Session 7: Binomial option pricing

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Fall, 2025

Overview

- We will now attempt to derive the fair price to pay for an option before it reaches expiry.
- The approach will be to assume a simplified movement for the underlying asset, in that at each point in time it can move to one of two values. This is the binomial model.
- The binomial approach is very useful in understanding some key concepts of option pricing such as risk-neutrality and replicating portfolios. It is also practically useful for pricing complex derivatives, especially when there are early exercise features (e.g. American options)

Simple problem

- Consider a simplified example, where we are attempting to calculate the value of some common stock with the following known payoffs (and associated probabilities) after one year:

Ending Stock Value ($)	Probability
150	0.10
130	0.20
110	0.40
90	0.20
70	0.10

- If the stock's required return is 10% (and no dividend payout) then the best estimate for today's value is:

$$(150 \times 0.1 + 130 \times 0.2 + 110 \times 0.4 + 90 \times 0.2 + 70 \times 0.1)/1.1 = \$100$$

Option example

- Imagine that we can perform a similar analysis for an option where the strike price is \$100. We have the following:

Ending Stock Value (\$)	Call Option value (\$)	Probability
150	50	0.10
130	30	0.20
110	10	0.40
90	0	0.20
70	0	0.10

- So,

$$E[C_T] = 50 \times 0.1 + 30 \times 0.2 + 10 \times 0.4 = \$15.$$

What is the price today??

Quiz

What do you imagine the appropriate discount rate to be?

- Higher than the one for the underlying
- Lower than the one for the underlying
- The same
- The risk-free rate
- It will depend upon the scenario

Quiz

What do you imagine the appropriate discount rate to be?

- Higher than the one for the underlying
- Lower than the one for the underlying
- The same
- The risk-free rate
- It will depend upon the scenario CORRECT

Problem

- It is not clear what the correct discount rate is for a call option. Clearly the option feels like it is more risky than the underlying asset - note the difference in the leverage of the two payoffs.
- Early (pre-1973) attempts at option pricing attempted to theoretically deduce the correct discount rate for options.
- However, the problem is different from the usual discounting problem as the risk will be different depending upon time to expiry as well as the current asset price value.
- We will start by trying to model the probability of payoffs, using the simplest possible distribution - the binomial with one time step (trial).

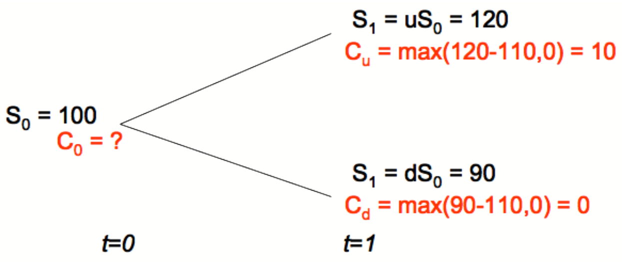
Introduction to binomial pricing

- Consider a hypothetical example, the current stock price $S_0 = \$100$ after one year the stock has either increased in value to $S_1 = \$120$ or has decreased in value to $S_1 = \$90$.
- We typically denote u to be the "up factor" which in this case is 1.2 and d to be the "down factor" , 0.9 in this case.
- If we also have a one-year call option on a non-dividend paying stock with an exercise price of \$110 and in the up-(down-)state we define the value of the call by $C_u(C_d)$ then we have the following:

$$\begin{aligned} C_u &= \max(120 - 110, 0) = 10 \\ C_d &= \max(90 - 110, 0) = 0 \end{aligned}$$

- The following diagram summarizes this information:

Example



$S_1 = uS_0 = 120$
$C_u = \max(120-110,0) = 10$
$S_0 = 100$
$C_0 = ?$
$S_1 = dS_0 = 90$
$C_d = \max(90-110,0) = 0$
$t=0$
$t=1$

No-arbitrage

- We will now use no arbitrage arguments to deduce the value of the call option at time $t = 0$.
- To do this we will construct a "replicating portfolio" which will have the same value as the call option in both possible states at time $t = 1$. As it will have the same value at $t = 1$ then it must also have the same value at $t = 0$ and so we will be able to price the option.
- The portfolio, Π, consists of being long an amount Δ of the underlying asset S and investing a \$ amount B at the risk-free rate.
- So,

$$\Pi = \Delta S + B$$

No-arbitrage

- Thus, for the relevant time period one needs to determine values of both Δ and B such that the portfolio is equal to the option value in both scenarios. In our example we have:

$$\begin{aligned} 120\Delta + 1.01B &= \$10 \\ 90\Delta + 1.01B &= \$0 \end{aligned}$$

- This assumes that the one-year risk-free rate is 1% and that your risk-free investment has grown by this amount.
- Solving these equations give:

$$\begin{aligned} \Delta &= 1/3 \\ B &= -\$29.70 \end{aligned}$$

so we should buy one third of a share and borrow \$29.70 at the risk-free rate to replicate the value of the option.

Replicating portfolio

- To verify this we see:

$$\begin{aligned}120 \times \frac{1}{3} - \$29.70 \times 1.01 &= \$10 \\ 90 \times \frac{1}{3} - \$29.70 \times 1.01 &= \$0\end{aligned}$$

- We can use this to value the option today ($t = 0$) as by the no-arbitrage principle it must have the same value as the replicating portfolio at $t = 0$, thus:

$$C_0 = \Delta S_0 + B = \$100 \times \frac{1}{3} - \$29.70 = \$3.63$$

Arrow–Debreu Securities

- **Arrow–Debreu (AD) securities:**
  - A_u: pays \$1 in the **up** state only; \$0 otherwise.
  - A_d: pays \$1 in the **down** state only; \$0 otherwise.
- Prices of AD securities are called **state prices**: $q_u =$ price of A_u, $q_d =$ price of A_d.

$$\boxed{\text{Price at } (t = 0) = q_u \times (\text{up payoff}) + q_d \times (\text{down payoff})}$$

Computing AD Prices for This Example

- Use two assets to *identify* q_u, q_d:
  - **Risk-free \$1 at** $t = 1$ has price $\frac{1}{1.01}$ today $\Rightarrow q_u + q_d = \frac{1}{1.01}$.
  - **Stock** price today must equal state-price weighted payoff: $S_0 = 120\, q_u + 90\, q_d$.
- Solve:

$$q_u + q_d = \frac{1}{1.01} \approx 0.990099,$$
$$120\, q_u + 90\, q_d = 100 \quad \Rightarrow \quad \boxed{q_u \approx 0.3630,\ \ q_d \approx 0.6271}.$$

- Check: $q_u + q_d \approx 0.990099 = \frac{1}{1.01}$.

Pricing a Put with AD Securities ($K = 110$)

- **Payoffs at $t = 1$:** $P_u = \max(110 - 120, 0) = 0$, $P_d = \max(110 - 90, 0) = 20$.
- **Put price at $t = 0$:**

$$P_0 \;=\; q_u P_u + q_d P_d \;=\; (0.3630) \cdot 0 + (0.6271) \cdot 20 \;=\; \boxed{\$12.54}.$$

- *Sanity check (put–call parity):* $C_0 - P_0 = S_0 - \frac{K}{1.01} \Rightarrow P_0 = C_0 + \frac{K}{1.01} - S_0$. With $C_0 = \$3.63$, $K = 110$: $P_0 \approx 3.63 + \frac{110}{1.01} - 100 = \$12.54$ (matches).

Quiz

How did the probability of up and down moves affect the outcome?

- It is implied that there is a 50-50 chance of each outcome.
- They are linked with u and d.
- We don't need them to determine the option price.

Quiz

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- It is implied that there is a 50-50 chance of each outcome.
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Notes

- Notice that in this calculation it was not necessary to define the probability of each of the outcomes.
- It was also not clear how u and d were formulated. However, given that we will develop a rule for u and d it is striking that we do not need to know the probabilities.
- This replicating portfolio arguments works well here as we need two securities to replicate the two possible outcomes at the end of the period.
- Before moving on to more realistic multi-period trees we will define the general one period model.

Notation

- Assume that the continuously compounded risk-free rate of interest is r, and that the up-factor is u and the down factor is d.
- Additionally, assume that $d < e^{rt} < u$.
- The current asset price is S_0 and the value of any option (call or put) at $t = 0$ is now denoted by V_0.
- The value of the option in the up-state is V_u and in the down state it is V_d.
- The asset tree (lattice) is given next:

Quiz

What would happen if $d < e^{rt} < u$ was not true?

- We cannot calculate Δ and B.
- Interest rates are negative.
- Arbitrage opportunities.

Quiz

What would happen if $d < e^{rt} < u$ was not true?

- We cannot calculate Δ and B.
- Interest rates are negative.
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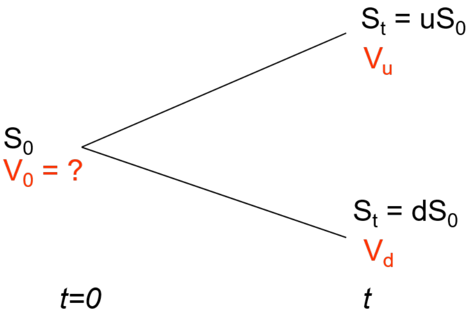
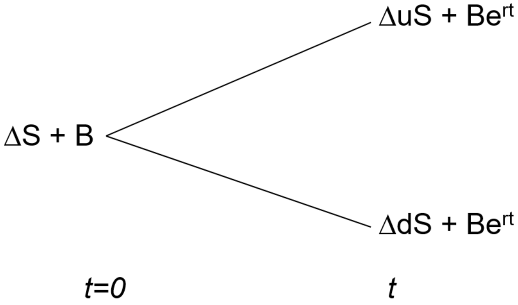
Set-up



Replication



Thus for the replication we need:

$$
\begin{aligned}
\Delta u S_0 + B e^{rt} &= V_u \\
\Delta d S_0 + B e^{rt} &= V_d
\end{aligned}
$$

Option pricing

- Rearranging these equations gives:

$$\Delta = \frac{V_u - V_d}{uS_0 - dS_0}$$
$$B = \frac{uV_d - dV_u}{(u-d)e^{rt}}$$

- Therefore, we can write the current value of the option as

$$\begin{aligned} V_0 &= \Delta S_0 + B = S_0 \frac{V_u - V_d}{uS_0 - dS_0} + \frac{uV_d - dV_u}{(u-d)e^{rt}} \\ &= e^{-rt}(\pi V_u + (1-\pi)V_d) \end{aligned}$$

where $\pi = \frac{e^{rt}-d}{u-d}$.

Quiz

How could we interpret Δ?

- Rate of change of the option price relative to time
- Rate of change of the option price relative to B
- Rate of change of the option price relative to S
- It depends

Quiz

How could we interpret Δ?

- Rate of change of the option price relative to time
- Rate of change of the option price relative to B
- Rate of change of the option price relative to S CORRECT
- It depends

Quiz

How could we interpret π?

- A discount factor.
- A forward price.
- A probability.
- A bond price.

Quiz

How could we interpret π?

- A discount factor.
- A forward price.
- A probability. CORRECT
- A bond price.

Interpreting π

- The expression for the option value, V, looks very similar to the expression for an expectation where π can be interpreted as the probability for V being worth V_u at expiry.
- Note also that as long as $d < e^{rt} < u$ then $0 \leq \pi \leq 1$.
- How can we interpret the value?
- Notice that if we apply this probability to the movement of S then we would have:

$$\begin{aligned}
\pi uS + (1-\pi)dS &= S\left(u\frac{e^{rt}-d}{u-d} + d\frac{u-e^{rt}}{u-d}\right) \\
&= S\left(\frac{ue^{rt}-ud}{u-d} + d\frac{ud-de^{rt}}{u-d}\right) \\
&= S\left(\frac{ue^{rt}-de^{rt}}{u-d}\right) \\
&= S_0e^{rt}
\end{aligned}$$

Interpreting π

- Thus if we interpret πas a real probability then it would say that the expected return on this risky security is the risk-free rate. This would mean that, if these were the true probabilities, investors were risk-neutral (they are neither risk-averse or risk-neutral but judge investments entirely on the expected value at expiry.

Quiz

Using our formula for the option price, if π is a probability then what is the expected return of the option

- Less than the risk-free rate
- More than the risk-free rate
- 0
- The risk-free rate

Quiz

Using our formula for the option price, if π is a probability then what is the expected return of the option

- Less than the risk-free rate
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- 0
- The risk-free rate CORRECT

Interpreting the risk neutral probabilities

- In our example we argued that call price was $3.63 using a replication argument:
  - All investors would agree on the price of $3.63, regardless of their risk aversion and beliefs.
  - Thus, given the stock price investors' risk aversion and beliefs are irrelevant.
  - Thus, we can assume whatever beliefs are convenient.
- It is convenient to assume investors are risk-neutral as then the expected return on option is the riskless rate and its price is the expected cash flow, discounted at riskless rate.

Interpreting the risk neutral probabilities

- So, if investors are risk-neutral the expected return on option is equal to riskless rate and so the price is the expected cash flow, discounted at riskless rate.
- That is, if investors are risk-neutral:

$$
\begin{aligned}
\$3.63 &= \frac{0.3667(10) + 0.6333(0)}{1.01} \\
&= \frac{\text{expected payoff}}{1.01}
\end{aligned}
$$

- In general we can view the option price, V_0 as

$$V_0 = \frac{E^{RN}[Payoff]}{1 + r_f} = \frac{E[payoff]}{1 + r_{true}}$$

where RN denotes a risk-neutral expectation, r_f is the risk-free rate and r_{true} is the real, risky, discount rate for the option.

Interpreting the risk neutral probabilities

- Risk-neutral probabilities can also be applied to stock price valuation. For a stock we know the risky discount rate (r_{true}) can be calculated from the CAPM or a 3/4 factor model.
- So, in these case we can view the current stock price, S_0 as

$$S_0 = \frac{E^{RN}[S_T]}{1 + r_f} = \frac{E[S_T]}{1 + r_{true}}$$

  where, again, RN denotes a risk-neutral expectation, r_f is the risk-free rate and r_{true} is the expected return/discount rate calculated from the CAPM or a 3/4 factor model.
- Generally for stocks, we know the risky discount rate and NOT the risk-neutral probabilities. However, for option pricing, the risk-neutral probabilities are easy to calculate and the risky discount rate is not. So, we use risk-neutral pricing.

Arbitrage

- We have generally referred to our argument as a no-arbitrage one but how would we exploit a mis-priced option in this one period world. Imagine in our previous example that the actual market price of the call option was \$4 (theoretical value = \$3.63).
- In this case as the value is too high you want to buy the replicating portfolio and short the call (buy low sell high).
- This is typically referred to as a "delta-neutral" hedge with reference to the number of shares you need to go long in.
- The next table shows one way of exploiting the arbitrage.

Arbitrage

	$t = 0$	$S_T = 120$	$S_T = 90$
Sell option	\$4	-\$10	\$0
Buy Δ shares	-\$33.33	\$40	\$30
Borrow shortfall	+\$29.33	-\$29.33(1.01)	-\$29.33(1.01)
Net	\$0	\$0.37	\$0.37

Quiz

What is your strategy if the option price is too low?

- Buy the call, short the portfolio
- Buy the call, long the portfolio
- Short call, long the portfolio
- Short call, short the portfolio
- There is no arbitrage

Quiz

What is your strategy if the option price is too low?

- Buy the call, short the portfolio CORRECT
- Buy the call, long the portfolio
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- Short call, short the portfolio
- There is no arbitrage

Other arbitrage

- So here the arbitrage is clearly exploitable, what happens if the option price is lower than the value obtained from the binomial model? In this example the call is trading for \$3.
- In this case we just reverse our above strategy, this time buying the call and shorting the replicating portfolio.
- The payoff is shown in the following table:

Arbitrage

	$t = 0$	$S_T = 120$	$S_T = 90$
Buy option	-\$3	\$10	\$0
Short Δ shares	+\$33.33	-\$40	-\$30
Invest difference	-\$30.33	+\$30.33(1.01)	+\$30.33(1.01)
Net	\$0	\$0.64	\$0.64

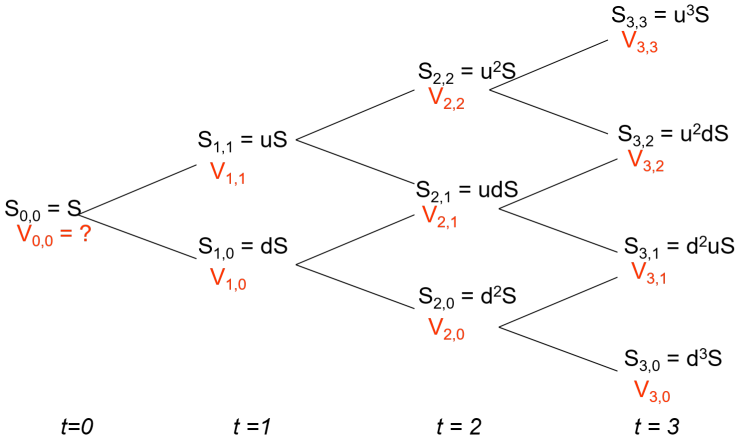
Replicating portfolio: puts

- If we have a put option rather than a call option then the above arguments still hold but we will see some slight differences in the strategies.
- For call options $V_u \geq V_d$ and so $\Delta \geq 0, B \leq 0$ and so to replicate the option we typically go long the underlying and fund this by borrowing money.
- For a put option $V_u \leq V_d$ and so $\Delta \leq 0, B \geq 0$ and so to replicate the option in this case we typically short the underlying and invest the money at the risk free rate.
- This will also reverse the above arbitrage strategies in the case of put options.

European put option prices

- Clearly, this simple one-step model will not be a decent approximation to reality, however, the same replicating portfolio argument will also apply for more than one time-step to expiry.
- Again assume that $u = 1.2$ and $d = 0.9$, only this time we consider a three year European call option, where $S_0 = S$.
- Thus at the end of the first year there are two possible values for the asset, Su and Sd, whereas at the end of the second year the value can take one of three values: Su^2, $Sud = Sdu$ and Sd^2 and four values after three years.
- Note that by keeping u and d constant across time periods we have ensured that the tree "recombines" i.e $ud = du$.
- The next diagram depicts this tree:

Three-step tree



Three-step tree

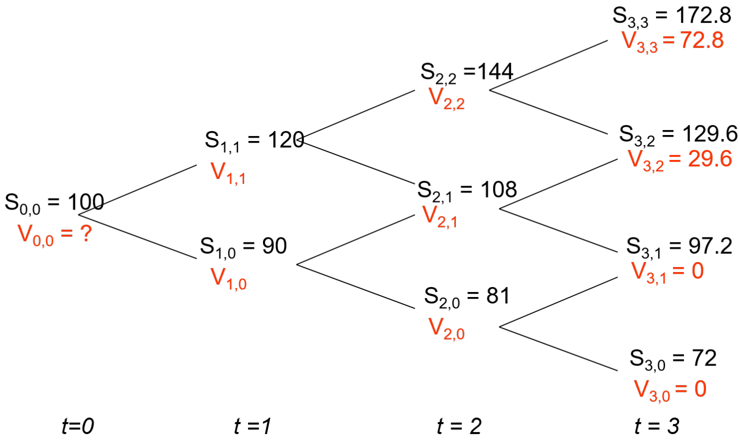
- Assume that $e^r = 1.01$ and that in this example the exercise price is $K = \$100$.
- We can put some numbers on some of these values as we know the option price at expiry ($t = 3$) as well as the values of u and d.
- Thus:

$$\begin{aligned} u^3S &= 1.2^3 \times 100 = 172.80 \\ u^2dS &= 129.6 \\ ud^2S &= 97.2 \\ d^3S &= 72.9\ldots \end{aligned}$$

- Also

$$\begin{aligned} V_{3,3} &= 72.8 \\ V_{3,2} &= 29.6 \\ V_{3,1} &= V_{3,0} = 0 \end{aligned}$$

Three-step tree



Valuation

- Valuation in a multi-step tree occurs backwards. We know the values at expiry and we then use these to value the option at the three possible states at $t = 2$. This size of the time step $\Delta T = T/N$ where N is the number of steps.
- At each of the values of S there will be a different Δ, B and V although their calculation will follow the same logic as that for the one period case.
- For example at $t = 2$ when $S_2 = 108$ then we need to find Δ and B such that

$$\begin{aligned} 129.6\Delta + 1.01B &= \$29.60 \\ 97.20\Delta + 1.01B &= \$0 \end{aligned}$$

which gives $\Delta = 0.914$ and $B = -\$87.921$.

Valuation

- If we introduce subscripts, i and j, such that $S_{i,j}$ denotes the share price at time $t = i\Delta t$ with j upstates (where at a given i $j = 0, 1, \ldots, i$) and $V_{i,j}$ denotes the option value at the same state then we by simply extending the one period results we have:

$$\begin{aligned} \Delta_{i,j} &= \frac{V_{i+1,j+1} - V_{i+1,j}}{uS_{i,j} - dS_{i,j}} \\ B_{i,j} &= \frac{uV_{i+1,j} - dV_{i+1,j+1}}{(u-d)e^{r\Delta t}} \end{aligned}$$

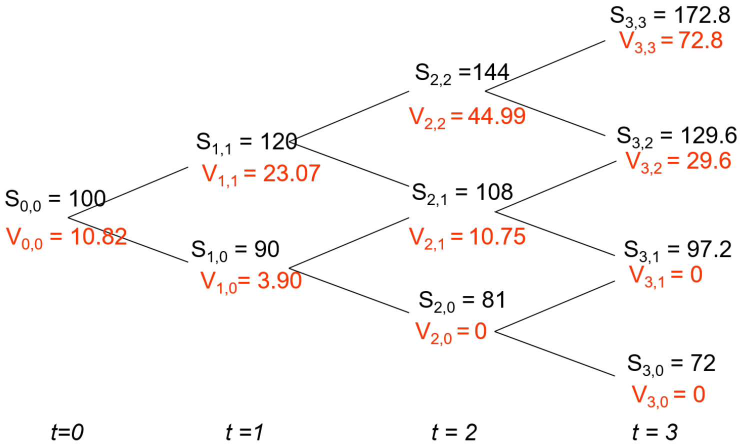
and to value the option

$$V_{i,j} = e^{-r\Delta t}(\pi V_{i+1,j+1} + (1-\pi)V_{i+1,j})$$

where

$$\pi = \frac{e^{r\Delta t} - d}{u - d}$$

Three-step tree



Option valuation

- Thus, we have $\pi = 0.3667 (1 - \pi = 0.6333)$ and so we can use this formula to calculate the option values, for example

$$\begin{aligned}
V_{2,1} &= \frac{(\pi V_{3,2} + (1-\pi) V_{3,1})}{1.01} \\
&= \frac{0.3667 \times 29.60 + 0.6333 \times 0}{1.01} = 10.746 \\
V_{1,1} &= \frac{(\pi V_{2,2} + (1-\pi) V_{2,1})}{1.01} \\
&= \frac{0.3667 \times 44.99 + 0.6333 \times 10.746}{1.01} = 23.071 \\
V_{0,0} &= \frac{(\pi V_{1,1} + (1-\pi) V_{1,0})}{1.01} \\
&= \frac{0.3667 \times 23.071 + 0.6333 \times 3.901}{1.01} = 10.822
\end{aligned}$$

Multiple steps: time

- This was a simplified example where each time step represented a year. It is of course possible to subdivide the time to maturity.
- For example if the option has a time to maturity T-years and you wish to use n time-steps then define $\Delta t = T/n$ and if r is the annual continuously compounded risk-free rate then we have

$$V_{i,j} = e^{-r\Delta t}(\pi V_{i+1,j+1} + (1-\pi)V_{i+1,j})$$

  where

$$\pi = \frac{e^{r\Delta t} - d}{u - d}.$$

- This is why it is often easier to use the continuously compounded rates for binomial tree examples.

Conclusions

- We have managed to value an option under the assumption that at each point in time the underlying asset can take one of two values uS or dS where we know the factors u and d.
- By creating a portfolio consisting of the underlying asset and a risk-free investment it is possible to replicate the option value and thus, by no-arbitrage arguments, value the option.
- This approach can also be interpreted as calculating the discounted expected option value assuming that the investor is risk-neutral.
- It is possible to extend this approach to consider more that one time period although we have not yet looked at early exercise of dividend payments.