TWO-DIMENSIONAL DOUBLE PENDULUM: STUDY OF THE CHAOTIC BEHAVIOR OF THE SYSTEM

PHY 312: NUMERICAL METHOD AND PROGRAMMING

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Guided by: Dr. Nirmal Ganguli

RAHUL BEHERA 20220

SAJAN DAHERIYA 20239

SNEHASIS DAS 20267

AIM OF THE PROJECT:

We solved the equation of motion of a double pendulum and displayed the motion of the system using animation, which explicitly shows the behaviour of the system at different instants of time, given different initial conditions, and hence study its chaotic behaviour and analyse the normal modes.

Also, we have used the numerical method in programming to solve the decoupled equation of motion of the double pendulum and to show how the numerical analysis is shown to solve the differential equations.

Why a Double Pendulum?

A double pendulum exhibits chaos but is simple enough to approximate and analyse. It is a nonlinear coupled oscillator with complicated motion, but it is relatively simple to interface with a computer. Double pendulums are familiar in nonlinear investigations and are studied in classical mechanics classes. So, we studied the double pendulum.

ASSUMPTIONS:

The assumptions taken into account to solve the problem are as follows:

- The bobs of the pendulum are considered point masses.
- The strings attached to the bobs are massless and inextensible.

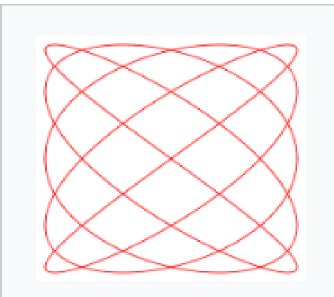
THEORY:

CHAOS SYSTEM:

- Chaos is the Irregular, predictable, apparently Random behaviour of Non-linear Dynamical Systems described mathematically by the Deterministic iterations of nonlinear difference equations or the evolution of non-linear ordinary or partial differential equations.
- In contrast to linear systems, which have a defined unique solution, a nonlinear system admits multiple solutions and thereby shows multiple scenarios of behaviour.
- These systems have regular behaviour under parametric consideration.
- Under other parametric situations, the system may exhibit random, aperiodic behaviour synonymous with a stochastic description. In simple terms, chaotic behaviour is an attribute of a nonlinear system. The mechanism of chaos is now well understood.
- We can say a famous quote by Edward Lorenz is when the present determines the future, but the approximate present does not approximately determine the future.

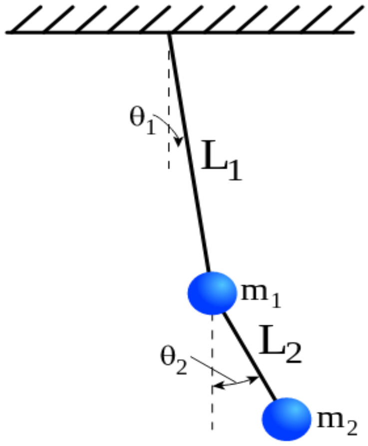
Lissajous Figures:

The Lissajous figure, also called BOWDITCH CURVE, is a pattern produced by the intersection of two sinusoidal curves, the axes at right angles.



DOUBLE PENDULUM:

- A double pendulum is a physical system that consists of two connected pendulums, where the motion of the second pendulum is affected by the motion of the first pendulum.



- The double pendulum is a classic example of a chaotic system, where a slight change in the initial conditions can lead to vastly different outcomes. This means that the motion of the double pendulum is predictable and exhibits complex patterns.

FIGURE OF THE DOUBLE PENDULUM

Deriving the equations of motion using Lagrange of the system:

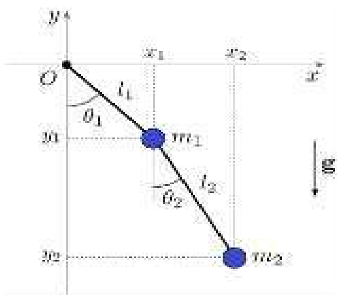


Fig 1. Schematic double pendulum setup

As we can see from the schematic diagram, we have two bobs of masses **m_1** and **m_2**, respectively, attached to strings of length **l_1** and **l_2**, respectively. The angles each pendulum makes with the vertical are **θ_1** and **θ_2.**

The expressions for kinetic energy and potential energy in terms of the polar coordinates and the co-ordinates are given below:

- $x_1 = l_1 \sin\theta_1$, $x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$
  $y_1 = -l_1 \cos\theta_1$, $y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$

- Kinetic energy = $\frac{1}{2}(m_1 + m_2) l_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_2 l_2{}^2 \dot{\theta}_2{}^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$
- Potential energy = $-(m_1 - m_2) g l_1 \cos\theta_1 - m_2 l_2 g \cos\theta_2$

where $\dot{\theta_1} = \frac{d\theta_1}{dt}$ and $\dot{\theta}_2 = \frac{d\theta_2}{dt}$ are the angular velocities of the two bobs respectively.

We can formulate the Lagrangian, L= K.E – P.E as follows:

$$L = \frac{1}{2}m_1 {l_1}^2 \dot{\theta_1}^2 + \frac{1}{2}m_2\left[{l_1}^2\dot{\theta_1}^2 + {l_2}^2\dot{\theta_2}^2 + 2l_1 l_2 \dot{\theta_1}\,\dot{\theta_2}\cos(\theta_1 - \theta_2)\right] + (m_1 + m_2)gl_1\cos\theta_1 + m_2 g l_2 \cos\theta_2$$

From the Euler Lagrange's Equations we get the following equations of motion:

- $\frac{d\theta_1}{dt} = \omega_1$
- $\frac{d\theta_2}{dt} = \omega_2$
- $\frac{d\omega_1}{dt} = \frac{-m_2 l_1 {\omega_1}^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) + m_2 g \sin\theta_2 \cos(\theta_1-\theta_2) - m_2 l_2 {\omega_2}^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1}{(m_1+m_2)l_1 - m_2 l_1 \cos(\theta_1-\theta_2)^2}$
- $\frac{d\omega_2}{dt} = \frac{-m_2 l_2 {\omega_2}^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) + (m_1+m_2)(g\sin\theta_1\cos(\theta_1-\theta_2) + l_1{\omega_1}^2\sin(\theta_1-\theta_2) - g\sin\theta_2)}{(m_1+m_2)l_2 - m_2 l_2 \cos(\theta_1-\theta_2)^2}$

Algorithm of RK4 Used To Solve Equations:

Here we are going to use well known Runge-Kutta method of fourth order. We chose RK4 because it provides greater and sufficient accuracy than Euler-Lagrange and RK2 method. The order of the Runge-Kutta method refers to the error at each time step being of order 5 O(t^5).

Let us have a differential equation,

$$\dot{y} = G(y, t)$$

To solve this differential equation using RK4 method, it is required to calculate the following k1,k2,k3,k4 for point (y_n, t_n)

$$k_1 = G(y_n, t_n)$$

$$k_2 = G\left(y_n + \frac{k_1 dt}{2}, t_n + \frac{dt}{2}\right)$$

$$k_3 = G\left(y_n + \frac{k_2 dt}{2}, t_n + \frac{dt}{2}\right)$$

$$k_4 = G(y_n + k_3 dt, t_n + dt)$$

Now we can write,

$$y_{n+1} = y_n + \frac{dt \cdot (k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

Where dt is our time step. If the range of t is from t_i to t_f and N is the number of steps to be taken then,

$$dt = \frac{t_f - t_i}{N}$$

This is an iterative process, and we get solution points by iterating over all time steps.

Modifying Double Pendulum Equations:

We need to modify double pendulum differential equations in order to apply RK4 method efficiently. We can see these equations are coupled and of second order.

$$(m_1 + m_2)l_1 \frac{d^2\theta_1}{dt^2} + m_2 l_2 \frac{d^2\theta_2}{dt^2}\cos(\theta_1 - \theta_2) + m_2 l_2 \left(\frac{d\theta_2}{dt}\right)^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)g\sin(\theta_1) = 0$$

$$l_2 \frac{d^2\theta_2}{dt^2} + l_1 \frac{d^2\theta_1}{dt^2}\cos(\theta_1 - \theta_2) - l_1 \left(\frac{d\theta_1}{dt}\right)^2 \sin(\theta_1 - \theta_2) + g\sin(\theta_2) = 0$$

We can write these equations as follows:

$$\begin{bmatrix} (m_1+m_2)\,l_1 & m_2 l_2 \cos(\theta_1-\theta_2) \\ l_1\cos(\theta_1-\theta_2) & l_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1 \\ l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) - g\sin\theta_2 \end{bmatrix}$$

we have represented as $[M]\ddot{\theta} = F$

$$\ddot{\theta} = M^{-1}F$$

Let $\vec{y} = \begin{bmatrix} \dot{\theta} \\ \theta \end{bmatrix} \Rightarrow \dot{Y} = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} M^{-1}F \\ \dot{\theta} \end{bmatrix} = \vec{G}(\vec{Y}, t)$

$$\Rightarrow \dot{Y} = \vec{G}(\vec{Y}, t)$$

Parameters Used:

$$m_1 = 1,\ m_2 = 1,\ l_1 = 1, l_2 = 1, g = 9.8$$

$$dt = 0.01\ and\ (t_i, t_f) = (0.0,\ 50.0)$$

Initial State:

$$\frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = 0$$

$$\theta_1 = 1.0\,rad,\ \theta_2 = 1.3\,rad$$

Python Code:

```
def G(y,t):
  a1d, a2d = y[0], y[1]
  a1, a2 = y[2], y[3]

  m11, m12 = (m1+m2)*l1, m2*l2*cos(a1-a2)
  m21, m22 = l1*cos(a1-a2), l2
  m = np.array([[m11, m12],[m21, m22]])

  f1 = -m2*l2*a2d*a2d*sin(a1-a2) - (m1+m2)*g*sin(a1)
  f2 = l1*a1d*a1d*sin(a1-a2) - g*sin(a2)
  f = np.array([f1, f2])

  accel = inv(m).dot(f)

  return np.array([accel[0], accel[1], a1d, a2d])
```

G(y,t) is the function which returns an array according to the modified equations of double pendulum.

```
def RK4_step(y, t, dt):
  k1 = G(y,t)
  k2 = G(y+0.5*k1*dt, t+0.5*dt)
  k3 = G(y+0.5*k2*dt, t+0.5*dt)
  k4 = G(y+k3*dt, t+dt)

  return dt * (k1 + 2*k2 + 2*k3 + k4) /6
```

RK4_step handles calculation of RK4 step to further calculate solution point.

```python
delta_t = 0.1
time = np.arange(0.0, 50.0, delta_t)
# initial state
y = np.array([0,0,1.0,1.3])   # [velocity, displacement]
Y1, Y2 = [], []
Y1d, Y2d = [], []

# time-stepping solution
for t in time:
  y = y + RK4_step(y, t, delta_t)

  Y1d.append(y[0])
  Y2d.append(y[1])
  Y1.append(y[2])
  Y2.append(y[3])
```

Code above for loop is just for declaring system parameters and initials, the for loop handles iterative process which iterates over time steps and calculates solution point using RK4_step and G function and stores them into 4 lists.

```python
def saveData(y1,y2,y1d,y2d,t):
    fields = ('t','a1','a2','a1dot','a2dot')
    file_name = 'rk4_data.csv'
    rows = np.transpose(np.array([t,y1,y2,y1d,y2d]))
    with open(file_name, 'w',newline='') as csvfile:
        csvwriter = csv.writer(csvfile)
        csvwriter.writerow(fields)
        # csvwriter.writerows(rows)
        for row in rows:
            csvwriter.writerow(row)
```

The function **saveData()** handles saving the data obtained in a csv file which can be used in analysis and simulation.

With the use of these functions and the Python module ''**pygame**'' we can-do real-time simulation of two double pendulums with different initial conditions and can compare how chaos prevails with even the slatiest change in initial conditions.

The code of the same is attached.

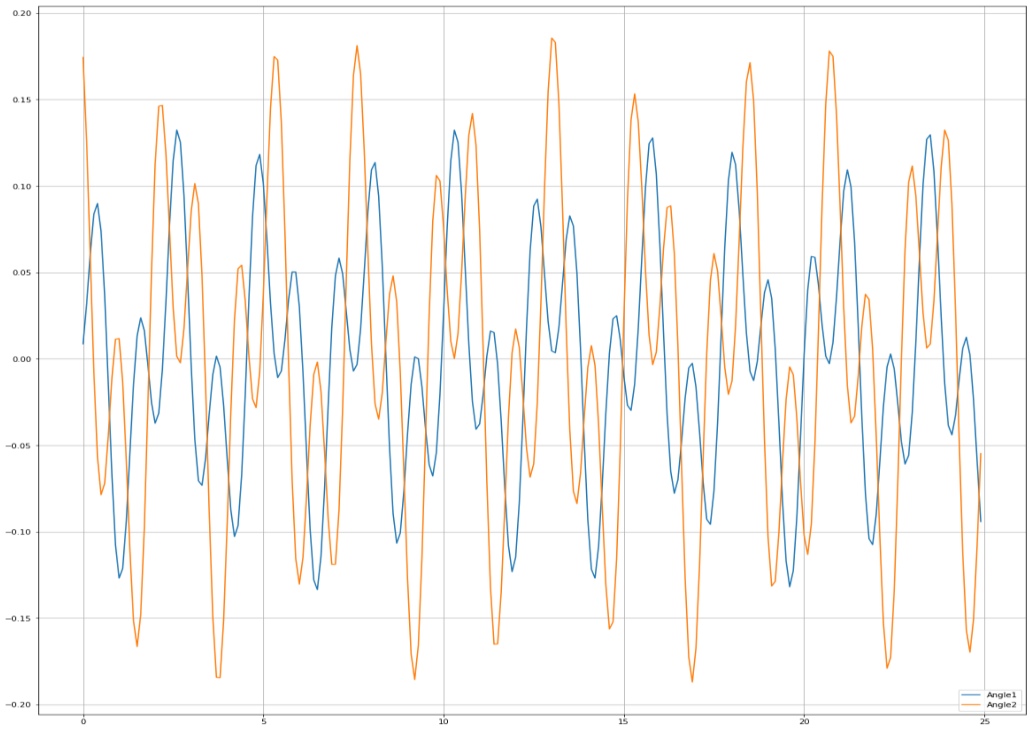
Discussion on Data and Analysis:

This is the data up to first 10 entries, we got from execution of the code for above given parameters and initials-

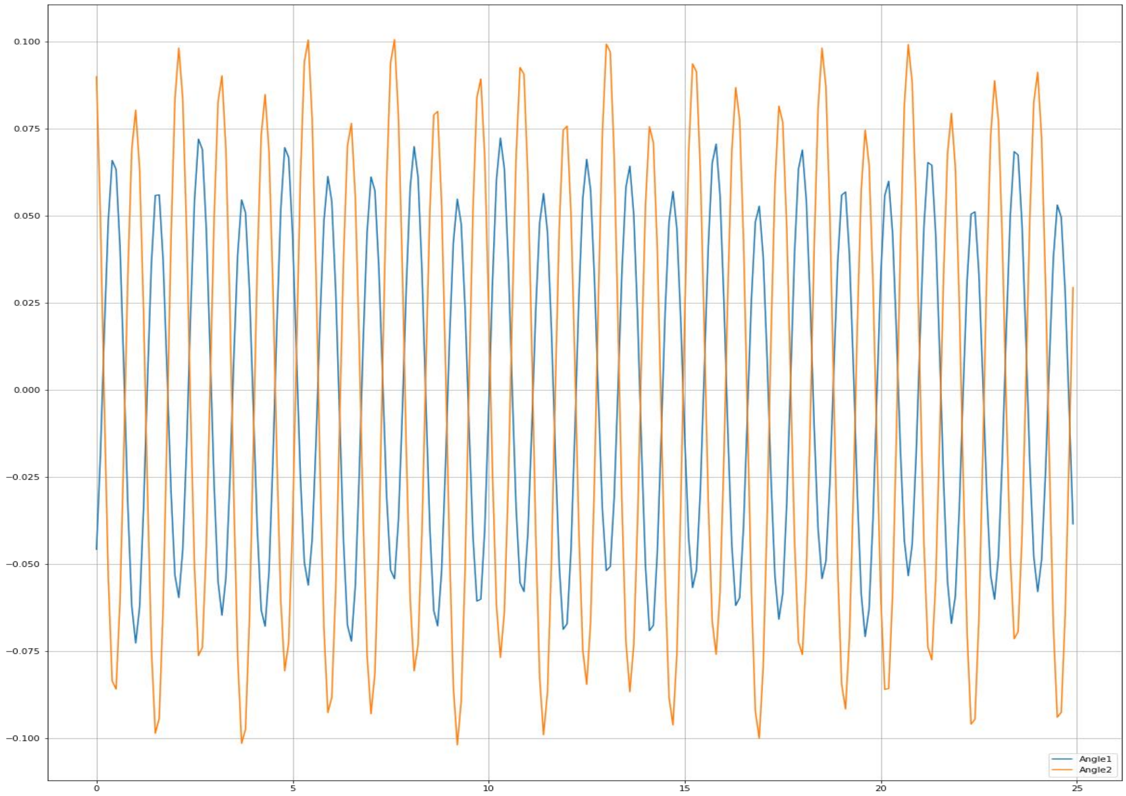
	t	a1	a2	a1dot	a2dot
0	0.0	0.966034	1.285115	-0.671447	-0.307511
1	0.1	0.868873	1.234600	-1.248292	-0.730619
2	0.2	0.722685	1.131911	-1.637785	-1.360862
3	0.3	0.550158	0.954788	-1.766840	-2.211410
4	0.4	0.378565	0.685775	-1.631597	-3.163477
5	0.5	0.226116	0.329613	-1.447241	-3.871374
6	0.6	0.075117	-0.060992	-1.647668	-3.795229
7	0.7	-0.114523	-0.403926	-2.151121	-2.991864
8	0.8	-0.349643	-0.652801	-2.499120	-1.993094
9	0.9	-0.601369	-0.809706	-2.464835	-1.204480

Here are 6 initial conditions on which we executed the double pendulum code and plotted angles Vs time-

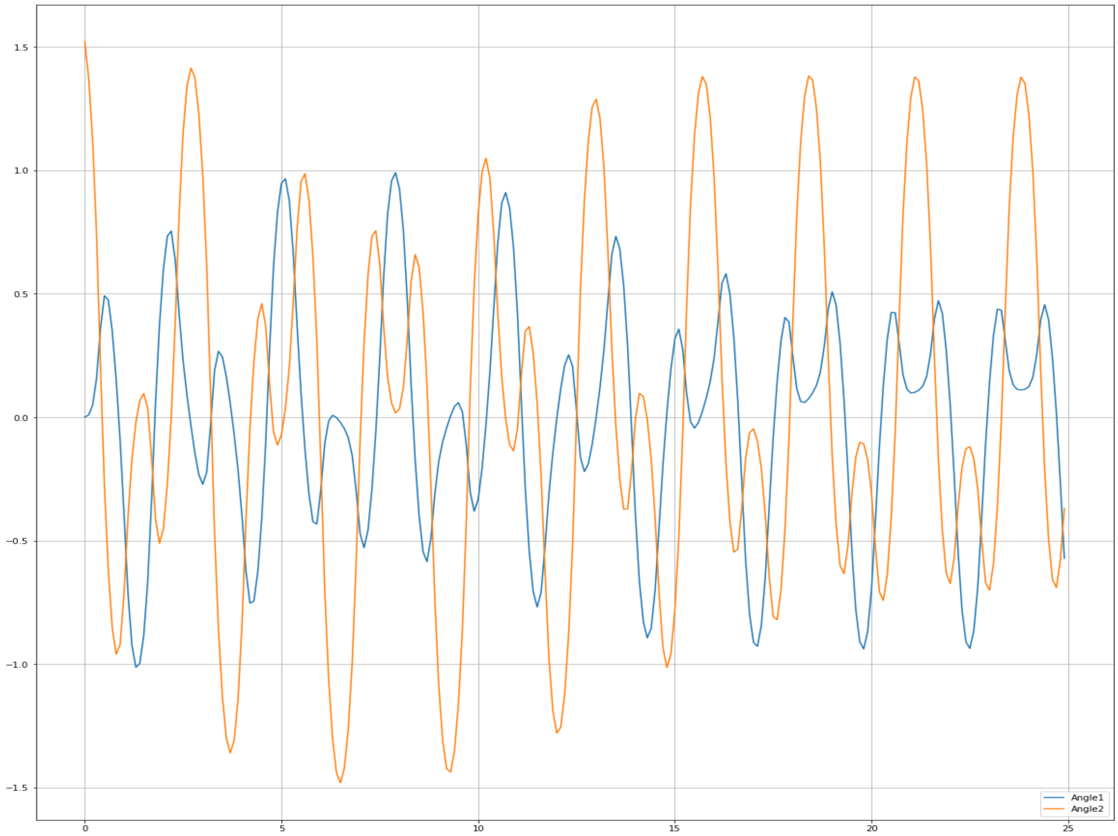
(1) $m1 = m2 = 1kg, l1 = l2 = 1m,\ \theta_1 = 0^o,\ \theta_2 = 11^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$



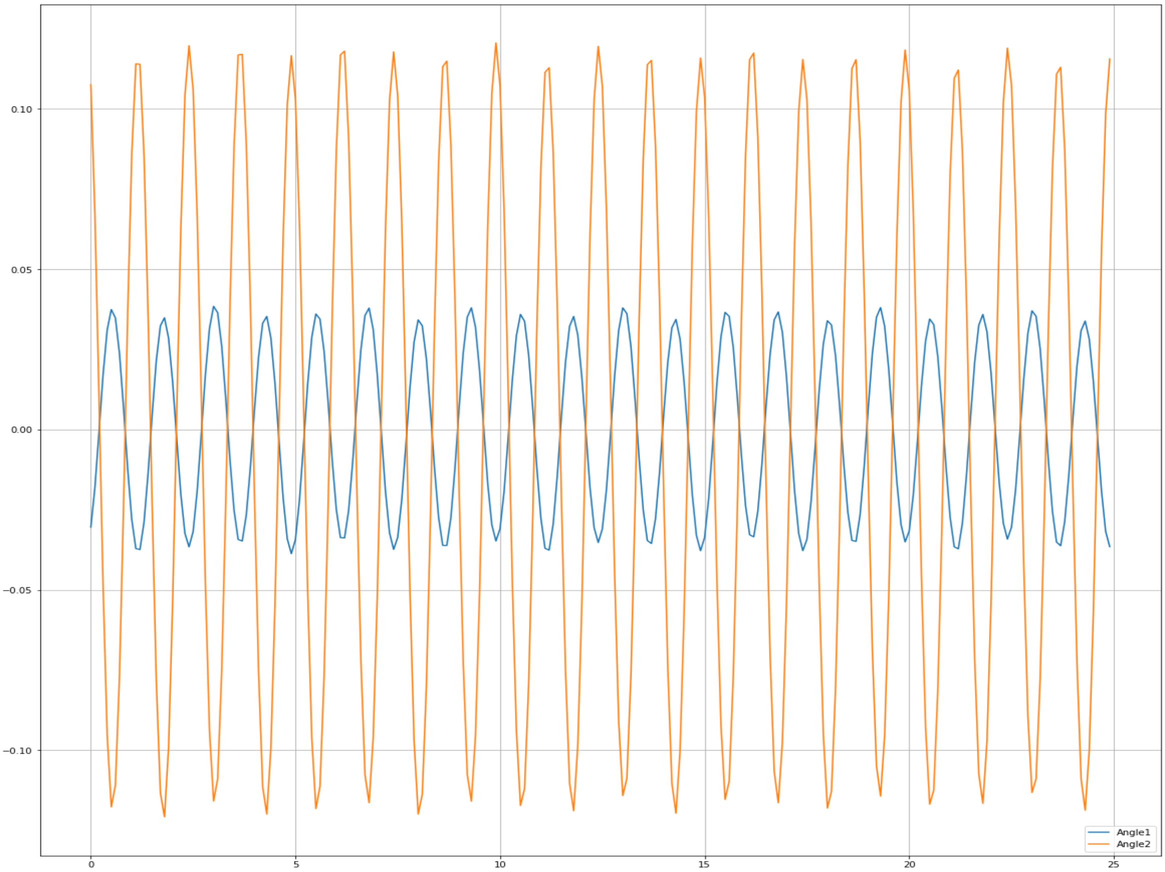
(2) $m1 = m2 = 1kg, l1 = l2 = 1m,\ \theta_1 = -3.2^o,\ \theta_2 = 6^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$



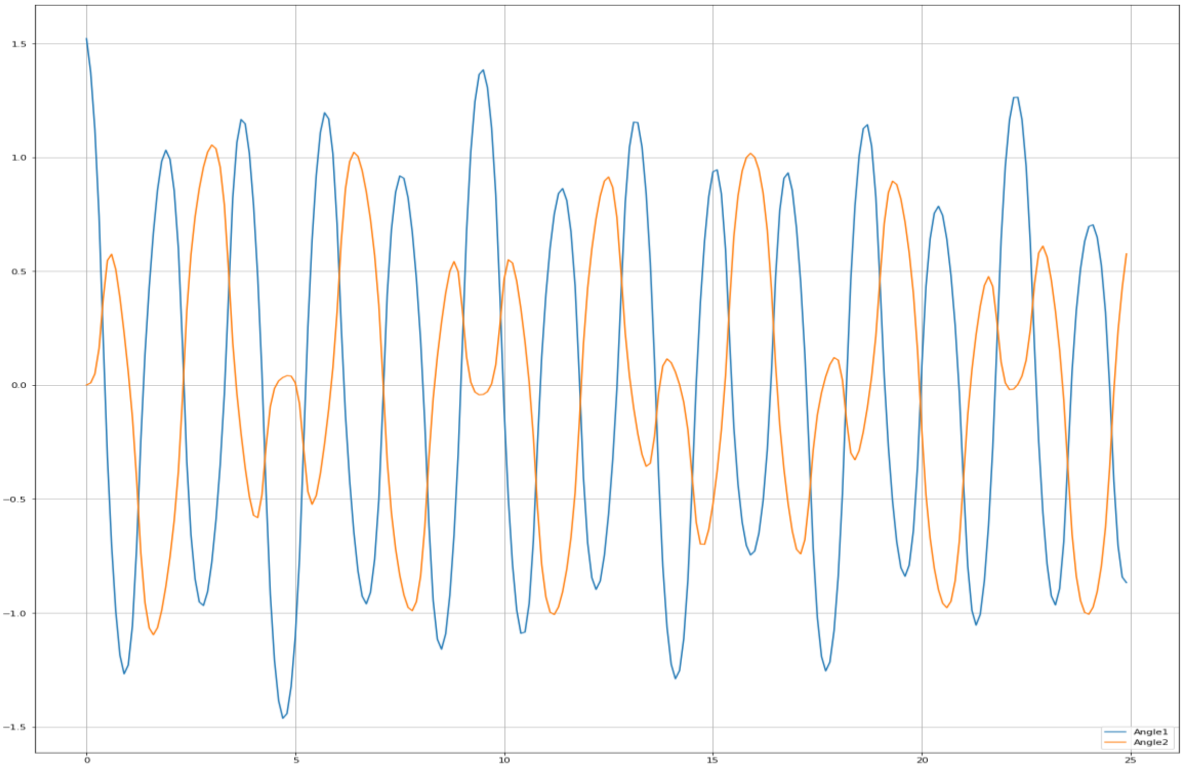
(3) $m1 = m2 = 1kg, l1 = l2 = 1m,\ \theta_1 = 0^o,\ \theta_2 = 90^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$



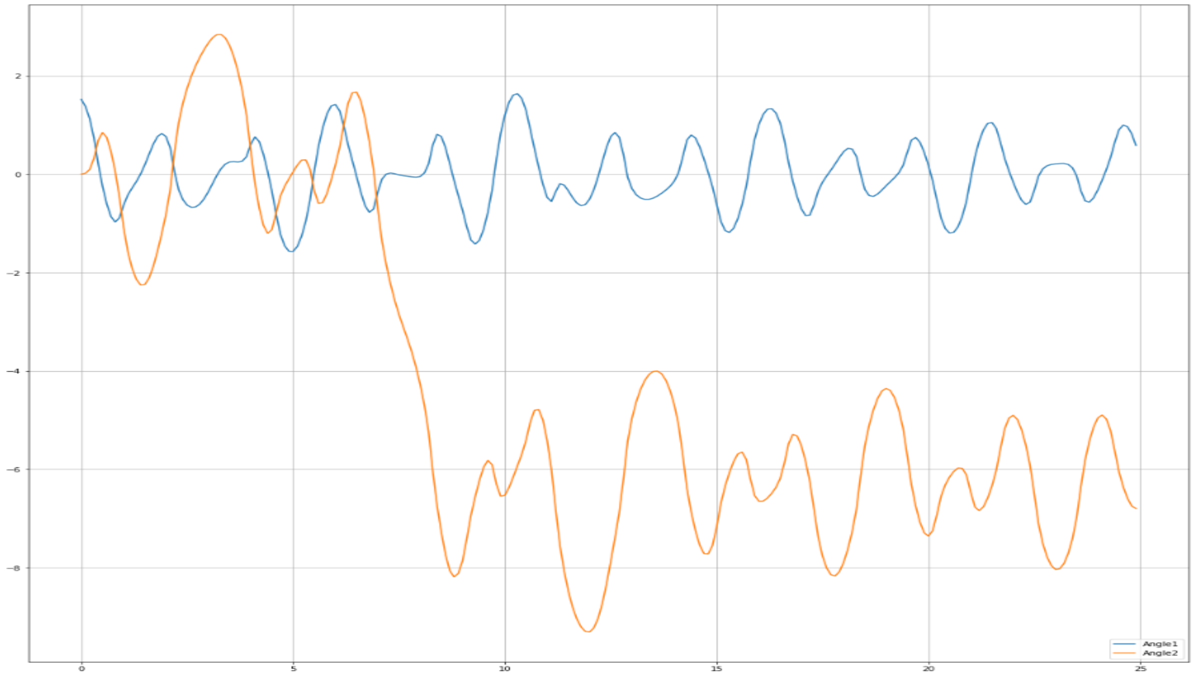
(4) $m1 = m2 = 1kg, l1 = 2m,\ l2 = 1m,\ \theta_1 = -2^o,\ \theta_2 = 7^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$



(5) $m1 = m2 = 1kg, l1 = 1m,\ l2 = 2m,\ \theta_1 = 90^o,\ \theta_2 = 0^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$



(6) $m1 = m2 = 1kg, l1 = 1m,\ l2 = 1m,\ \theta_1\ =\ 90^o,\ \theta_2\ =\ 0^o,\ \omega_1\ =\ \omega_2\ =\ 0\ \frac{deg}{sec}$

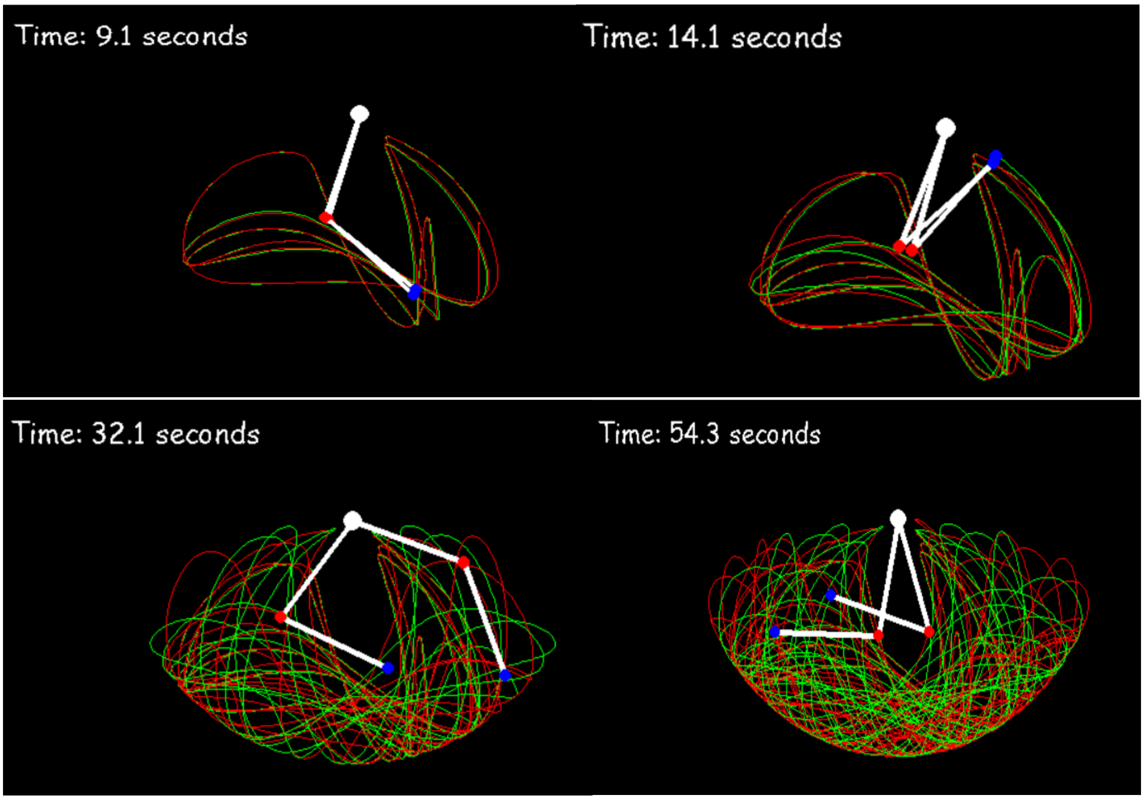


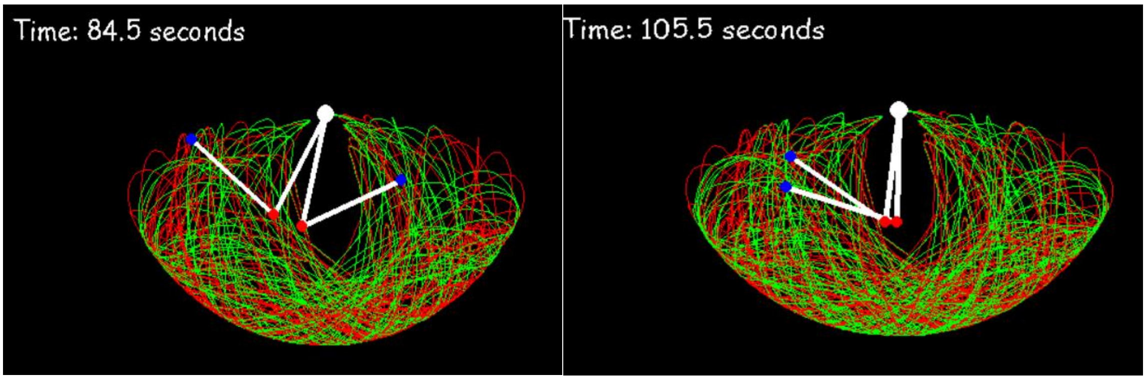
Chaotic Nature:

We can see that for small angles the plot of θ_1 *and* θ_2 *Vs t* shows periodic behaviour

While when angles are large enough it produces chaotic system.

Here we put real time simulation results for the last case (6).





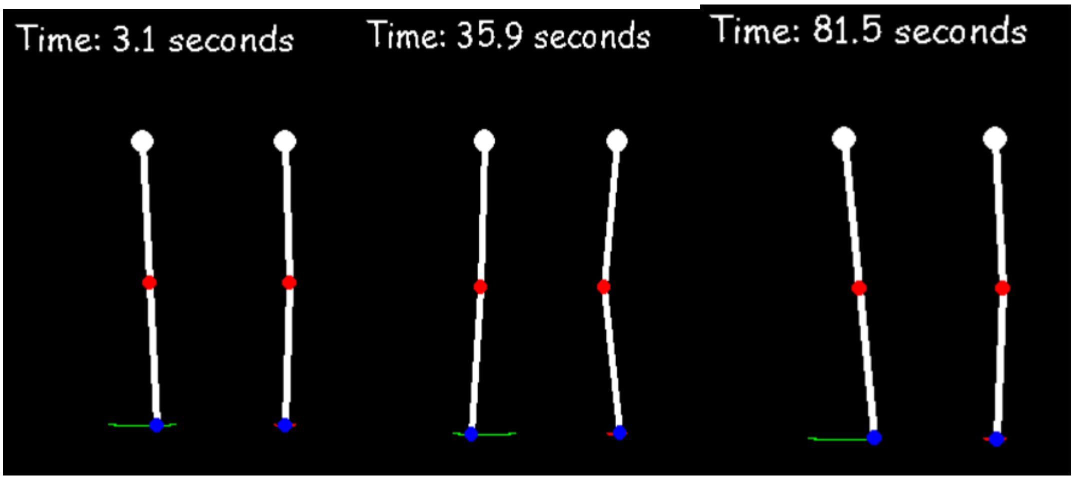
Green trajectory is for actual case (6) while **Red** trajectory is for slight variation in angle θ_1 by 0.1 degrees.

(7) $m1 = m2 = 1kg, l1 = l2 = 1m,\ \theta_1 = 6.37^o,\ \theta_2 = 6.37^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$

&

(8) $m1 = m2 = 1kg, l1 = l2 = 1m,\ \theta_1 = 6.37^o,\ \theta_2 = -6.37^o,\ \omega_1 = \omega_2 = 0\ \frac{deg}{sec}$

The out-phase mode-



First pendulum is for 7th case and second is for 8th case. For small angles this is very periodic.

Insights From Realtime Simulation:

We can clearly see that when initial condition is slightly changed the pendulum trajectory significantly deviates from each other. This is what chaos is in Double Pendulum System. For small angles, it takes more time to get to the visible changes in trajectories while for large angles the system rapidly becomes chaotic.

CONCLUSION AND RESULT:

- We derived the equation of motion of the motion of the double pendulum by Euler-Lagrange equation and decoupled them to write 4 set of linear ordinary differential equation.
- We used RK-4 method to solve for $\theta_1,\ \theta_2,\ \dot{\theta}_1,\ \dot{\theta}_2$ numerically.
- We put the above values in the expression for energy and plotted it as function of time to validate our solution.
- We have formulated our code in a manner to make it flexible to a user who inputs their choice of bob masses, string lengths, initial angles, initial angular velocities, the time step and the total time for the simulation to run
- We have plotted evolution of angles with time and an animation showing the trajectory of two bobs to realize the nature of the system's behaviour is sensitive to initial conditions. Thus, we studied both, the periodic and chaotic nature of the double pendula system.

FURTHER STUDY:

The study of double pendulum and its application will be of immense benefit to the physics and mathematics departments.

The entire population under the umbrella of the double pendulum, the factors affecting the performance of the double pendulum, the demonstration of the normal modes in the double pendulum and also the demonstration on the appearance of the chaos in the double pendulum will serve as a repository of information to other researchers that desire to carry similar research on the above topic.

One could do a full energy analysis to determine the quality factor of the coupled oscillators in the Double Pendulum. Determine a modification to the Equations of motion in presence of friction (is it constant, linear, quadratic...?).

We can study the non-linear dynamics abide by this as a basis.

Finally, the study will contribute to the body of existing literature and knowledge in the field of study and provide a basis for further research.

ACKNOWLEDGEMENT:

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