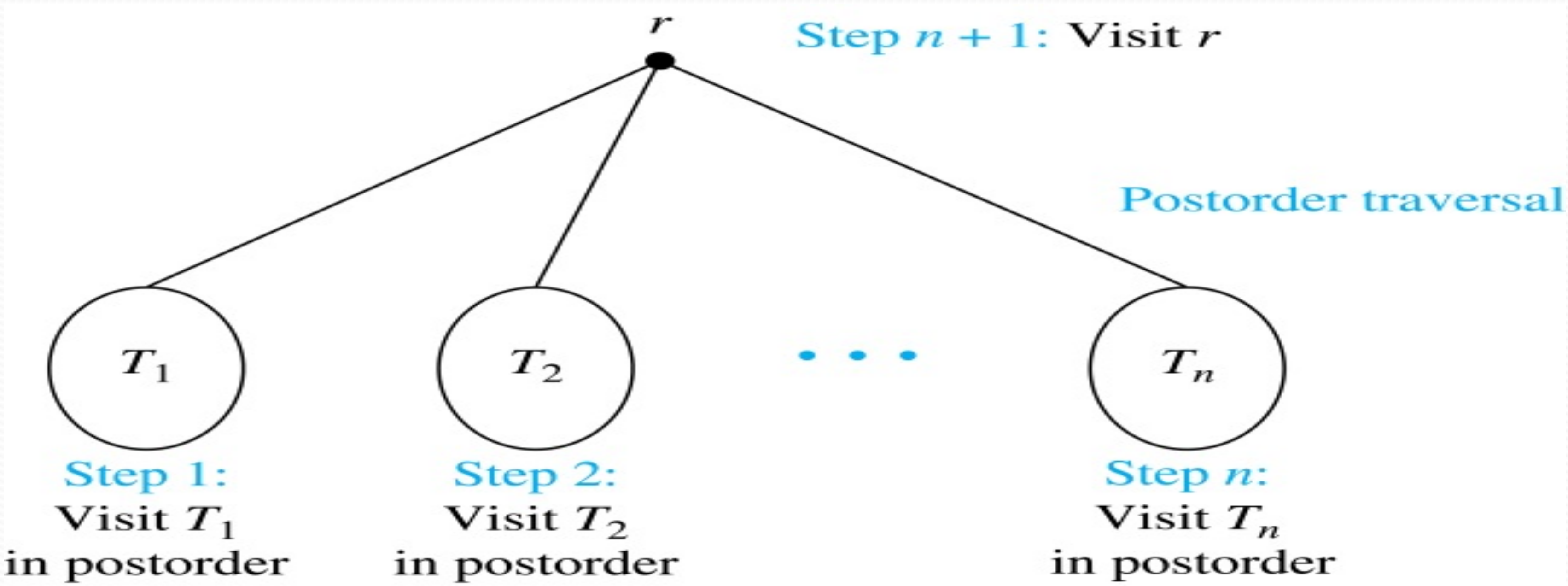
Inorder Traversal (*continued*)

procedure *inorder* (*T*: ordered rooted tree)
r := root of *T*
if *r* is a leaf **then** list *r*
else
l := first child of *r* from left to right
T(*l*) := subtree with *l* as its root
inorder(*T*(*l*))
list(*r*)
for each child *c* of *r* from left to right
T(*c*) := subtree with *c* as root
inorder(*T*(*c*))



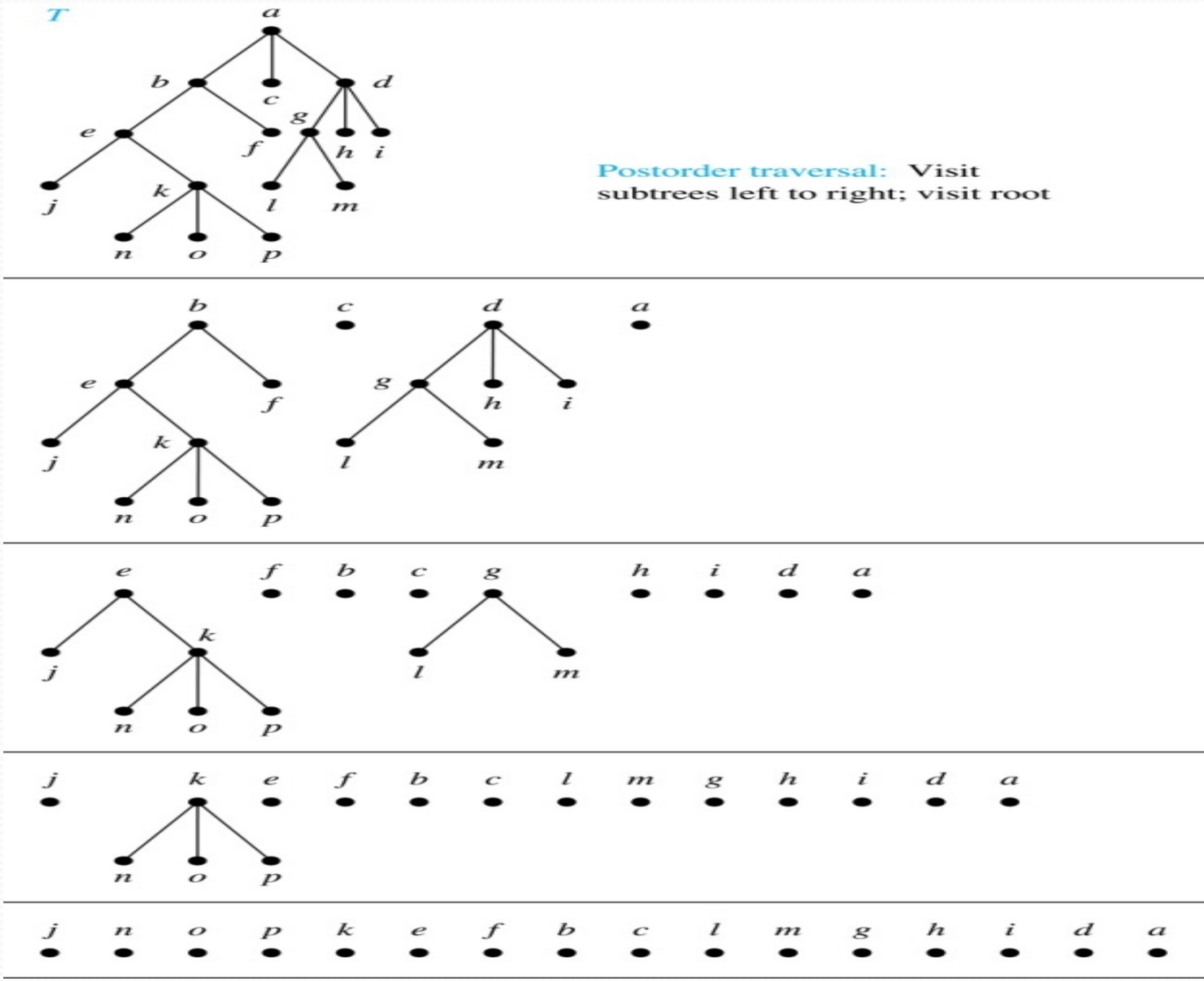
Postorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *postorder traversal* of T. Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The postorder traversal begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.



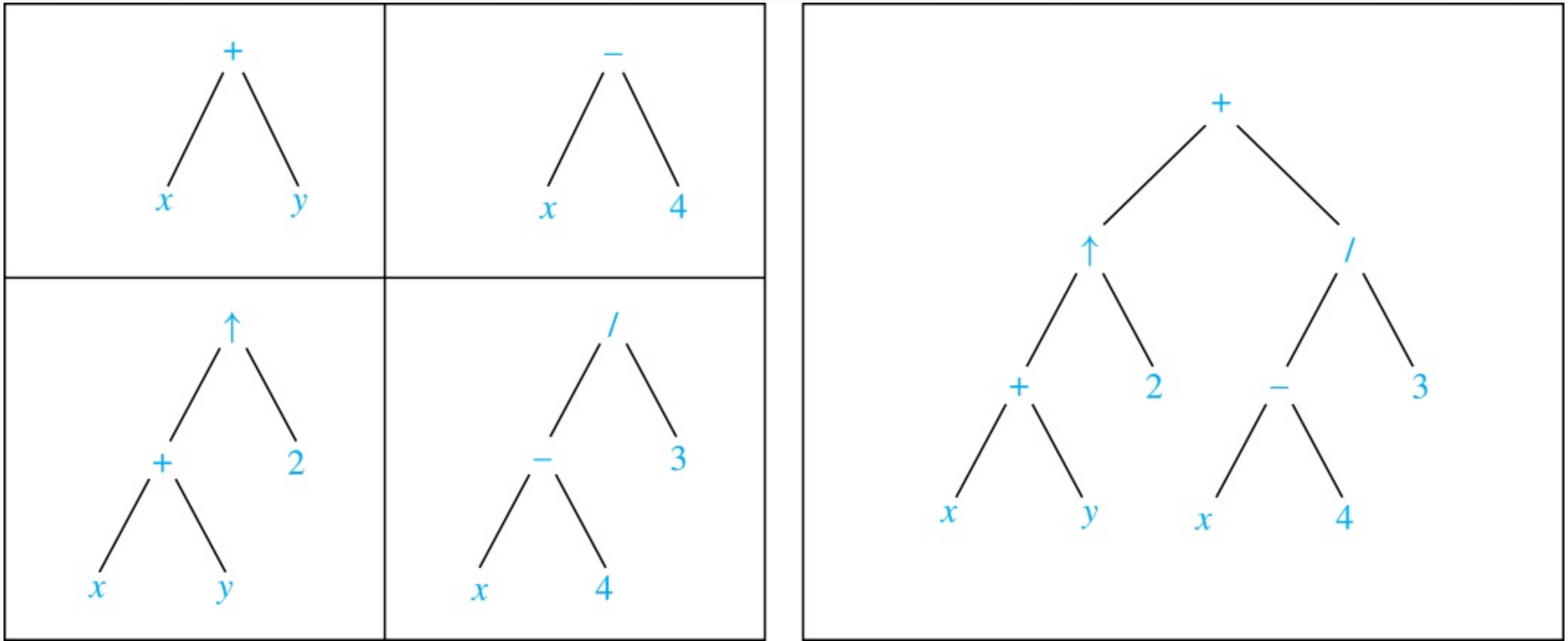
Post order Traversal (*continued*)

procedure *postordered* (*T*: ordered rooted tree)
r := root of *T*
for each child *c* of *r* from left to right
  T(*c*) := subtree with *c* as root
  postorder(*T*(*c*))
list *r*



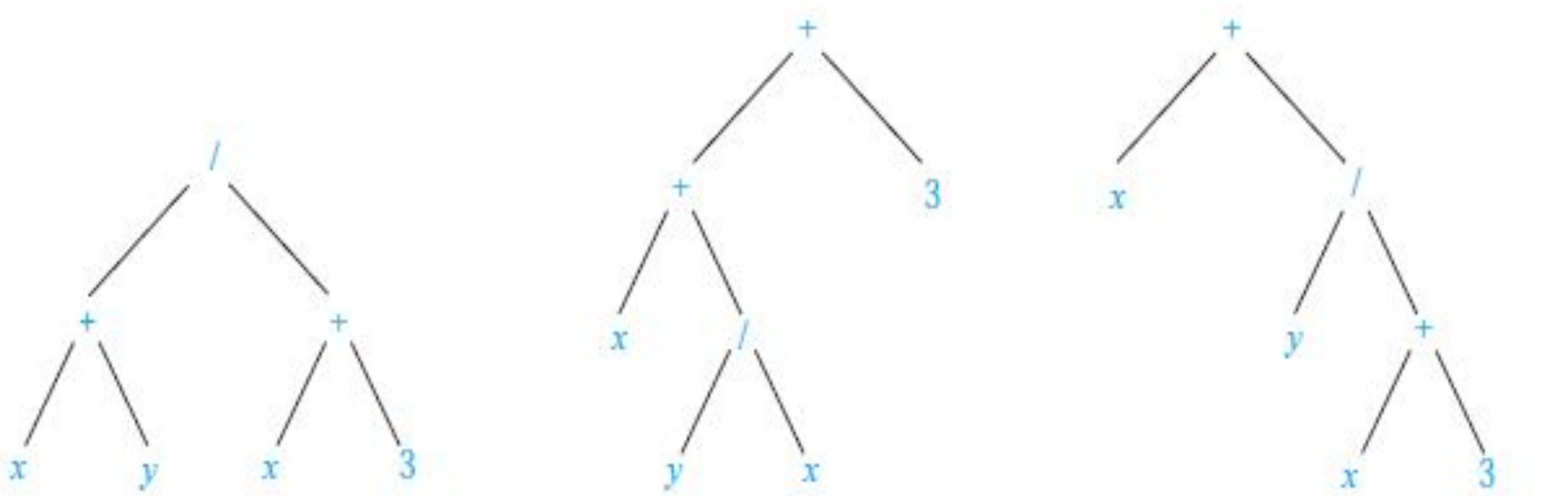
Expression Trees

- Complex expressions can be represented using ordered rooted trees.
- Consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$.
- A binary tree for the expression can be built from the bottom up, as is illustrated here.

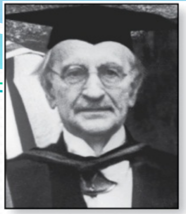


Infix Notation

- An inorder traversal of the tree representing an expression produces the original expression when parentheses are included except for unary operations, which now immediately follow their operands.
- We illustrate why parentheses are needed with an example that displays three trees all yield the same infix representation.



Rooted Trees Representing $(x + y)/(x + 3)$, $(x + (y/x)) + 3$, and $x + (y/(x + 3))$.



Jan Łukasiewicz
(1878-1956)

Prefix Notation

- When we traverse the rooted tree representation of an expression in preorder, we obtain the *prefix* form of the expression. Expressions in prefix form are said to be in *Polish notation*, named after the Polish logician Jan Łukasiewicz.
- Operators precede their operands in the prefix form of an expression. Parentheses are not needed as the representation is unambiguous.
- The prefix form of $((x + y) \uparrow 2) + ((x - 4)/3)$
  is $+ \uparrow + x\ y\ 2\ / - x\ 4\ 3$.
- Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the corresponding operation with the two operations to the right.

Prefix Notation

- **Example**: We show the steps used to evaluate a particular prefix expression:

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad / \quad \underbrace{\uparrow \quad 2 \quad 3}_{2 \uparrow 3 = 8} \quad 4$$

$$+ \quad - \quad * \quad 2 \quad 3 \quad 5 \quad \underbrace{/ \quad 8 \quad 4}_{8 / 4 = 2}$$

$$+ \quad - \quad \underbrace{* \quad 2 \quad 3}_{2 * 3 = 6} \quad 5 \quad 2$$

$$+ \quad \underbrace{- \quad 6 \quad 5}_{6 - 5 = 1} \quad 2$$

$$\underbrace{+ \quad 1 \quad 2}_{1 + 2 = 3}$$

Value of expression: 3

Postfix Notation

- We obtain the *postfix form* of an expression by traversing its binary trees in postorder. Expressions written in postfix form are said to be in *reverse Polish notation*.
- Parentheses are not needed as the postfix form is unambiguous.
- $x\ y + 2 \uparrow x\ 4 - 3 / +$ is the postfix form of $((x + y) \uparrow 2) + ((x - 4)/3)$.
- A binary operator follows its two operands. So, to evaluate an expression one works from left to right, carrying out an operation represented by an operator on its preceding operands.

Postfix Notation

- **Example**: We show the steps used to evaluate a particular postfix expression.

$$7\quad \underbrace{2\quad 3\quad *}_{2 * 3 = 6}\quad -\quad 4\quad \uparrow\quad 9\quad 3\quad /\quad +$$

$$\underbrace{7\quad 6\quad -}_{7 - 6 = 1}\quad 4\quad \uparrow\quad 9\quad 3\quad /\quad +$$

$$\underbrace{1\quad 4\quad \uparrow}_{1^4 = 1}\quad 9\quad 3\quad /\quad +$$

$$1\quad \underbrace{9\quad 3\quad /}_{9 / 3 = 3}\quad +$$

$$\underbrace{1\quad 3\quad +}_{1 + 3 = 4}$$

Value of expression: 4

Spanning Trees

Section 11.4

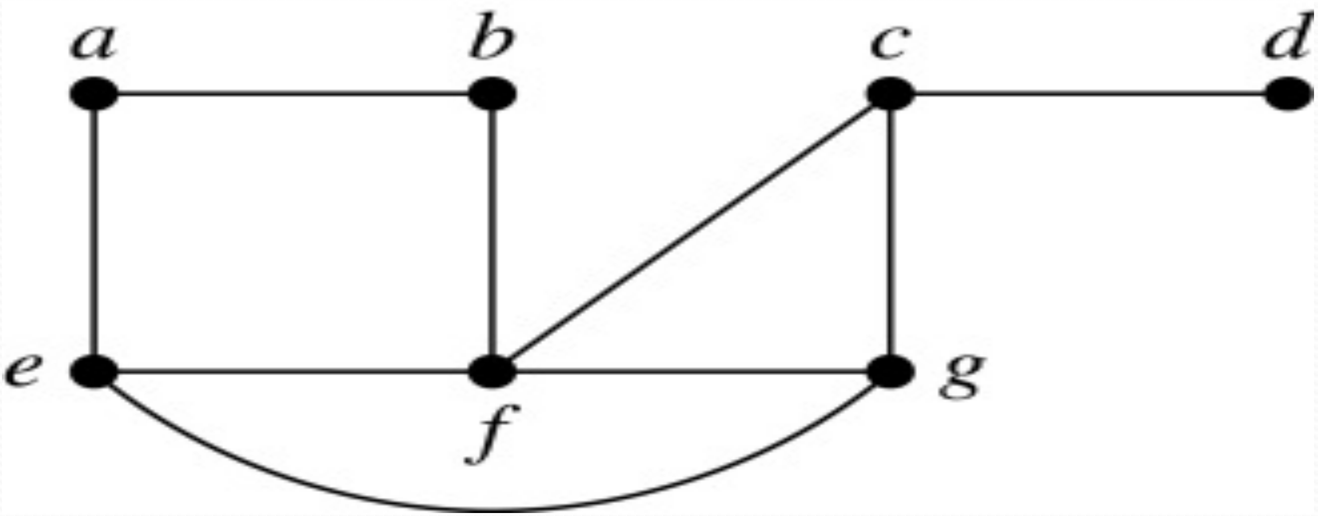
Section Summary

- Spanning Trees
- Prim’s Algorithm
- Kruskal Algorithm

Spanning Trees

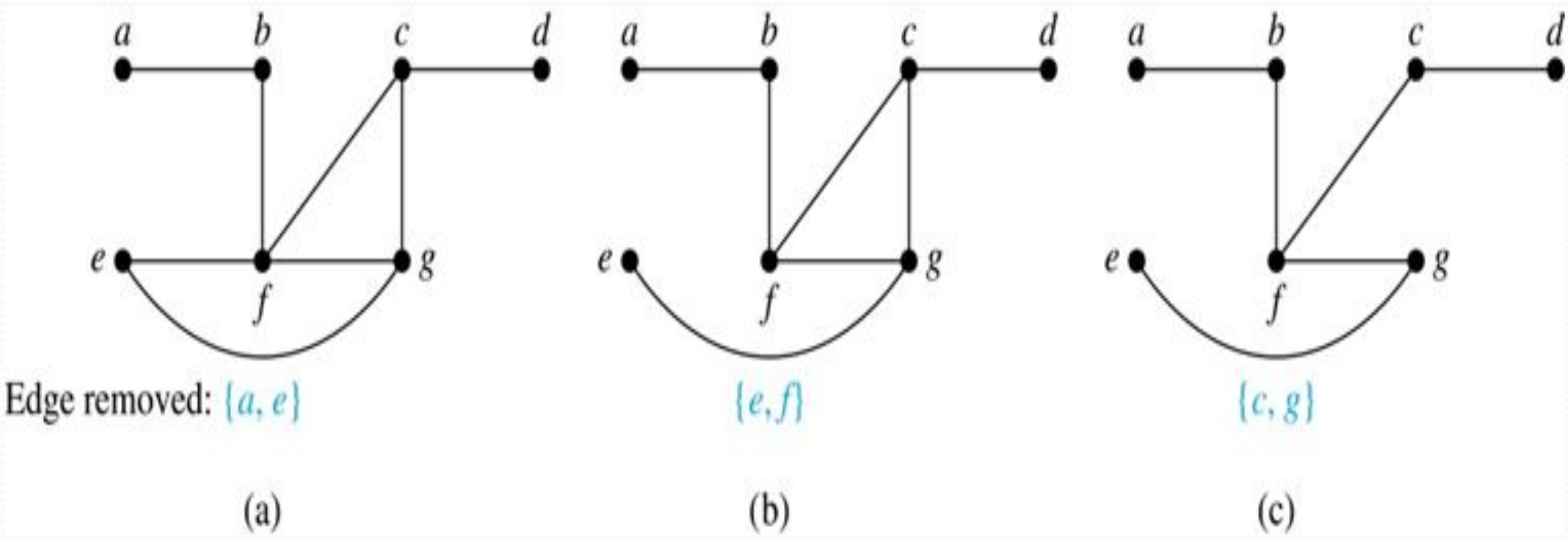
Definition: Let G be a simple graph. A spanning tree of G is a subgraph of G that is a tree containing every vertex of G.

Example: Find the spanning tree of the simple graph:



Spanning Trees

Solution: The graph is connected, but is not a tree because it contains simple circuits. Remove the edge $\{a, e\}$. Now one simple circuit is gone, but the remaining subgraph still has a simple circuit. Remove the edge $\{e, f\}$ and then the edge $\{c, g\}$ to produce a simple graph with no simple circuits. It is a spanning tree, because it contains every vertex of the original graph.



Minimum Spanning

Section 11.5

Minimum spanning tree

- A *minimum spanning tree* in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.
- **Example:** A company plans to build a communications network connecting its five computer centers. Any pair of these centers can be linked with a leased telephone line. Which links should be made to ensure that there is a path between any two computer centers so that the total cost of the network is minimized?

Minimum spanning tree

- **Solution:** We can model this problem using the weighted graph shown in Figure 1, where vertices represent computer centers, edges represent possible leased lines, and the weights on edges are the monthly lease rates of the lines represented by the edges. We can solve this problem by finding a spanning tree so that the sum of the weights of the edges of the tree is minimized. Such a spanning tree is called a **minimum spanning tree**.

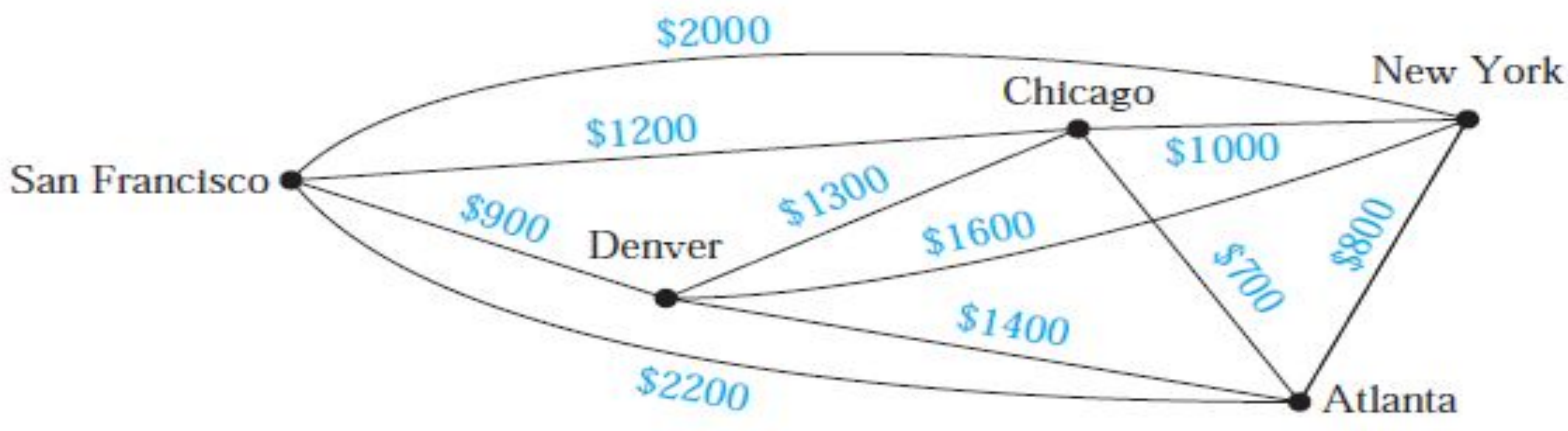


FIGURE 1 A Weighted Graph Showing Monthly Lease Costs for Lines in a Computer Network.

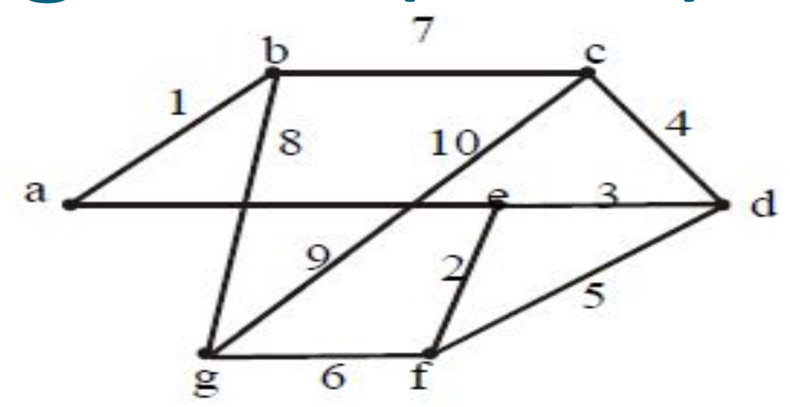
PRIM'S ALGORITHM

ALGORITHM 1 Prim's Algorithm.

procedure *Prim*(G: weighted connected undirected graph with n vertices)
T := a minimum-weight edge
for $i := 1$ **to** $n - 2$
    e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T
    $T := T$ with e added
return T {T is a minimum spanning tree of G}

Minimal spanning tree (MST)

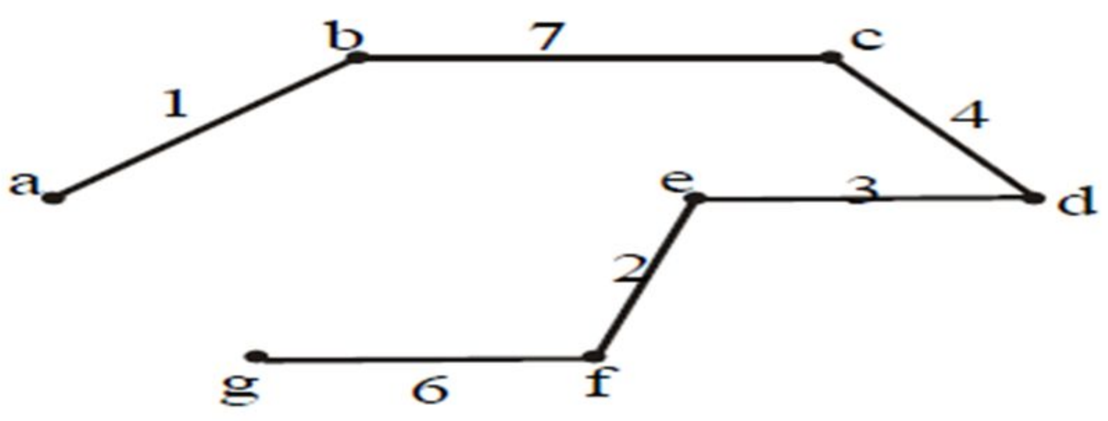
Example: Use Prims algorithm to find a minimal spanning tree for the graph below. Indicate the order in which edges are added to form the tree.



Order of adding the edges:

{a , b}, {b , c}, {c , d}, {d , e}, {e , f}, {f , g}

MST COST = 23



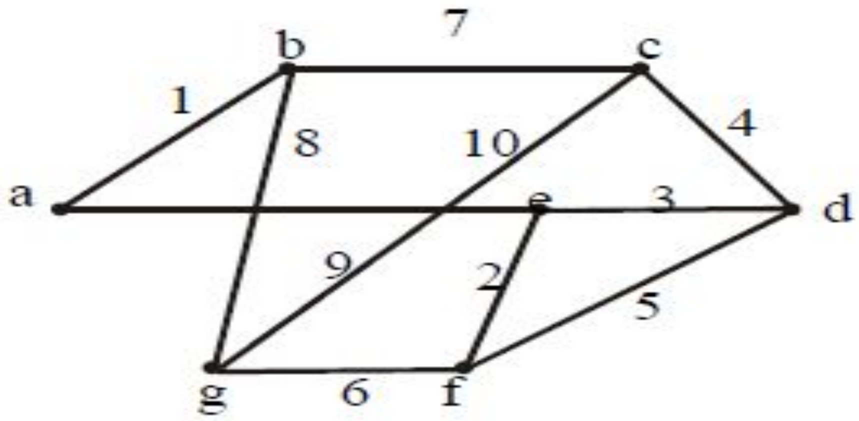
KRUSKAL'S ALGORITHM

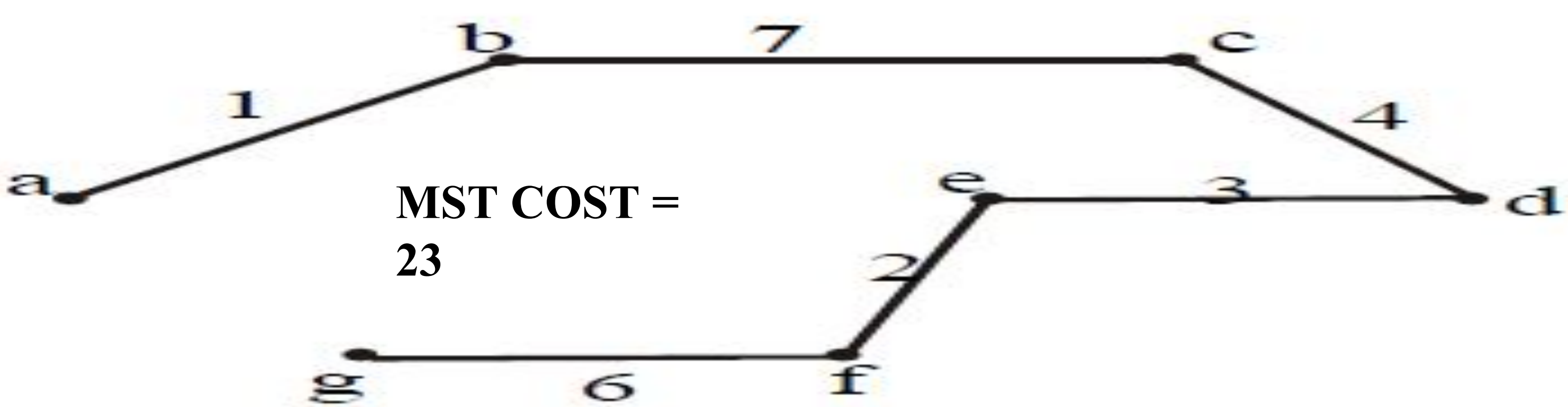
ALGORITHM 2 Kruskal's Algorithm.

procedure *Kruskal*(G: weighted connected undirected graph with n vertices)
T := empty graph
for $i := 1$ **to** $n - 1$
    e := any edge in G with smallest weight that does not form a simple circuit
        when added to T
    $T := T$ with e added
return T $\{T$ is a minimum spanning tree of $G\}$

Minimal spanning tree (MST)

Example: Use Kruskal's algorithm to find a minimal spanning tree for the graph below. Indicate the order in which edges are added to form the tree.





MST COST = 23

Order of adding the edges:
{a, b}, {e, f}, {e, d}, {c, d}, {g, f}, {b, c}

The Foundations: Logic and Proofs

Proofs

- A proof is a valid argument that establishes the truth of a mathematical statement.

- Ingredients:
  - hypotheses of the theorem
  - axioms assumed to be true
  - previously proven theorems
  - rules of inference

You get:
truth of the
statement
being proved

Usefulness

- Computer Science
  - Verifying that computer programs are correct.
  - Establishing that operating systems are secure.
  - Making inferences in artificial intelligence.
  - Showing that system specifications are consistent.

- Mathematics
  - Defining Formalism.
  - Providing specification in a common language.
  - Justification for the results.

Definitions

. An integer n is even if, and only if, n = 2k for some integer k.

2. An integer n is odd if, and only if, n = 2k + 1 for some integer k.

3. An integer n is prime if, and only if, n > 1 and for all positive integers r and s, if n = r·s, then r = 1 or s = 1.

4. An integer n > 1 is composite if, and only if, n = r·s for some positive integers r and s with r ≠ 1 and s ≠ 1.

5. A real number r is rational if, and only if, r= $\frac{a}{b}$ for some integers a and b with b ≠ 0.

6. If n and d are integers and d ≠ 0, then d divides n, written d|n if, and only if, n = d.k for some integers k.

7. An integer n is called a perfect square if, and only if, n = k^2 for some integer k.

Types of Proofs

- **Proving conditional Statements**
  - Direct Proofs
  - Indirect Proofs
    - **Proof by Contraposition**
    - **Proofs by Contradiction**
- **Proving Non-conditional Statements**
  - Indirect Proofs
  - If-And-Only-If Proof
  - Constructive Versus Non-constructive Proofs
  - Existence Proofs; Existence and Uniqueness Proofs
  - Disproofs (Counterexample, Contradiction, Existence Statement)
  - Proofs Involving Sets
- **Mathematical Induction**

Direct Proofs

- $p \rightarrow q$
  - first step is the assumption that *p is true*
  - subsequent steps constructed using rules of inference.
  - final step showing that *q must also be true*

showing that if *p is true,*
then q must also be true,
so that the combination
p true and q false never occurs

Outline for Direct Proof

Proposition If P, then Q.

Proof. Suppose P.

$\vdots$

Therefore Q. $\blacksquare$

Activity Time

Prove that the sum of two odd integers is even.

Prove that the sum of two odd integers is even.

Let **m** and **n** be two odd integers. Then by definition of odd numbers

$m = 2k + 1$ for some $k \in Z$

$n = 2l + 1$ for some $l \in Z$

$$
\begin{aligned}
\text{Now } m + n &= (2k + 1) + (2l + 1) \\
&= 2k + 2l + 2 \\
&= 2(k + l + 1) \\
&= 2r \quad \text{where } r = (k + l + 1) \in Z
\end{aligned}
$$

Hence $m + n$ is even.

EXERCISE:

Prove that if n is any even integer, then $(-1)^n = 1$

SOLUTION:

Suppose n is an even integer. Then n = 2k for some integer k.

Now

$$\begin{aligned}(-1)^n &= (-1)^{2k}\\ &= [(-1)^2]^k\\ &= (1)^k\\ &= 1 \qquad \text{(proved)}\end{aligned}$$

EXERCISE:

Prove that the product of an even integer and an odd integer is even.

SOLUTION:

Suppose m is an even integer and n is an odd integer. Then

$m = 2k$ for some integer k

and $n = 2l + 1$ for some integer l

Now

$$
\begin{aligned}
m \cdot n &= 2k \cdot (2l + 1) \\
&= 2 \cdot k\,(2l + 1) \\
&= 2 \cdot r \qquad \text{where } r = k(2l + 1) \text{ is an integer}
\end{aligned}
$$

Hence $m \cdot n$ is even. (Proved)

EXERCISE:

Prove that the square of an even integer is even.

SOLUTION:

Suppose n is an even integer. Then $n = 2k$

Now

$$\begin{aligned} \text{square of } n = n^2 &= (2 \cdot k)^2 \\ &= 4k^2 \\ &= 2 \cdot (2k^2) \\ &= 2 \cdot p \quad \text{where } p = 2k^2 \in Z \end{aligned}$$

Hence, n^2 is even. (proved)

proved that if *n is an odd integer, then n^2 is an odd integer*

- We assume that the hypothesis of this conditional statement is true, namely, we assume that *n is odd*.
- By the definition of an odd integer, it follows that $n = 2k + 1$, *where k is some integer.*
- *Square both sides* $n^2 = (2k + 1)^2$
  - $4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$
- Consequently, we have proved that if *n is an odd integer, then n^2 is an odd integer*

EXERCISE:

Prove that if n is an odd integer, then $n^3 + n$ is even.

SOLUTION:

Let n be an odd integer, then $n = 2k + 1$ for some $k \in Z$

Now $n^3 + n = n(n^2 + 1)$

$= (2k + 1)((2k+1)^2 + 1)$

$= (2k + 1)(4k^2 + 4k + 1 + 1)$

$= (2k + 1)(4k^2 + 4k + 2)$

$= (2k + 1)\, 2.\, (2k^2 + 2k + 1)$

$= 2\cdot(2k + 1)(2k^2 + 2k + 1) \qquad k \in Z$

$=$ an even integer

Proposition If x is an even integer, then x^2-6x+5 is odd.

Proof. Suppose x is an even integer.

Then $x=2a$ for some $a \in \mathbb{Z}$, by definition of an even integer.

So $x^2-6x+5=(2a)^2-6(2a)+5=4a^2-12a+5=4a^2-12a+4+1=2(2a^2-6a+2)+1$.

Therefore we have $x^2-6x+5=2b+1$, where $b=2a^2-6a+2 \in \mathbb{Z}$.

Consequently x^2-6x+5 is odd, by definition of an odd number. $\blacksquare$

EXERCISE:

Prove that, if the sum of any two integers is even, then so is their difference.

SOLUTION:

Suppose m and n are integers so that $m + n$ is even. Then by definition of even numbers

$$m + n = 2k \qquad \text{for some integer } k$$

$$\Rightarrow \quad m = 2k - n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now $m - n = (2k - n) - n$ using (1)

$= 2k - 2n$

$= 2(k - n) = 2r$ where $r = k - n$ is an integer

Hence $m - n$ is even.

EXERCISE:

Prove that the sum of any two rational numbers is rational.

SOLUTION:

Suppose r and s are rational numbers.

Then by definition of rational

$$r = \frac{a}{b} \quad \text{and} \quad s = \frac{c}{d}$$

for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$

Now

$$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = \frac{p}{q}$$

where $p = ad + bc \in Z$ and $q = bd \in Z$

and $q \neq 0$

Hence r + s is rational.

EXERCISE:

Given any two distinct rational numbers r and s with $r < s$. Prove that there is a rational number x such that $r < x < s$.

SOLUTION:

Given two distinct rational numbers r and s such that

$r < s$ ....(1)

Adding r to both sides of (1), we get

$$r + r < r + s$$
$$2r < r + s$$

$$\Rightarrow \quad r < \frac{r+s}{2} \quad \text{.......................(2)}$$

Next adding s to both sides of (1), we get

$$r + s < s + s$$

$$\Rightarrow \quad r + s < 2s$$

$$\Rightarrow \quad \frac{r+s}{2} < s \quad \text{.............................(3)}$$

Combining (2) and (3), we may write

$$r < \frac{r+s}{2} < s \quad \text{.........................(4)}$$

Since the sum of two rationals is rational, therefore $r + s$ is rational. Also the quotient of a rational by a non-zero rational, is rational, therefore $\frac{r+s}{2}$ is rational and by (4) it lies between r & s.
Hence, we have found a rational number
such that $r < x < s$. (proved)

EXERCISE:

Prove that the sum of any three consecutive integers is divisible by 3.

PROOF:

Let n, n + 1 and n + 2 be three consecutive integers.

Now

$$n + (n + 1) + (n + 2) = 3n + 3$$
$$= 3(n + 1)$$
$$= 3 \cdot k \quad \text{where } k=(n+1) \in Z$$

Hence, the sum of three consecutive integers is divisible by 3.

Activity Time

Give a direct proof that if *m and n are both perfect squares, then nm is also a perfect square.*

Proof

- We assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares.
- By the definition of a perfect square, It follows that there are integers s and t such that $m = s^2$ and $n = t^2$.
- Multiplying both m and n to get s^2t^2.
- Hence, $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$, using commutativity and associativity of multiplication.
- By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer.
- We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Activity Time

Give a direct proof that if *n is an integer and n is odd, then 3n + 2 is odd.*

Indirect Proofs

- Direct proof begin with the premises, continue with a sequence of deductions, and end with the conclusion.
- Attempts at direct proofs often reach dead ends
- Proofs that **do not** start with the premises and end with the conclusion, are called **indirect proofs**

PROOF BY CONTRAPOSITION:

A proof by contraposition is based on the logical equivalence between a statement and its contrapositive. Therefore, the implication $p \to q$ can be proved by showing that its contrapositive $\sim q \to \sim p$ is true. The contrapositive is usually proved directly.

The method of proof by contrapositive may be summarized as:

1. Express the statement in the form if p then q.
2. Rewrite this statement in the contrapositive form if not q then not p.
3. Prove the contrapositive by a direct proof.

Outline for Contrapositive Proof

Proposition If P, then Q.

Proof. Suppose $\sim Q$.

$\vdots$

Therefore $\sim P$. ■

Prove that if *n is an integer and 3n + 2 is odd, then n is odd.*

PROOF:

The contrapositive of the given conditional statement is
"if n is even then $3n + 2$ is even"

Suppose n is even, then

$n = 2k$ for some $k \in Z$

Now $3n + 2 = 3(2k) + 2$

$= 2.(3k + 1)$

$= 2.r$ where $r = (3k + 1) \in Z$

Hence $3n + 2$ is even. We conclude that the given statement is true since its contrapositive is true.

EXERCISE:

Prove that for all integers n, if n^2 is even then n is even.

PROOF:

The contrapositive of the given statement is:

"if n is not even (odd) then n^2 is not even (odd)"

We prove this contrapositive statement directly.

Suppose n is odd. Then $n = 2k + 1$ for some $k \in Z$

Now $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$$= 2\cdot(2k^2 + 2k) + 1$$

$$= 2\cdot r + 1 \qquad \text{where } r = 2k^2 + 2k \in Z$$

Hence n^2 is odd. Thus the contrapositive statement is true and so the given statement is true.

EXERCISE:

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even.

PROOF:

Suppose n is an odd integer. Since, a product of two odd integers is odd, therefore $n^2 = n.n$ is odd; and $n^3 = n^2.n$ is odd.

Since a sum of two odd integers is even therefore $n^2 + 5$ is even.

Thus we have prove that if n is odd then $n^3 + 5$ is even.

Since this is the contrapositive of the given conditional statement, so the given statement is true.

EXERCISE:

Prove that if n^2 is not divisible by 25, then n is not divisible by 5.

SOLUTION:

The contra positive statement is:

"if n is divisible by 5, then n^2 is divisible by 25"

Suppose n is divisible by 5. Then by definition of divisibility

$$n = 5 \cdot k \qquad \text{for some integer } k$$

Squaring both sides

$$n^2 = 25 \cdot k^2 \qquad \text{where } k^2 \in Z$$

n^2 is divisible by 25

Proofs by Contradiction

A proof by contradiction is based on the fact that either a statement is true or it is false but not both. Hence the supposition, that the statement to be proved is false, leads logically to a contradiction, impossibility or absurdity, then the supposition must be false. Accordingly, the given statement must be true.

The method of proof by contradiction may be summarized as follows:

1. Suppose the statement to be proved is false.

2. Show that this supposition leads logically to a contradiction.

3. Conclude that the statement to be proved is true.

Basic Idea

- Assume that the statement we want to prove is *false, and then show* that this assumption leads to nonsense!

We are then led to conclude that we were wrong to assume the statement was false, so the statement must be true.

Outline for Proof by Contradiction

Proposition P.

Proof. Suppose $\sim P$.

$\vdots$

Therefore $C \wedge \sim C$. ■

THEOREM:

There is no greatest integer.

PROOF:

Suppose there is a greatest integer N. Then $n \leq N$ for every integer n.

Let $M = N + 1$

Now M is an integer since it is a sum of integers.

Also $M > N$ since $M = N + 1$

Thus M is an integer that is greater than the greatest integer, which is a contradiction. Hence our supposition is not true and so there is no greatest integer.

EXERCISE:

Give a proof by contradiction for the statement:

"If n^2 is an even integer then n is an even integer."

PROOF:

Suppose n^2 is an even integer and n is not even, so that n is odd.

Hence $n = 2k + 1$ for some integer k.

Now $n^2 = (2k + 1)^2$

$= 4k^2 + 4k + 1$

$= 2\cdot(2k^2 + 2k) + 1$

$= 2r + 1$ where $r = (2k^2 + 2k) \in Z$

This shows that n^2 is odd, which is a contradiction to our supposition that n^2 is even. Hence the given statement is true.

EXERCISE:

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even using contradiction method.

SOLUTION:

Suppose that $n^3 + 5$ is odd and n is not even (odd). Since n is odd and the product of two odd numbers is odd, it follows that n^2 is odd and $n^3 = n^2 . n$ is odd. Further, since the difference of two odd numbers is even, it follows that

$$5 = (n^3 + 5) - n^3$$

is even. But this is a contradiction. Therefore, the supposition that $n^3 + 5$ and n are both odd is wrong and so the given statement is true.

THEOREM:

The sum of any rational number and any irrational number is irrational.

PROOF:

We suppose that the negation of the statement is true. That is, we suppose that there is a rational number r and an irrational number s such that r + s is rational. By definition of ration

$$r = \frac{a}{b} \qquad \ldots\ldots\ldots\ldots\ldots(1) \qquad \text{and} \qquad r + s = \frac{c}{d} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

for some integers a, b, c and d with b≠ 0 and d ≠ 0.

Using (1) in (2), we get

$$\frac{a}{b} + s = \frac{c}{d}$$

$$\Rightarrow \qquad s = \frac{c}{d} - \frac{a}{b}$$

$$s = \frac{bc - ad}{bd} \qquad (bd \neq 0)$$

Now bc - ad and bd are both integers, since products and difference of integers are integers. Hence s is a quotient of two integers bc-ad and bd with bd ≠ 0. So by definition of rational, s is rational.

This contradicts the supposition that s is irrational. Hence the supposition is false and the theorem is true.

EXERCISE:

Prove that $\sqrt{2}$ is irrational.

PROOF:

Suppose $\sqrt{2}$ is rational. Then there are integers m and n with no common factors so

$$\sqrt{2} = \frac{m}{n}$$

that

Squaring both sides gives

$$2 = \frac{m^2}{n^2}$$

Or $\qquad m^2 = 2n^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

This implies that m^2 is even (by definition of even). It follows that m is even. Hence

$m = 2k \qquad$ for some integer k $\qquad (2)$

Substituting (2) in (1), we get

$$(2k)^2 = 2n^2$$

$$\Rightarrow \quad 4k^2 = 2n^2$$

$$\Rightarrow \quad n^2 = 2k^2$$

This implies that n^2 is even, and so n is even. But we also know that m is even. Hence both m and n have a common factor 2. But this contradicts the supposition that m and n have no common factors. Hence our supposition is false and so the theorem is true.

PROOF BY COUNTER EXAMPLE

Disprove the statement by giving a counter example.

For all real numbers a and b, if $a < b$ then $a^2 < b^2$.

SOLUTION:

Suppose $a = -5$ and $b = -2$

then clearly $-5 < -2$

But $a^2 = (-5)^2 = 25$ and $b^2 = (-2)^2 = 4$

But $25 > 4$

This disproves the given statement.

EXERCISE:

Prove or give counter example to disprove the statement.

For all integers n, $n^2 - n + 11$ is a prime number.

SOLUTION:

The statement is not true

For $n = 11$

we have , $n^2 - n + 11 = (11)^2 - 11 + 11$

$$= (11)^2$$

$$= (11)(11)$$

$$=121$$

which is obviously not a prime number.

Mathematical Induction

Shoaib Raza

Conjecture: The sum of the first n odd natural numbers equals n^2.

n	sum of the first n odd natural numbers	n^2
1	$1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	1
2	$1+3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	4
3	$1+3+5 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	9
4	$1+3+5+7 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	16
5	$1+3+5+7+9 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	25
$\vdots$	$\vdots$	$\vdots$
n	$1+3+5+7+9+11+\cdots+(2n-1) = \ldots\ldots\ldots$	n^2
$\vdots$	$\vdots$	$\vdots$

An infinite ladder

- Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder.

- We know two things:
  1. We can reach the first rung of the ladder.
  2. If we can reach a particular rung of the ladder, then we can reach the next rung.

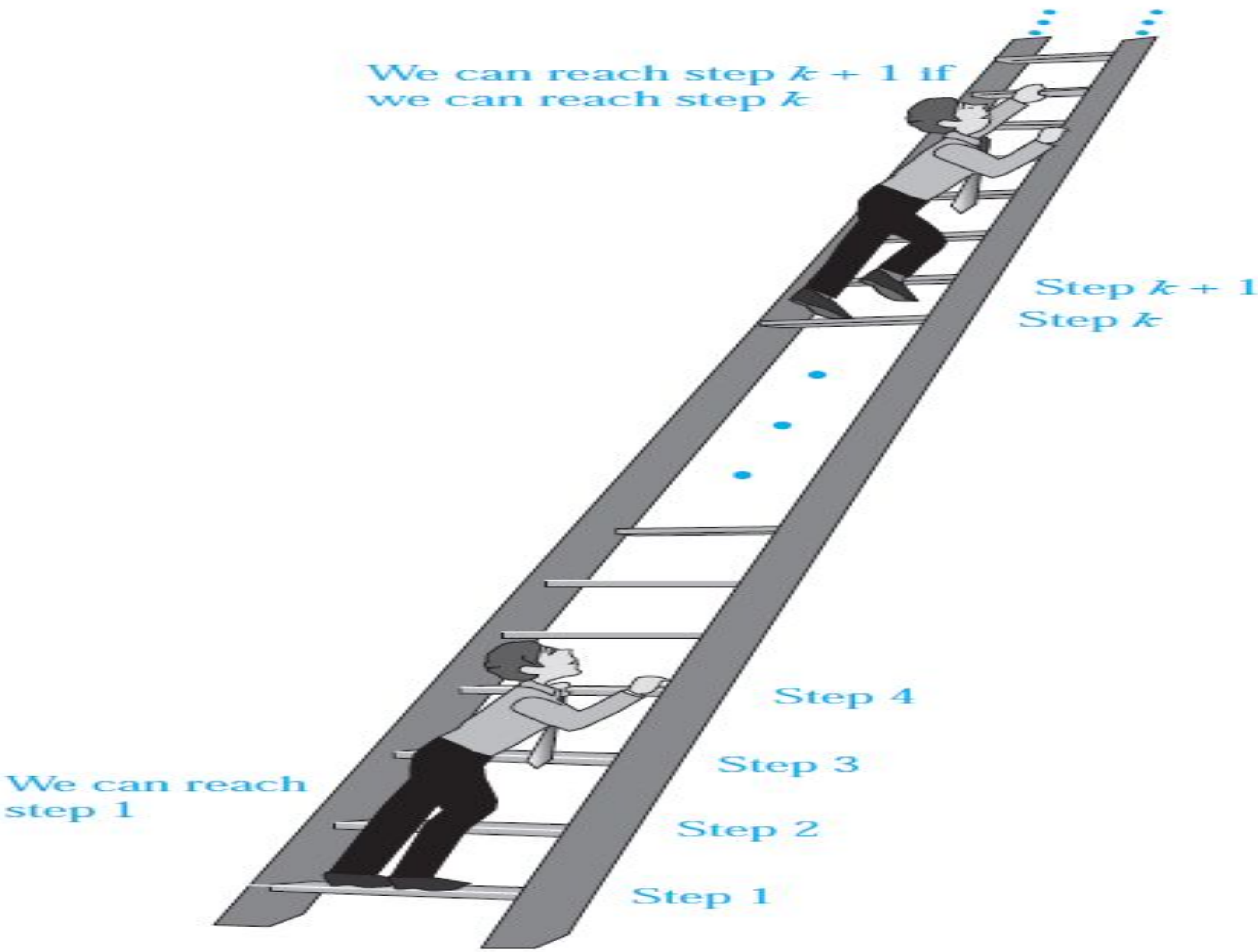


FIGURE 1 **Climbing an Infinite Ladder.**

Mathematical Induction

- Mathematical statements assert that a property is true for all positive integers.

- Proofs using mathematical induction have two parts.
  - First, they show that the statement holds for the positive integer 1 (base case).
  - Second, they show that if the statement holds for a positive integer then it must also hold for the next larger integer. (inductive case)

- The method can be extended to prove statements about more general well-founded structures, such as trees; this generalization, known as structural induction, is used in mathematical logic and computer science.

NOTE

- It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way.
- It is *not a tool for discovering formulae or* theorems.
- Mathematicians sometimes find proofs by mathematical induction unsatisfying because they do not provide insights as to why theorems are true.
- You can prove a theorem by mathematical induction even if you do not have the slightest idea why it is true!

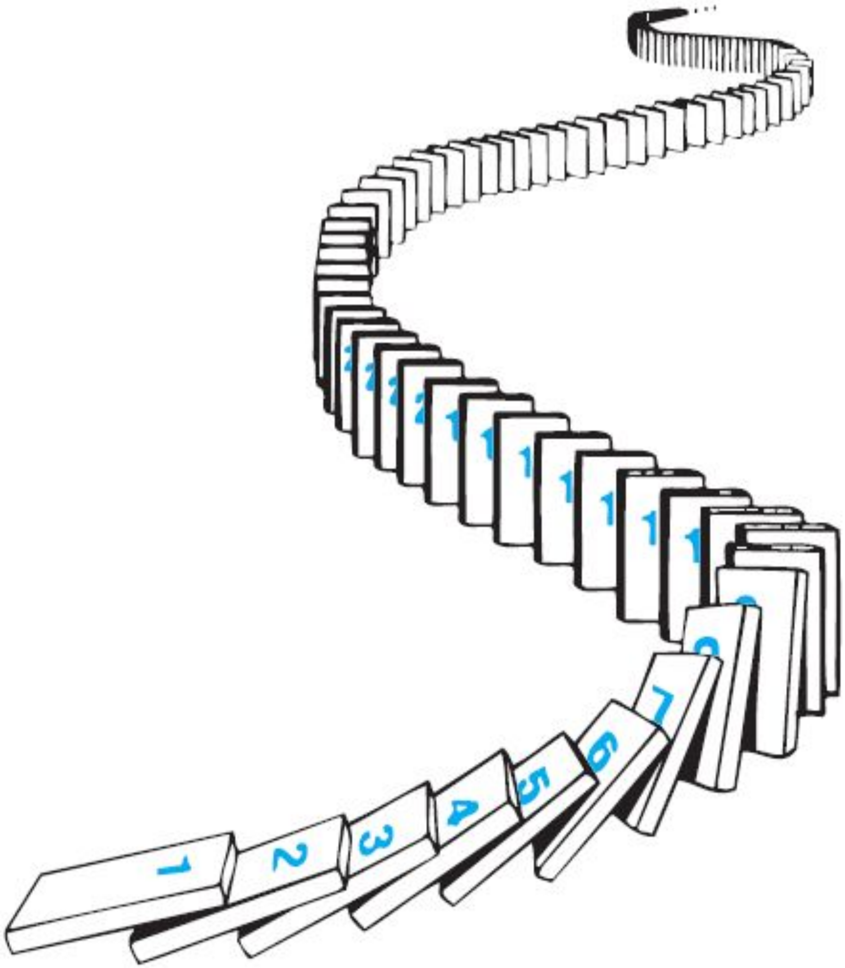
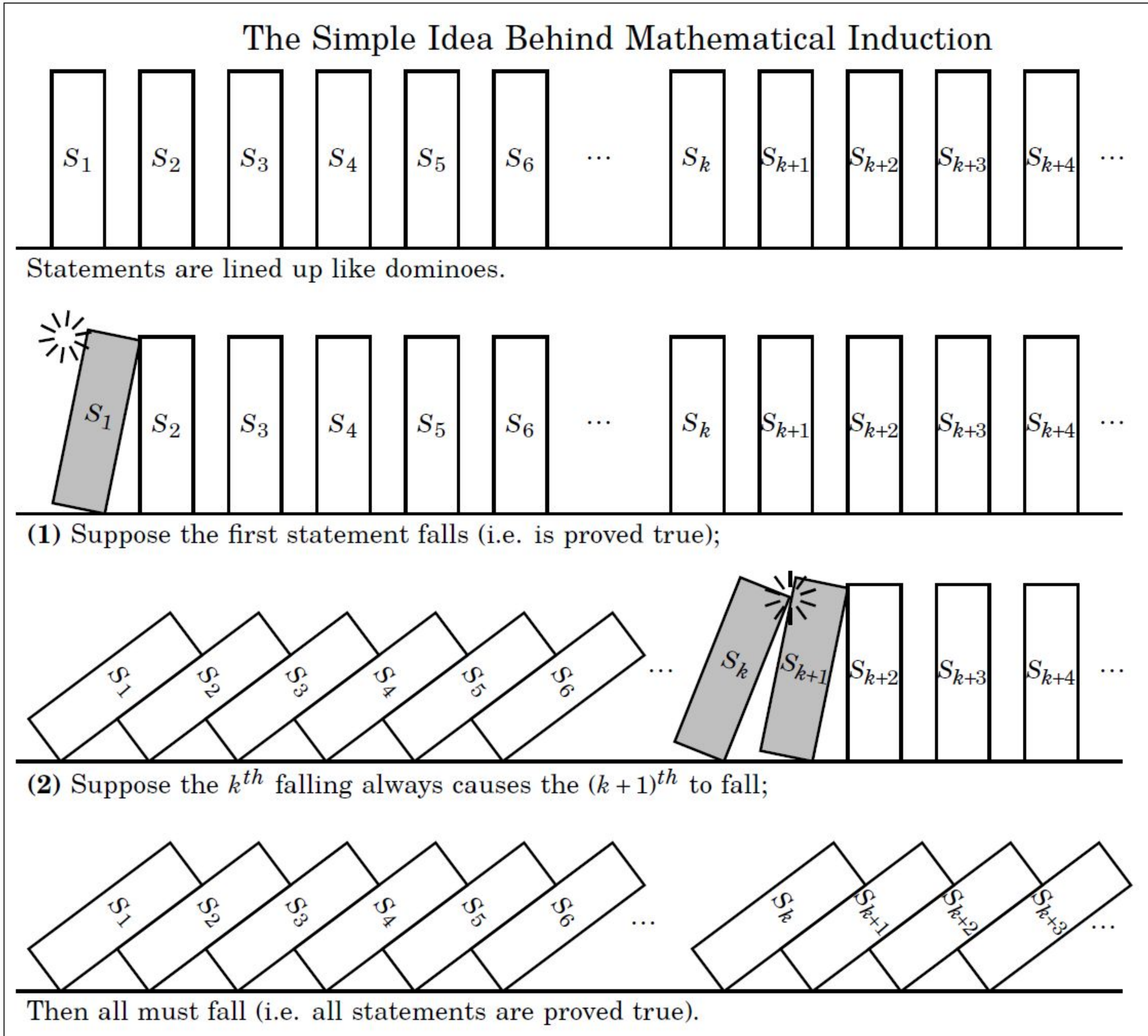


FIGURE 2 Illustrating How Mathematical Induction Works Using Dominoes.


The Simple Idea Behind Mathematical Induction
S_1 S_2 S_3 S_4 S_5 S_6 $\cdots$ S_k S_{k+1} S_{k+2} S_{k+3} S_{k+4} $\cdots$
Statements are lined up like dominoes.
S_1 S_2 S_3 S_4 S_5 S_6 $\cdots$ S_k S_{k+1} S_{k+2} S_{k+3} S_{k+4} $\cdots$
(1) Suppose the first statement falls (i.e. is proved true);
S_1 S_2 S_3 S_4 S_5 S_6 $\cdots$ S_k S_{k+1} S_{k+2} S_{k+3} S_{k+4} $\cdots$
(2) Suppose the k^{th} falling always causes the $(k+1)^{th}$ to fall;
S_1 S_2 S_3 S_4 S_5 S_6 $\cdots$ S_k S_{k+1} S_{k+2} S_{k+3} $\cdots$
Then all must fall (i.e. all statements are proved true).

PRINCIPLE OF MATHEMATICAL INDUCTION:

Let P(n) be a propositional function defined for all positive integers n. P(n) is true for every positive integer n if

1.Basis Step:

The proposition P(1) is true.

2.Inductive Step:

If P(k) is true then P(k + 1) is true for all integers $k \geq 1$.

i.e. $\forall k \quad p(k) \rightarrow P(k+1)$

EXAMPLE:

Use Mathematical Induction to prove that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2} \quad \text{for all integers n} \geq 1$$

SOLUTION:

Let $P(n): 1+2+3+\cdots+n=\frac{n(n+1)}{2}$

1.Basis Step:

P(1) is true.

For n = 1, left hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1, so LHS = 1 and RHS is

$$R.H.S=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

so the proposition is true for n = 1.

2. Inductive Step: Suppose P(k) is true for, some integers k ≥ 1.

(1) $1+2+3+\cdots+k=\frac{k(k+1)}{2}$

To prove P(k + 1) is true. That is,

$$(2) \qquad 1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$$

Consider L.H.S. of (2)

$$\begin{aligned}
1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) \qquad \text{using (1)} \\
&= (k+1)\left[\frac{k}{2}+1\right] \\
&= (k+1)\left[\frac{k+2}{2}\right] \\
&= \frac{(k+1)(k+2)}{2} = \text{RHS of (2)}
\end{aligned}$$

Hence by principle of Mathematical Induction the given result true for all integers greater or equal to 1.

EXERCISE:

Use mathematical induction to prove that
$1+3+5+\ldots+(2n-1) = n^2$ for all integers $n \geq 1$.

SOLUTION:

Let $P(n)$ be the equation $1+3+5+\ldots+(2n-1) = n^2$

1. **Basis Step:**

   $P(1)$ is true
   For $n = 1$, L.H.S of $P(1) = 1$ and
   R.H.S $= 1^2 = 1$
   Hence the equation is true for $n = 1$

2. **Inductive Step:**

   Suppose $P(k)$ is true for some integer $k \geq 1$. That is,
   $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ ....(1)

To prove $P(k+1)$ is true; i.e.,
$1 + 3 + 5 + \ldots + [2(k+1)-1] = (k+1)^2$ ....(2)

Consider L.H.S. of (2)

$$\begin{aligned}
1+3+5+\cdots+[2(k+1)-1] &= 1+3+5+\cdots+(2k+1) \\
&= 1+3+5+\cdots+(2k-1)+(2k+1) \\
&= k^2+(2k+1) \qquad \text{using (1)} \\
&= (k+1)^2 \\
&= \text{R.H.S. of (2)}
\end{aligned}$$

Thus $P(k+1)$ is also true. Hence by mathematical induction, the given equation is true for all integers $n \geq 1$.

Exercise (cont.)

Proof.

1. $P(n)$: $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$
2. Basis step $P(0)$: $2^0 = 1 = 2^{0+1} - 1$.
3. Inductive step:
   Inductive hypothesis $P(k)$: $2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$
   Let's prove $P(k+1)$:

$$
\begin{aligned}
2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} && \text{(by IH)} \\
&= 2(2^{k+1}) - 1 && \text{(by arithmetic)} \\
&= 2^{k+2} - 1 && \text{(by arithmetic)}
\end{aligned}
$$

Counting

Chapter 6

Mr. Shoaib Raza

Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations

The Basics of Counting

Section 6.1

COMBINATORICS

- Combinatorics is the mathematics of counting and arranging objects. Counting of objects with certain properties (enumeration) is required to solve many different types of problem.
- Applications, include topics as diverse as codes, circuit design and algorithm complexity [and gambling]

Counting

- Enumeration, the counting of objects with certain properties, is an important part of combinatorics.
- We must count objects to solve many different types of problems. For example, counting is used to:

1. Determine number of ordered or unordered arrangement of objects.
2. Generate all the arrangements of a specified kind which is important in computer simulations.
3. Compute probabilities of events.
4. Analyze the chance of winning games, lotteries etc.
5. Determine the complexity of algorithms.

Section Summary

- The Sum Rule
- The Product Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways to do the second task, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \varnothing \text{ for all } i, j.$$

- The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Basic Counting Principles: The Sum Rule

Example:

Suppose there are 7 different optional courses in Computer Science and 3 different optional courses in Mathematics. How many ways student can choose a course.

Solution: By the sum rule it follows that there are $7 + 3 = 10$ choices for a student who wants to take one optional course.

Basic Counting Principles: The Sum Rule

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

Basic Counting Principles: The Sum Rule

Example: A student can choose a computer project from one of the three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from?

Solution: The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways. Hence, there are

$23 + 15 + 19 = 57$ projects to choose from.

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, …, and an element in A_m.
- By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

The Product Rule

Example: How many ways a student can choose one optional course each from computer science and mathematics courses if there are 7 different optional courses in Computer Science and 3 different optional courses in Mathematics.

Solution:

A student who wants to take one optional course of each subject, there are $7 \times 3 = 21$ choices.

The Product Rule

Example: The chairs of an auditorium are to be labeled with two characters, a letter followed by a digit. What is the largest number of chairs that can be labeled differently?

Solution:

The procedure of labeling a chair consists of two events, namely,

Assigning one of the 26 letters: A, B, C, …, Z and

Assigning one of the 10 digits: 0, 1, 2, …, 9

By product rule, there are $26 \times 10 = 260$ different ways that a chair can be labeled by both a letter and a digit.

The Product Rule

Example: Find the number n of ways that an organization consisting of 15 members can elect a president, treasurer, and secretary. (assuming no person is elected to more than one position)

Solution:

The president can be elected in 15 different ways; following this, the treasurer can be elected in 14 different ways; and following this, the secretary can be elected in 13 different ways. Thus, by product rule, there are

$$n = 15 \times 14 \times 13 = 2730$$

different ways in which the organization can elect the officers.

The Product Rule

Example: There are four bus lines between A and B; and three bus lines between B and C.

Find the number of ways a person can travel:

a) By bus from A to C by way of B;

b) Round trip by bus from A to C by way of B;

c) Round trip by bus from A to C by way of B, if the person does not want to use a bus line more than once.

The Product Rule

a) By bus from A to C by way of B;

Solution:

$$A \xrightarrow{4} B \xrightarrow{3} C.$$

There are 4 ways to go from A to B and 3 ways to go from B to C; hence there are $4 \times 3 = 12$ ways to go from A to C by way of B.

The Product Rule

b) Round trip by bus from A to C by way of B;

Solution:

The person will travel from A to B to C to B to A for the round trip. i.e. ($A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$)

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{3} B \xrightarrow{4} A$$

The person can travel 4 ways from A to B and 3 way from B to C and back.

Thus there are $4 \times 3 \times 3 \times 4 = 144$ ways to travel the round trip.

The Product Rule

c) Round trip by bus from A to C by way of B, if the person does not want to use a bus line more than once.

Solution:

$$A \xrightarrow{4} B \xrightarrow{3} C \xrightarrow{2} B \xrightarrow{3} A$$

The person can travel 4 ways from A to B and 3 ways from B to C, but only 2 ways from C to B and 3 ways from B to A, since bus line cannot be used more than once. Hence there are

$4 \times 3 \times 2 \times 3 = 72$ ways

to travel the round trip without using a bus line more than once.

The Product Rule

Example: A bit string is a sequence of 0's and 1's.How many bit string are there of length 4?

Solution:

Each bit (binary digit) is either 0 or 1.

Hence, there are 2 ways to choose each bit. Since we have to choose four bits therefore,

$$2 \times 2 \times 2 \times 2 = 2^4 = 16$$

the product rule shows, there are a total of different bit strings of length four.

The Product Rule

Example: How many bit strings of length 8:

(i) begin with a 1?

(ii) begin and end with a 1?

Solution:

(i) If the first bit (left most bit) is a 1, then it can be filled in only one way. Each of the remaining seven positions in the bit string can be filled in 2 ways (i.e., either by 0 or 1). Hence, there are

$$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$$

different bit strings of length 8 that begin with a 1.

The Product Rule

(ii) begin and end with a 1?

Solution:

If the first and last bit in an 8 bit string is a 1, then only the intermediate six bits can be filled in 2 ways, i.e. by a 0 or 1. Hence there are

$$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 1 = 2^6 = 64$$

different bit strings of length 8 that begin and end with a 1.

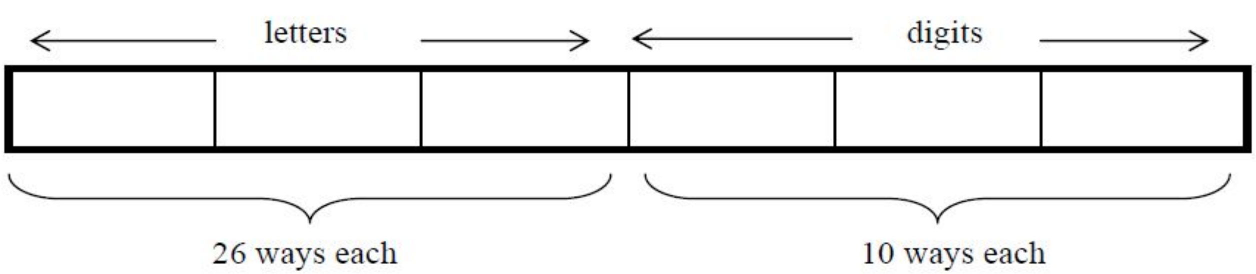
The Product Rule

Example: Suppose that an automobile license plate has three letters followed by three digits.

(a) How many different license plates are possible?

Solution:

Each of the three letters can be written in 26 different ways, and each of the three digits can be written in 10 different ways.



$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$

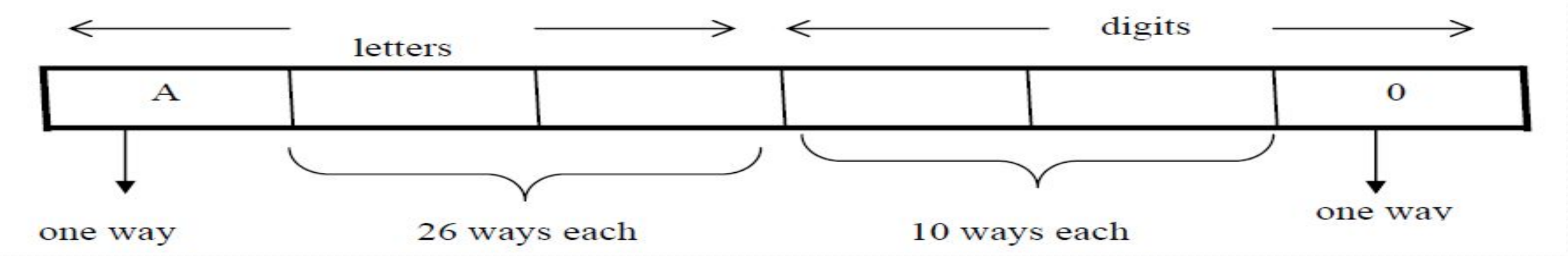
different License plates possible.

The Product Rule

(b) How many license plates could begin with A and end on 0?

Solution:

The first and last place can be filled in one way only, while each of second and third place can be filled in 26 ways and each of fourth and fifth place can be filled in 10 ways.



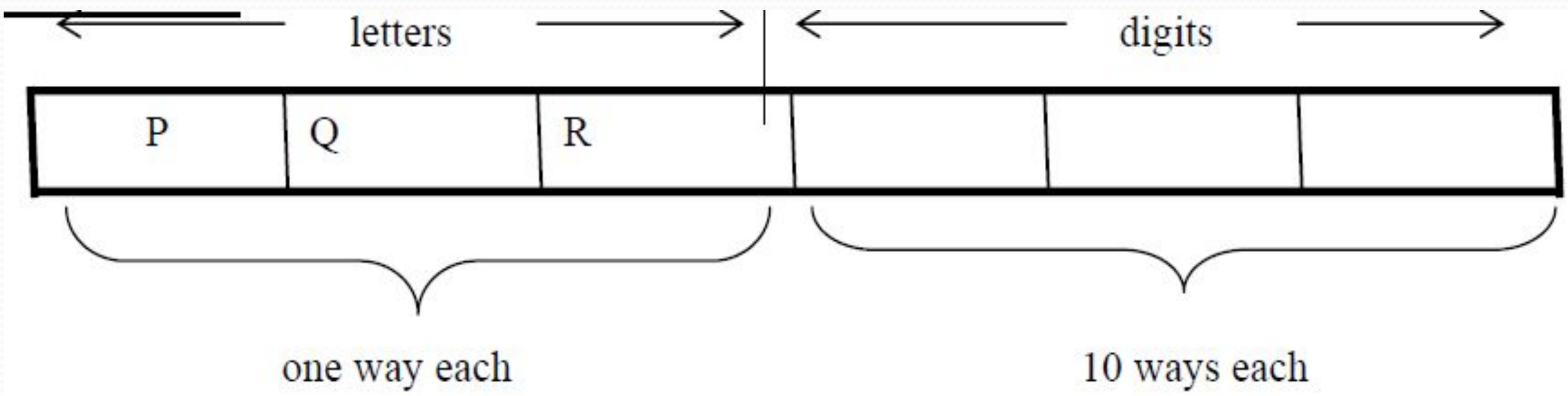
Number of license plates that begin with A and end in 0 are

$$1 \times 26 \times 26 \times 10 \times 10 \times 1 = 67600$$

The Product Rule

(c) How many license plates begin with PQR.

Solution:



Number of license plates that begin with PQR are

$$1 \times 1 \times 1 \times 10 \times 10 \times 10 = 1000 \text{ ways.}$$

The Product Rule

(d) How many license plates are possible in which all the letters and digits are distinct?

Solution:

The first letter place can be filled in 26 ways. Since, the second letter place should contain a different letter than the first, so it can be filled in 25 ways. Similarly, the third letter place can be filled in 24 ways. And the digits can be respectively filled in 10, 9, and 8 ways.

Hence;

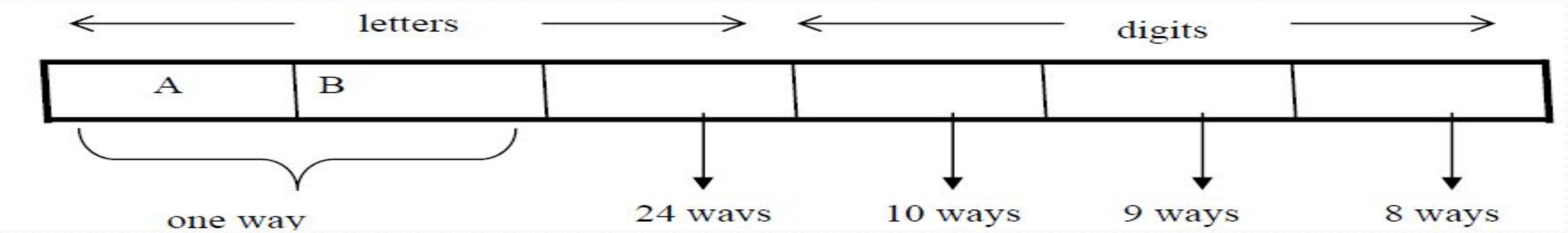
number of license plates in which all the letters and digits are distinct are

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$$

The Product Rule

(e) How many license plates could begin with AB and have all three letters and digits distinct.

Solution:



The first two letters places are fixed (to be filled with A and B), so there is only one way to fill them. The third letter place should contain a letter different from A & B, so there

are 24 ways to fill it.

The three digit positions can be filled in 10 and 8 ways to have distinct digits. Hence, desired number of license plates are

$$1 \times 1 \times 24 \times 10 \times 9 \times 8 = 17280$$

Telephone Numbering Plan

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was NYX-NNX-$XXXX$.
- In the new plan, the format is NXX-NXX-$XXXX$.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format NYX.
- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX.
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format NNX.
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format $XXXX$.

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.

Number of new plan telephone numbers: $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

NUMBER OF ITERATIONS OF A NESTED LOOP

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run

For i: = 1 to 4

For j : = 1 to 3

[Statement in body of inner loop. None contain branching statements that lead out of the inner loop.]

next j

next i

Solution:

The outer loop is iterated four times, and during each iteration of the outer loop, there are three iterations of the inner loop.

Hence, by product rules the total number of iterations of inner loop is $4 \cdot 3 = 12$

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

for i = 5 to 50

for j: = 10 to 20

[Statement in body of inner loop. None contain branching statements that lead out of the inner loop.]

next j

next i

Solution:

The outer loop is iterated 50 - 5 + 1 = 46 times and during each iteration of the outer loop there are 20 - 10 + 1 = 11 iterations of the inner loop. Hence by product rule, the total number of iterations of the inner loop is $46 \times 11 = 506$.

Example: Determine how many times the inner loop will be iterated when the following algorithm is implemented and run.

for i: = 1 to 4

for j: = 1 to i

[Statements in body of inner loop. None contain branching statements that lead outside the loop.]

next j

next i

Solution:

The outer loop is iterated 4 times, but during each iteration of the outer loop, the inner loop iterates different number of times.

For first iteration of outer loop, inner loop iterates 1 times.

For second iteration of outer loop, inner loop iterates 2 times.

For third iteration of outer loop, inner loop iterates 3 times.

For fourth iteration of outer loop, inner loop iterates 4 times.

Hence, total number of iterations of inner loop = 1 + 2 + 3 + 4 = 10.

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

- First consider variable names one character in length. Since such names consist of a single letter, there are 26 variable names of length 1.
- Next, consider variable names two characters in length. Since the first character is a letter, there are 26 ways to choose it. The second character is a digit, there are 10 ways to choose it. Hence, to construct variable name of two characters in length, there are $26 \times 10 = 260$ ways.
- Finally, by sum rule, there are $26 + 260 = 286$ possible variable names in the programming language.

Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$

$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$

$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Internet Addresses

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- **Class A Addresses**: used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses**: used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses**: used for the smallest networks, a 110,followed by a 21-bit netid and a 8-bit hostid.
  - Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
  - 1111111 is not available as the netid of a Class A network.
  - Hostids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

Example: How many different IPv4 addresses are available for computers on the internet?

Solution: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find, x_A: $2^7 - 1 = 127$ netids. $2^{24} - 2 = 16{,}777{,}214$ hostids.
  $$x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178.$$
- To find, x_B: $2^{14} = 16{,}384$ netids. $2^{16} - 2 = 16{,}534$ hostids.
  $$x_B = 16{,}384 \cdot 16{,}534 = 1{,}073{,}709{,}056.$$
- To find, x_C: $2^{21} = 2{,}097{,}152$ netids. $2^8 - 2 = 254$ hostids.
  $$x_C = 2{,}097{,}152 \cdot 254 = 532{,}676{,}608.$$
- Hence, the total number of available IPv4 addresses is

$$\begin{aligned} x &= x_A + x_B + x_C \\ &= 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608 \\ &= 3{,}737{,}091{,}842. \end{aligned}$$

Not Enough Today !!
The newer IPv6 protocol solves the problem of too few addresses.

Combining the Sum and Product Rule

- **Example**: A computer access code word consists of from one to three letters of English alphabets with repetitions allowed. How many different code words are possible.

Solution:

Number of code words of length 1 = 26^1

Number of code words of length 2 = 26^2

Number of code words of length 3 = 26^3

Hence, the total number of code words =

$$26^1 + 26^2 + 26^3 = 18{,}278$$

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

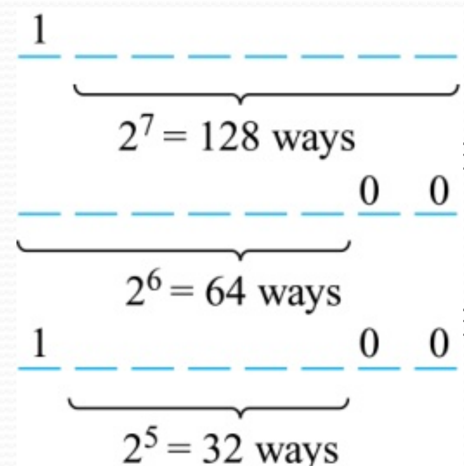
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight start either with a 1 bit and end with the two bits 00?

Solution: Use the subtraction rule.



- Number of bit strings of length eight start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

Basic Counting Principles: Division Rule

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

- Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.
- In terms of functions: If f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

Basic Counting Principles: Division Rule

Example: How many ways are there to seat four people around a circular table, where two seating's are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2 for seat 3, and one way for seat 4. Thus there are $4! = 24$ ways to order the four people. But since two seating's are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdot \cdot \cdot \; n = n^m$ such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?

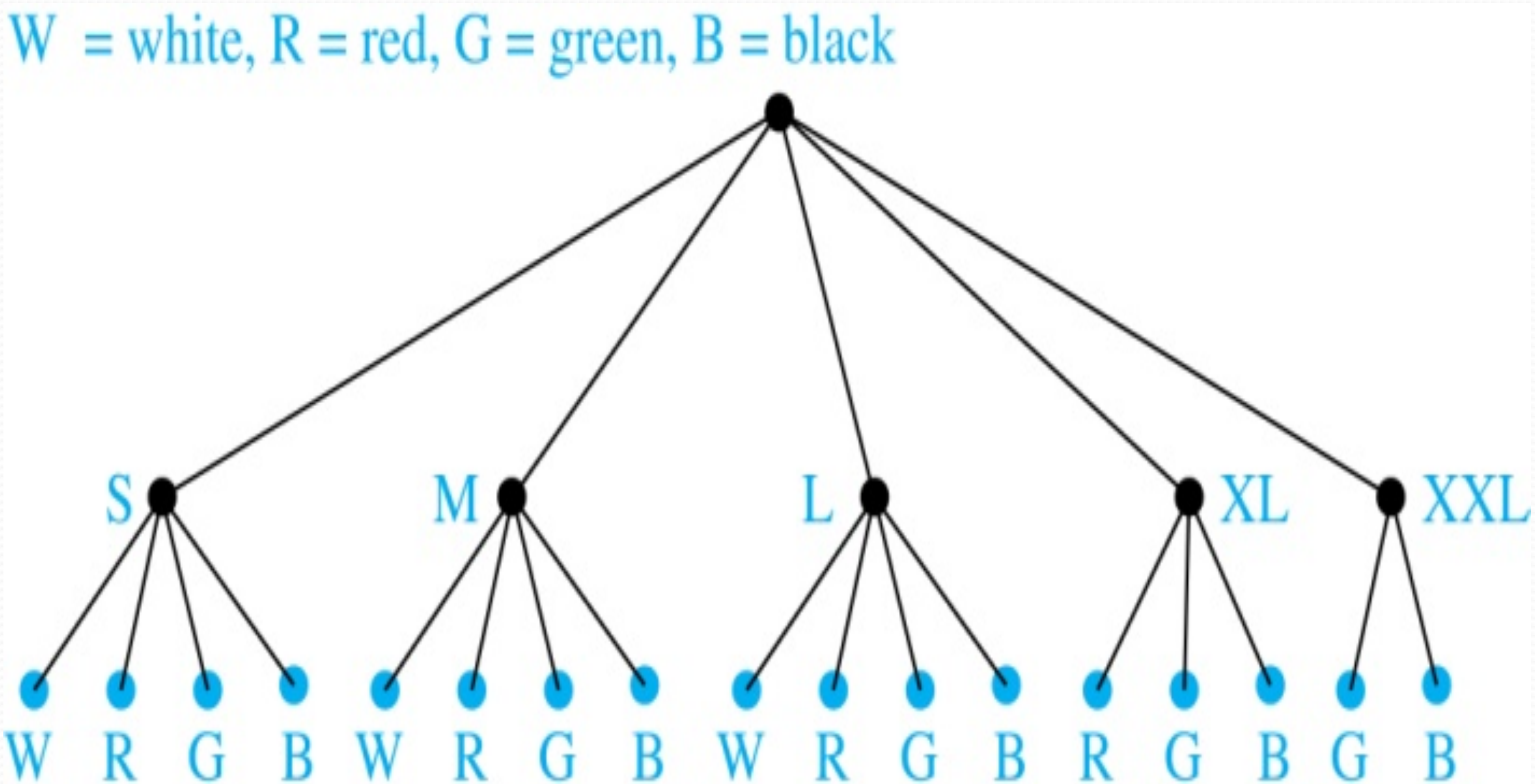
Solution: Suppose the elements in the domain are a_1, $a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Tree Diagrams

- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

Tree Diagrams

- **Solution**: Draw the tree diagram.



- The store must stock 17 T-shirts.

The Pigeonhole Principle

Section 6.2

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

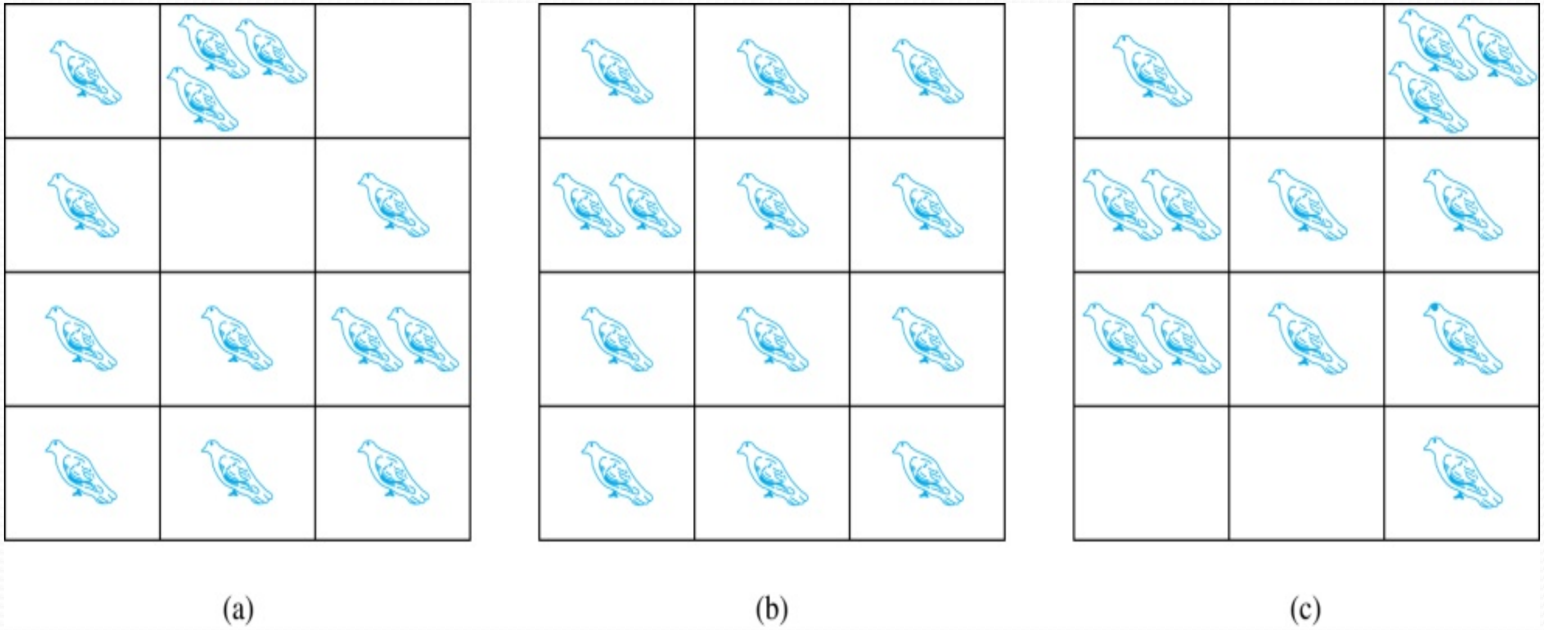
The Pigeonhole Principle

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



The Pigeonhole Principle

Corollary 1: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one. ◀

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < \lceil N/k \rceil + 1$ has been used. This is a contradiction because there are a total of n objects. ◀

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays. $\lceil 367/366 \rceil = 2$

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example: In any set of 27 English , must be at least two that begin with the same letter, since there are 26 letters in the English alphabet. $\lceil 27/26 \rceil = 2$

The Generalized Pigeonhole Principle

Example: What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F.

Solution:

The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer N such that $\lceil N/K \rceil = \lceil N/5 \rceil = 6$. The smallest such integer is

$$N = K(\lceil N/K \rceil - 1) + 1 = 5(6-1)+1 = 5 \cdot 5 + 1 = 26.$$

Thus 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grades.

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

- The number of r-permutations of a set with n elements is denoted by $P(n,r)$.
- The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations (*continued*)

- **Example**: Suppose that there are eight runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?
- **Solution**: The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are

$$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$$

possible ways to award the medals.

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$P(7,7) = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$P(6,6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

- The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

- The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section*. 6.4)

Combinations

Example:

- Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.
- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of r-combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$
$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 26\cdot 17\cdot 10\cdot 49\cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. $\rightarrow$

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} .$$

Hence, $C(n, r) = C(n, n - r)$. ◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

- **Definition 1**: A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements. ◀
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◀

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775\ .$$

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)\,(x + y)\,(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x\,y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two. There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x\,y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Using the Binomial Theorem

Example:

What is the expansion of $(x+y)^4$?

Solution: From the binomial theorem it follows that

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Using the Binomial Theorem

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$.
By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

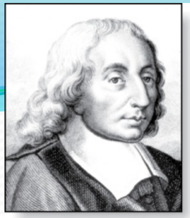
$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}. \quad \blacktriangleleft$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is $\sum_{k=0}^{n} \binom{n}{k}$.

Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \quad \blacktriangleleft$$

Blaise Pascal
(1623-1662)



Pascal’s Identity

Pascal’s Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \blacktriangleleft$$

See Exercise 19 *for an algebraic proof.*

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \; \binom{1}{1} \\
\binom{2}{0} \; \binom{2}{1} \; \binom{2}{2} \\
\binom{3}{0} \; \binom{3}{1} \; \binom{3}{2} \; \binom{3}{3} \\
\binom{4}{0} \; \binom{4}{1} \; \binom{4}{2} \; \binom{4}{3} \; \binom{4}{4} \\
\binom{5}{0} \; \binom{5}{1} \; \binom{5}{2} \; \binom{5}{3} \; \binom{5}{4} \; \binom{5}{5} \\
\binom{6}{0} \; \binom{6}{1} \; \binom{6}{2} \; \binom{6}{3} \; \binom{6}{4} \; \binom{6}{5} \; \binom{6}{6} \\
\binom{7}{0} \; \binom{7}{1} \; \binom{7}{2} \; \binom{7}{3} \; \binom{7}{4} \; \binom{7}{5} \; \binom{7}{6} \; \binom{7}{7} \\
\binom{8}{0} \; \binom{8}{1} \; \binom{8}{2} \; \binom{8}{3} \; \binom{8}{4} \; \binom{8}{5} \; \binom{8}{6} \; \binom{8}{7} \; \binom{8}{8} \\
\cdots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Discrete Probability

Probability

The probability of an event occurring is a number between 0 and 1, and represents essentially how often that event occurs. For example:

- The probability of flipping a coin and it landing on heads is ½.
- The probability of rolling a 6-sided die and getting the number 3 is 1/6.

Sample Space: A sample space is the set of all possible outcomes of a random process.

Event: An event is a subset of the sample space.

The probability of an event E is

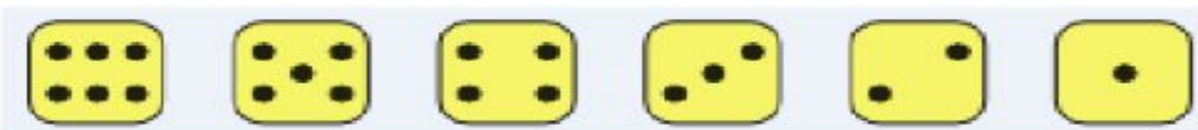
$$P(E) = \frac{\text{Number of outcomes in E}}{\text{Number of outcomes in the sample space}} = \frac{n(E)}{n(S)}$$

In "Discrete Probability", we focus on finite and countable sample spaces.

<u>Example</u>: What is the sample space for one flip of a coin?

Heads, Tails

<u>Example</u>: Suppose I roll two six-sided dice. What is the sample space for the possible outcomes?



1, 2, 3, 4, 5, 6

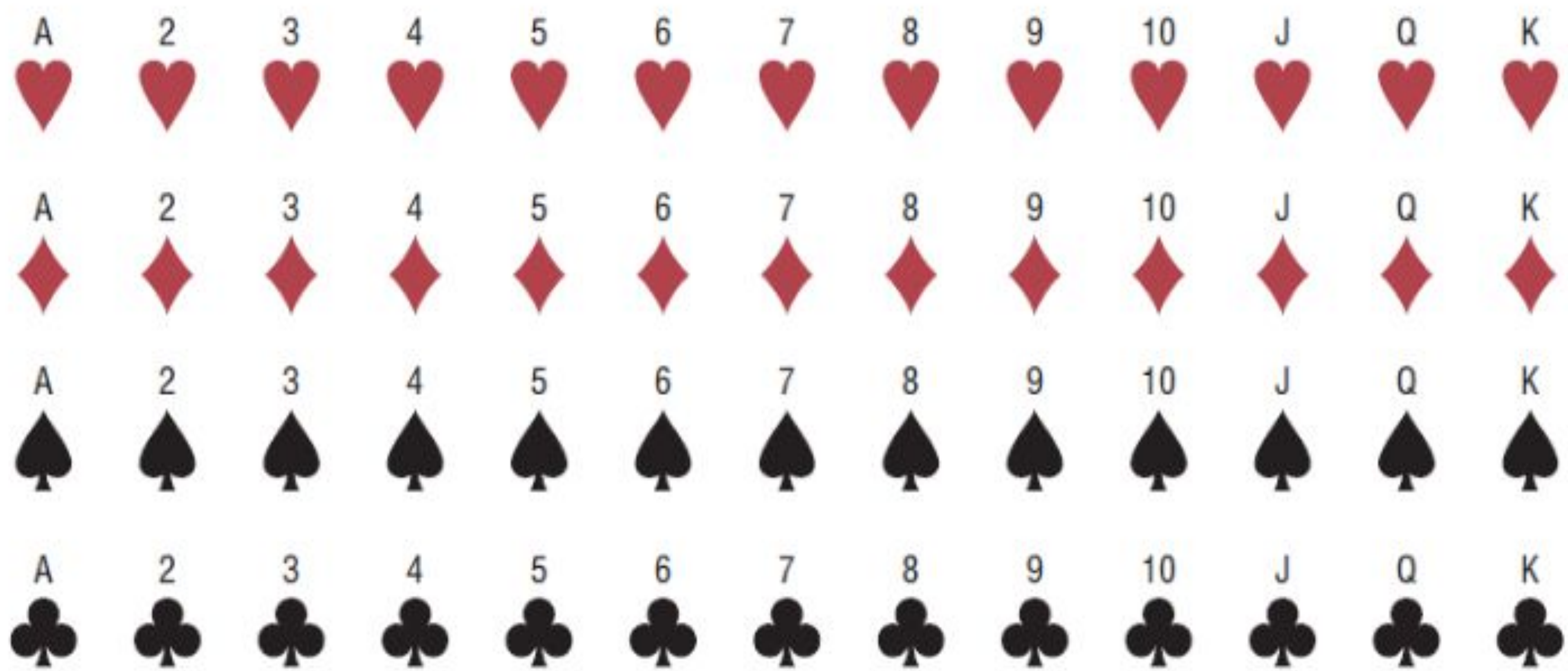
An **event** may contain one, many, all or none of the sample points in U.

- **Simple event** – an event with one outcome.
- **Compound event** – an event with more than one outcome.

<u>Example</u>: Roll a die and get a 6 (**simple event**).

<u>Example</u>: Roll a die and get an even number (**compound event**).

<u>Example</u>: Find the sample space for drawing one card from an ordinary deck of cards.



Sample space consists of all possible 13x4=52 outcomes:

A♥, 2♥,…,K♥,…, A♣, 2♣,…,K♣

The collection of all events is non-empty and satisfies the following:

1. If A is an event, so is A^c, the event that A doesn't happen.

2. If A and B are events, so is $A \cup B$, the event that (either or both) A or B happens.

3. If A and B are events, so is $A \cap B$, the event that A and B happen.

Example, let U be the sample space of all sequences of three coin tosses described above, and consider the following events:

$A = \{\text{HTT, HTH, HHT, HHH}\}$	The first flip was heads.
$A^c = \{\text{TTT, TTH, THT, THH}\}$	The first flip was not a head.
$B = \{\text{TTH, THH, HTH, HHH}\}$	The third coin flip is heads.
$A \cup B = \{\text{TTH, THH, HTT, HTH, HHT, HHH}\}$	The first or third flip was heads.
$A \cap B = \{\text{HTH, HHH}\}$	The first and third flip were heads.
$C = \{\text{THH, HTH, HHT, HHH}\}$	Their were more heads than tails.

Probability Rules

1. The Probability of an event E must be a number between 0 and 1. i.e., $0 \leq P(E) \leq 1$.

2. If an event E **cannot** occur, then its probability is **0**.

3. If an event E **must** occur, then its probability is **1**.

4. The sum of all probabilities of all the outcomes in the sample space is 1.

Example: Consider a standard deck of 52 cards:

Find the probability of selecting a queen

$$P(\text{queen}) = \frac{4}{52} = \frac{1}{13} = 0.077$$

Complementary Events

Complement of an event *E* - the set of outcomes in the sample space that are **not** included in the outcomes of event *E*. The complement of E is denoted by $\overline{E}$ ("E bar").

<u>Example</u>: What is the complement of the following events?

Rolling a six-sided die and getting a 4?

Complement = Rolling a die and getting 1 ,2, 3, 5 or 6.

Rolling a die and getting a multiple of 3?

Rule for Complementary Events:

$$P(\bar{E}) = 1 - P(E) \text{ or } P(E) = 1 - P(\bar{E}) \text{ or } P(E) + P(\bar{E}) = 1.$$

Example: The probability of purchasing a defective light bulb is 12%. What is the probability of not purchasing a defective light bulb?

P(not defective) = 1 – P(defective) = 1-0.12=0.88

Example: What is the probability of not selecting a club in a standard deck of 52 cards?

Mutually exclusive - Two events are mutually exclusive (disjoint) if they cannot occur at the same time.

Looking ahead: If we have mutually exclusive events, then their probabilities will add. Let's make sure we understand what it means for events to be mutually exclusive.

Example:

Which events are mutually exclusive and which are not, when a single die is rolled?

1. Getting an odd number and getting an even number

   Mutually exclusive! You can't have a roll be both.

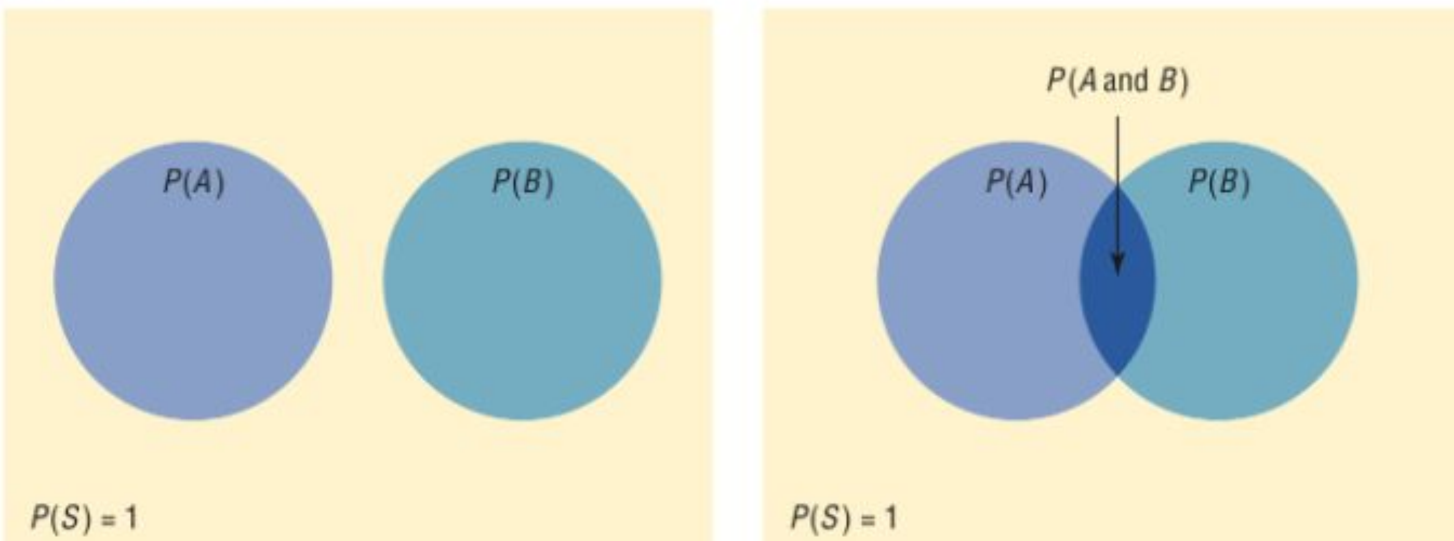
2. Getting a 3 and getting an odd number

3. Getting an odd number and getting a number less than 4

4. Getting a number greater than 4 and getting a number less than 4

Intersection – the intersection of events A and B are the outcomes that are in both A and B. If A and B have outcomes intersecting each other than we say that they are **non-mutually exclusive.**

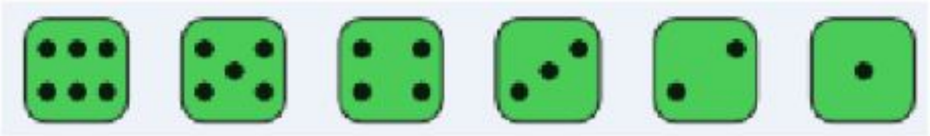
Union – the union of events A and B are all the outcomes that are in A, B, or both.



(a) Mutually exclusive events
$P(A \text{ or } B) = P(A) + P(B)$

(b) Nonmutually exclusive events
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

<u>Example</u>: Suppose we roll a six-sided die. Let A be that we roll an even number. Let B be that we roll a number greater than 3.



A A A

B B B

What is the intersection between A and B?

Rolling a 6 or 4

What is the union of A and B?

Rolling a 6, 5, 4, or 2

Addition Rules (These apply to "or" statements.)

Rule 1: If two events A and B are mutually exclusive, then:

$$P(A \text{ or } B) = P(A) + \mathrm{P}(B)$$

Rule 2: For **ANY** two outcomes A and B,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Note: In probability "A or B" denotes that A occurs, or B occurs, <u>or both occur</u>!

Example: At a political rally, there are 20 Republicans, 13 Democrats, and 6 Independents. If a person is selected at random, find the probability that he or she is either a Democrat or an Independent.

Event A = a person is a democrat
Event B = a person is an independent
These are mutually exclusive since you can NOT be both.

$$\begin{aligned}
P(\text{a person is a Democrat or an Independent}) &= P(A \text{ or } B) \\
&= P(A) + P(B) \\
&= \frac{13}{20+13+6} + \frac{6}{20+13+6} \\
&= \frac{13}{39} + \frac{6}{39} \\
&= \frac{19}{39} \approx 0.487
\end{aligned}$$

Example

- If you choose a number between 1 and 100, what is the probability that it is divisible by 2 or 5 or both?
- Let n be the number chosen
  - $p(2|n) = 50/100$ (all the even numbers)
  - $p(5|n) = 20/100$
  - $p(2|n)$ and $p(5|n) = p(10|n) = 10/100$
  - $p(2|n)$ or $p(5|n) = p(2|n) + p(5|n) - p(10|n)$

    $= 50/100 + 20/100 - 10/100$

    $= 3/5$

Independent - two events A and B are independent events if the fact that A occurs does not affect the probability of B occurring.

<u>Example</u>: Rolling one die and getting a six, rolling a second die and getting a three.

<u>Example</u>: Draw a card from a deck and replacing it, drawing a second card from the deck and getting a queen.

In each example, the first event has no effect on the probability of the second event.

Multiplication Rule for Independent Events

Multiplication Rule 1: When two events A and B are independent, then $P(A \text{ and } B) = P(A)P(B)$

That is, when events are independent, their probabilities multiply in an "and" statement.

Example: The New York state lottery uses balls numbered 0-9 circulating in 3 separation bins. To select the winning sequence, one ball is chosen at random from each bin. What is the probability that the sequence 9-1-1 would be the one selected?

$$\mathrm{P(Sequence\ 9-1-1)} = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1000} = 0.001$$

Actually, this is the same probability of any of the equally likely 1000 draws

Dependent - Two outcomes are said to be *dependent* if knowing that one of the outcomes has occurred affects the probability that the other occurs.

Examples:

- Drawing a card from a deck, not replacing it, and then drawing a second card.
- Being a lifeguard and getting a suntan
- Having high grades and getting a scholarship
- Parking in a no-parking zone and getting a ticket

The Conditional Probability of an event B in relationship to an event A is the probability that event B occurs after event A has already occurred.

- This probability is denoted as $P(B \mid A)$.

Multiplication Rule for Dependent Events

Multiplication Rule 2: When two events are dependent, the probability of both occurring is $P(A \text{ and } B) = P(A)P(B \mid A)$.

<u>Example</u>: What is the probability of getting an Ace on the first draw and a king on a second draw?

$$P(\text{Ace then King}) = P(\text{Ace})P(\text{King} \mid \text{Ace})$$

$$= \frac{4}{52} \times \frac{4}{51} \approx 0.006$$

First draw from full deck of 52 cards has 4 Aces

Second draw from a deck of 51 cards (which is missing a single Ace) has 4 Kings

Example: World Wide Insurance Company found that 53% of the residents of a city had homeowner's insurance (H) with the company. Of these clients, 27% also had automobile insurance (A) with the company. If a resident is selected at random, find the probability that the resident has both homeowner's insurance and automobile insurance with World Wide Insurance Company.

$$
\begin{aligned}
P(H \text{ and } A) &= P(H)P(A \mid H) \\
&= 0.53 \times 0.27 \\
&= 0.1431
\end{aligned}
$$

Formula for Conditional Probability

The probability that the second event B occurs given that the first event A has occurred can be found by dividing the probability that both events have occurred by the probability that the first event has occurred. For events A and B, the conditional probability of event B given A occurred is

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example: A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip and a white chip is 15/56, and the probability of selecting a black chip in the first draw is 3/8, find the probability of selecting the white chip on the second draw, given that the first chip selected was a black chip.

Want to compute: $P(\text{White chip on second draw}|\text{First chip was black})$

Know: $P(\text{Selecting black and white chip})=15/56$

$P(\text{Selecting black chip on first draw})=3/8$

Applying formula for conditional probability:

$$P(\text{White chip on second draw}|\text{First chip was black})$$

$$=\frac{P(\text{Selecting black and white chip})}{P(\text{First chip was black})}=\frac{15/56}{3/8}\approx 0.714$$

Example: A game is played by drawing 4 cards from an ordinary deck and replacing each card after it is drawn. Find the probability that at least 1 ace is drawn.

$$P(\text{at least 1 Ace}) = 1 - P(\text{no aces drawn})$$ ← Complementation

$$= 1 - \frac{48}{52} \times \frac{48}{52} \times \frac{48}{52} \times \frac{48}{52}$$ ← Multiplication Rule

$$= 1 - 0.726025$$

$$\approx 0.274$$ ← Note rounding to 3 decimal places.

Variable

A variable is a characteristic or attribute that can assume different values.

A random variable is a variable whose values are determined by chance.

Discrete variables are countable.

Example: Roll a die and let X represent the outcome

so $X = \{1,2,3,4,5,6\}$

Discrete probability distribution - the values a random variable can assume and the corresponding probabilities of the values.

- They **can be displayed by a graph or a table.**

Example: Create a probability distribution for the number of girls out of 3 children.

We previously used a tree diagram to construct the sample space which consisted of 8 possible outcomes:

BBB	*X=0*
BBG, BGB, GBB	*X=1*
BGG, GBG, GGB	*X=2*
GGG	*X=3*

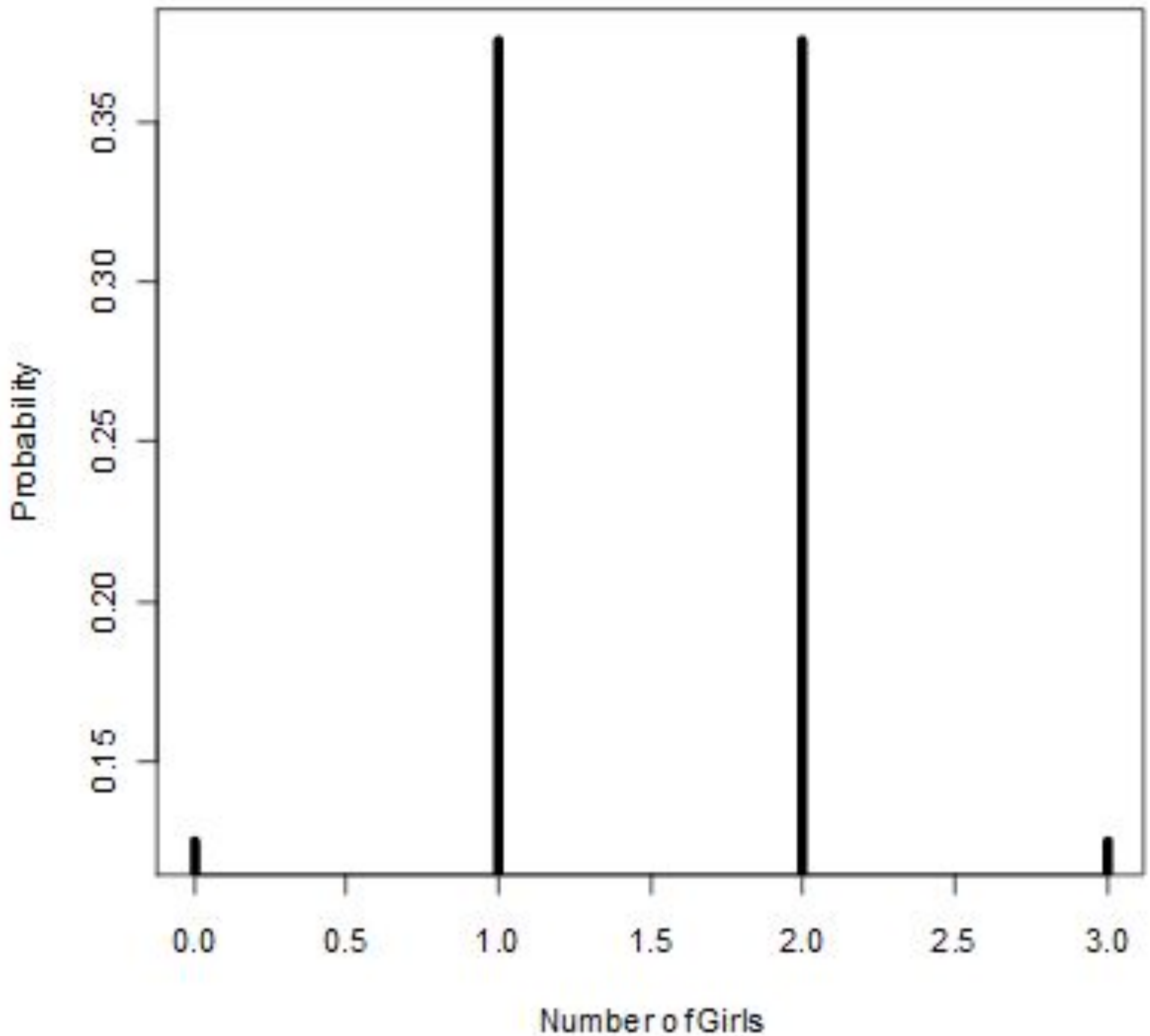
The corresponding (discrete) probability distribution is:

Number of Girls X	0	1	2	3
Probability $P(X)$	1/8	3/8	3/8	1/8

(2) Probability Distribution for number of tattoos each student has in a population of students

Tattoos	0	1	2	3	4
Probability	0.850	0.120	0.015	0.010	0.005

Graph the probability distribution above.



Two Requirements for a Probability Distribution

1. The sum of the probabilities of all the outcomes in the sample space must be 1; that is $\sum P(X) = 1$.

2. The probability of each outcome in the sample space must be between or equal to 0 and 1; that is $0 \leq P(X) \leq 1$.

Example: Determine whether each distribution is a probability distribution. Explain.

X	1	2	3	4
$P(X)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{9}{16}$

X	2	3	7
$P(X)$	0.5	0.3	0.4