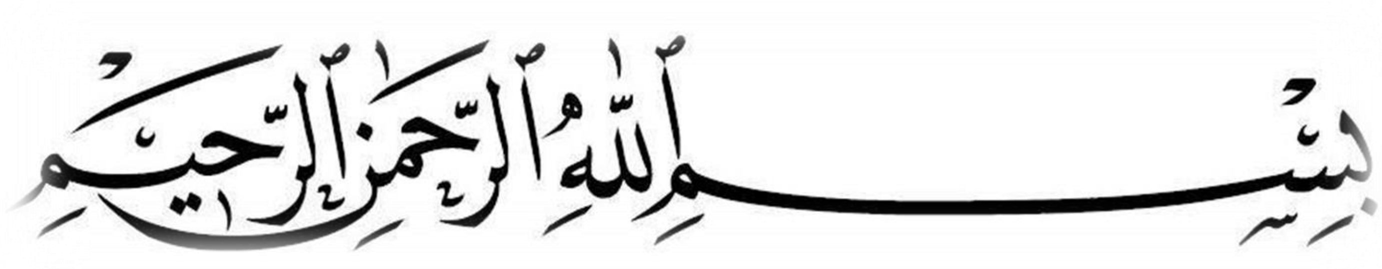
بسم الله الرحمن الرحيم

Course Name: Numerical Methods
Course Code: MTH-351
Enrollment Code: 039417862
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Instructions For This Course

1. Attendance will be marked in the first 5 minutes of the lecture. (so be on time)
2. Use of mobile phone is strictly prohibited.
3. Every student will perform class tasks.
4. Leaving the class before time will be marked absent.
5. Your questions related to the topic will be appreciated.

Course Outlines

In this course we will study

1. Error, types of error and its computation
2. Numerical methods for solving Algebraic and Transcendental equations
3. Interpolation for constructing polynomials
4. Solving system of linear equations by Numerical Methods
5. Numerical differentiation
6. Numerical Integration
7. Numerical solutions of ODEs and PDEs

Books Recommended

E. Kreyszing: Advanced Engineering Mathematics (9th Ed)

J. Douglas Faires, Richard Burden: Numerical Methods (3rd Ed)

Vidamurthy, Numerical Methods

Shanker Rao, Numerical Methods

Marks Distribution

Total Marks: 100

6-Quizzes Marks: 10

3-Assignment Marks: 10

2-OHT's Marks: 30

ESE Marks: 50

Mathematical Methods

All the mathematical problems which can be solve through mathematical tools (arithmetic's) are called mathematical methods.

1. Analytical Methods
2. Numerical Methods

Analytical Methods: These are normal of conventional ways to solve mathematical problem algebraically by analysis (thinking).

Numerical Methods: Some times, some mathematical problem can not be solved analytically or can be solved but are very difficult, tedious and time consuming. Facing such situations we goes toward numerical methods. In numerical methods we feed input and get output.

Errors and Their Computations

Solving any mathematical problem analytically yields very exact solution and there is no error but some time when we solve the same problem numerically (by any numerical method), the obtained solution is accurate and precise but not exact.

Example: Solve $x^2 + 4x + 4$=0.
Then $x = -2$ is the solution of this equation by analytical method and it is exact solution because it satisfy the given equation. Now if we solve the same problem by numerical method we will get $x = -1.9999$ or $x = -2.0001$. So these solutions are near to 2 but not exactly 2. **Thus demerits of numerical methods are that it involves errors.**

Why we use numerical methods if it has errors

Real word problems yields very lengthy and non-linear equations (during mathematical modeling) which can not be tackled analytically, so we goes toward **numerical methods** which gives guess towards the exact solutions.

Error Analysis

(i) Inherent Errors.
(ii) Round-off Errors.
(iii) Truncation Errors.
(iv) Absolute Errors.
(v) Relative Errors.
(vi) Percentage Errors.

Accuracy of numbers:

We have two types of numbers.

(i) Exact numbers (ii) Approximate numbers.

Exact numbers: The numbers which are Integers or terminated decimal numbers.

e.g: $8, 1, \frac{1}{2}, \frac{3}{4}, 9.75$ etc.

Approximate numbers: The numbers which can't terminate after decimal point or numbers don't having finite ~~number~~ decimal expansions.

Numerical Methods

eg: $\frac{1}{3} = 0.333\ldots\ldots$, $\pi = 3.141592\ldots\ldots$

Thus these numbers are approximated to some finite number of digits called as significant digits for the purpose of calculations.

eg: $\frac{1}{3} \approx 0.33$ (two significant digits)

$\pi \approx 3.1415$ (5. ~~dig~~ significant digits).

Errors in Numerical Computations

Inherent Errors: This Error is due to approximate data and this error can be reduced by taking better data.

Round-off Errors: These errors occurs due to rounding of numbers during process of computations. These Errors can be reduced by taking more significant digits in computations.

example: let we have have a number (floating number or decimal number)

7.5846712.

Numerical Methods

rounding this number ~~to~~ upto 3-decimal point

then 7·585

Round-off error = 7·5846712 − 7585 = 0·000329

rounding the same number upto 4-decimal point

7·5847

Round-off error = 7·5846712 − 7·5847 = 0·000029

* **Truncation Errors**:- These errors araises due to approximate formulla in computations or by truncating the infinite series to some approximate finite terms.

Numerical Methods

eg. $y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + + \cdots$

$y(0.1) \approx$

$\cancel{y(x) \approx 1 + x + \frac{x^2}{2!}}$

$(0.1)^3, (0.1)^4$ and all other powers are neglected $(0.1)^4 \approx 0$

Thus the error will occur in computations and such error is Truncation Error.

Absolute Errors: If x is the true value of a quantity and $\bar{x}$ is its approximate value then:

$$E_A = |x - \bar{x}|$$

Relative Errors: $$E_R = \frac{E_A}{|x|} = \frac{|x-\bar{x}|}{|x|} = \left|\frac{x-\bar{x}}{x}\right|$$

Numerical Methods

Percent Errors: $E_P = E_R \times 100 = \frac{E_A}{|X|} \times 100.$

Relative Errors are used for Comparison b/w two Errors.

eg: $X = 10$, $\bar{X} = 5$ $\quad E_R = \bar{X} - \bar{X} = 5$

$E_R = \frac{|X - \bar{X}|}{|X|} = \frac{5}{10} = \frac{1}{2} = 0.5$

$X = 50$, $\bar{X} = 45$, $E_A = 50 - 45 = 5$

$E_R = \frac{5}{45} = 0.1111\ldots$

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Errors in Numerical solutions

1. **Round-off errors**
2. **Truncation errors**

Round-off errors: These errors occur due to finite number of bits or due to short storage in computers (machines) for storing of floating point numbers (decimal numbers). Due to which these number are either chopped or rounded after so many significant digits (digits in decimal part are chopped or rounded).

Truncation errors: These errors occur due to fixing number of term of an infinite series or reducing terms of finite series in order to approximate it to a certain function.

Example: Let $S = x + x^2 + x^3 + x^4 + \cdots$(infinite series).
If we neglect x^3 and other higher power of x, then $f(x) = x + x^2$. This function is approximation for infinite series given above. Now under certain value of x this function will be differ from the infinite series. This difference is called truncation error.

Quantifying errors

Error

Exact value (True value)-Approximate value

$$E = X - \bar{X}$$

Here X represents exact value and $\bar{X}$ represents approximate value.

Absolute Errors:

Absolute error is the numerical difference between the true value of a quantity and its approximate value. Thus, if X is the true value of a quantity and X_1 is its approximated value, then the absolute error E_A is given by

$$E_A = |X - X_1|.$$

Relative Error:

The relative error is defined by

$$E_R = \frac{|X - X_1|}{X} = \frac{E_A}{X}$$

Percentage Error:

The percentage error is defined by

$$E_P = \frac{|X - X_1|}{X} \times 100 = 100E_R$$

Errors and Their Computations

There are two types of numbers, exact numbers and approximate numbers. Examples of exact numbers are 1, 2, 3, ..., 1/4, 2/3, π, etc. Approximate numbers are those that represent the numbers a certain degree of accuracy. Thus, an approximate value of π is 3.1416.

Significant Digits:

The digits that are used to express a number are called significant digits or significant figures. Thus, the numbers 3.1416, 0.66667 and 4.0687 contain five significant digits each. The number 0.00023 has, however, only two significant digits, 2 and 3, since the zeros serve only to fix the position of the decimal point.

- The digits 1, ..., 9 are always called significant digits.
- If 0 appears between any two significant digits, then 0 is also called significant digit.
- If 0 appears on the rightest sides of the significant digits, then 0 is called a significant digit.

Errors and Their Computations

Rounding or rounding-off a number to n significant digits:

To round-off a number to n significant digits, then discard (drop) all digits to the right of the nth digit. The nth digit and digits on its left are called retaining digits (RD) while digits on right of nth digit are called dropping digits (DD). The nth significant digit is call last retaining digit (LRD) while the digit to very right of nth LRD is called first dropping digit (FDR). Thus during rounding you have to focus on LRD and FDD. According to FDD we have the following cases

1. If FDD$>$5 then LRD+1.
2. If FDD$<$ 5 then LRD+0.
3. If FDD=5 and LRD is even then LRD+0.
4. If FDD=5 and is followed by a non-zero digit (at least) or zero digits then make LRD even.

The number thus round-off is said to be correct to n significant figures.

Errors and Their Computations

Example: Round-off the following to first decimal place

1. $79.7213 = \mathbf{79.7}$
2. $79.7813 = 79.8$
3. $79.7513 = 79.8$
4. $79.7500 = 79.8$
5. $79.6513 = 79.6$

Rounding –off error due to chopping a number to n significant digits:

To chop a number to n significant digits, then discard (drop) all digits to the right of the nth digit.

Example: Approximate following numbers to first decimal place using chopping

1. $79.7213 = \mathbf{79.7}$
2. $79.7813 = 79.7$
3. $79.7513 = 79.7$
4. $79.7500 = 79.7$
5. $79.6513 = 79.6$

Errors and Their Computations

Example:

An approximate value of π is given by $X_1 = 3.1428571$ and its true value is X=3.1415926. Find the absolute and Relative errors.

We have

$$E_A = |X - X_1| = |-0.0012645| = 0.0012645$$

$$E_R = \frac{|X - X_1|}{X} = \frac{|-0.0012645|}{3.1415926} = 0.000402$$

Example 1 Determine the five-digit **(a)** chopping and **(b)** rounding values of the irrational number π.

Solution The number π has an infinite decimal expansion of the form $\pi = 3.14159265\ldots$

Errors and Their Computations

Example 1.14 *If $X = 2.536$, find the absolute error and relative error when*

(i) X is rounded and

(ii) X is truncated to two decimal digits.

Errors and Their Computations

Solution

(*i*) Here $X = 2.536$

Rounded-off value of X is $x = 2.54$

The Absolute Error in X is

$$E_A = |2.536 - 2.54|$$

$$= |-0.004| = 0.004$$

$$\text{Relative Error} = E_R = \frac{0.004}{2.536} = 0.0015772$$

$$= 1.5772 \times 10^{-3}.$$

(*ii*) Truncated Value of X is $x = 2.53$

Absolute Error $E_A = |2.536 - 2.53| = |0.006| = 0.006$

$$\therefore \quad \text{Relative Error} = E_R = \frac{E_A}{X} = \frac{0.006}{2.536} = 0.0023659$$

$$= 2.3659 \times 10^{-3}.$$

Errors and Their Computations

Example 1.15 *If* $\pi = \frac{22}{7}$ *is approximated as 3.14, find the absolute error, relative error and relative percentage error.*

Errors and Their Computations

Solution Absolute Error $= E_A = \left|\frac{22}{7} - 3.14\right| = \left|\frac{22 - 21.98}{7}\right|$

$$= \left|\frac{0.02}{7}\right| = 0.002857.$$

$$\text{Relative Error } E_R = \left|\frac{0.002857}{22/7}\right| = 0.0009$$

$$\text{Relative Percentage Error } E_P = E_R \times 100 = 0.0009 \times 100$$

$$= 0.09$$

$$\therefore \quad E_P = 0.09\%.$$

Example 1.16 *The number $x = 37.46235$ is rounded off to four significant figures. Compute the absolute error, relative error and the percentage error.*

Errors and Their Computations

Solution We have $X = 37.46235; \quad x = 37.46000$

$$\text{Absolute error} = |X - x| = |37.46235 - 37.46000|$$

$$E_A = 0.00235$$

$$E_r = \left|\frac{X - x}{x}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E_P = E_r \times 100 = 6.27 \times 10^{-3}$$

Example 1.12 *If* $\Delta x = 0.005$, $\Delta y = 0.001$ *be the absolute errors in* $x = 2.11$ *and* $y = 4.15$, *find the relative error in the computation of* $x + y$.

Solution $x = 2.11,\ y = 4.15$

$\therefore$ $x + y = 2.11 + 4.15 = 6.26,$

and $\Delta x = 0.005,\ \Delta y = 0.001$

$\Rightarrow\ \Delta x + \Delta y = 0.005 + 0.001 = 0.006.$

$\therefore$ The relative error in $(x + y)$ is

$$E_R = \frac{\Delta x + \Delta y}{(x + y)} = \frac{0.006}{6.26}$$

$$= 0.000958.$$

The relative error in $(x + y) = 0.001$ (approximately).

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Example 1.13 Given that $u = \dfrac{5xy^2}{z^3}$ $\Delta x, \Delta y$ and Δz denote the errors in x, y and z respectively such that $x = y = z = 1$ and $\Delta x = \Delta y = \Delta z = 0.001$, find the relative maximum error in u.

Errors and Their Computations

Solution We have

$$\frac{\partial u}{\partial x} = \frac{5y^2}{z^3}, \frac{\partial u}{\partial y} = \frac{10xy}{z^3}, \frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$$

$$\therefore \ \Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$\Rightarrow \ (\Delta u)_{\max} = \left|\frac{\partial u}{\partial x}\Delta x\right| + \left|\frac{\partial u}{\partial y}\Delta y\right| + \left|\frac{\partial u}{\partial z}\Delta z\right|$$

$$= \left|\frac{5y^2}{z^3}\Delta x\right| + \left|\frac{10xy}{z^3}\Delta y\right| + \left|\frac{-15xy^2}{z^4}\Delta z\right|$$

Substituting the given values in (1) and using the formula to find the relative maximum error we get

$$(E_R)_{\max} = \frac{(\Delta u)_{\max}}{u} = \frac{0.03}{5} = 0.006.$$

Errors and Their Computations

1. Round-off the following numbers to two decimal places.

   (*a*) 52.275 (*b*) 2.375 (*c*) 2.385 (*d*) 81.255 (*e*) 2.375

2. Round-off the following numbers to three decimal places.

   (*a*) 0.4699 (*b*) 1.0532 (*c*) 0.0004555 (*d*) 0.0028561 (**e**) 0.0015

3. Round-off the following numbers to four decimal places.

   (*a*) 0.235082 (*b*) 0.0022218 (*c*) 4.50089 (*d*) 2.36425 (*e*) 1.3456

5. Find the relative error in computation of

   $x - y$ for $x = 12.05$ and $y = 8.02$ having absolute errors $\Delta x = 0.005$ and $\Delta y = 0.001$.

6. Find the relative error in computation of $x - y$ for $x = 9.05$ and $y = 6.56$ having absolute errors $\Delta x = 0.001$ and $\Delta y = 0.003$ respectively.

7. Find the relative error in computation of $x + y$ for $x = 11.75$ and $y = 7.23$ having absolute errors $\Delta x = 0.002$ and $\Delta y = 0.005$.

8. If $y = 4x^6 - 5x$, find the percentage error in y at $x = 1$, if the error is $x = 0.04$.

9. If $\frac{5}{6}$ be represented approximately by 0.8333, find (a) relative error and (b) percentage error.

10. If $f(x) = 4 \cos x - 6x$, find the relative percentage error in $f(x)$ for $x = 0$ if the error in $x = 0.005$.

11. Find the relative percentage error in the approximate representation of $\frac{4}{3}$ by 1.33.

15. If $\frac{2}{3}$ is approximated to 0.6667. Find

(*a*) absolute error

(*b*) relative error and

(*c*) percentage error

16. Given $X = 66.888$. If x is rounded to 66.89 find the absolute error.

17. If $\frac{1}{3}$ is approximated by 0.333 find

(*a*) absolute error

(*b*) relative error and

(*c*) relative percentage error

18. If $u = \frac{5xy^2}{z^3}$ and error in x, y, z be 0.001, 0.002, and 0.003, compute the relative error in u. Where $x = y = z = 1$.

19. If the true value of a number is 2.546282 and 2.5463 is its approximate value; find the absolute error, relative error and the percentage error in the number.

Errors and Their Computations

22. If u = $4x^2y^3/z^4$ and errors in x, y, z be 0.001 compute the relative maximum error in u when $x = y = z = 1$.

23. If x = 865 250 is rounded off to four significant figures compute the absolute error, relative error and the percentage error in x.

26. If $u = 10x^3\ y^2\ z^2$ and errors in x, y, z are 0.03, 0.01, 0.02 respectively at $x = 3$, $y = 1$, $z = 2$. Calculate the asolute error and relative error and percentage erro in u.

27. If the number $X = 3.1416$ is correct to 4 decimal places; then find the error in X.

28. If $u = \dfrac{5xy^2}{z^2}$ and $\Delta x = \Delta y = \Delta z = 0.1$, compute the maximum relative error in u where $x = y = z = 1$.